CS109 - lecture 3

Start working on Problem Sets

Review: We did counting operations on n objects

- **Sort, order matters (permutations)**
  - Distinct: $n!$
  - Some indistinct: $\frac{n!}{n_1!\, n_2! \cdots}$
- **Choose k (combinations)**
  - Distinct: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
- **Put in r Buckets**
  - Distinct: r^n
  - None Distinct: $\frac{(n+r-1)!}{n!\,(r-1)!}$

Review question

n animals, how many calculations for every pair

$1, 2, \ldots n$

calc. calc.

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!}$$

Remember: combination does not care about ordering

And $\binom{n}{k}$ also always gives distinct items

(k)

so no pair of the same item

(example: ~~(dog, dog)~~)

Another way to think about it:

	A	B	C
A	X	O	O
B	X	X	O
C	X	X	X

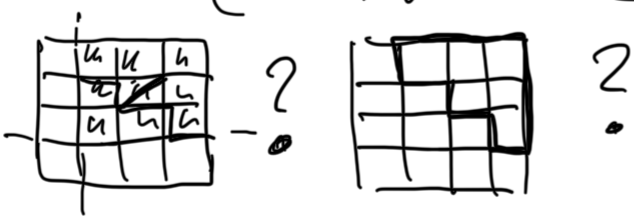
n

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!}$$

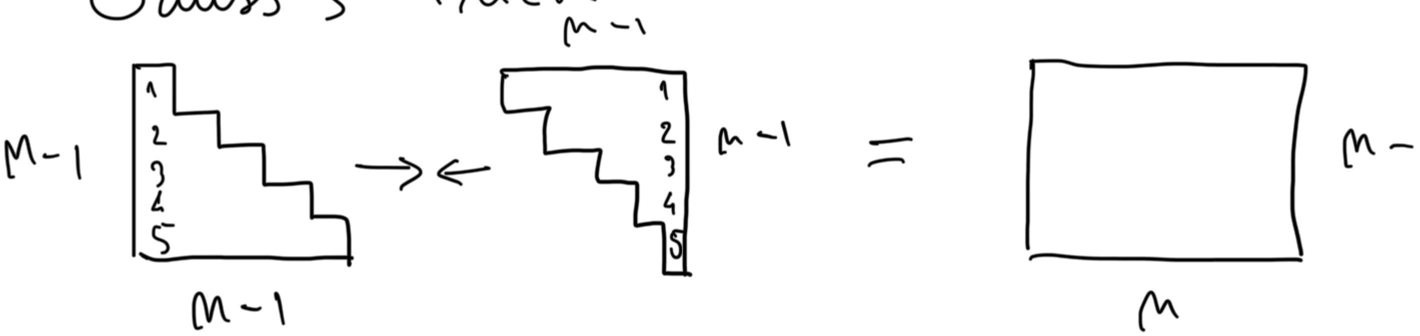
$$= \frac{n\cdot(n-1)\cdot\cancel{(n-2)}\cdot\cancel{\cdots}}{(2\cdot 1)\,\cancel{(n-2)}\cdot\cancel{(n-3)}}$$

$$= \frac{n\cdot(n-1)}{2}$$

$(n-1)(n-1)/2$



Gauss's trick



So each is $\frac{n\cdot(n-1)}{2}$

So $\sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$

Summary of Patterns:

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Formula	Value	Visual Trick
$\sum k$	$\frac{n(n+1)}{2}$	Triangle + triangle = rectangle
$\sum(2k-1)$	n^2	Build squares from concentric layers
$\sum 2k$	$n(n+1)$	Factor 2 out of natural sum
$\sum k^2$	$\frac{n(n+1)(2n+1)}{6}$	Stack squares = pyramid
$\sum k^3$	$\left(\frac{n(n+1)}{2}\right)^2$	Sum of k, then square it!

Sample Space, S, is set of all possible outcomes of an experiment

- coin flip $S = \{\text{heads}, \text{tails}\}$ Size of $S = 2$
- flip 2 coins $S = \{(H,T), (H,H), (T,T), (T,H)\}$
- roll of die $S = \{1, 2, 3, 4, 5, 6\}$

$$S = \{a \mid a \in \mathbb{Z} \text{ or } \mathbb{R}\}$$

Event Space, E, is some subset of S $\{E \subseteq S\}$

- coin flip $E = \{\text{head}\}$
- roll die is 3 or less $E = \{1, 2, 3\}$

What is probability? Number between 0 and 1

$Pr(E)$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

where n is the number of trials

$\boxed{\text{Axioms of Probability}}$ Kolmogorov

E = the event

S = all outcomes

1: $0 \leq P(E) \leq 1$

2: $P(S) = 1$

3: If events E and F are mutually exclusive

$$P(E \cup F) = P(E) + P(F)$$

↳ Identity: $\underbrace{P(E^c)}_{\text{Probability of } E \text{ not happening}} = 1 - P(E)$

E complement

Equally likely Outcomes

↳ Some sample spaces have equally likely outcomes

Then $P\{\text{each outcome}\} = \frac{1}{|S|}$ Example coin $P(\text{out}) = \frac{1}{2}$

Therefore $P(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{|E|}{|S|}$

Example:

P(Sum of 2 dice = 7) ?

S = [(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)*
(2,1) (2,5)*
(3,1) (3,4)*
(4,1) (4,3)*
(5,1) (5,2)*
(6,1)* (,) (,) (,) (,) (6,6)]

All should be equally likely

Size of |S| = 6x6 = 36
Size of |E| = 6

$$P(E) = \frac{6}{36} = \frac{1}{6} = 0{,}1666$$

Sum of two die : three options for the sample space

Think of the die as Distinct	Think of the die as Indistinct
[5 , 5]	{5 , 5}
↑ Value of dice 1 ↑ Value of dice 2	↑ value of any dice ↑ value of any dice
Done ✓	

S = { (1,1) (1,2) (1,3) (1,4) (1,5) (1,6) 7

~~(2,1)~~ (2,2) (2,3) (2,4) (2,5) (2,6)

7 7

same as (1,2) → ~~(3,1)~~ ~~(3,2)~~ ← same as (2,3)

(3,3) (3,4)

(4,4)

(5,5)

(6,6)

P(Sum = 7) ?

$$P(E) = \frac{|E|}{|S|} = \frac{3}{\underbrace{6}_{6}+\underbrace{5+4}_{9}+\underbrace{3+2+1}_{6}} = \frac{3}{21}$$

Bug! however the sample space is not equally likely because (1,1) is half as likely as (1,2) because of (1,2) and (2,1)

The choice of Sample Space is yours!

S = 4 cows, 3 pigs

$\boxed{\text{choose 3}}$	Distinct	Indistinct
Unordered	(a) $\{C_1\ P_3\ P_2\}$ $\{C_1, C_2, C_3\}$	(b) {2 cows, 1 pig} {3 cows}
Ordered	(c) $[C_1\ P_3\ P_2]$	(d) [cow, pig, pig]

Ordered | $[c_1, c_2, c_3]$ | [cow, pig, pig]
| $[c_2, c_1, c_3]$ | [cow, cow, cow]

Think: which choice will lead to equally likely outcomes?

b) {2 cows, 1 pig} are not equally likely than {3 cows}

d) [cow, pig, pig] is also not equally likely as [cow, cow, cow]

Because we have more cows than pigs

So we must use a and c (Distinct items)

a $P(1\text{cow}, 2\text{pigs}) = \dfrac{\text{Event space} \quad ?}{\text{Sample Space} \quad \binom{7}{3}}$

? $P(1\text{cow})$ Just take 1 cow out of 4 cows

$P(2\text{pigs})$ just take 2 pigs out of 3 pigs

$? = \binom{4}{1}\binom{3}{2}$

So $P(1 \text{ pig and } 2 \text{ cows}) = \dfrac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}}$

$$= \frac{12}{35}$$

(c) Ordered Sample Space
and distinct

Pick 3 ordered items : $|S| = 7 \cdot 6 \cdot 5$
$= 210$

$|E|$ = pick a cow as a 1st, 2nd or 3rd

$$= (\underbrace{4}_{\text{choices as cow}} \times \underbrace{3}_{\text{choices as pig}} \times \underbrace{2}_{\text{choices pig}}) + (3 \times \underbrace{4}_{\text{cow}} \times 2) + (3 \times 2 \times \underbrace{4}_{\text{cow}})$$

OR rule (mutually exclusive) yes!

$$= 24 + 24 + 24 = 72$$

Cow order is important!

$P(\text{ordered}, 1 \text{ cow}, 2 \text{ pigs}) = \dfrac{72}{210} = 12/35$

So we were able to calculate it via (a) or (c) - Distinct

Unordered   Ordered

Pro tip: Make indistinct items "distinct" to get equally likely sample space outcomes — Trick to be used!

P(straight) in a 52 card deck

$$P = \frac{|E| \; ?}{\binom{52}{5}}$$

E = 2, 3, 4, 5, 6

3, 4, 5, 6, 7

4, 5, 6, 7, 8

of any suit: 4 suits

2 can be $\begin{cases} \text{clubs} \\ \text{diamonds} \\ \text{hearts} \\ \text{Spades} \end{cases}$

$\boxed{2}$ → 4 suits

→ forces the other cards to be 3, 4, 5, 6

however they can be of any suit!

5

13 ranks

(2) $4 \times 4 \times 4 \times 4 \times 4 = 4$

(3)

(4)

(10) (J) (Q) (K) (A) → cannot have more straights!

(13)

$$|E| = 10 \times 4^5$$

$$P_{\text{straight}} = \frac{10 \times 4^5}{\binom{52}{5}} = \frac{10 \times \binom{4}{1}^5}{\binom{52}{5}}$$

<u>Protip</u> Write the Event Spaces and Sample Spaces

$n = 7$ chips and 1 is failed

k sample $= 3$ tested chips

P of 1 failed chip in k → Question: how can we arrange k so we can get a defective chip?

$$P = \frac{1\,\text{defective} \cdot \binom{7-1}{3-1}^{\text{*the rest}}}{\binom{7}{3}} \quad \text{generally} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}}$$

we fix

the event space so we select the defective

$$P(\text{defective chip is in } k \text{ selected chips})$$

$$= \frac{\binom{1}{1}\binom{m-1}{k-1}}{\binom{m}{k}} = \frac{\dfrac{(m-1)!}{(k-1)!\,\cancel{(m-k)!}}}{\dfrac{m!}{k!\,\cancel{(m-k)!}}} = \frac{k!\,(m-1)!}{m!\,(k-1)!}$$

$$= \frac{k\cdot\cancel{(k-1)!}\,\cancel{(m-1)!}}{m\cdot\cancel{(m-1)!}\cdot\cancel{(k-1)!}}$$

Wow!

$$= \frac{k}{m}$$