Stellar Structure on the Zero-Age Main Sequence

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1. Introduction

Stellar structure has been a topic of deep study for decades, in part because of its complexity. Stellar structure is a complicated interplay among a variety of different physical fields: quantum mechanics, statistical mechanics, thermodynamics, hydrodynamics, radiative transport, particle physics, nuclear processes, and electromagnetism, to name a few. For these reasons, modeling stellar structure is difficult, even numerically. In this project, I will attempt to build a model of stellar structure for main sequence stars, specifically considering the test case of the Sun. In particular, I will be attempting to model the zero-age main sequence (ZAMS); this allows for the simplification that the chemical composition is isotropic throughout the star. At later stages in a star's life, this assumption is no longer valid (for example, the core will become enriched in heavier elements).

3. Equations of Stellar Structure and Boundary Conditions

Considering the star as a series of mass shells, we can write the primary differential equations as

$$\frac{\partial r(m)}{\partial m} = \frac{1}{4\pi\, r^2(m)\, \rho(m)}$$

$$\frac{\partial P(m)}{\partial m} = -\frac{Gm}{4\pi\, r^4(m)}$$

$$\frac{\partial L(m)}{\partial m} = \epsilon(m)$$

$$\frac{\partial T(m)}{\partial m} = \begin{cases} -\dfrac{3\kappa\, L(m)}{256\pi^2 \sigma_{SB}\, r^4(m)\, T^3(m)} & \text{(radiative/conductive)} \\ \left(1 - \dfrac{1}{\gamma}\right)\dfrac{T(m)}{P(m)}\dfrac{\partial P(m)}{\partial m} & \text{(convective)} \end{cases}$$

These are governed by the equations of state

$$P = P(\vec{X}, \rho, T) = P_{\text{ions}} + P_e + P_{\text{rad}} \Rightarrow \rho = \rho(\vec{X}, P, T)$$

$$\epsilon = \epsilon(\vec{X}, \rho, T) = \epsilon_{pp} + \epsilon_{\text{CNO}} + \epsilon_{3\alpha}$$

$$\kappa = \kappa(\vec{X}, \rho, T) = \left[\frac{1}{\kappa_{e,\text{rad}} + \kappa_{\text{f-f}} + \kappa_{\text{b-f}} + \kappa_{\text{H}^-}} + \frac{1}{\kappa_{e,\text{cond}} + \kappa_{\gamma,\text{cond}}}\right]^{-1}$$

I assume the following ZAMS boundary conditions:

Outer	Inner
$m = M$	$\lim_{m\to 0} m = M_0 \ll M$
$L(M) = \left(\frac{M}{M_\odot}\right)^{3.5} L_\odot + \delta L$	$r(M_0) = \left(\frac{3}{4\pi}\frac{M_0}{\rho(M_0)}\right)^{1/3}$
$r(M) = \begin{cases} \left(\frac{M}{M_\odot}\right)^{0.8} R_\odot, M \le M_\odot \\ \left(\frac{M}{M_\odot}\right)^{0.57} R_\odot, M > M_\odot \end{cases} + \delta r$	$P(M_0) = 7.701\frac{GM^2}{r^4(M)} + \delta P$
$T(M) = \left(\frac{L(M)}{4\pi\sigma_{SB}\, r^2(M)}\right)^{1/4}$	$T(M_0) = \frac{2}{3}\frac{Gm_u}{k_B}\frac{M}{r(M)} + \delta T$
$P(M) = \frac{2}{3}\frac{GM}{\bar{\kappa}\, r^2(M)} + \frac{2}{3c}\frac{L(M)}{4\pi\, r^2(M)}$	$L(M_0) = M_0\, \epsilon\left(\vec{X}, \rho(M_0), T(M_0)\right)$

2. Variables and Constants

The variables used in this project include

Symbol	Meaning	Units
m	Mass coordinate	kg
r	Radius	m
P	Pressure	N/m^2
L	Luminosity	W
T	Temperature	K
$\vec{X}$	Mass fractions (chemical composition)	–
ρ	Density	kg/m^3
ϵ	Reduced luminosity	W/kg
κ	Opacity	$\text{m}^2\text{/kg}$
γ	Adiabatic index	–
μ	Mean molecular weight	Atomic Mass Units (AMU)

with constants

Constant	Meaning
$G = 6.673 \cdot 10^{-11}\ \text{m}^3/\text{kg s}^2$	Gravitational constant
$\sigma_{SB} = 5.6704 \cdot 10^{-8}\ \text{W/m}^2\ \text{K}^4$	Stefan-Boltzmann constant
$\gamma = \begin{cases} 5/3 \text{ (non-relativistic)} \\ 4/3 \text{ (relativistic)} \end{cases}$	Adiabatic index
$M_\odot = 1.9891 \cdot 10^{30}$ kg	Solar mass
$L_\odot = 3.84 \cdot 10^{26}$ W	Solar luminosity
$R_\odot = 6.95508 \cdot 10^{8}$ m	Solar radius
$c = 2.99792458 \cdot 10^{8}$ m/s	Speed of light
$m_u = 1.660538782 \cdot 10^{-27}$ kg	Nucleon mass
$k_B = 1.3806504 \cdot 10^{-23}$ J/K	Boltzmann constant

4. Computational Methods

This is a boundary value problem that we can solve with relaxation. However, we only know 4 of the 8 boundary conditions – we need to guess the other 4. Therefore, we can integrate in both directions until our solutions meet in the middle. If they don't match, we can perturb our guessed boundary conditions to determine how we should update them for a better-matching solution. Therefore, the computational algorithm is:

1. Specify total mass M and chemical composition $\vec{X} = [X_{\text{H}}, X_{\text{He}}, X_{\text{Li}}, \dots]$

Perturbations Pass

Integration Pass

2. Specify internal ($m = M_0 \ll M$) and surface ($m = M$) boundary conditions
3. Initialize variable arrays $\vec{z}_1$ (outward) and $\vec{z}_2$ (inward)

Integration Step

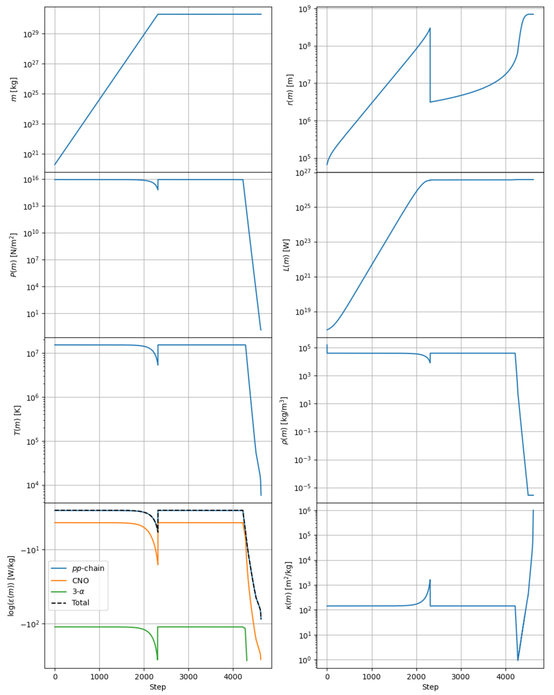
4. Calculate intermediate variables (e.g., ρ, κ, ϵ) using EOS
5. Calculate derivatives $\frac{\partial \vec{z}_i}{\partial m}$
6. Set stepsizes $h_i \sim \min\left(\frac{\partial \ln(\vec{z}_i)}{\partial m}\right)$ with $h_2 < 0$
7. Update solutions using 4th-order Runge-Kutta integration
8. Integration step until $m_1 = m_2 = m_{mid}$
9. Calculate residuals $\vec{\mathcal{R}} = \vec{z}_2(m_{mid}) - \vec{z}_1(m_{mid})$
10. Perturb each guessed boundary condition and calculate new residuals (integration pass for each)
11. Calculate updates to boundary conditions ($\delta\vec{z}$) by solving matrix equation $\sum_j \frac{\partial R_i}{\partial z_j}\, \delta z_j = -\mathcal{R}_i$
12. Perturbations pass until residuals are below some desired error tolerance

5. Results

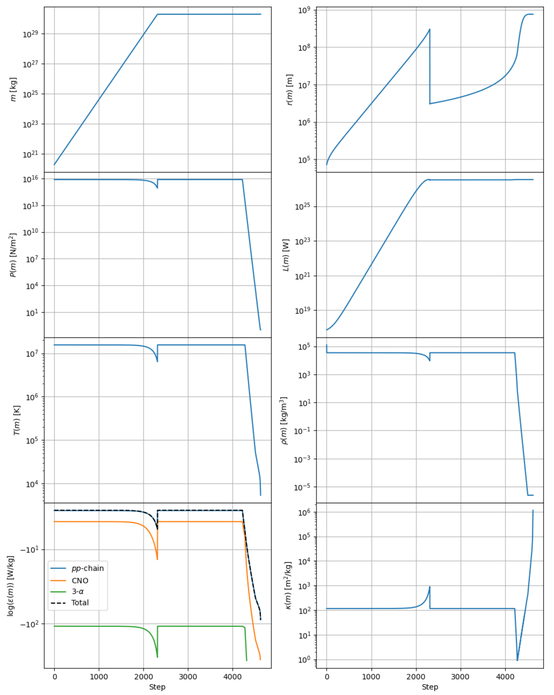
The results from the initial (unperturbed) integration pass are shown below as **Pass 0**. The mass increases substantially when integrating outward, but sees very little change when integrating inward. For the outward integration, the limiting variable appears to be luminosity; for the inward integration, the limiting variable appears to be radius. There are clear discontinuities in all variables at the meeting point except luminosity. There appear to be additional discontinuities early in the inward integration; these may be due to a transition from convective to radiative transport in the outer envelope.

After 6 perturbation passes, the results do not appear noticeably different. I can confirm that they are not identical, just very similar. Part of this is scale – since all variables span multiple orders of magnitude in this problem, discerning differences of $\mathcal{O}(\lesssim 10\%)$ (like the luminosity residual) is very difficult. There is an additional effect in that I imposed the constraint $\left|\frac{\delta z_i}{z_i}\right| \le 10\%$ when updating the boundary conditions to ensure the integration was still well-behaved, which likely hindered the overall convergence rate.

Pass 0



Pass 6



6. Conclusions

Based on these results, integrating the equations of stellar structure for a continuous solution is possible, but it is computationally expensive and slow (especially in `Python`). The simulation correctly reproduces:

- Proper directional behavior for all variables
  - m, r, L, κ increase outward
  - P, T, ρ, ϵ increase inward (and all display similar behavior)
- Energy production is dominated by the pp-chain at all points in the Sun

7. Future Work

There are a handful of ways in which this work could be improved, including but not limited to:

- More sophisticated perturbation calculations to improve convergence
- Running for additional masses/chemical compositions (does it recover the zero-age main sequence?)
- Extending to non-main sequence stars:
  - Giant stars
    - Requires new boundary conditions (different scaling relations)
    - Chemical composition will vary throughout the star (e.g., He-burning core with H-burning shell and H envelope)
  - White dwarfs
    - Requires new boundary conditions (different scaling relations)
    - No internal energy production – would need a new way to calculate luminosity