# G-K Equation for Nanowires {Microscale considerations}

$$\vec{q} + \lambda \nabla T = m^2 l^2 \nabla^2 \vec{q}$$

$$\nabla T = -\frac{\Delta T}{L} \qquad \vec{q} = q(r) \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Cylindrical Laplacian

$$m^2 l^2 \frac{d^2 q}{dr^2} + \frac{m^2 l^2}{r}\frac{dq}{dr} - q = -\frac{\lambda \Delta T}{L}$$

$$\frac{d^2 q}{dr^2} + \frac{1}{r}\frac{dq}{dr} - \frac{q}{m^2 l^2} = \underbrace{\frac{-\lambda \Delta T}{m^2 l^2 L}}_{\to c}$$

Variation of Parameters

$$t^2 + \frac{t}{r} - \frac{1}{m^2 l^2} = 0$$

$$t_1, t_2 = \frac{-1/r \pm \sqrt{1/r^2 + 4/m^2 l^2}}{2}$$

$$\therefore q_{gen} = Ae^{t_1 r} + Be^{t_2 r}$$

Since RHS is a constant

let $q_p = k$

$$\frac{-k}{m^2 l^2} = -\frac{\lambda \Delta T}{m^2 l^2 L}$$

$$\Rightarrow q_p(r) = \frac{\lambda \Delta T}{L}$$

$$\therefore q(r) = A e^{t_1 r} + B e^{t_2 r} + \frac{\lambda \Delta T}{L}$$

BC are:

$$q(R) = 0 \quad \text{and} \quad \left.\frac{dq}{dr}\right|_{r=0} = 0$$

$$q(r) = A e^{-1 + \sqrt{1 + \frac{4r^2}{ml^2}}} + B e^{-1 - \sqrt{1 + \frac{4r^2}{ml^2}}} + \frac{\lambda \Delta T}{L}$$

$$q(r) = \frac{1}{e} \left\{ A e^{\sqrt{\quad}} + B e^{-\sqrt{\quad}} + \frac{\lambda \Delta T}{L} \right\}$$

Solve with Bessel functions.

$\longrightarrow$

Using Bessel Functions:

$$\frac{d^2 q}{dr^2} + \frac{1}{r}\frac{dq}{dr} - \frac{q}{m^2 l^2} = \frac{-\lambda \Delta T}{m^2 l^2 L}$$

$\hookrightarrow c$

General form:

$$r^2 \frac{d^2 q}{dr^2} + r\frac{dq}{dr} + \frac{r^2 q^2}{(-m^2 l^2)} = 0$$

$\Rightarrow$ General Solution

$$q_{gen}(r) = A J_0(r) + B Y_0(r)$$

Scale the differential $x \rightarrow \frac{r}{iml}$

$$\therefore q_h(r) = A J_0\left(\frac{r}{iml}\right) + B Y_0\left(\frac{r}{iml}\right)$$

$$q_p(r) = \frac{\lambda \Delta T}{L}$$

$$\therefore q_b(r) = A J_0\left(\frac{r}{iml}\right) + B Y_0\left(\frac{r}{iml}\right) + \frac{\lambda \Delta T}{L}$$

BCs

$$\left.\frac{\partial q_b}{\partial r}\right|_{r=0} = 0 \qquad q_b(R) = 0$$

$$\frac{\partial q_b}{\partial r} = \frac{-A}{im\ell} J_1\left(\frac{r}{im\ell}\right) - \frac{B}{im\ell} Y_1\left(\frac{r}{im\ell}\right)$$

at $r = 0$ $\quad Y_1(0) \longrightarrow -\infty$

$\Rightarrow B = 0$

$J_1(0) = 0$

$$q_b(R) = A J_0\left(\frac{R}{im\ell}\right) + \frac{\lambda \Delta T}{L} = 0$$

$$A = -\frac{\lambda \Delta T}{L} \times \frac{1}{J_0\left(R/im\ell\right)}$$

$$\therefore q_b(r) = -\frac{\lambda \Delta T}{L} \times \frac{1}{J_0\left(R/im\ell\right)} \times J_0\left(r/im\ell\right) + \frac{\lambda \Delta T}{L}$$

$$q_b(r) = \frac{\lambda \Delta T}{L} \left\{ 1 - \frac{J_0\left(r/im\ell\right)}{J_0\left(R/im\ell\right)} \right\}$$

$\because J_0(x)$ is an even fn

$$J_0(-x) = J_0(x)$$

$$q_b(r) = \frac{\lambda \Delta T}{L}\left\{1 - \frac{J_0(ir/m\ell)}{J_0(iR/m\ell)}\right\}$$

$$q_w = C\ell \left.\frac{\partial q_b}{\partial r}\right|_{r=R}$$

$\uparrow$ property of wall

[to accomodate for slip condition at wall]

$$q_w = \frac{\lambda \Delta T}{L}\left\{\frac{J_1(ir/m\ell)}{J_0(iR/m\ell)} \times i/m\cancel{\ell}\right\} \times C\cancel{\ell}\,\Big|_{r=R} \times (-1)$$

$$= \frac{-i\lambda \Delta T C}{mL}\left\{\frac{J_1(iR/m\ell)}{J_0(iR/m\ell)}\right\}$$

Additional factor of $-1$ since $q_w$ is actually derivative of wall normal $\frac{\partial q_b}{\partial \xi}$.

$r$ and $\xi$ have opposing directions.

$$q(r) = q_b(r) + q_w$$

$$q(r) = \frac{\lambda \Delta T}{L}\left\{1 - \frac{J_0(ir/m\ell)}{J_0(iR/m\ell)} - \frac{iC}{m}\frac{J_1(iR/m\ell)}{J_0(iR/m\ell)}\right\}$$

$$q(r) = \frac{\lambda \Delta T}{L}\left[1 - \frac{J_0\left(ir/m\ell\right)}{J_0\left(iR/m\ell\right)} - \frac{iC}{m}\,\frac{J_1\left(iR/m\ell\right)}{J_0\left(iR/m\ell\right)}\right]$$

Derived under the assumptions;

(1) $\nabla \cdot \vec{q} = 0$

From balance equation of phonon energy

$$\frac{\partial E}{\partial t} + \nabla \cdot \vec{q} = 0$$

Invoking steady state condition.

(ii) $\dfrac{\partial \vec{q}}{\partial t} = 0$ Steady state condition.

(iii) $\nabla T = \nabla_z T = \dfrac{-\lambda \Delta T}{L}$

Temperature gradient only along the length of nanowire.

# Calculating Effective Thermal Conductivity

$$\lambda_{eff} A |\nabla_z T| = \int_A q(r)\, dA$$

$$= \int_0^R 2\pi r\, q(r)\, dr$$

$$\frac{\lambda_{eff} \cancel{\pi} R^2 |\cancel{\nabla_z T}|}{2\cancel{\pi}} = \int_0^R \cancel{\frac{\lambda \Delta T}{L}} \left[ r - \frac{r J_0 (ir/ml)}{J_0 (\ )} - \frac{iCr}{m} \frac{J_1 (iR/ml)}{J_0 (\ )} \right] dr$$

$$\lambda_{eff} \frac{R^2}{2} = \lambda \int_0^R \left[ r - r \frac{J_0 (ir/ml)}{J_0 (iR/ml)} - \frac{iCr}{m} \frac{J_1 (iR/ml)}{J_0 (iR/ml)} \right] dr$$

$$\lambda_{eff} \frac{\cancel{R^2}}{\cancel{2}} = \lambda \left[ \frac{\cancel{R^2}}{\cancel{2}} - \frac{2\cancel{R} ml}{Ri} \frac{J_1 (iR/ml)}{J_0 (iR/ml)} - \frac{i\cancel{R^2} c}{\cancel{2} m} \frac{J_1 (iR/ml)}{J_0 (iR/ml} \right]$$

Let $Kn = \frac{l}{R}$

$$\lambda_{eff} = \lambda \left[ 1 - 2Kn\, m \frac{J_1\left(i/_{m}Kn\right)}{i\, J_0\left(i/_{m}Kn\right)} + \frac{c}{m} \frac{J_1\left(i/_{m}Kn\right)}{i\, J_0\left(i/_{m}Kn\right)} \right]$$