Linear Time Series Analysis and Its Applications

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4 April 2020

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First, we will look at the theoretical properties of these models.

2 Autoregressive process

2.1 Properties of AR(1) model

Consider the following $AR(1)$ model.

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \epsilon_t \tag{1}$$

where ϵ_t is assumed to be a white noise process with mean zero and variance σ^2.

2.1.1 Mean

Assuming that the series is weak stationary, we have $E(Y_t) = \mu$, $Var(Y_t) = \gamma_0$, and $Cov(Y_t, Y_{t-k}) = \gamma_k$, where μ and γ_0 are constants. Given that ϵ_t is a white noise, we have $E(\epsilon_t) = 0$. The mean of $AR(1)$ process can be computed as follows:

$$\begin{aligned} E(Y_t) &= E(\phi_0 + \phi_1 Y_{t-1}) \\ &= E(\phi_0) + E(\phi_1 Y_{t-1}) \\ &= \phi_0 + \phi_1 E(Y_{t-1}). \end{aligned}$$

Under the stationarity condition, $E(Y_t) = E(Y_{t-1}) = \mu$. Thus we get

$$\mu = \phi_0 + \phi_1 \mu.$$

Solving for μ yields

$$E(Y_t) = \mu = \frac{\phi_0}{1 - \phi_1}. \tag{2}$$

The results has two constraints for Y_t. First, the mean of Y_t exists if $\phi \neq 1$. The mean of Y_t is zero if and only if $\phi_0 = 0$.

2.1.2 Variance and stationary condition of AR (1) process

First take variance of both sides of Eq. (1)

$$Var(Y_t) = Var(\phi_0 + \phi_1 Y_{t-1} + \epsilon_t)$$

The Y_{t-1} occurred before time t. The ϵ_t does not depend on any past observation. Hence, $cov(Y_{t-1}, \epsilon_t) = 0$. Furthermore, ϵ_t is a white noise and hence

$$Var(Y_t) = \phi_1^2 Var(Y_{t-1}) + \sigma^2.$$

Under the stationarity condition, $Var(Y_t) = Var(Y_{t-1})$, so that,

$$Var(Y_t) = \frac{\sigma^2}{1 - \phi_1^2}.$$

provided that $\phi_1^2 < 1$ or $|\phi| < 1$ (The variance of a random variable is bounded and non-negative). The necessary and sufficient condition for the $AR(1)$ model in Eq. (1) to be weakly stationary is $|\phi| < 1$. This condition is equivalent to saying that the root of $1 - \phi_1 B = 0$ must lie outside the unit circle. This can be explained as below

Using the backshift notation we can write $AR(1)$ process as

$$Y_t = \phi_0 + \phi_1 B Y_t + \epsilon_t.$$

Then we get

$$(1 - \phi_1 B) Y_t = \phi_0 + \epsilon_t.$$

The $AR(1)$ process is said to be stationary if the roots of $(1 - \phi_1 B) = 0$ lie outside the unit circle.

2.2 Covariance

The covariance $\gamma_k = Cov(Y_t, Y_{t-k})$ is called the lag-k autocovariance of Y_t. The two main properties of γ_k: (a) $\gamma_0 = Var(Y_t)$ and (b) $\gamma_{-k} = \gamma_k$.

The lag-k autocovariance of Y_t is

$$\begin{aligned}
\gamma_k &= Cov(Y_t, Y_{t-k}) \\
&= E[(Y_t - \mu)(Y_{t-k} - \mu)] \\
&= E[Y_t Y_{t-k} - Y_t \mu - \mu Y_{t-k} + \mu^2] \\
&= E(Y_t Y_{t-k}) - \mu^2.
\end{aligned} \tag{3}$$

Now we have

$$E(Y_t Y_{t-k}) = \gamma_k + \mu^2 \tag{4}$$

2.2.1 Autocorrelation function of an AR(1) process

To derive autocorrelation function of an AR(1) process we first multiply both sides of Eq. (1) by Y_{t-k} and take expected values:

$$E(Y_t Y_{t-k}) = \phi_0 E(Y_{t-k}) + \phi_1 E(Y_{t-1} Y_{t-k}) + E(\epsilon_t Y_{t-k})$$

Since ϵ_t and Y_{t-k} are independent and using the results in Eq. (4)

$$\gamma_k + \mu^2 = \phi_0 \mu + \phi_1(\gamma_{k-1} + \mu^2)$$

Substituting the results in Eq. (2) to Eq. (4) we get

$$\gamma_k = \phi_1 \gamma_{k-1}. \tag{5}$$

The autocorrelation function is defined as

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

.

Setting $k = 1$, we get $\gamma_1 = \phi_1 \gamma_0$. Hence,

$$\rho_1 = \phi_1.$$

Similarly with $k = 2$, $\gamma_2 = \phi_1 \gamma_1$. Dividing both sides by γ_0 and substituting with $\rho_1 = \phi_1$ we get

$$\rho_2 = \phi_1^2.$$

Now it is easy to see that in general

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k$$

for $k = 0, 1, 2, 3, ...$

Since $|\phi_1| < 1$, the autocorrelation function is an exponentially decreasing as the number of lags k increases. There are two features in the ACF of AR(1) process depending on the sign of ϕ_1. They are,

1. If $0 < \phi_1 < 1$, all correlations are positive.
2. if $-1 < \phi_1 < 0$, the lag 1 autocorrelation is negative ($\phi_1 = \phi_1$) and the signs of successive autocorrelations alternate from positive to negative with their magnitudes decreasing exponentially.

2.3 Properties of AR(2) model

Now consider an second-order autoregressive process (AR(2))

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t. \tag{6}$$

2.3.1 Mean

Question 1: Using the same technique as that of the AR(1), show that

$$E(Y_t) = \mu = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

and the mean of Y_t exists if $\phi_1 + \phi_2 \neq 1$.

2.3.2 Variance

Question 2: Show that

$$Var(Y_t) = \frac{(1 - \phi_2)\sigma^2}{(1 + \phi_2)((1 + \phi_2)^2 - \phi_1^2)}.$$

Here is a guide to the solution

Start with

$$Var(Y_t) = Var(\phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t)$$

Solve it until you obtain the Eq. (a) as shown below.

$$\gamma_0(1 - \alpha_1^2 - \alpha_1^2) = 2\alpha_1\alpha_2\gamma_1 + \sigma^2. \tag{a}$$

Next multiply both sides of Eq. (6) by Y_{t-1} and obtain a expression for γ_1. Let's call this Eq. (b).

Solve Eq. (a) and (b) for γ_0.

2.3.3 Stationarity of AR(2) process

To discuss the stationarity condition of the $AR(2)$ process we use the roots of the characteristic polynomial. Here is the illustration.

Using the backshift notation we can write $AR(2)$ process as

$$Y_t = \phi_0 + \phi_1 B Y_t + \phi_2 B^2 Y_t + \epsilon_t.$$

Furthermore, we get

$$(1 - \phi_1 B - \phi_2 B^2)Y_t = \phi_0 + \epsilon_t.$$

The **characteristic polynomial** of $AR(2)$ process is

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2.$$

and the corresponding **AR characteristic equation**

$$1 - \phi_1 B - \phi_2 B^2 = 0.$$

For stationarity, the roots of AR characteristic equation must lie outside the unit circle. The two roots of the AR characteristic equation are

$$\frac{\phi_1}{-2\phi_2}$$

Using algebraic manipulation, we can show that these roots will exceed 1 in modulus if and only if simultaneously $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $|\phi_2| < 1$. This is called the stationarity condition of $AR(2)$ process.

2.3.4 Autocorrelation function of an AR(2) process

To derive autocorrelation function of an AR(2) process we first multiply both sides of Eq. (6) by Y_{t-k} and take expected values:

$$E(Y_t Y_{t-k}) = E(\phi_0 Y_{t-k} + \alpha_1 Y_{t-1} Y_{t-k} + \alpha_2 Y_{t-2} Y_{t-k}) + \epsilon_t Y_{t-k} \quad (7)$$

$$= \phi_0 E(Y_{t-k}) + \phi_1 E(Y_{t-1} Y_{t-k}) + \phi_2 E(X_{t-2} X_{t-k}) + E(\epsilon_t X_{t-k}). \quad (8)$$

Using the independence between ϵ_t and Y_{t-1}, $E(\epsilon_t X_{t-k}) = 0$ and the results in Eq. 3 (This valid for AR(2)) we have

$$\gamma_k + \mu^2 = \gamma_0 \mu + \alpha_1(\gamma_{k-1} + \mu^2) + \phi_2(\gamma_{k-2} + \mu^2).$$

(Note that $E(X_{t-1} X_{t-k}) = E(X_{t-1} X_{(t-1)-(k-1)} = \gamma_{k-1})$)

Solving for γ_k we get

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}. \quad (9)$$

By dividing the both sides of Eq. (9) by γ_0, we have

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}. \quad (10)$$

for $k > 0$.

Setting $k = 1$ and using $\rho_0 = 1$ and $\rho_{-1} = \rho_1$, we get **the Yule-Walker equation for $AR(2)$ process.**

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

or

$$\rho_1 = \frac{\phi_1}{1-\phi_2}$$

.

Similarly, we can show that

$$\rho_2 = \frac{\phi_2(1-\phi_2)+\phi_1^2}{(1-\phi_2)}.$$

2.4 Properties of AR(p) model

The pth order autoregressive model can be written as

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \epsilon_t. \tag{11}$$

The AR characteristic equation is

$$1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p = 0.$$

For stationarity of $AR(p)$ process, the p roots of the AR characteristic must lie outside the unit circle.

2.4.1 Mean

Question: Find $E(Y_t)$ of $AR(p)$ process.

2.4.2 Variance

Question: Find $Var(Y_t)$ of $AR(p)$ process.

2.4.3 Autocorrelation function of an AR(p) process

Question: Assuming stationarity and multiplying both sides of Eq. (11) obtain the following recursive relationship.

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + ... + \phi_p\rho_{k-p}. \tag{12}$$

Setting $k = 1, 2, ..., p$ into Eq. (11) and using $\rho_0 = 1$ and $\rho_{-k} = \rho_k$, we get the Yule-Walker equations for $AR(p)$ process

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + ... + \phi_p\rho_{p-1} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + ... + \phi_p\rho_{p-2} \\
&... \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + ... + \phi_p
\end{aligned} \tag{13}$$

3 Moving average (MA) models

We first derive the properties of $MA(1)$ and $MA(2)$ models and then give the results for the general $MA(q)$ model.

3.1 Properties of MA(1) model

3.1.1 Mean

3.1.2 Variance

3.1.3 Autocorrelation function of an MA(1) process

3.1.4 Partial autocorrelation function of an MA(1) process

3.2 Properties of MA(2) model

3.2.1 Mean

3.2.2 Variance

3.2.3 Autocorrelation function of an MA(2) process

3.2.4 Partial autocorrelation function of an MA(2) process

3.3 Properties of MA(p) model

3.3.1 Mean

3.3.2 Variance

3.3.3 Autocorrelation function of an MA(q) process

3.3.4 Partial autocorrelation function of an MA(q) process

4 Autoregressive and Moving-average (ARMA) models

5 Unit root nonstationarity

6 ARIMA models

7 SARIMA models

8 Forecasting model building process using R

9 Appendix

Some important rules in statistics

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X+Y)$$