MA 5124 Financial Time Series Analysis & Forecasting

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2021-02-07

Contents

Course Syllabus

Module Code: MA 5124

Title: Financial Time Series Analysis & Forecasting

Credits: 4

Pre-requiites

None

Learning Objectives

- The purpose of this course is to provide students with introductory tools for the time series analysis of financial time series.
- Analyze of data series based on stochastic and non stochastic models

Learning Outcomes

- On successful completion of this course, students will be able to provide more than an introductory treatment of the topics.
- Students are encouraged to pursue further study in this area if they find that the topics covered in this course.

Outline Syllabus

- Definition and examples of time series
- back-shift and differencing-operators, - strong and weak stationarity, definition of ACF, PACF.

- Definitions and properties of the $MA(q), MA(\infty), AR(p), AR(\infty)$ and $ARMA(p,q)$,in particualr their acf's
- causal stationarity of AR
- invertibility of MA models and causal stationarity and invertibility of ARMA; - concept of spectral density function and its applications
- definition and properties of integrated $ARIMA(p,d,q)$ processes
- definition and properties of random walks with or without drift.
- Model selection following the AIC and BIC
- brief introduction to linear prediction and calculation of forecasting intervals for normal ARMA models
- point and interval forecasts for normal random walks with or without drift.
- Definition and properties of the VAR (vector autoregressive) model, arrange a univariate time series as a multivariate Markov model.
- Nonlinear properties of financial time series
- definition and properties of the well known ARCH, GARCH etc.
- Cointegration in Single Equations, Modeling and Forecasting Financial Time Series.

Method of Assessment

- Assignment 30%
- End-semester examination 70%

Lecturer

Dr. Priyanga D. Talagala

Schedule

Lectures:

- Sunday [9.00am -12.00 noon]

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Chapter 1

Intordution to Time Series Forecasting

Chapter 2

ARIMA models

- **AR**: autoregressive (lagged observations as inputs)
- **I**: integrated (differencing to make series stationary)
- **MA**: moving average (lagged errors as inputs)

An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

2.1 Stationarity and differencing

2.1.1 Stationarity

Definition

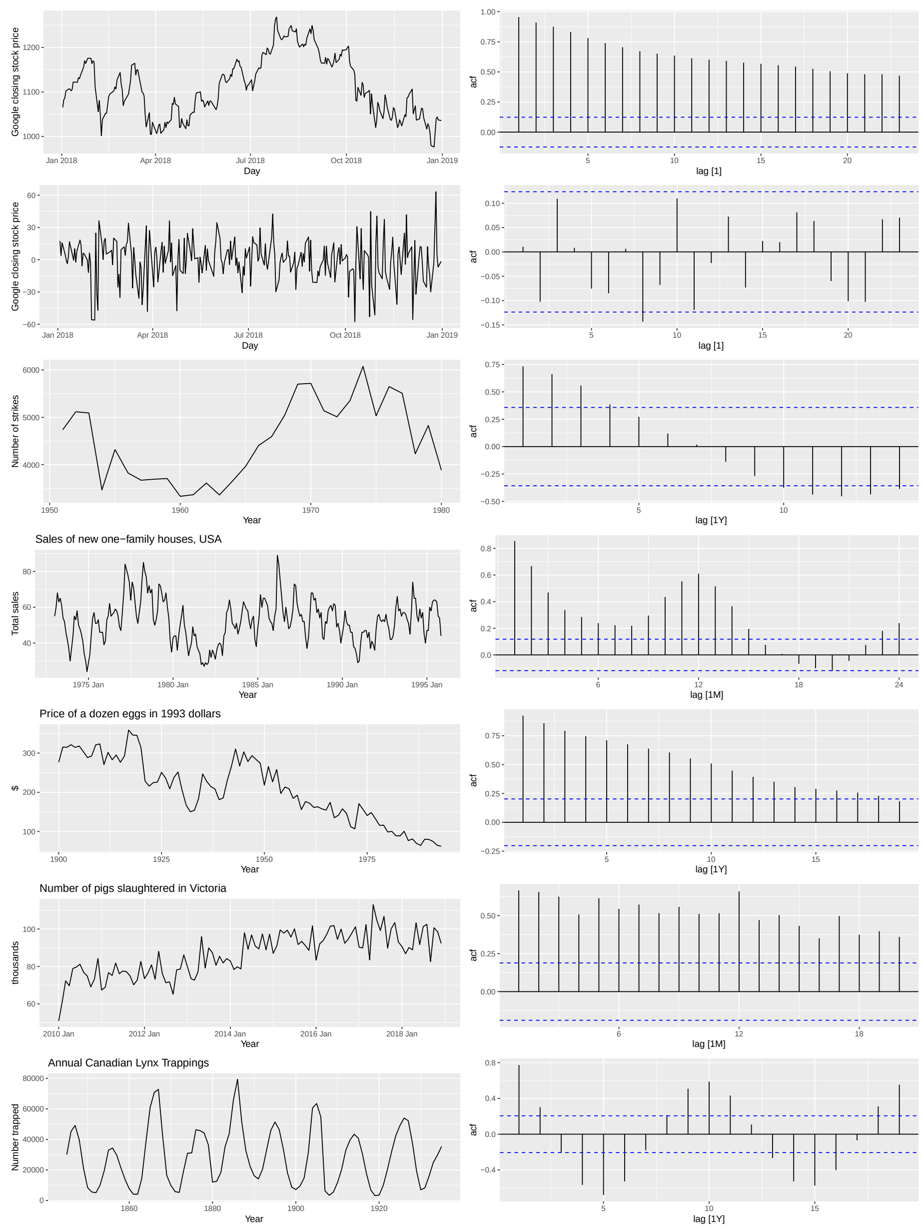
If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \dots, y_{t+s})$ does not depend on t.

A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term
- Transformations help to stabilize the variance.
- For ARIMA modelling, we also need to stabilize the mean.

Identifying non-stationary series

- time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of r1 is often large and positive.



- A time series, $\{Y_t, t = 0, \pm 1, ...\}$ is said to be **strict stationary**, if $(Y_1, ..., Y_n)$ and $(Y_{1+h}, ..., Y_{n+h})$ have the same joint distribution for all integers h and $n > 0$.

2.1.1.1 Weak Stationarity

Definition: Covariance function (in (Brockwell et al., 2016), p. 15; the notations have been changed for consistency within this note)

Let $\{Y_t\}$ be a time series with $E(Y_t^2) < \infty$. The **mean function** of $\{Y_t\}$ is

$$\mu_Y(t) = E(Y_t)$$

The **covariance function** of $\{Y_t\}$ is

$$\gamma_Y(r, s) = Cov(Y_r, Y_s) = E[(Y_r - \mu_Y(r))(Y_s - \mu_Y(s))]$$

for all intergers r and s.

Definition: Weakly stationary (in (Brockwell et al., 2016), p. 15; the notations have been changed for consistency within this note)

$\{Y_t\}$ is **weakly stationary** if

1. $\mu_Y(t)$ is independent of t,

and

2. $\gamma_Y(t + h, t)$ is independent of t for each h.

- Unless specifically indicate otherwise, whenever we use the term *stationary* we shall mean *weakly stationary*.

2.1.2 Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series: $y'_t = y_t - y_{t-1}$.
- The differenced series will have only $T - 1$ values since it is not possible to calculate a difference y'_1 for the first observation.

2.1.2.1 Second-order differencing

Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time:

$$
\begin{aligned}
y''_t &= y'_t - y'_{t-1} \\
&= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\
&= y_t - 2y_{t-1} + y_{t-2}.
\end{aligned}
$$

- y''_t will have $T-2$ values.
- In practice, it is almost never necessary to go beyond second-order differences.

2.1.2.2 Seasonal differencing

A seasonal difference is the difference between an observation and the corresponding observation from the previous year.

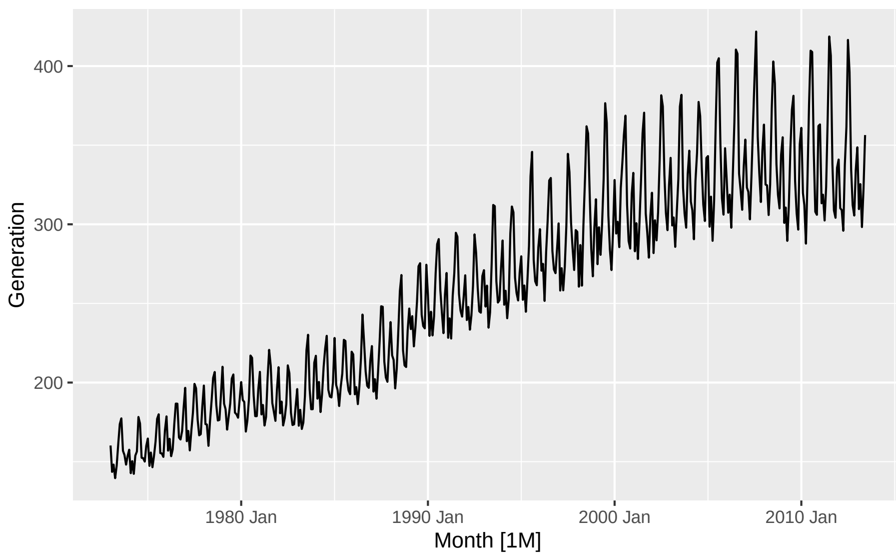
$$y'_t = y_t - y_{t-m}$$

where $m =$ number of seasons.

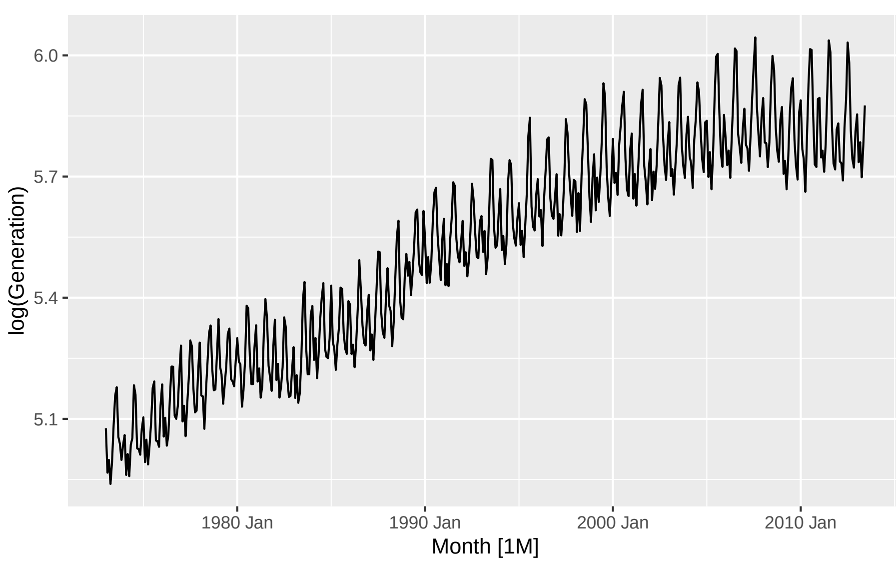
- For monthly data $m = 12$.
- For quarterly data $m = 4$.

Example : Electricity production

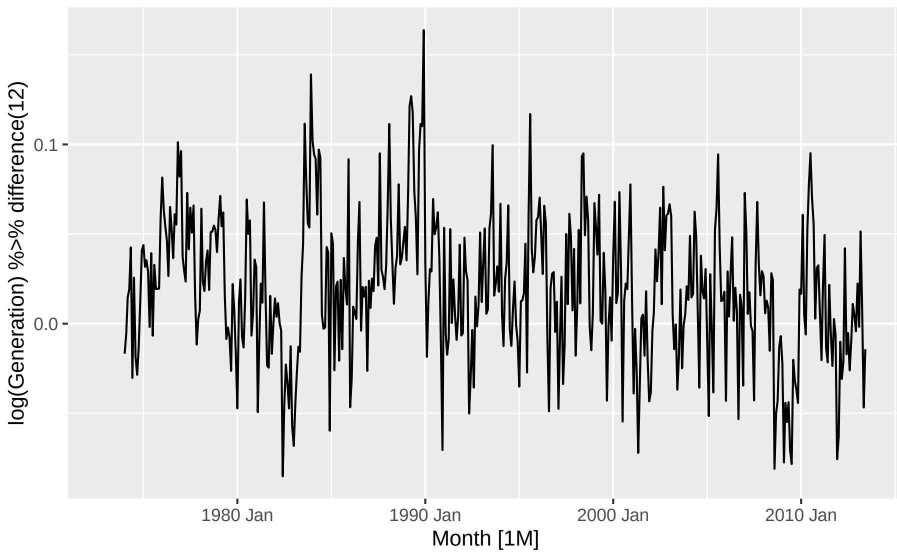
```
usmelec %>% autoplot(Generation)
```



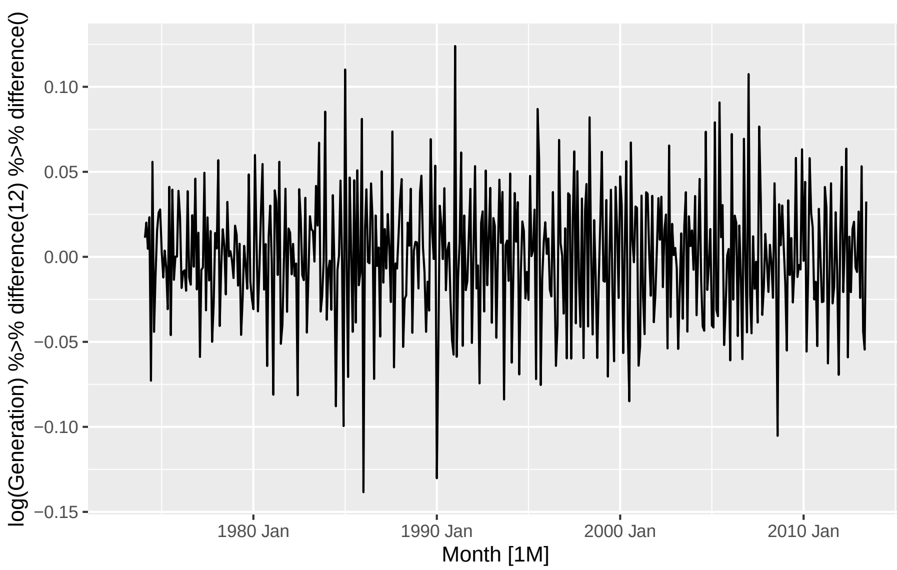

```
usmelec %>% autoplot(log(Generation))
```



```
usmelec %>% autoplot(log(Generation) %>% difference(12))
```




```
usmelec %>% autoplot(log(Generation) %>% difference(12) %>% difference())
```



- Seasonally differenced series is closer to being stationary.
- Remaining non-stationarity can be removed with further first difference.

If $y'_t = y_t - y_{t-12}$ denotes seasonally differenced series, then twice-differenced series is

$$\begin{aligned} y^*_t &= y'_t - y'_{t-1} \\ &= (y_t - y_{t-12}) - (y_{t-1} - y_{t-13}) \\ &= y_t - y_{t-1} - y_{t-12} + y_{t-13}. \end{aligned}$$

When both seasonal and first differences are applied ...

- it makes no difference which is done first—the result will be the same.
- If seasonality is strong, we recommend that seasonal differencing be done first because sometimes the resulting series will be stationary and there will be no need for further first difference.

It is important that if differencing is used, the differences are interpretable.

2.1.2.3 Interpretation of differencing

- first differences are the change between one observation and the next;
- seasonal differences are the change between one year to the next.

But taking lag 3 differences for yearly data, for example, results in a model which cannot be sensibly interpreted.

2.1.3 Backshift notation

A very useful notational device is the backward shift operator, B, which is used as follows:

$$By_t = y_{t-1}$$

In other words,

- B, operating on y_t, has the effect of **shifting the data back one period**.
- Two applications of B to y_t **shifts the data back two periods**:

$$B(By_t) = B^2 y_t = y_{t-2}$$

- For monthly data, if we wish to shift attention to "the same month last year", then B^{12} is used, and the notation is

$$B^{12} y_t = y_{t-12}$$

.

- The backward shift operator is convenient for describing the process of *differencing*.
- A first difference can be written as

$$y'_t = y_t - y_{t-1} = y_t - By_t = (1-B)y_t$$

- Note that a first difference is represented by $(1-B)$.
- Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1-B)^2 y_t$$

- Second-order difference is denoted $(1-B)^2$.
- *Second-order difference* is not the same as a *second difference*, which would be denoted $1-B^2$;
- In general, a dth-order difference can be written as

$$(1-B)^d y_t$$

* A seasonal difference followed by a first difference can be written as

$$(1-B)(1-B^m)y_t$$

- The "backshift" notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1-B)(1-B^m)y_t &= (1-B-B^m+B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

- For monthly data, $m = 12$ and we obtain the same result as earlier.

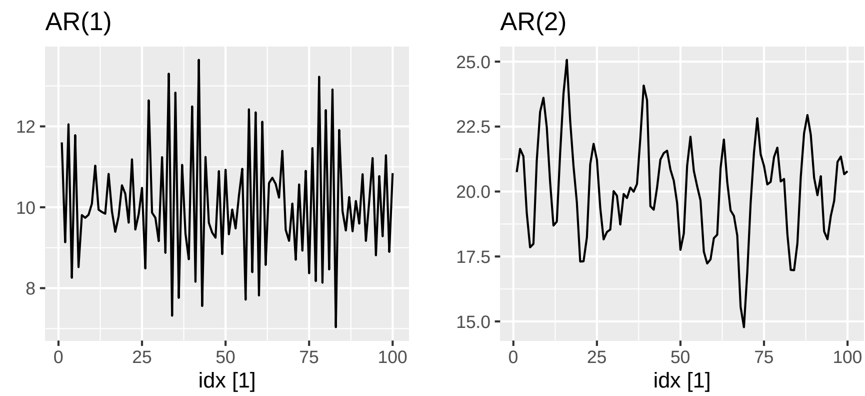
2.2 Non-seasonal ARIMA models

2.2.1 Autoregressive models

Autoregressive (AR) models:

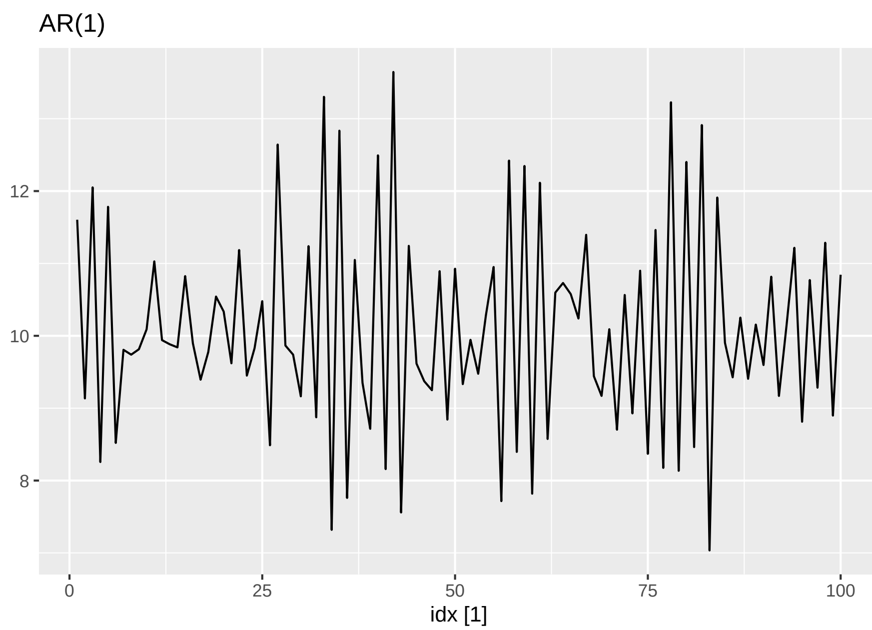
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.



2.2.1.1 AR(1) model

$$y_t = 18 - 0.8y_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim N(0,1), \quad T = 100.$$



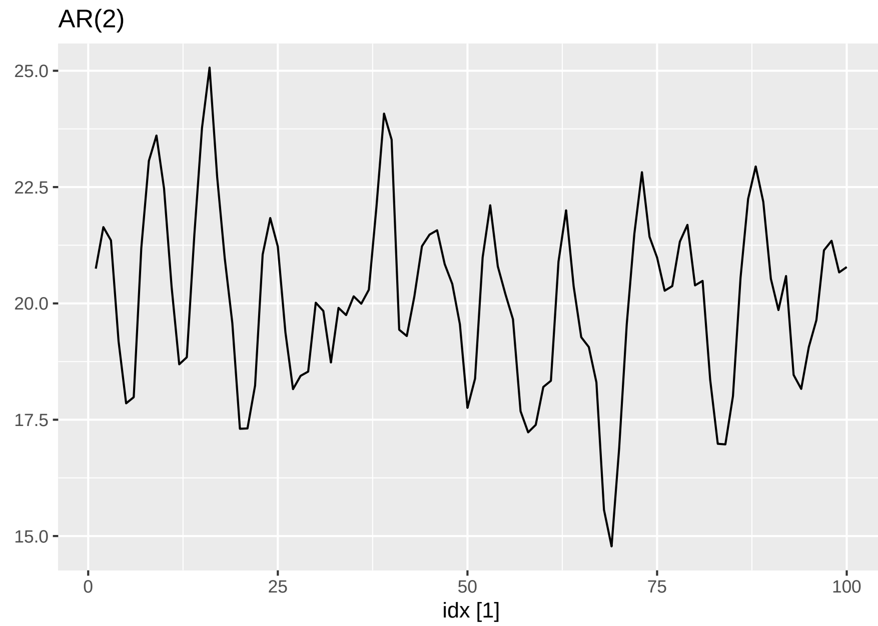
$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$

- When $\phi_1 = 0$, y_t is **equivalent to WN**

- When $\phi_1 = 1$ and $c = 0$, y_t is **equivalent to a RW**
- When $\phi_1 = 1$ and $c \neq 0$, y_t is **equivalent to a RW with drift**
- When $\phi_1 < 0$, y_t tends to **oscillate between positive and negative values**.

2.2.1.2 AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + \varepsilon_t$$
$$\varepsilon_t \sim N(0,1), \qquad T = 100.$$



2.2.1.3 Stationarity conditions

- We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ lie outside the unit circle on the complex plane.

- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:
  $-1 < \phi_2 < 1 \qquad \phi_2 + \phi_1 < 1 \qquad \phi_2 - \phi_1 < 1$.

- More complicated conditions hold for $p \geq 3$.
- Estimation software takes care of this.

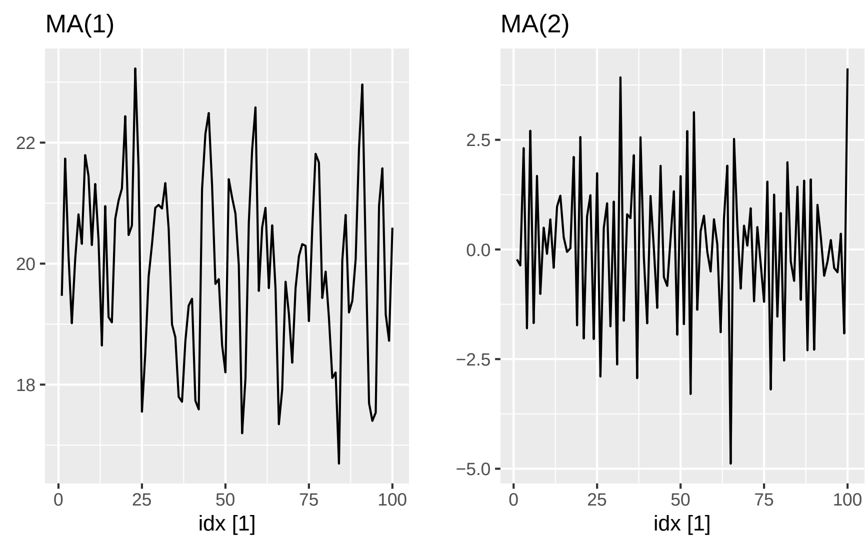
2.2.2 Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where ε_t is white noise. This is a multiple regression with **past errors** as predictors.

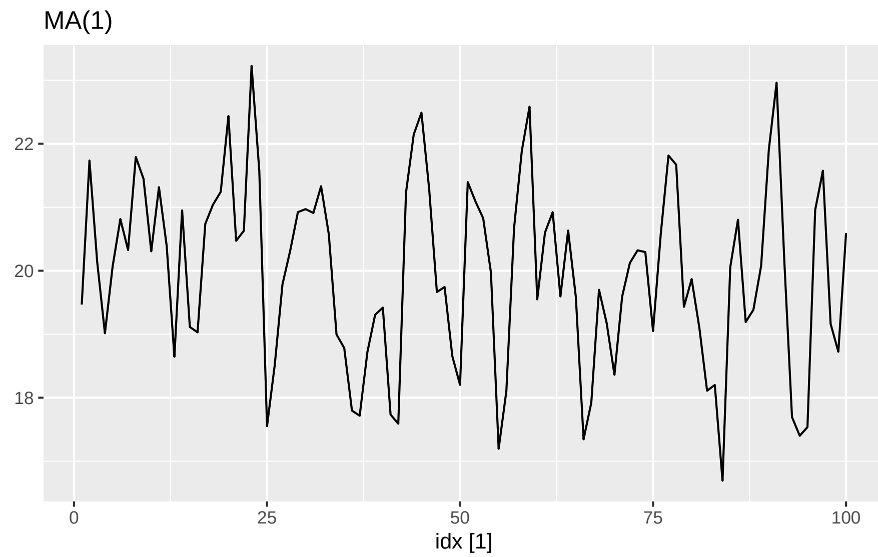
- Don't confuse this with moving average smoothing!



2.2.2.1 MA(1) model

$$y_t = 20 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

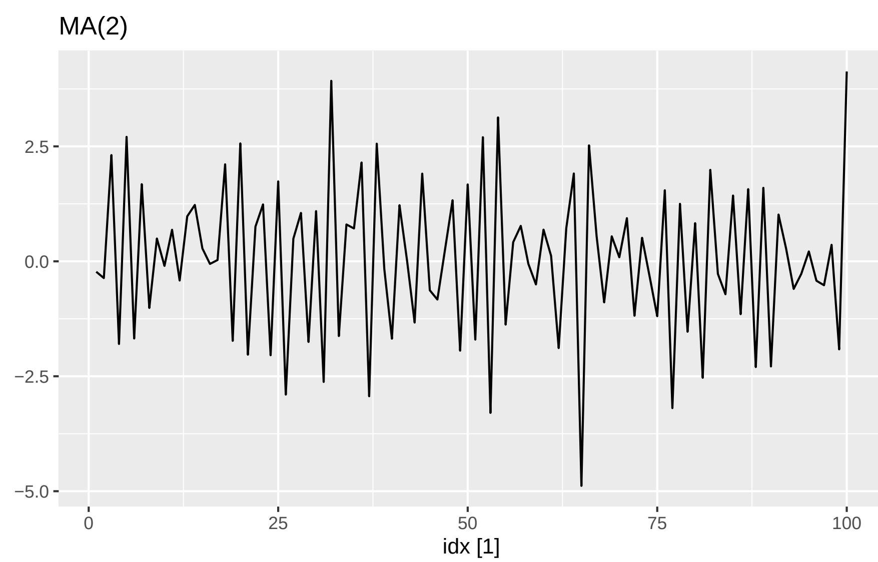
$$\varepsilon_t \sim N(0,1), \quad T = 100.$$



2.2.2.2 MA(2) model

$$y_t = \varepsilon_t - \varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

$$\varepsilon_t \sim N(0,1), \quad T = 100.$$



2.2.2.3 MA(∞) models

It is possible to write any stationary AR(p) process as an MA(∞) process.

Example: AR(1)

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \varepsilon_t \\ &= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\ &= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\ &\dots \end{aligned}$$

Provided $-1 < \phi_1 < 1$:

$$y_t = \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \phi_1^3 \varepsilon_{t-3} + \cdots$$

2.2.3 Invertibility

- Any MA(q) process can be written as an AR(∞) process if we impose some constraints on the MA parameters.
- Then the MA model is called "invertible".
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

General condition for invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle on the complex plane.

- For $q = 1$: $-1 < \theta_1 < 1$.
- For $q = 2$:
  $-1 < \theta_2 < 1 \qquad \theta_2 + \theta_1 > -1 \qquad \theta_1 - \theta_2 < 1$.
- More complicated conditions hold for $q \geq 3$.
- Estimation software takes care of this.

2.2.4 ARIMA models

Autoregressive Moving Average models:

$$\begin{aligned} y_t &= c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ &+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t \end{aligned}$$

- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing**.
- $(1-B)^d y_t$ follows an ARMA model.

Autoregressive Integrated Moving Average models

ARIMA(p, d, q) model

- **AR:** $p =$ order of the autoregressive part
- **I:** $d =$ degree of first differencing involved
- **MA:** $q =$ order of the moving average part.
  - White noise model: ARIMA(0,0,0)
  - Random walk: ARIMA(0,1,0) with no constant
  - Random walk with drift: ARIMA(0,1,0) with const.
  - AR(p): ARIMA(p,0,0)
  - MA(q): ARIMA(0,0,q)

2.2.5 Backshift notation for ARIMA

- **ARMA model:**

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

$$\text{or} \quad (1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q)\varepsilon_t$$

ARIMA(1,1,1) model:

$$(1 - \phi_1 B)(1 - B) y_t = c + (1 + \theta_1 B)\varepsilon_t$$

NOTE:

Written out:

$$y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

2.3 Estimation and order selection

2.3.1 Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters c, $\phi_1, \dots, \phi_p$, $\theta_1, \dots, \theta_q$.

- MLE is very similar to least squares estimation obtained by minimizing

$$\sum_{t-1}^{T} e_t^2$$

- The `ARIMA()` function allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.

2.3.2 Partial autocorrelations

Partial autocorrelations measure relationship between y_t and y_{t-k}, when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

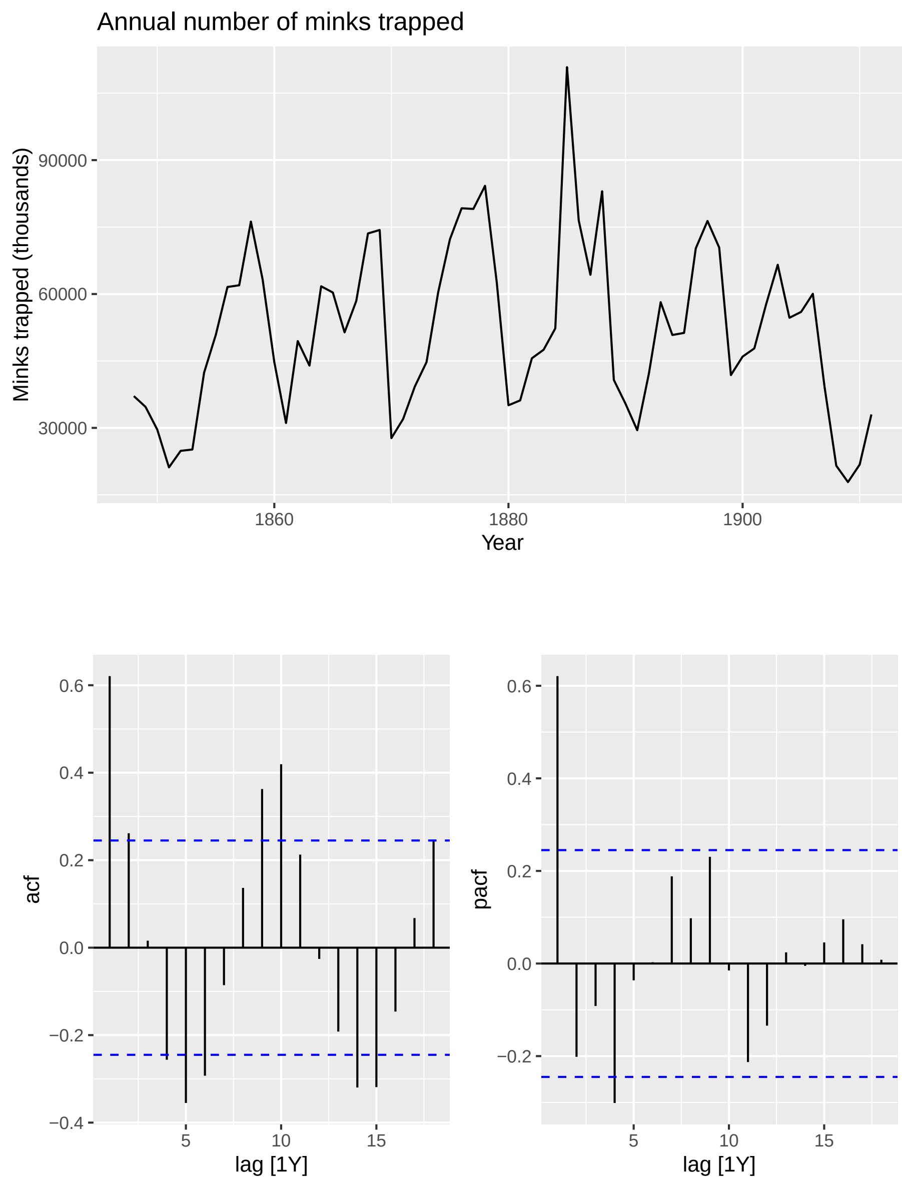
$$\alpha_k = k\text{th partial autocorrelation coefficient}$$

$$= \text{equal to the estimate of } \phi_k \text{ in regression:}$$

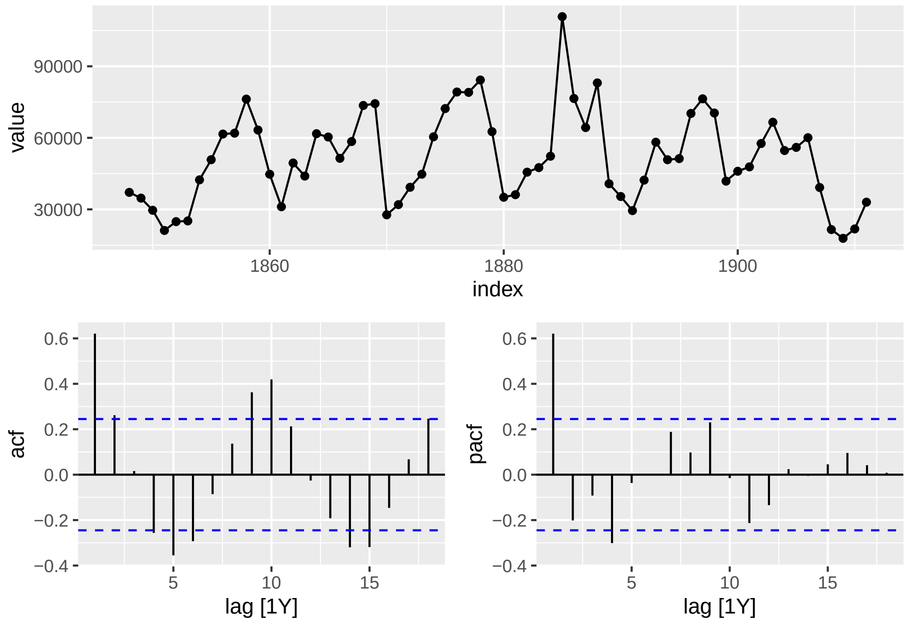
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}.$$

- Varying number of terms on RHS gives α_k for different values of k.
- $\alpha_1 = \rho_1$
- same critical values of $\pm 1.96/\sqrt{T}$ as for ACF.
- Last significant α_k indicates the order of an AR model.

2.3.2.1 Example: Mink trapping




```
mink %>% gg_tsdisplay(value, plot_type='partial')
```



2.3.3 ACF and PACF interpretation

AR(1)

$$\rho_k = \phi_1^k \qquad \text{for } k = 1, 2, \ldots;$$

$$\alpha_1 = \phi_1 \qquad \alpha_k = 0 \qquad \text{for } k = 2, 3, \ldots.$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

AR(p)

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the pth spike

So we have an AR(p) model when

- the ACF is exponentially decaying or sinusoidal

- there is a significant spike at lag p in PACF, but none beyond p

MA(1)

$$\rho_1 = \theta_1 \qquad \rho_k = 0 \qquad \text{for } k = 2, 3, \dots;$$

$$\alpha_k = -(-\theta_1)^k$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF

MA(q)

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the qth spike

So we have an MA(q) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag q in ACF, but none beyond q

2.3.4 Information criteria

Akaike's Information Criterion (AIC)

$$\text{AIC} = -2\log(L) + 2(p + q + k + 1),$$

where L is the likelihood of the data, $k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p + q + k + 1).$$

- Good models are obtained by minimizing either the AIC, AICc or BIC.
- Our preference is to use the AICc.

2.4 Seasonal ARIMA models

ARIMA	$\underbrace{(p,d,q)}$	$\underbrace{(P,D,Q)_m}$
	$\uparrow$ Non-seasonal part of the model	$\uparrow$ Seasonal part of of the model

where m = number of observations per year.

Example: ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t \;=\; (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$y_t = (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6}$$

$$-\Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-4} + \theta_1\Theta_1\varepsilon_{t-5}.$$

2.4.1 Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA(0,1,1)(0,1,1)$_m$	with log transformation
ARIMA(0,1,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,0)(0,1,1)$_m$	with log transformation
ARIMA(0,2,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,2)(0,1,1)$_m$	with no transformation

2.4.2 Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)$_{12}$ will show:

- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ….

ARIMA(0,0,0)(1,0,0)$_{12}$ will show:

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

2.5 Theoretical properties of the models

2.5.1 Autoregressive (AR) models

2.5.1.1 Properties of AR(1) model

Consider the following $AR(1)$ model.

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \epsilon_t \tag{2.1}$$

where ε_t is white noise.

2.5.1.1.1 Mean

Assuming that the series is weak stationary, we have $E(Y_t) = \mu$, $Var(Y_t) = \gamma_0$, and $Cov(Y_t, Y_{t-k}) = \gamma_k$, where μ and γ_0 are constants. Given that ϵ_t is a white noise, we have $E(\epsilon_t) = 0$. The mean of $AR(1)$ process can be computed as follows:

$$
\begin{aligned}
E(Y_t) &= E(\phi_0 + \phi_1 Y_{t-1}) \\
&= E(\phi_0) + E(\phi_1 Y_{t-1}) \\
&= \phi_0 + \phi_1 E(Y_{t-1}).
\end{aligned}
$$

Under the stationarity condition, $E(Y_t) = E(Y_{t-1}) = \mu$. Thus we get

$$\mu = \phi_0 + \phi_1 \mu.$$

Solving for μ yields

$$E(Y_t) = \mu = \frac{\phi_0}{1 - \phi_1}. \tag{2.2}$$

The results has two constraints for Y_t. First, the mean of Y_t exists if $\phi_1 \neq 1$. The mean of Y_t is zero if and only if $\phi_0 = 0$.

2.5.1.1.2 Variance and the stationary condition of AR (1) process

First take variance of both sides of Equation (2.1)

$$Var(Y_t) = Var(\phi_0 + \phi_1 Y_{t-1} + \epsilon_t)$$

The Y_{t-1} occurred before time t. The ϵ_t does not depend on any past observation. Hence, $cov(Y_{t-1}, \epsilon_t) = 0$. Furthermore, ϵ_t is a white noise. This gives

$$Var(Y_t) = \phi_1^2 Var(Y_{t-1}) + \sigma^2.$$

Under the stationarity condition, $Var(Y_t) = Var(Y_{t-1})$. Hence,

$$Var(Y_t) = \frac{\sigma^2}{1 - \phi_1^2}.$$

provided that $\phi_1^2 < 1$ or $|\phi_1| < 1$ (The variance of a random variable is bounded and non-negative). The necessary and sufficient condition for the $AR(1)$ model in Equation (2.1) to be weakly stationary is $|\phi_1| < 1$. This condition is equivalent to saying that the root of $1 - \phi_1 B = 0$ must lie outside the unit circle. This can be explained as below

Using the backshift notation we can write $AR(1)$ process as

$$Y_t = \phi_0 + \phi_1 B Y_t + \epsilon_t.$$

Then we get

$$(1 - \phi_1 B) Y_t = \phi_0 + \epsilon_t.$$

The $AR(1)$ process is said to be stationary if the roots of $(1 - \phi_1 B) = 0$ lie outside the unit circle.

2.5.1.1.3 Covariance

The covariance $\gamma_k = Cov(Y_t, Y_{t-k})$ is called the lag-k autocovariance of Y_t. The two main properties of γ_k: (a) $\gamma_0 = Var(Y_t)$ and (b) $\gamma_{-k} = \gamma_k$.

The lag-k autocovariance of Y_t is

$$\begin{aligned}\gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= E[(Y_t - \mu)(Y_{t-k} - \mu)] \\ &= E[Y_t Y_{t-k} - Y_t \mu - \mu Y_{t-k} + \mu^2] \\ &= E(Y_t Y_{t-k}) - \mu^2.\end{aligned} \tag{2.3}$$

Now we have

$$E(Y_t Y_{t-k}) = \gamma_k + \mu^2 \tag{2.4}$$

2.5.1.1.4 Autocorrelation function of an AR(1) process

To derive autocorrelation function of an AR(1) process we first multiply both sides of Equation (2.1) by Y_{t-k} and take expected values:

$$E(Y_t Y_{t-k}) = \phi_0 E(Y_{t-k}) + \phi_1 E(Y_{t-1} Y_{t-k}) + E(\epsilon_t Y_{t-k})$$

Since ϵ_t and Y_{t-k} are independent and using the results in Equation (2.4)

$$\gamma_k + \mu^2 = \phi_0 \mu + \phi_1 (\gamma_{k-1} + \mu^2)$$

Substituting the results in Equation (2.2) to Equation (2.4) we get

$$\gamma_k = \phi_1 \gamma_{k-1}. \tag{2.5}$$

The autocorrelation function, ρ_k, is defined as

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

.

Setting $k = 1$, we get $\gamma_1 = \phi_1\gamma_0$. Hence,

$$\rho_1 = \phi_1.$$

Similarly with $k = 2$, $\gamma_2 = \phi_1\gamma_1$. Dividing both sides by γ_0 and substituting with $\rho_1 = \phi_1$ we get

$$\rho_2 = \phi_1^2.$$

Now it is easy to see that in general

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k \tag{2.6}$$

for $k = 0, 1, 2, 3, ...$

Since $|\phi_1| < 1$, the autocorrelation function is an exponentially decreasing as the number of lags k increases. There are two features in the ACF of AR(1) process depending on the sign of ϕ_1. They are,

1. If $0 < \phi_1 < 1$, all correlations are positive.
2. if $-1 < \phi_1 < 0$, the lag 1 autocorrelation is negative ($\rho_1 = \phi_1$) and the signs of successive autocorrelations alternate from positive to negative with their magnitudes decreasing exponentially.

2.5.1.2 Properties of AR(2) model

Now consider a second-order autoregressive process (AR(2))

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t. \tag{2.7}$$

2.5.1.2.1 Mean

Question 1: Using the same technique as that of the AR(1), show that

$$E(Y_t) = \mu = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

and the mean of Y_t exists if $\phi_1 + \phi_2 \neq 1$.

2.5.1.2.2 Variance

Question 2: Show that

$$Var(Y_t) = \frac{(1-\phi_2)\sigma^2}{(1+\phi_2)((1+\phi_2)^2-\phi_1^2)}.$$

Here is a guide to the solution

Start with

$$Var(Y_t) = Var(\phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t)$$

Solve it until you obtain the Eq. (a) as shown below.

$$\gamma_0(1-\phi_1^2-\phi_2^2) = 2\phi_1\phi_2\gamma_1 + \sigma^2. \tag{a}$$

Next multiply both sides of Equation (2.7) by Y_{t-1} and obtain an expression for γ_1. Let's call this Eq. (b).

Solve Eq. (a) and (b) for γ_0.

2.5.1.2.3 Stationarity of AR(2) process

To discuss the stationarity condition of the $AR(2)$ process we use the roots of the characteristic polynomial. Here is the illustration.

Using the backshift notation we can write $AR(2)$ process as

$$Y_t = \phi_0 + \phi_1 BY_t + \phi_2 B^2 Y_t + \epsilon_t.$$

Furthermore, we get

$$(1-\phi_1 B - \phi_2 B^2)Y_t = \phi_0 + \epsilon_t.$$

The **characteristic polynomial** of $AR(2)$ process is

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2.$$

and the corresponding **AR characteristic equation**

$$1 - \phi_1 B - \phi_2 B^2 = 0.$$

For stationarity, the roots of AR characteristic equation must lie outside the unit circle. The two roots of the AR characteristic equation are

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

Using algebraic manipulation, we can show that these roots will exceed 1 in modulus if and only if simultaneously $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $|\phi_2| < 1$. This is called the stationarity condition of $AR(2)$ process.

2.5.1.2.4 Autocorrelation function of an AR(2) process

To derive autocorrelation function of an AR(2) process we first multiply both sides of Equation (2.7) by Y_{t-k} and take expected values:

$$\begin{aligned} E(Y_tY_{t-k}) &= E(\phi_0 Y_{t-k} + \theta_1 Y_{t-1}Y_{t-k} + \theta_2 Y_{t-2}Y_{t-k}) + \epsilon_t Y_{t-k} && (2.8) \\ &= \phi_0 E(Y_{t-k}) + \phi_1 E(Y_{t-1}Y_{t-k}) + \phi_2 E(Y_{t-2}Y_{t-k}) + E(\epsilon_t Y_{t-k}). && (2.9) \end{aligned}$$

Using the independence between ϵ_t and Y_{t-1}, $E(\epsilon_t Y_{t-k}) = 0$ and the results in Equation (2.4) (This is valid for AR(2)) we have

$$\gamma_k + \mu^2 = \gamma_0 \mu + \theta_1(\gamma_{k-1} + \mu^2) + \phi_2(\gamma_{k-2} + \mu^2).$$

(Note that $E(X_{t-1}X_{t-k}) = E(X_{t-1}X_{(t-1)-(k-1)} = \gamma_{k-1}))$

Solving for γ_k we get

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}. \tag{2.10}$$

By dividing the both sides of Equation (2.10) by γ_0, we have

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}. \tag{2.11}$$

for $k > 0$.

Setting $k = 1$ and using $\rho_0 = 1$ and $\rho_{-1} = \rho_1$, we get **the Yule-Walker equation for $AR(2)$ process.**

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

or

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}.$$

Similarly, we can show that

$$\rho_2 = \frac{\phi_2(1-\phi_2)+\phi_1^2}{(1-\phi_2)}.$$

2.5.1.3 Properties of AR(p) model

The pth order autoregressive model can be written as

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t. \tag{2.12}$$

The AR characteristic equation is

$$1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p = 0.$$

For stationarity of $AR(p)$ process, the p roots of the AR characteristic must lie outside the unit circle.

2.5.1.3.1 Mean

Question 3: Find $E(Y_t)$ of $AR(p)$ process.

2.5.1.3.2 Variance

Question 4: Find $Var(Y_t)$ of $AR(p)$ process.

2.5.1.3.3 Autocorrelation function (ACF) of an AR(p) process

Question 5: Similar to the results in Equation (2.11) for $AR(2)$ process, obtain the following recursive relationship for $AR(p)$.

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + ... + \phi_p \rho_{k-p}. \tag{2.13}$$

Setting $k = 1, 2, ..., p$ into Equation (2.13) and using $\rho_0 = 1$ and $\rho_{-k} = \rho_k$, we get the Yule-Walker equations for $AR(p)$ process

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2 \rho_1 + ... + \phi_p \rho_{p-1} \\
\rho_2 &= \phi_1 \rho_1 + \phi_2 + ... + \phi_p \rho_{p-2} \\
&... \\
\rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + ... + \phi_p
\end{aligned} \tag{2.14}$$

The Yule-Walker equations in (2.14) can be written in matrix form as below.

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_p \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{p-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & . & . & . & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ . \\ \phi_p \end{bmatrix}$$

or

$$_{\mathbf{p}} = \mathbf{P}_{\mathbf{p}} \; .$$

where,

$$_{\mathbf{p}} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_p \end{bmatrix}, \mathbf{P}_{\mathbf{p}} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{p-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & . & . & . & 1 \end{bmatrix}, \; = \begin{bmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ . \\ \phi_p \end{bmatrix}$$

The parameters can be estimated using

$$= \mathbf{P}_{\mathbf{p}}^{-1} \;_{\mathbf{p}}.$$

Question 6: Obtain the parameters of an $AR(3)$ process whose first autocorrelations are $\rho_1 = 0.9$; $\rho_2 = 0.9$; $\rho_3 = 0.5$. Is the process stationary?

2.5.1.3.4 The partial autocorrelation function (PACF)

Let ϕ_{ki}, the jth coefficient in an $AR(k)$ model. Then, ϕ_{kk} is the last coefficient. From Equation (2.13), the ϕ_{kj} satisfy the set of equations

$$\rho_j = \phi_{k1}\rho_{j-1} + ... + \phi_{k(k-1)}\rho_{j-k+1} + \phi_{kk}\rho_{j-k}, \tag{2.15}$$

for $j = 1, 2, ...k$, leading to the Yule-Walker equations which may be written

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_k \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{k-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & . & . & . & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ . \\ . \\ . \\ \phi_{kk} \end{bmatrix} \tag{2.16}$$

or

$$_{\mathbf{k}} = \mathbf{P_k}\,_{\mathbf{k}}.$$

where

$$_{\mathbf{k}} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_k \end{bmatrix}, \mathbf{P_k} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{k-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & . & . & . & 1 \end{bmatrix}, \,_{\mathbf{k}} = \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ . \\ . \\ . \\ \phi_{kk} \end{bmatrix}$$

For each k, we compute the coefficients ϕ_{kk}. Solving the equations for $k = 1, 2, 3...$ successively, we obtain

For $k = 1$,

$$\phi_{11} = \rho_1. \tag{2.17}$$

For $k = 2$,

$$\phi_{22} = \frac{\begin{bmatrix} 1 & \rho_2 \\ \rho_1 & \rho_2 \end{bmatrix}}{\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} \tag{2.18}$$

For $k = 3$,

$$\phi_{33} = \frac{\begin{bmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{bmatrix}}{\begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix}} \tag{2.19}$$

The quantity ϕ_{kk} is called the partial autocorrelation at lag k and can be defined as

$$\phi_{kk} = Corr(Y_t Y_{t-k} | Y_{t-1}, Y_{t-2}, ..., Y_{t-k+1}).$$

The partial autocorrelation between Y_t and Y_{t-k} is the correlation between Y_t and Y_{t-k} after removing the effect of the intermediate variables $Y_{t-1}, Y_{t-2}, ..., Y_{t-k+1}$.

In general the determinant in the numerator of Equations (2.17), (2.18) and (2.19) has the same elements as that in the denominator, but replacing the last column with $_{\mathbf{k}} = (\rho_1, \rho_2, ...\rho_k)$.

2.5.1.3.5 PACF for AR(1) models

From Equation (2.6) we have

$\rho_k = \phi_1^k$ for $k = 0, 1, 2, 3, ...$

Hence, for $k = 1$, the first partial autocorrelation coefficient is

$$\phi_{11} = \rho_1 = \phi_1.$$

From (2.18) for $k = 2$, the second partial autocorrelation coefficient is

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\phi_1^2 - \phi_1^2}{1 - \phi_1^2} = 0$$

.

Similarly, for $AR(1)$ we can show that $\phi_{kk} = 0$ for all $k > 0$. Hence, for $AR(1)$ process the partial autocorrelation is non-zero for lag 1 which is the order of the process, but is zero for lags beyond the order 1.

2.5.1.3.6 PACF for AR(2) model

Question 7: For $AR(2)$ process show that $\phi_{kk} = 0$ for all $k > 2$. Sketch the PACF of $AR(2)$ process.

2.5.1.3.7 PACF for AR(P) model

In general for $AR(p)$ precess, the partial autocorrelation function ϕ_{kk} is non-zero for k less than or equal to p (the order of the process) and zero for all k greater than p. In other words, the partial autocorrelation function of a $AR(p)$ process has a cut-off after lag p.

2.5.2 Moving average (MA) models

We first derive the properties of $MA(1)$ and $MA(2)$ models and then give the results for the general $MA(q)$ model.

2.5.2.1 Properties of MA(1) model

The general form for $MA(1)$ model is

$$Y_t = \theta_0 + \theta_1 \epsilon_{t-1} + \epsilon_t \tag{2.20}$$

where θ_0 is a constant and ϵ_t is a white noise series.

2.5.2.1.1 Mean

Question 8: Show that $E(Y_t) = \theta_0$.

2.5.2.1.2 Variance

Question 9: Show that $Var(Y_t) = (1 + \theta_1^2)\sigma^2$.

We can see both mean and variance are time-invariant. MA models are finite linear combinations of a white noise sequence. Hence, MA processes are always weakly stationary.

2.5.2.1.3 Autocorrelation function of an MA(1) process

Method 1

To obtain the autocorrelation function of $MA(1)$, we first multiply both sides of Equation (2.20) by Y_{t-k} and take the expectation.

$$\begin{aligned} E[Y_t Y_{t-k}] &= E[\theta_0 Y_{t-k} + \theta_1 \epsilon_{t-1} Y_{t-k} + \epsilon_t Y_{t-k}] \\ &= \theta_0 E(Y_{t-k}) + \theta_1 E(\epsilon_{t-1} Y_{t-k}) + E(\epsilon_t Y_{t-k}) \end{aligned} \tag{2.21}$$

Using the independence between ϵ_t and Y_{t-k} (future error and past observation) $E(\epsilon_t Y_{t-k}) = 0$. Now we have

$$E[Y_t Y_{t-k}] = \theta_0^2 + \theta_1 E(\epsilon_{t-1} Y_{t-k}) \tag{2.22}$$

Now let's obtain an expression for $E[Y_t Y_{t-k}]$.

$$\begin{aligned} \gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= E[(Y_t - \theta_0)(Y_{t-k} - \theta_0)] \\ &= E[Y_t Y_{t-k} - Y_t \theta_0 - \theta_0 Y_{t-k} + \theta_0^2] \\ &= E(Y_t Y_{t-k}) - \theta_0^2. \end{aligned} \tag{2.23}$$

Now we have

$$E(Y_t Y_{t-k}) = \gamma_k + \theta_0^2. \tag{2.24}$$

Using the Equations (2.22) and (2.24) we have

$$\gamma_k = \theta_0^2 - \theta_0^2 + \theta_1 E(\epsilon_{t-1} Y_{t-k}). \tag{2.25}$$

Now let's consider the case $k = 1$.

$$\gamma_1 = \theta_0^2 - \theta_0^2 + \theta_1 E(\epsilon_{t-1} Y_{t-1}) \tag{2.26}$$

Today's error and today's value are dependent. Hence, $E(\epsilon_{t-1} Y_{t-1}) \neq 0$. We first need to identify $E(\epsilon_{t-1} Y_{t-1})$.

$$E(\epsilon_{t-1} Y_{t-1}) = E(\theta_0 \epsilon_{t-1} + \theta_1 \epsilon_{t-2} \epsilon_{t-1} + \epsilon_{t-1}^2) \tag{2.27}$$

Since, $\{\epsilon_t\}$ is a white noise process $E(\epsilon_{t-1}) = 0$ and $E(\epsilon_{t-2}\epsilon_{t-1}) = 0$. Hence, we have

$$E(\epsilon_{t-1} Y_{t-1}) = E(\epsilon_{t-1}^2) = \sigma^2 \tag{2.28}$$

Substituting (2.28) in (2.26) we get

$$\gamma_1 = \theta_1 \sigma^2$$

.

Furthermore, $\gamma_0 = Var(Y_t) = (1 + \theta_1^2)\sigma^2$. Hence

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1 + \theta_1^2}.$$

When $k = 2$, from Equation (2.26) and $E(\epsilon_{t-1} Y_{k-2}) = 0$ (future error and past observation) we get $\gamma_2 = 0$. Hence $\rho_2 = 0$. Similarly, we can show that

$$\gamma_k = \rho_k = 0$$

for all $k \geq 2$.

We can see that the ACF of $MA(1)$ process is zero, beyond the order of 1 of the process.

Method 2: By using the definition of covariance

$$\begin{aligned} \gamma_1 = Cov(Y_t, Y_{t-1}) &= Cov(\epsilon_t + \theta_1 \epsilon_{t-1} + \theta_0, \epsilon_{t-1} + \theta_1 \epsilon_{t-2} + \theta_0) \\ &= Cov(\theta_1 \epsilon_{t-1}, \epsilon_{t-1}) \\ &= \theta_1 \sigma^2. \end{aligned} \tag{2.29}$$

$$\begin{aligned} \gamma_2 = Cov(Y_t, Y_{t-2}) &= Cov(\epsilon_t + \theta_1 \epsilon_{t-1} + \theta_0, \epsilon_{t-2} + \theta_1 \epsilon_{t-3} + \theta_0) \\ &= 0. \end{aligned} \tag{2.30}$$

We have $\gamma_0 = \sigma^2(1 + \theta_1^2)$, (Using the variance).

Hence

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1}{1+\theta_1^2}.$$

Similarly we can show $\gamma_k = \rho_k = 0$ for all $k \geq 2$.

2.5.2.2 Properties of MA(2) model

An $MA(2)$ model is in the form

$$Y_t = \theta_0 + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \epsilon_t \tag{2.31}$$

where θ_0 is a constant and ϵ_t is a white noise series.

2.5.2.2.1 Mean

Question 10: Show that $E(Y_t) = \theta_0$.

2.5.2.2.2 Variance

Question 11: Show that $Var(Y_t) = \sigma^2(1+\theta_1^2+\theta_2^2)$.

2.5.2.2.3 Autocorrelation function of an MA(2) process

Question 12: For $MA(2)$ process show that,

$$\rho_1 = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2},$$

$$\rho_2 = \frac{\theta_2}{1+\theta_1^2+\theta_2^2},$$

and $\rho_k = 0$ for all $k \geq 3$.

2.5.2.3 Properties of MA(q) model

$$Y_t = \theta_0 + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_q \epsilon_{t-q} + \epsilon_t \tag{2.32}$$

where θ_0 is a constant and ϵ_t is a white noise series.

2.5.2.3.1 Mean

Question 13: Show that the constant term of an MA model is the mean of the series (i.e. $E(Y_t) = \theta_0$).

2.5.2.3.2 Variance

Question 14: Show that the variance of an MA model is

$$Var(Y_t) = (1 + \theta_1^2 + \theta_2^2 + ... + \theta_q^2)\sigma^2.$$

2.5.2.3.3 Autocorrelation function of an MA(q) process

Question 15: Show that the autocorrelation function of a $MA(q)$ process is zero, beyond the order of q of the process. In other words, the autocorrelation function of a moving average process has a cutoff after lag q.

2.5.2.3.4 Partial autocorrelation function of an MA(q) process

The partial autocorrelation functions for $MA(q)$ models behave very much like the autocorrelation functions of $AR(p)$ models. The PACF of MA models decays exponentially to zero, rather like ACF for AR model.

2.5.3 Dual relation between AR and MA process

Dual relation 1

First we consider the relation AR(p) <−> MA(∞)

Let $AR(p)$ be a **stationary** AR model with order p. Then,

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t,$$

where $\epsilon_t \sim WN(0, \sigma^2)$.

Using the backshift operator we can write the $AR(p)$ model as

$$(1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^P)Y_t = \epsilon_t.$$

Then

$$\phi(B)Y_t = \epsilon_t,$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$. Furthermore, Y_t can be written as infinite sum of previous ϵ's as below

$$Y_t = \phi^{-1}(B)\epsilon_t,$$

where $\phi(B)\psi(B) = 1$ and $\psi(B) = 1 + \Psi_1 B + \psi_2 B^2 + ...$ Then

$$Y_t = \psi(B)\epsilon_t.$$

This is a representation of $MA(\infty)$ process.

Next, we consider the relation MA(q) <–> AR(∞)

Let $MA(q)$ be **invertible** moving average process

$$Y_t = \epsilon_t + \theta_t \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_p \epsilon_{t-q}.$$

Using the backshift operator we can write the $MA(q)$ process as

$$Y_t = (1 + \theta_1 B + \theta_2 B^2 - ... + \theta_q B^q)\epsilon_t.$$

Then,

$$Y_t = \theta(B)\epsilon_t,$$

where $\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + ... + \theta_1 B^q$. Hence, for an **invertible** moving average process, Y_t can be represented as a finite weighted sum of previous error terms, ϵ. Furthermore, since the process is invertible ϵ_t can be represented as an infinite weighted sum of previous Y's as below

$$\epsilon_t = \theta^{-1}(B)Y_t,$$

where $\pi(B)\theta(B) = 1$, and $\pi(B) = 1 + \pi_1 B + \pi B^2 + ...$ Hence,

$$\epsilon_t = \pi(B)Y_t.$$

This is an representation of a $AR(\infty)$ process.

Dual relation 2

An $MA(q)$ process has an ACF function that is zero beyond lag q and its PACF is decays exponentially to 0. Consequently, an $AR(p)$ process has an PACF that is zero beyond lag-p, but its ACF decays exponentially to 0.

Dual relation 3

For an $AR(p)$ process the roots of $\phi(B) = 0$ must lie outside the unit circle to satisfy the condition of stationarity. However, the parameters of the $AR(p)$ are not required to satisfy any conditions to ensure invertibility. Conversely, the parameters of the MA process are not required to satisfy any condition to ensure stationarity. However, to ensure the condition of invertibility, the roots of $\theta(B) = 0$ must lie outside the unit circle.

2.5.4 Autoregressive and Moving-average (ARMA) models

current value = linear combination of past values + linear combination of past error + current error

The $ARMA(p, q)$ can be written as

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_q \epsilon_{t-q} + \epsilon_t,$$

where $\{\epsilon_t\}$ is a white noise process.

Using the back shift operator

$$\phi(B)Y_t = \theta(B)\epsilon_t,$$

where $\phi(.)$ and $\theta(.)$ are the pth and qth degree polynomials,

$$\phi(B) = 1 - \phi_1 \epsilon - ... - \phi_p \epsilon^p,$$

and

$$\theta(B) = 1 + \theta_1 \epsilon + ... + \theta_q \epsilon^q.$$

2.5.4.1 Stationary condition

Roots of

$$\phi(B) = 0$$

lie outside the unit circle.

2.5.4.2 Invertible condition

Roots of

$$\theta(B) = 0$$

lie outside the unit circle.

2.5.4.3 Autocorrelation function and Partial autocorrelation function

The ACF of an ARMA model exhibits a pattern similar to that of an AR model. The PACF of ARMA process behaves like the PACF of a MA process. Hence, the ACF and PACF are not informative in determining the order of an ARMA model.

2.6 Thiyanga Section 2.3.4

2.7 Unit Root Test

slide 29 to 33

Note: Test whwre to put

References:

- Brockwell, P. J., Brockwell, P. J., Davis, R. A., & Davis, R. A. (2016). Introduction to time series and forecasting. springer.

Chapter 3

Exponential Smoothing

Bibliography

Brockwell, P. J., Brockwell, P. J., Davis, R. A., and Davis, R. A. (2016). *Introduction to time series and forecasting.* Springer.