Turbulent Energy Production

Boltzmann Production Term Estimation for Turbulent Energy Production

The Boltzmann Equation

To directly study the evolution of moments of the Boltzmann distribution, i.e. the macroscopic quantities of interest, the Boltzmann equation can be modified to yield a general equation of transfer of the form Eq(1)

$$\frac{\partial \rho \langle \xi \rangle}{\partial t} + \frac{\partial \rho \langle \xi c_k \rangle}{\partial x_k} = \rho \left\langle \frac{\partial \xi}{\partial t} \right\rangle + \rho \left\langle c_k \frac{\partial \xi}{\partial x_k} \right\rangle + F_k \rho \left\langle \frac{\partial \xi}{\partial c_k} \right\rangle + P_\xi \tag{1}$$

The production term in the above equation, P_ξ, represents the variation of molecular property ξ due to collisions and is defined in terms of the collision integral $J(f, f_1)$ as Eq(2)

$$P_\xi = m \int \xi J(f, f_1) d\mathbf{c} \tag{2}$$

As opposed to using approximations to calculate the collision integral and subsequently evaluate the production term, we can instead evaluate the remaining terms of Eq(1) and express P_ξ in terms of them.

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Applying the Reynolds Decomposition on the standard Navier Stokes equation and using time averages to convert all velocity terms to the mean flow velocity yields the Reynolds-Averaged Navier-Stokes Eq(3)

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} (T_{ij}) \tag{3}$$

$$T_{ij} = -P\delta_{ij} + 2\mu S_{ij} - \overline{\rho C_i C_j} \tag{4}$$

The last term on the RHS is the Reynolds stress. It is a symmetric tensor that represents the contribution of turbulent motion to the mean flow stress.
The mean momentum of the turbulent fluctuations is zero, and so to better observe the effects of mean flow on turbulence we instead analyze the respective kinetic energy equations Eq(5) and Eq(6)

$$u_j \frac{\partial}{\partial x_j} (\frac{1}{2} u_i u_i) = \frac{1}{\rho} \frac{\partial}{\partial x_j} (-P\delta_{ij} + 2\mu S_{ij} - \overline{\rho C_i C_j}) + 2\nu S_{ij} S_{ij} + \overline{C_i C_j} S_{ij} \tag{5}$$

$$u_j \frac{\partial}{\partial x_j} (\frac{1}{2} \overline{C_i C_i}) = -\frac{1}{\rho} \frac{\partial}{\partial x_j} (\overline{C_j p} + \frac{\rho}{2} \overline{C_i C_i C_j} - 2\mu \overline{C_i s_{ij}}) - 2\nu s_{ij} s_{ij} - \overline{C_i C_j} S_{ij} \tag{6}$$

The **turbulence production term** $-\overline{C_i C_j} S_{ij}$ occurs in both equations with opposite signs, indicative of the energy exchange between the mean flow and the turbulence. The last term in the turbulence energy budget Eq(6) represents the viscous dissipation, which is always a drain in energy. Unlike in the mean flow energy budget Eq(5), the viscous dissipation term is not negligible. On the whole, the two equations describe a dynamical mechanism that transfers energy from large scales to small scales and subsequently dissipates it.
The turbulence production can be estimated in terms of characteristic velocity (u) and characteristic length (ℓ) of turbulence as Eq(7)[1]

$$\overline{C_i C_j} S_{ij} = K u \ell S_{ij} S_{ij} \tag{7}$$

[1] *A First Course in Turbulence*. Section 3.1, Chapter 3.

Substitution of $\mathscr{P}$ in the Boltzmann Transfer Equation

We substitute the turbulence energy production as estimated in Eq(7) into the Boltzmann transfer equation Eq(1).

$$\mathscr{P} = Ku\ell S_{ij}S_{ij} \tag{8}$$

$S_{ij}S_{ij}$ in tensor notation represents the sum of the square of each element in the mean strain rate matrix $\mathbf{S}$. This value is therefore a function of the mean flow velocity gradients.

$$\begin{aligned} S_{ij}S_{ij} = \frac{1}{2}&\left[\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right] \\ &+ \left[\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x} + \frac{\partial u_y}{\partial z}\frac{\partial u_z}{\partial y}\right] + \left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] \end{aligned} \tag{9}$$

As per Reynolds decomposition, $\mathbf{c} = \mathbf{u}(\mathbf{x}, t) + \mathbf{C}(\mathbf{c}, \mathbf{x}, t)$, i.e. the mean flow velocity is not a function of $\mathbf{c}$.

$$\langle \mathscr{P} Ku\ell S_{ij}S_{ij}\rangle = \frac{1}{n}Ku\ell S_{ij}S_{ij}\int c_k f d\mathbf{c} = Ku\ell S_{ij}S_{ij}u_k \tag{10}$$

$$\frac{\partial \rho\langle \mathscr{P} c_k\rangle}{\partial x_k} = \rho Ku\ell S_{ij}S_{ij}\frac{\partial u_k}{\partial c_k} = 0 \tag{11}$$

Since the mean flow velocity is not a function of $\mathbf{c}$, the gradients of the mean flow velocity also are not a function of $\mathbf{c}$.

$$F_k\rho\left\langle \frac{\partial \mathscr{P}}{\partial c_k}\right\rangle = F_k\rho Ku\ell\left\langle \frac{\partial S_{ij}S_{ij}}{\partial c_k}\right\rangle = 0 \tag{12}$$

The first terms on the LHS and the RHS contain derivatives of time and due to the underlying steady state assumption used in the derivation/estimation of turbulence energy production term, these terms evaluate to 0.

$$\frac{\partial \rho Ku\ell\langle S_{ij}S_{ij}\rangle}{\partial t} = 0 \quad \text{and} \quad \rho\left\langle Ku\ell\frac{\partial S_{ij}S_{ij}}{\partial t}\right\rangle = 0 \tag{13}$$

This leaves on the second RHS term as non-zero besides the Boltzmann production term.

$$P_{\mathscr{P}} = -\rho Ku\ell\left\langle c_k\frac{\partial S_{ij}S_{ij}}{\partial x_k}\right\rangle = -\rho Ku\ell\frac{\partial S_{ij}S_{ij}}{\partial x_k}\langle c_k\rangle \tag{14}$$

$$P_{\mathscr{P}} = -\rho Ku\ell u_k\frac{\partial S_{ij}S_{ij}}{\partial x_k} \tag{15}$$

Eq(15) can now be represented in non-tensorial notation and further subjected to numerical methods to study the variation of the Boltzmann turbulent energy production term in a flow.

$$P_{\mathscr{P}} = -\rho Ku\ell\left(u_x\frac{\partial S_{ij}S_{ij}}{\partial x} + u_y\frac{\partial S_{ij}S_{ij}}{\partial y} + u_z\frac{\partial S_{ij}S_{ij}}{\partial z}\right) \tag{16}$$

The expanded form of $S_{ij}S_{ij}$ from Eq(9) can be substituted into Eq(16) to obtain the full final expression.

Glossary of Notation

ρ	Volumetric mass density
μ	Dynamic viscosity
ν	Kinematic viscosity
f	Single-particle distribution function
$\mathbf{c}$	Microscopic velocity
$\mathbf{u}$	Mean flow velocity
$\mathbf{C}$	Local velocity fluctuation
$\mathbf{F}$	External force per unit mass
P	Mean pressure
p	Local pressure fluctuation
$\tilde{\mathbf{S}}$	Mean strain rate
$\tilde{\mathrm{s}}$	Local strain rate fluctuations
u	Turbulent representative velocity
ℓ	Turbulent characteristic length
$\mathscr{P}$	Turbulent energy production

References

[1] Tennekes, H., & Lumley, J. L. (1972). *A First Course in Turbulence*. The MIT Press.

[2] Agrawal, A. et al. (2020). *Microscale Flow and Heat Transfer*. Springer Nature Switzerland.