

1 Analytical treatment

We start from following Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{e}{\hbar} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{k}} + \mathbf{v}(\mathbf{k}) \frac{\partial f}{\partial r} = \left(\frac{\partial f}{\partial t} \right)_{st} \quad (1)$$

We consider case when $f(\dots)$ is spatially homogeneous, i.e. it depends only on \mathbf{k} , then

$$\frac{\partial f}{\partial t} + \frac{e}{\hbar} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{k}} = \frac{f_0 - f}{\tau} \quad (2)$$

Electric field is along x-axis and magnetic field is along z-axis.

$$\mathbf{E} = (E, 0, 0) \quad (3)$$

$$\mathbf{B} = (0, 0, B) \quad (4)$$

$$\mathbf{v} \times \mathbf{B} = (v_y B, -v_x B, 0) \quad (5)$$

And will make following substitution

$$t \rightarrow \tau t \quad (6)$$

which gives following form to the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{ed\tau}{\hbar} (E + v_y B) \frac{\partial f}{\partial (k_x d)} - \frac{ed\tau}{\hbar} v_x B \frac{\partial f}{\partial (k_y d)} = f_0 - f \quad (7)$$

where

$$\mathbf{v} = \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \quad (8)$$

and ε is an energy of an electron in the zone, which in tight-binding approximation takes form in the first zone

$$\varepsilon = \frac{\hbar^2 k_y^2}{2m} - \frac{\Delta_1}{2} \cos(k_x d) \quad (9)$$

therefore

$$v_x = \frac{\Delta_1 d}{2\hbar} \sin(dp_x/h) \quad (10)$$

$$v_y = \frac{\hbar}{md} \phi_y \quad (11)$$

And Boltzmann equations now takes following form

$$\frac{\partial f}{\partial t} + \left(\frac{ed\tau}{\hbar} E + \frac{ed\tau}{m} k_y B \right) \frac{\partial f}{\partial (k_x d)} - \frac{\Delta_1 ed^2 \tau}{2\hbar^2} B \sin(k_x d) \frac{\partial f}{\partial (k_y d)} = f_0 - f \quad (12)$$

Now we can define

$$E_* = \frac{\hbar}{ed\tau} \quad (13)$$

$$m_x = \frac{2\hbar^2}{\Delta_1 d^2} \quad (14)$$

$$(15)$$

$$\boxed{\alpha = m/m_x} \quad (16)$$

Then, with cyclotron frequency defined as

$$\omega_c = \frac{eB}{\sqrt{mm_x}} \quad (17)$$

we get following

$$\frac{\partial f}{\partial t} + \left(\frac{E}{E_*} + \frac{\omega_c \tau}{\sqrt{\alpha}} k_y d \right) \frac{\partial f}{\partial(k_x d)} - \sqrt{\alpha} \omega_c \tau \sin(k_x d) \frac{\partial f}{\partial(k_y d)} = f_0 - f \quad (18)$$

And if we know define

$$\phi_x = k_x d \quad (19)$$

$$\phi_y = \frac{k_y d}{\sqrt{\alpha}} \quad (20)$$

$$\tilde{E} = E/E_* \quad (21)$$

$$\tilde{B} = \omega_c \tau \quad (22)$$

Boltzmann equations takes form

$$\boxed{\frac{\partial f}{\partial t} + (\tilde{E} + \tilde{B} \phi_y) \frac{\partial f}{\partial \phi_x} - \tilde{B} \sin(\phi_x) \frac{\partial f}{\partial \phi_y} = f_0 - f} \quad (23)$$

Alternatively, if we define Bloch frequency as

$$\omega_B = \frac{edE}{\hbar} \quad (24)$$

it can be written as

$$\frac{\partial f}{\partial t} + (\omega_B \tau + \omega_c \tau \phi_y) \frac{\partial f}{\partial \phi_x} - \omega_c \tau \sin(\phi_x) \frac{\partial f}{\partial \phi_y} = f_0 - f \quad (25)$$

Ratio of Bloch and cyclotron oscillation frequencies will be important to us down the road. For f_0 we use Boltzmann distribution

$$f_0 \propto e^{-\frac{\varepsilon}{k_b T}} \quad (26)$$

which in our variables ϕ_x and ϕ_y takes form

$$f_0 = C \exp \left\{ \mu \cos(\phi_x) - \frac{\mu}{2} \phi_y^2 \right\} \quad (27)$$

$$\boxed{\mu = \frac{\Delta_1}{2k_b T}} \quad (28)$$

Now, constant C must be such that dimensionless norm of f_0 is 1. Therefore

$$\frac{1}{C} = \frac{d^2}{\hbar^2} \int_{-\infty}^{+\infty} dp_y \int_{p_x \in \text{BZ}} dp_x \exp \left\{ \mu \cos(\phi_x) - \frac{\mu}{2} \phi_y^2 \right\} \quad (29)$$

$$= \sqrt{\alpha} \int_{-\infty}^{+\infty} e^{-\mu \phi_y^2 / 2} d\phi_y \int_{-\pi}^{\pi} e^{\mu \cos(\phi_x)} d\phi_x \quad (30)$$

$$= 2\pi I_0(\mu) \sqrt{\frac{2\pi\alpha}{\mu}} \quad (31)$$

Thus full form of equilibrium distribution is

$$f_0 = \frac{1}{2\pi I_0(\mu)} \sqrt{\frac{\mu}{2\pi\alpha}} \exp \left\{ \mu \cos(\phi_x) - \frac{\mu}{2} \phi_y^2 \right\} \quad (32)$$

This way, in total, we have four free parameters defining our system, μ and α that characterize lattice parameters such as lattice period and band width and temperature, and \tilde{E} , \tilde{B} that specify external fields. One of the parameters that we will be calculating is drift velocity v_{dr} , which we will define like this

$$v_{dr} = \frac{2d}{\Delta_1 \hbar} \iint \frac{\partial \varepsilon}{\partial p_x} f(p_x, p_y) dp_x dp_y \quad (33)$$

$$= \sqrt{\alpha} \int_{-\pi}^{\pi} d\phi_x \int_{-\infty}^{+\infty} d\phi_y \sin(\phi_x) f(\phi_x, \phi_y) \quad (34)$$

When magnetic field is 0 and E is constant, i.e. E_{dc} , analytic solution Boltzmann equation is well known and as well as expression for drift velocity

$$v_{dr} = \left\{ \sqrt{\alpha} \int_{-\pi}^{\pi} d\phi_x \int_{-\infty}^{+\infty} d\phi_y \sin(\phi_x) f_0(\phi_x, \phi_y) \right\} \frac{E_{dc}/E_*}{1 + (E_{dc}/E_*)^2} \quad (35)$$

$$= \frac{I_1(\mu)}{I_0(\mu)} \frac{E_{dc}/E_*}{1 + (E_{dc}/E_*)^2} \quad (36)$$

From which follows that peak value of v_{dr} is at $E_{dc}/E_* = 1$ and is

$$v_p = \frac{I_1(\mu)}{2I_0(\mu)} \quad (37)$$

Later, down the road, we will be plotting not v_{dr} , but v_{dr}/v_p , which for dc electric field only take very simple form, known as "Esaki-Tsu" equation

$$\frac{v_{dr}}{v_p} = 2 \frac{E_{dc}/E_*}{1 + (E_{dc}/E_*)^2} \quad (38)$$

In general we will be applying a/c emf in the form

$$\tilde{E} = \tilde{E}_{dc} + \tilde{E}_{\omega} \cos(\omega t) \quad (39)$$

and in case when magnetic field is not applied we also have analytic expression for v_{dr} , which is known as "Tien-Gordon" equation. Analytic expression, "Taker formulae", is also known for absorption, which we will define like this

$$A = \left\langle \frac{v_{dr}}{v_p} \cos(\omega t) \right\rangle_t \quad (40)$$

In case when magnetic field is applied, however, analytic expression for absorption is not known. And that is the quantity we are most interested in.

Now due to periodicity of $f(\phi_x, \phi_y)$ along ϕ_x with period 2π and additionally $f_0(-\phi_x, \phi_y) = f_0(\phi_x, \phi_y)$, it makes sense for us to expand f and f_0 into Fourier series.

$$f_0 = \sum_{n=0}^{\infty} a_n^{(0)} \cos(n\phi_x) \quad (41)$$

$$f = \sum_{n=0}^{\infty} a_n \cos(n\phi_x) + b_n \sin(n\phi_x) \quad (42)$$

where coefficients $a_n^{(0)}$, a_n and b_n in general will depend on ϕ_y and last two also on time t . Note, that $a_{n<0} \equiv 0$ and $b_{n<1} \equiv 0$. And with the form of f_0 , as selected above, $a_n^{(0)}$ becomes

$$a_n^{(0)} = \frac{\sigma(n)}{\pi} \int_{-\pi}^{\pi} f_0(\phi_x, \phi_y) \cos(n\phi_x) d\phi_x \quad (43)$$

$$= \frac{\sigma(n)I_n(\mu)}{\pi I_0(\mu)} \sqrt{\frac{\mu}{2\pi\alpha}} \exp\left\{-\frac{\mu}{2}\phi_y^2\right\} \quad (44)$$

where

$$\sigma(n) = \begin{cases} 1/2 & : n = 0 \\ 1 & : n \neq 1 \end{cases} \quad (45)$$

Norm of $f(\phi_x, \phi_y)$ must always be one, i.e.

$$\sqrt{\alpha} \int_{-\pi}^{\pi} d\phi_x \int_{-\infty}^{+\infty} d\phi_y f(\phi_x, \phi_y) = 1 \quad (46)$$

which in fourier representation takes form

$$\boxed{2\pi\sqrt{\alpha} \int_{-\infty}^{+\infty} a_0(\phi_y) d\phi_y = 1} \quad (47)$$

Once we move on to the numerical calculations this equation can be used to check correctness. And from equation (34) it is clear that only b_1 will survive. And calculation of v_{dr} is done through following equation

$$v_{dr} = \sqrt{\alpha} \int_{-\infty}^{+\infty} d\phi_y \int_{-\pi}^{\pi} d\phi_x \sin(\phi_x) b_1(\phi_y) \sin(\phi_y) \quad (48)$$

$$= \pi\sqrt{\alpha} \int_{-\infty}^{+\infty} b_1(\phi_y) d\phi_y \quad (49)$$

and in view of definition of peak value of v_{dr} in eq. (37)

$$\boxed{\frac{v_{dr}}{v_p} = \frac{2I_0(\mu)\pi\sqrt{\alpha}}{I_1(\mu)} \int_{-\infty}^{+\infty} b_1(\phi_y) d\phi_y} \quad (50)$$

In addition to drift velocity along x -axis we can look at drift velocity along y -axis, which we will define like this

$$v_y = \frac{2}{\Delta_1} \iint \frac{\partial \varepsilon}{\partial p_y} f(\phi_x, \phi_y) dp_x dp_y \quad (51)$$

$$= \int_{-\pi}^{\pi} d\phi_x \int_{-\infty}^{+\infty} \phi_y f(\phi_x, \phi_y) d\phi_y \quad (52)$$

$$= 2\pi \int_{-\infty}^{+\infty} a_0(\phi_y) \phi_y d\phi_y \quad (53)$$

However just as with drift velocity along x -axis we will be working with ratio of v_y and peak velocity v_p .

$$\boxed{\frac{v_y}{v_p} = \frac{4\pi I_0(\mu)}{I_1(\mu)} \int_{-\infty}^{+\infty} a_0(\phi_y) \phi_y d\phi_y} \quad (54)$$

This way we accounted for meaning of a_0 and b_1 . Let us now take a look at a_1 . Effective mass of an electron in the x direction is given by

$$m_{x,\mathbf{k}}^{-1} = \frac{1}{\hbar^2} \frac{\partial^2 \varepsilon}{\partial k_x^2} \quad (55)$$

We will be working however with ratio of electron mass to $m_{x,\mathbf{k}}$. And in our approximation of energy (9) that takes form

$$\frac{m}{m_{x,\mathbf{k}}} = \frac{\Delta_1 d^2 m}{2\hbar^2} \cos(k_x d) \quad (56)$$

$$= \alpha \cos(\phi_x) \quad (57)$$

To gather back this value from $f(\phi_x, \phi_y)$ we have to integrate over $\{p_x, p_y\}$ and to maintain dimensionlessness of ratio of these masses, this integration will take form

$$\frac{m}{m_{x,\mathbf{k}}} = \alpha^{3/2} \frac{d^2}{\hbar^2} \iint f(\phi_x, \phi_y) \cos(k_x d) dp_x dp_y \quad (58)$$

$$= \alpha^{3/2} \int_{-\pi}^{\pi} d\phi_x \int_{-\infty}^{+\infty} d\phi_y f(\phi_x, \phi_y) \cos(\phi_x) \quad (59)$$

$$= \alpha^{3/2} \int_{-\pi}^{\pi} d\phi_x \int_{-\infty}^{+\infty} d\phi_y a_1(\phi_y) \cos(\phi_x) \cos(\phi_x) \quad (60)$$

Finally giving us following

$$\boxed{\frac{m}{m_{x,\mathbf{k}}} = \pi \alpha^{3/2} \int_{-\infty}^{+\infty} a_1(\phi_y) d\phi_y} \quad (61)$$

Now using fourier representation of $f(\phi_x, \phi_y)$ and $f_0(\phi_y)$ (equations 41, 42) we can rewrite Boltzmann equation (23) like this

$$\sum_{(n)} \left\{ \frac{\partial a_n}{\partial t} \cos(n\phi_x) + \frac{\partial b_n}{\partial t} \sin(n\phi_x) = a_n^{(0)} \cos(n\phi_x) - a_n \cos(n\phi_x) - b_n \sin(n\phi_x) + \right. \\ \left. n(\tilde{E} + \tilde{B}\phi_y)(a_n \sin(n\phi_x) - b_n \cos(n\phi_x)) + \tilde{B} \frac{\partial a_n}{\partial \phi_y} \sin(\phi_x) \cos(n\phi_x) + \tilde{B} \frac{\partial b_n}{\partial \phi_y} \sin(\phi_x) \sin(n\phi_x) \right\} \quad (62)$$

In absence of magnetic field there is no mixing of different harmonic, however when \tilde{B} is not 0 then harmonics will be come mixed due to presence of $\sin(\phi_x) \cos(n\phi_x)$ and $\sin(\phi_x) \sin(n\phi_x)$, since

$$\sin(\phi_x) \cos(n\phi_x) = \{\sin((n+1)\phi_x) - \sin((n-1)\phi_x)\} / 2 \quad (63)$$

$$\sin(\phi_x) \sin(n\phi_x) = \{\cos((n-1)\phi_x) - \cos((n+1)\phi_x)\} / 2 \quad (64)$$

And using this equations, after some manipulation of symbols, combining elements with the same harmonics, and noting special treatment of b_1 we get

$$\frac{\partial a_n}{\partial t} = a_n^{(0)} - a_n - n(\tilde{E} + \tilde{B}\phi_y)b_n + \tilde{B} \left(\frac{\partial b_{n+1}}{\partial \phi_y} - \frac{\partial b_{n-1}}{\partial \phi_y} \right) \quad (65)$$

$$\frac{\partial b_n}{\partial t} = -b_n - n(\tilde{E} + \tilde{B}\phi_y)a_n + \tilde{B} \left(\chi(n) \frac{\partial a_{n-1}}{\partial \phi_y} - \frac{\partial a_{n+1}}{\partial \phi_y} \right) \quad (66)$$

where

$$\chi(n) = \begin{cases} 2 & : n = 1 \\ 1 & : n \neq 1 \end{cases} \quad (67)$$

The quantity we are most intersted is absorption defined like this

2 Numerical solution

Straightforward application of method of finite differences to (65) and (66) leads to either unstable or difficult, i.e. computationally intensive, equations. To combat this problem I am using several methods at once. First, we are going to discretize a_n and b_n along time and ϕ_y axes.

$$\begin{aligned} t &\leftarrow \text{time step} \\ a_{n,m} &\leftarrow \phi_y \text{ lattice step} \end{aligned} \tag{68}$$

and n is "harmonic number".

TODO: Complete this section

3 Results

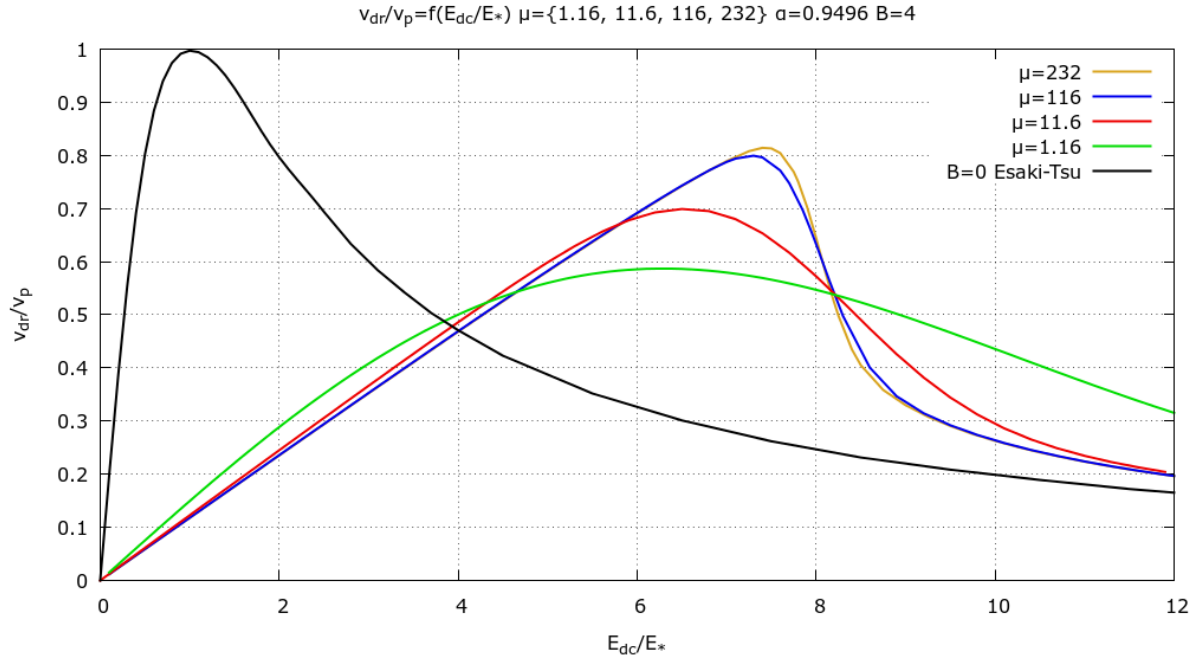


Figure 1:

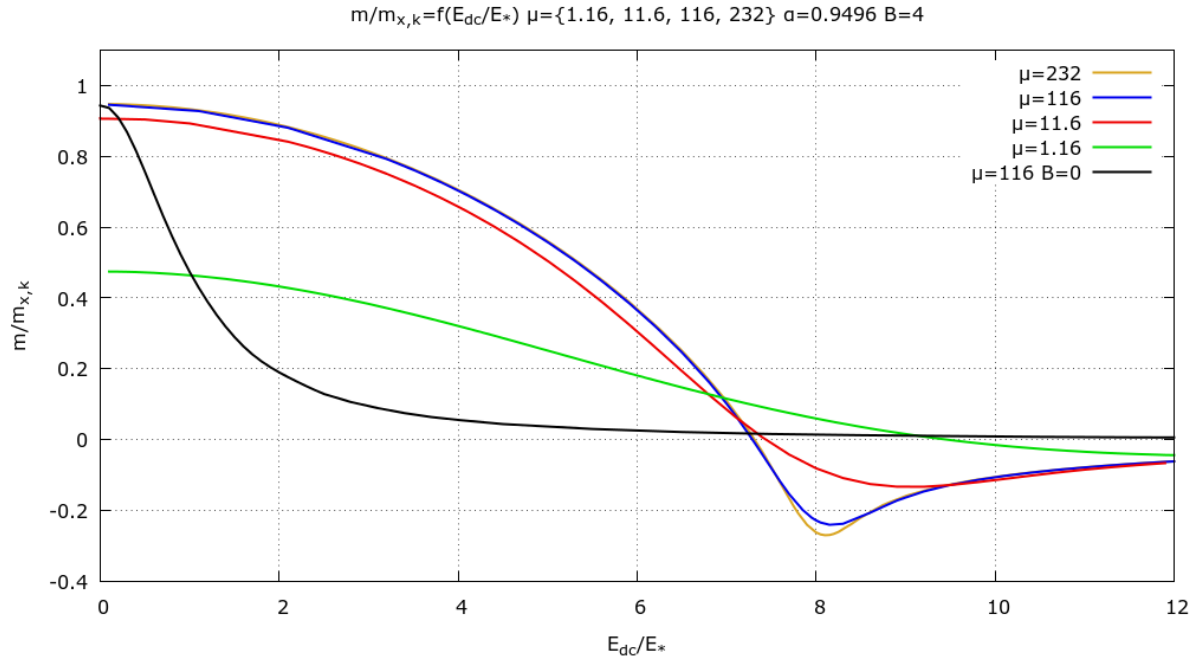


Figure 2:

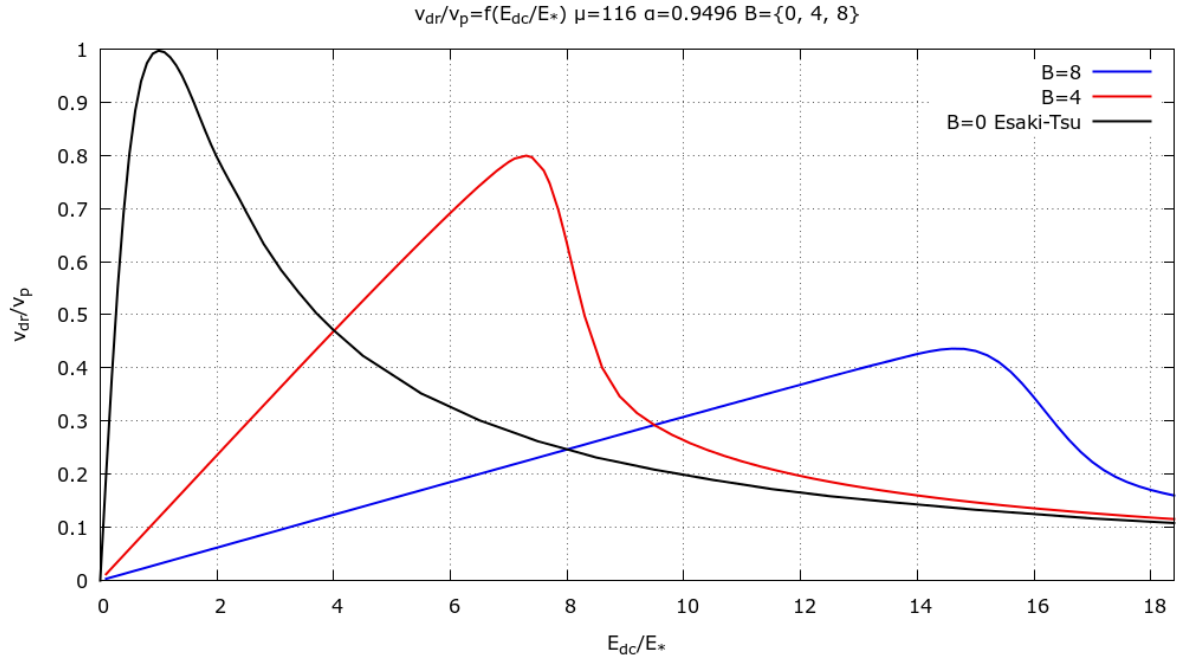


Figure 3:

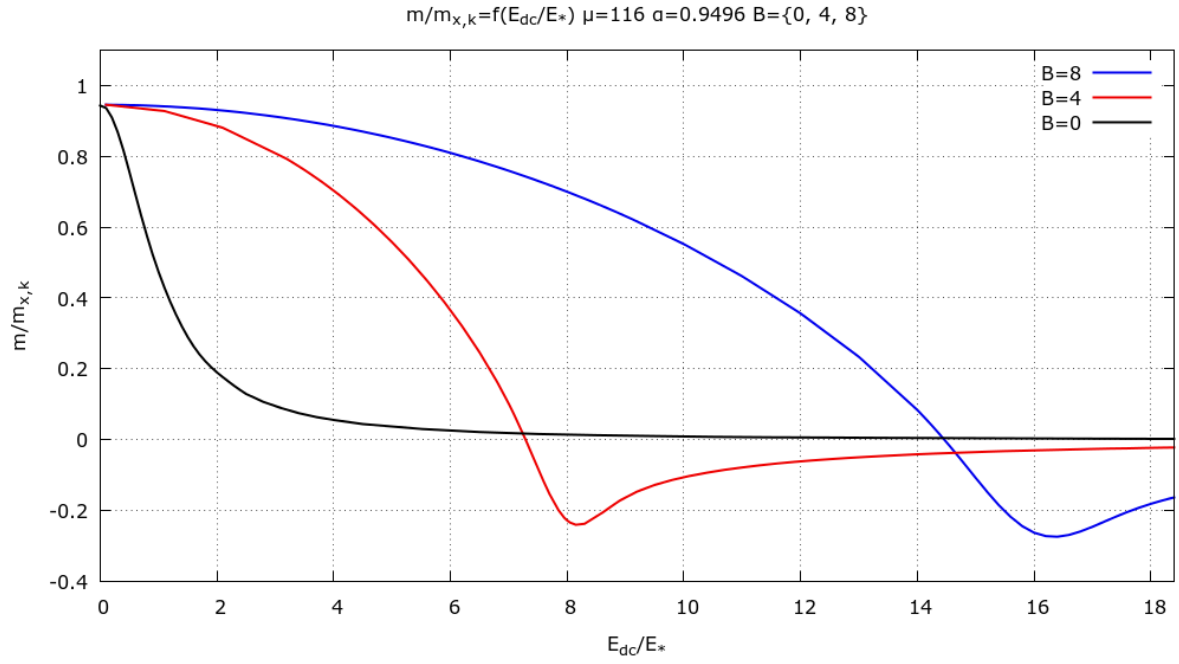


Figure 4: