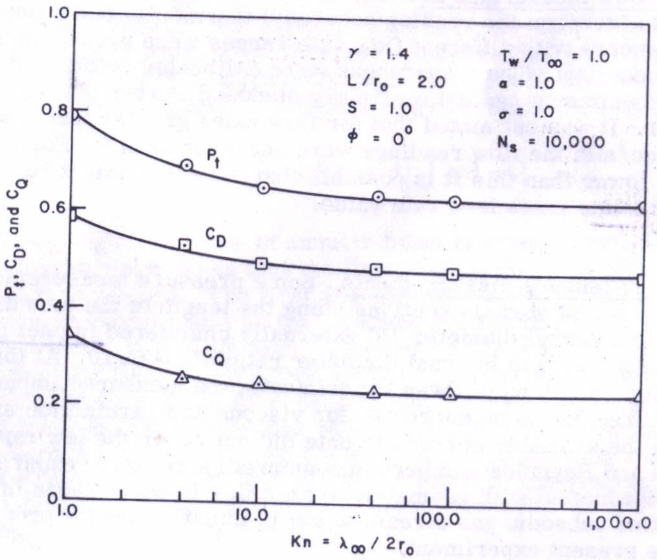

Fig. 6. Total Transmission Probability, Drag Coefficient, and Heat Transfer Coefficient vs Knudsen Number.

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(1) Long (2" dia) circular tube → N_2 at room temp.

Kn = 0.007 to 0.24

pr. ratio = 1.2 to 90

SLIP FLOW THROUGH LONG CIRCULAR TUBES*

by

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ABSTRACT

Rarefied flow of nitrogen gas through long circular tubes was experimentally studied. Measurements were made of the mass flow, pressure drop and cross-sectional velocity profiles. The Knudsen number at the entrance to the tube ranged from 0.24 to 0.007. The overall pressure ratios across the length of the tube varied from 1.2 to 90. The experiments showed that in the slip flow regime there was excellent agreement between experiment and theory based on the assumptions of isothermal and locally fully developed flow. Comparisons with theory using second order boundary conditions were also made at low pressures.

INTRODUCTION

The flow of rarefied gases through tubes and ducts has aroused considerable interest in recent years. Almost all the work done since the time of Knudsen has been confined to flows in which the overall pressure drop across the tube is small. Some experiments at large pressure ratios have been reported recently but these are specialized to geometries like nozzles, short tubes or orifices. Lack of any theoretical work for these types of flow geometries has prevented the comparison of the measured data

* The experimental part of this work was conducted at the Boeing Scientific Research Laboratories during author's stay there.

with theory and consequently has restricted the usefulness of the data, since one is not sure of the applicability of these for flow geometries or flow conditions different from those of the tests.

The flow of rarefied gases through tubes at very large pressure ratio levels is an interesting problem as it would permit some or all of the flow regimes (slip, transition, and free molecule) to coexist as the degree of rarefaction increases along the length of the tube due to a pressure gradient. Apart from its usefulness in predicting flow rates and pressure drops in a similarly encountered flow system, a detailed and accurate experimental study of this problem should provide limits for the applicability of various theories, since these theories do not by themselves give the range of Knudsen numbers for which they are valid. With this in mind the present work was undertaken.

EXPERIMENTAL ARRANGEMENT AND PROCEDURE

All the experiments were conducted using nitrogen at room temperature ($\simeq$ 23°C) in a continuous flow low density gas dynamics facility. A schematic diagram of the experimental arrangement is shown in Fig. 1. A polished brass tube connected two large vacuum chambers. A constant pressure differential across the two chambers maintained steady flow through the tube. Flow through three different tube geometries was studied. All tubes had the same smooth entrance section and a 2 inch internal diameter but the lengths were 9.54, 19.32, and 30.06 inches, respectively (Fig. 2). For each tube geometry the mass flow through the tube and the pressure drop along the length of the tube were measured at various upstream pressure values, and for a given upstream pressure at various pressure ratios. In addition some impact pressure measurements were made at various cross sections for the 30.06-inch long tube.

Pressure Measurements. Pressures were measured using thermistor gauges calibrated against a precision mercury McLeod gauge. It was estimated that the overall accuracy in the pressure measurements was ±1% for pressure values greater than 1 x 10^{-3} torr and ±2% for pressures lower than this. Before each run the whole system was evacuated, checked for leaks, and kept below 10^{-6} torr for several hours to eliminate outgassing effects on pressure readings.

Mass-flow measurements. Commercially available (slightly modified to increase the reading accuracy) thermopile mass flow transducers with different flow rate ranges were used to measure the mass flow. The transducers were calibrated before and after the experiments against a primary standard Porter low flow calibrator. It was estimated that for flow rates greater than 3.0 standard cc/min the flow readings were accurate to ±1%. For flow rates lower than this it is possible that the error might be ±2% of the absolute mass flow rate value.

Impact pressure measurements. Some pressure measurements were made at various sections along the length of the tube using a 0.0415" internal diameter 10° externally chamfered impact probe having a length to internal diameter ratio of about 10. At the density conditions of these experiments, the measured impact pressures had to be corrected for viscous and rarefaction effects. Since the available correction data did not cover the low ranges of Mach and Reynolds numbers encountered, a separate experiment was conducted[1] with geometrically similar impact probes in a rarefied subsonic gas stream to aid in interpreting the pressures of the present experiment.

THEORETICAL CONSIDERATIONS

Based on the Knudsen number, the flow can be roughly divided into three regimes; namely, slip flow (0.1 > Kn > 0.001), transition flow (10 > Kn > 0.1), and free molecular flow (Kn ≥ 10). Then, depending on the pressure in the present tests, flow conditions such as slip flow throughout, slip flow at entrance with free molecule flow at exit, etc., existed inside the tubes. Lack of any theory for the flow through long tubes in the transition and in near free molecular regimes prevented comparison of the present experimental data with theory in these flow regimes. For this reason most of the discussions are confined to comparisons with slip flow theory (based on continuum equations) briefly outlined below.

Using the boundary condition that the slip velocity at the wall is given by $u_w = -C_1\lambda\,(\partial u/\partial r)_w$, it is possible to show for a fully developed flow that the velocity ratio and the shear stress at the wall are given by

$$\frac{u}{\bar{u}} = \frac{1 + 4C_1\mathrm{Kn} - r^2/R^2}{1/2 + 4C_1\mathrm{Kn}} \quad \text{and} \quad \bar{\tau}_w = \frac{8\,\mathrm{Re}\,\mu^2}{\rho D^2\,(1 + 8C_1\mathrm{Kn})} \qquad (1)$$

where $\bar{u}$ = mean velocity, $Re = \rho\bar{u}D/\mu$; $Kn = \lambda/D$; $R = D/2$ = tube radius. Neglecting viscous compressive stresses the momentum balance for the case of compressible flow on an elemental volume between two cross sections of a tube with axial length dz is given by

$$-\frac{\pi D^2}{4}\, dp - \bar{\tau}_w \pi D\, dz = d\left[\int_A \rho u^2\, dA\right] \tag{2}$$

(A similar analysis for flow in annular tubes is given in ref. 2.) Assumptions are now made that the flow is (i) isothermal and (ii) locally fully developed. The first assumption was nearly satisfied in the present experiments as the tube was not insulated and the flow speed inside it was not high. The second assumption means that the velocity field at any cross section is the same as that for fully developed flow at the local density and the wall shear stress takes on the local fully developed value. An expression relating the pressure difference between some downstream position and the reference position z_0 (pressure p_0) can be found by integrating Eq. (2) after substituting values for u and $\bar{\tau}_w$ from Eq. (1) and noting that $Kn \propto 1/p$. The result is

$$\left(\frac{p}{p_0}\right)^2 - 1 + 16\, C_1 Kn_0 \left(\frac{p}{p_0} - 1\right) + \frac{8}{3} Re^2 \beta \left[\frac{2C_1 Kn_0\, (p/p_0 + 24\, C_1 Kn_0)}{(p/p_0 + 8C_1 Kn_0)\, p/p_0} - \frac{2C_1 Kn_0\, (1 + 24 C_1 Kn_0)}{1 + 8C_1 Kn_0} + \frac{1}{4} \log \frac{1 + 8C_1 Kn_0}{p/p_0 + 8C_1 Kn_0} - \frac{3}{4} \log p/p_0\right] = -64\, Re\, \beta\, \frac{z - z_0}{D} \tag{3}$$

where

$$\beta = \frac{\mu^2 RT}{p_0^2 D^2}\,; \quad R = \text{Gas Constant.}$$

This is an exact expression as compared to the first approximation result; namely

$$(p/p_0)^2 - 1 + 16\, C_1 Kn_0\, (p/p_0 - 1) + 2\, Re^2 \beta\, \chi \left[8C_1 Kn_0\, (p_0/p - 1) - \log p/p_0\right] = -\, 64\, Re\, \beta\, \frac{z - z_0}{D} \tag{4}$$

where

$$\chi = \frac{1}{A} \int_A \left(\frac{u}{\bar{u}}\right)^2 dA = \frac{1/3 + 4C_1 Kn + 16 C_1^2 Kn^2}{(0.5 + 4C_1 Kn)^2} \tag{5}$$

obtained from the general solution for flow through annular ducts given in ref. 2 where the assumption χ = constant was made during the integration of Eq. (2). Comparison between the exact and the approximate solutions showed very little difference for most of the pressure conditions covered in these tests, particularly when an average value of Knudsen number is used in Eq. 5. (The average used was one half the value of the sum of the Knudsen numbers at the upstream and downstream reference stations).

In order to increase the range of applicability of the slip theory, second order slip boundary conditions were derived by Deissler[3] and Cercignani[4]. For flows in which there is only a normal velocity gradient both obtain the same form for the velocity jump; namely

$$u_w = -C_1 \lambda \left(\frac{\partial u}{\partial r}\right)_w - C_2 \lambda^2 \left(\frac{\partial^2 u}{\partial r^2}\right)_w \tag{6}$$

However these authors differ slightly in the value of the coefficients. According to Deissler $C_1 = 1.00$ and $C_2 = 1.6875$ while according to Cercignani $C_1 = 1.1466$ and $C_2 = 0.9756$. Using Eq. (6) and the same assumptions (i.e., isothermal and locally fully developed flow), the following expression, similar in form to Eq. (4), can be derived

$$(p/p_0)^2 - 1 + 16\, C_1 Kn_0 \left(\frac{p}{p_0} - 1\right) + 32\, C_2 Kn_0^2 \log \frac{p}{p_0} + 2\, Re^2 \beta\, \chi \left\{8\, C_1 Kn_0 \left(\frac{p_0}{p} - 1\right) + 8\, C_2 Kn_0^2 \left[\left(\frac{p_0}{p}\right)^2 - 1\right] - \log \frac{p}{p_0}\right\} = -\, 64\, Re\, \beta\, \frac{z - z_0}{D} \tag{7}$$

where

$$\chi = \frac{\frac{1}{3} + 4\, C_1 Kn + 16\, C_1^2 Kn^2 + 8\, C_2 Kn^2 + 64\, C_2^2 Kn^4 + 64\, C_1 C_2 Kn^3}{\left(\frac{1}{2} + 4\, C_1 Kn + 8\, C_2 Kn^2\right)^2} \tag{8}$$

At low pressures there is hardly any difference between the exact result and that obtained with the approximation χ = constant and as such the latter is used to derive Eq. (7). (Cercignani has pointed out that his second order boundary conditions (Eq. (6)) are only to be used for evaluating the fluid states far from the wall and not for evaluating space integrals which extend to regions close to or up to the wall. The correct rule for evaluating such integrals has been given in ref. 4. Applying this rule it was found that Eq. (7) is still correct for the pressure drop but the constant C_2 takes on the modified value of 1.356; $C_1 = 1.1466$.) In calculating the pressure drop as a function of distance (Eqs. 3, 4 & 7) the experimentally measured mass flow values were used to determine the Reynolds number of the flow. Further, the reference position was chosen sufficiently far downstream of the entrance to eliminate entrance effects.

RESULTS AND DISCUSSIONS

Experiments were conducted at seven different upstream pressure values, viz. 4.1, 14.6, 31.4, 63.7, 95.7, 112.8 and 130.4 x 10^{-3} torr for each tube geometry. The Knudsen number values (defined as the ratio of the local molecular mean free path λ to the tube diameter D; the mean free path is calculated from $\lambda = \mu/\rho\sqrt{2RT/\pi}$) ?? corresponding to these pressures were 0.237, 0.066, 0.031, 0.015, 0.010, 0.009 and 0.007 respectively. For fixed upstream pressures, the downstream pressures were varied to give different overall pressure ratios ranging between 1.2 and 90. Out of a total of 129 experiments, only a very few representative ones, which lead to some general conclusions, are considered and discussed herein.

Pressure drop results. Before a comparison of the theoretical and measured pressure drop results can be made, we have to know the value of the slip coefficient C_1 in Eq. (3). Maxwell[5] showed $C_1 = C\ 2\text{-}f/f$ where C is a constant close to 1.0 and f is the fraction of molecules diffusely reflected. Albertoni et al[6] give a value of $C_1 = 1.1466$ by numerically solving the linearized form of Boltzman equation.

For low Knudsen numbers ($Kn \leq 0.03$) the pressure drop predicted by Eq. (3) for $C_1 = 1.1466$ and $C_1 = 1.0$ are fairly close and there was very good agreement between theoretical and measured pressure drop results. A closer examination showed that for Knudsen numbers < 0.05 there was better agreement between experiment and theory for $C_1 = 1.0$ rather than 1.1466. This is illustrated in Fig. 3. It is to be noted that if any specular reflection was present, then the value of C_1 would be higher than 1.00 in the Maxwell's slip formula. However, at lower pressures corresponding to $Kn > .05$ the agreement was better with $C_1 = 1.1466$ as shown in Fig. 4. Since no effort was made to keep the tube surface extra clean, the increase in slip coefficient value with de-

creasing pressure suggests that the coefficient may be a function of the Knudsen number rather than simply related to specular reflection alone.

The effects of compressibility and momentum change on the flow are illustrated in Fig. 5 by comparing cases in which (i) it was assumed that the density is constant (fully developed incompressible flow), (ii) the flow is locally developed but momentum changes are neglected (right side of (2) is neglected), and (iii) the flow is locally fully developed and the momentum changes are included (Eq. 3). The results show that the gas compressibility significantly affects the pressure drop through changes in both momentum and viscous shear. Further, as can be seen from Fig. 5 the experiments confirm the observations made in Ref. 2, that as the pressure level decreases the contribution of momentum change plays a minor role and that the pressure drop is affected primarily through an increase in viscous shear rather than through increase in momentum flux.

Excellent agreement between theoretical and experimental pressure drops for all the three tube geometries tested at $Kn < 0.13$ not only justifies the assumption of locally fully developed flow in the theoretical analysis but also indicates the adequacy of using such a theoretical model to describe similar flow problems.

As the gas rarefaction increased ($Kn \geq 0.13$), the agreement between experiment and theory deviated, suggesting that the first order slip theory, based on continuum equations, was not adequate to describe the flow at these low pressures. To see whether the slip flow theory could be further extended by using second order slip velocity boundary conditions, to cover even lower pressures, comparisons of the experimental pressure drop data with second order theory (Eq. 7) were made. Such a comparison is shown in Fig. 6. As can be seen the second order slip velocity both due to Deissler and Cercignani overestimates the second order effect. If one assumes that the form of the slip velocity as given by Eq. (6) is correct, then the experiments show that the values of C_2 should be close to 0.14 (with $C_1 = 1.1466$) to make Eq. (7) agree with experimental values.

Velocity Measurements. The cross section velocity profile was calculated from the measured impact pressures after applying viscous and rarefaction corrections assuming that the wall static pressure was constant across the section. It was not possible to reduce all the measured pressure probe data to velocity due to the fact that the available viscous effects correction data[1] covered only limited Knudsen number and speed ratio ranges. However, when these effects were known, the resulting measured velocity profiles agreed well with theoretical locally fully developed values given by Eq. (1). A typical comparison is shown in Fig. (7). The slight discrepancy between theory and experiment may be attributed to the effects of tube wall on the probe readings and to the probe being in a shear flow.

Flow development. All the three tube geometries used had the same smooth contoured entrance section and as such had a region at the entrance where there was a velocity profile development. An analysis has been made by Hanks[7] on the velocity development in the entrance region of a circular cylinder. The distance, z, necessary for the centerline velocity to become 0.99 of the fully developed value was given by the expression

$$\frac{z}{D} = \frac{KRe}{4} \tag{9}$$

where K has a value between 0.15 and 0.2 depending on the Knudsen number. A direct comparison between the theory and experiment was not possible since in the experiments the flow entrance was a contoured section whereas the theory is for a uniform circular cylinder. However, a reference section far downstream from the entrance was chosen and the theoretical pressure drop upstream of the reference section was computed from Eq. (3) and compared with the measurements. The positions at which the theory and experiment started deviating was taken as the end of the velocity development region and the entrance region lengths thus obtained agreed with values predicted by Eq. (9).

Mass Flow Measurements. Equation (3) is a relation between the pressure drop, distance, and Reynolds number. The comparisons shown in Figs. 3-6 are between measured and theoretical pressure differences, the latter based on Reynolds number calculated from measured mass flows. Alternatively, one could compare the experimentally measured mass flows with theoretical predictions by making use of the measured pressure drop. Such comparisons also show good agreement between experiment and theory.

Mass flow data obtained at lower pressure levels were found to asymptotically approach Clausing's[9] theoretical free molecule mass transfer values. Further, for the same upstream Knudsen

number and the same pressure drop across the length of the tube, the approach to theoretical free molecule mass flow values was very much faster for the long tubes than for short tubes. This is illustrated in Fig. 8. A similar trend has also been observed previously[8] for the case of flow through orifices and short tubes.

CONCLUSIONS

The results of the rarefied gas flow through tubes showed that in the case of compressible flows, the slip flow theory based on the assumptions of isothermal and locally fully developed flow is able to predict the flow phenomena very well until Knudsen number reaches a value of about 0.13. At higher Knudsen number values comparison with the same theory using second order boundary conditions showed that the values of the second order slip coefficients as reported by Deissler and Cercignani (1.6875 and 0.9756 respectively) were too high. If it is assumed that the form of the second order slip velocity at the wall as given by the above authors is correct, the value for the slip coefficient would have to be about .14 to agree with the experimental measurements. Using this value it was found that the second order theory could be extended to predict flows having a Knudsen number as high as 1.5. There were also indications from the experiments that the slip coefficient may be a function of the Knudsen number rather than being a constant, and that its value changes from 1.00 to 1.1466 as the Knudsen number increases. Limited velocity profile measurements also suggested that the flow at any cross section is locally fully developed. At low pressures the measured mass flow was found to approach free molecule flow theoretical values much faster in long tubes than in short tubes.

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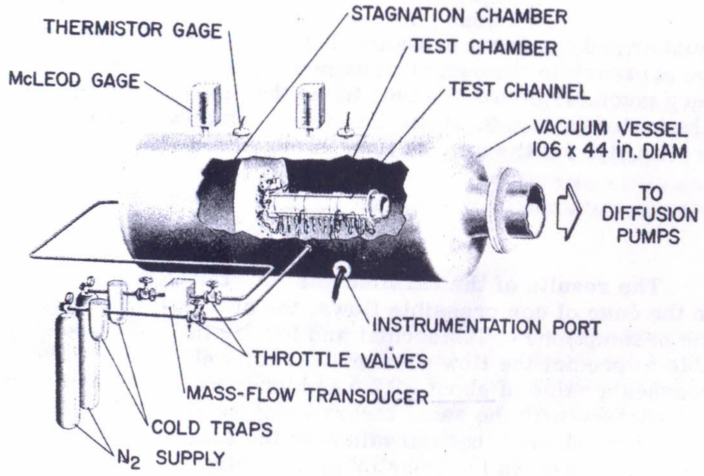

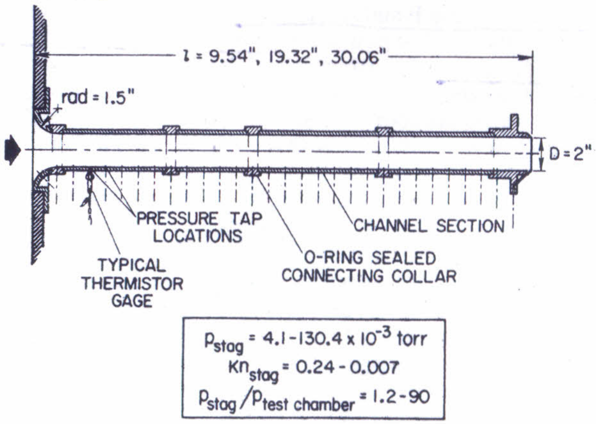

Fig. 2. Test Channel Details.

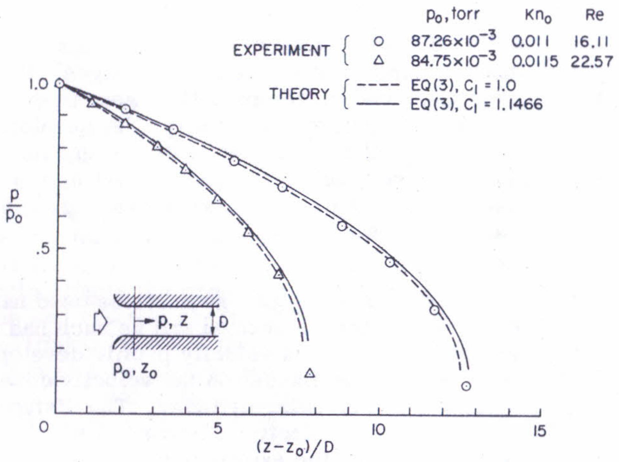

Fig. 3. Pressure Drop Data Compared with Slip Flow Theory.

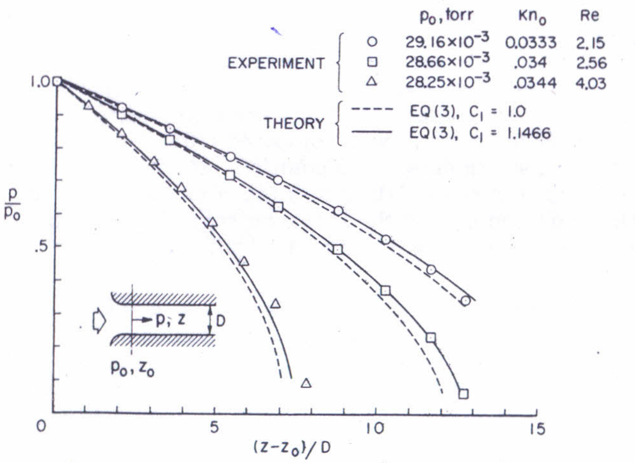

Fig. 4. Pressure Drop Data Compared with Slip Flow Theory.

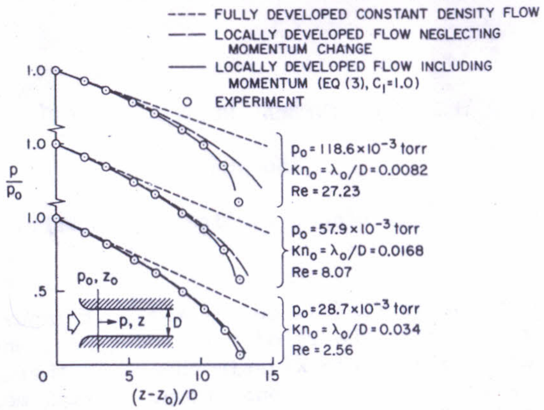

Fig. 5. Illustration of the Effects of Compressibility and Momentum Change on the Pressure Drop Values.

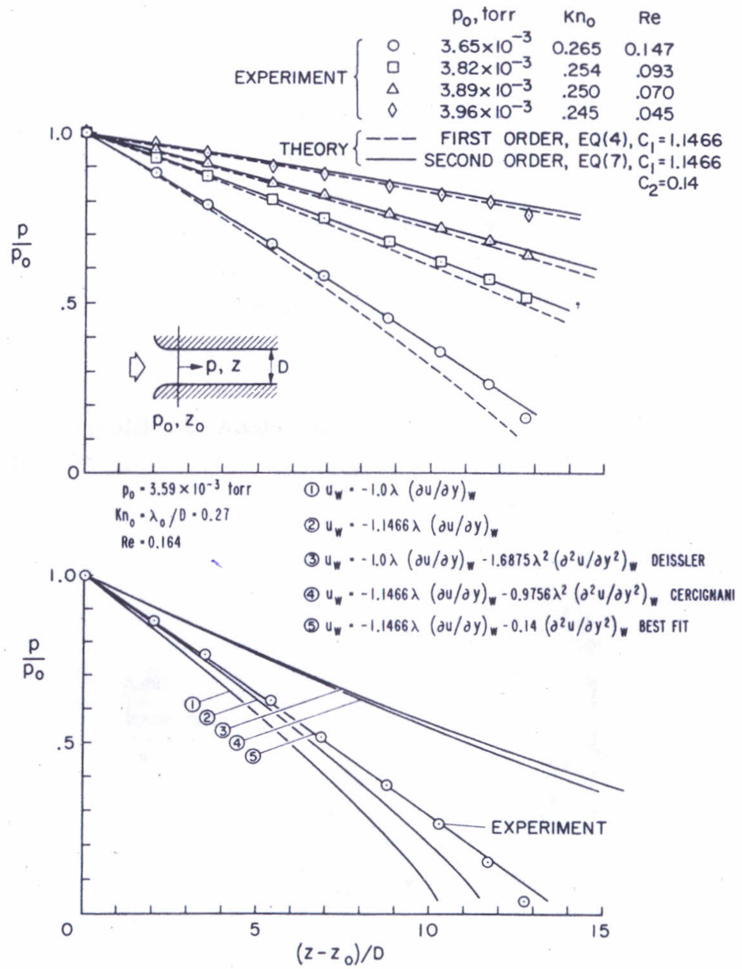

Fig. 6. Measured Pressure Drop Compared with Slip Theory Based on Several Second Order Boundary Conditions.

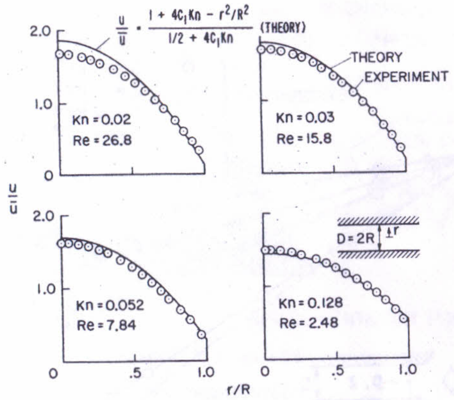

Fig. 7. Cross-Sectional Velocity Profiles.

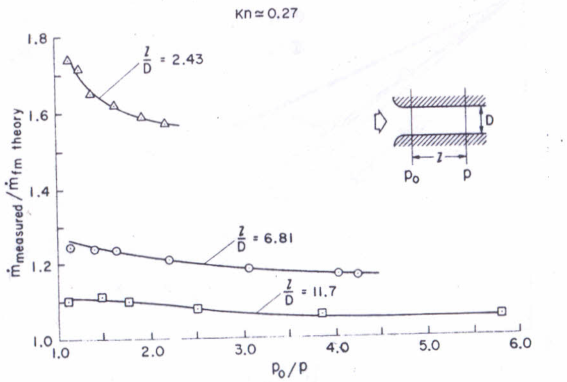

Fig. 8. Nondimensionalized Mass Flow vs Pressure Ratio for Three Tube Geometries.

THEORY OF THERMAL MOLECULAR FLOW†

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INTRODUCTION

Thermal transpiration is one of the most important phenomena in the kinetic theory of rarefied gases; and it performs a major role in the kinetic theory of thermal molecular flow; such as, thermal conduction, thermal molecular pressure and molecular effusion, etc. The phenomenon of thermal transpiration, discovered by Neumann, Feddersen and Reynolds, has been treated by Maxwell and Knudsen by Kinetic theory of gases. Recently, it has been shown by the author that the theories of Maxwell and Knudsen do not give an exact description of the microstate; and these theories are only valid in a near-Maxwellian distribution regions. However, the revised theory (Wu) contains a set of new invariants which deviate from the classical theories by a geometrical factor (I). This factor, defined as the isotropy, is a measure of the anisotropic characteristics of the velocity distribution function in a thermal transpiration system. The new invariants of thermal transpiration in the Knudsen regime are proved to be $PI\sqrt{T}$ and $n\sqrt{T}I$. In this note, the revised theory and several important phenomena of the thermal molecular flow will be reviewed briefly. The detail discussion has been presented in the references [1,2,3,4,].

ANISOTROPIC DISTRIBUTION FUNCTION (KNUDSEN GAS)

In an arbitrary closed system with given temperature distribution on the boundary T(S), it is shown that the velocity distribution function $F(\vec{x},\vec{v})$ of a Knudsen gas can be obtained by introducing a directional density $n_{\phi\theta}$ and temperature $T_{\phi\theta}$, in the form of a directional description,

$$F = F_{\phi\theta} = n_{\phi\theta}f_{\phi\theta} = n_{\phi\theta}\ (2\pi RT_{\phi\theta})^{-3/2}\ \exp\ (-C^2_{\phi\theta}/2RT_{\phi\theta}) \quad (1)$$

† This work was primarily executed at Princeton University.