

Structure and Cohomology of 3-Lie Algebras Induced by Lie Algebras

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Abstract The aim of this paper is to compare the structure and the cohomology spaces of Lie algebras and induced 3-Lie algebras.

1 Introduction

Lie algebras have held a very important place in mathematics and physics for a long time. Ternary Lie algebras appeared first in Nambu's generalization of Hamiltonian mechanics [11] which uses a generalization of Poisson algebra with a ternary bracket. The algebraic formulation is due to Takhtajan. The structure of n -Lie algebra was studied by Filippov [8] and Kasymov [10].

The Lie algebra cohomology complex is well known under the name of Chevalley-Eilenberg cohomology complex. The cohomology of n -Lie algebras was first introduced by Takhtajan [13] in its simplest form, later a complex adapted to the study of formal deformations was introduced by Gautheron [9], then reformulated by Daletskii and Takhtajan [5] using the notion of base Leibniz algebra of a n -Lie algebra.

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In [3], the authors introduce a realization of the quantum Nambu bracket in terms of matrices (using the commutator and the trace of matrices). This construction is generalized in [1] to the case of any Lie algebra where the commutator is replaced by the Lie bracket, and the matrix trace is replaced by linear forms having similar properties, which we call 3-Lie algebras induced by Lie algebras. This construction is generalized to the case of n -Lie algebras in [2]. We investigate the connections between the structural properties (Solvability, nilpotency,...) and the cohomology of a Lie algebra and an induced 3-Lie algebra.

The paper is organized as follows: in Section 2 we recall main definitions and results concerning n -Lie algebras, and construction of $(n+1)$ -Lie algebras induced by n -Lie algebras. In Section 3 we study some structural properties of 3-Lie algebras induced by Lie algebras, in particular: common subalgebras and ideals, solvability and nilpotency. In Section 4, we recall the cohomology complexes for Lie algebras and 3-Lie algebras, then we study relations between 1 and 2 cocycles of a Lie algebra and the induced 3-Lie algebra. In Section 5, we give definitions of central extensions of Lie algebras and n -Lie algebras, then we study the relation between central extension of a Lie algebra and those of a 3-Lie algebra it induces. In Section 6 we give a method to recognize 3-Lie algebras that are induced by some Lie algebra, and applying it, we can determine all 3-Lie algebras induced by Lie algebras up to dimension 5, based on classifications given in [4] and [8], then we give a list of Lie algebras up to dimension 4 and all the possible induced 3-Lie algebras. In Section 7 we present compute on 4 chosen Lie algebras and one trace map each, the set of 1-cocycles and 1 coboundaries of the Lie algebras and the induced 3-Lie algebras using the computer algebra software Mathematica, the algorithm is briefly explained there too.

2 n -Lie Algebras

In this paper, all considered vector spaces are over a field \mathbb{K} of characteristic 0. n -Lie algebras were introduced in [8], then deeper investigated in [10]. Let us recall of some basic definitions.

Definition 1. A n -Lie algebra $(A, [\cdot, \dots, \cdot])$ is a vector space together with a skew-symmetric n -linear map $[\cdot, \dots, \cdot] : A^n \rightarrow A$ such that :

$$[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = \sum_{i=1}^n [y_1, \dots, [x_1, \dots, x_{n-1}, y_i], \dots, y_n]. \quad (1)$$

for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in A$. This condition is called the fundamental identity or Filippov identity. For $n = 2$ (1) becomes the Jacobi identity and we get the definition of a Lie algebra.

Definition 2. Let $(A, [\cdot, \dots, \cdot])$ be a n -Lie algebra, and I a subspace of A . We say that I is an ideal of A if, for all $i \in I, x_1, \dots, x_{n-1} \in A$, it holds that $[i, x_1, \dots, x_{n-1}] \in I$.

Lemma 1. Let $(A, [\cdot, \dots, \cdot])$ be a n -Lie algebra, and I_1, \dots, I_n be ideals of A , then $I = [I_1, \dots, I_n]$ is an ideal of A .

Definition 3. Let $(A, [\cdot, \dots, \cdot])$ be a n -Lie algebra, and I an ideal of A . Define the derived series of I by:

$$D^0(I) = I \text{ and } D^{p+1}(I) = [D^p(I), \dots, D^p(I)].$$

and the central descending series of I by:

$$C^0(I) = I \text{ and } C^{p+1}(I) = [C^p(I), I, \dots, I].$$

Definition 4. Let $(A, [\cdot, \dots, \cdot])$ be a n -Lie algebra, and I an ideal of A . I is said to be solvable if there exists $p \in \mathbb{N}$ such that $D^p(I) = \{0\}$. It is said to be nilpotent if there exists $p \in \mathbb{N}$ such that $C^p(I) = \{0\}$.

Definition 5. A n -Lie algebra $(A, [\cdot, \dots, \cdot])$ is said to be simple if $D^1(A) \neq \{0\}$ and if it has no ideals other than $\{0\}$ and A . A direct sum of simple n -Lie algebras is said to be semi-simple.

In [1] and [2] a construction of a 3-Lie algebra from a Lie algebra, and more generally a $(n+1)$ -Lie algebra from a n -Lie algebra was introduced. We recall the main definitions and results.

Definition 6. Let $\phi : A^n \rightarrow A$ be a n -linear map and let τ be a linear map from A to \mathbb{K} . Define $\phi_\tau : A^{n+1} \rightarrow A$ by:

$$\phi_\tau(x_1, \dots, x_{n+1}) = \sum_{k=1}^{n+1} (-1)^k \tau(x_k) \phi(x_1, \dots, \hat{x}_k, \dots, x_{n+1}), \quad (2)$$

where the hat over \hat{x}_k on the right hand side means that x_k is excluded, that is ϕ is calculated on $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1})$.

We will not be concerned with just any linear map τ , but rather maps that have a generalized trace property. Namely:

Definition 7. For $\phi : A^n \rightarrow A$ we call a linear map $\tau : A \rightarrow \mathbb{K}$ a ϕ -trace (or trace) if $\tau(\phi(x_1, \dots, x_n)) = 0$ for all $x_1, \dots, x_n \in A$.

Lemma 2. Let $\phi : A^n \rightarrow A$ be a skew-symmetric n -linear map and τ a linear map $A \rightarrow \mathbb{K}$. Then ϕ_τ is a $(n+1)$ -linear totally skew-symmetric map. Furthermore, if τ is a ϕ -trace then τ is a ϕ_τ -trace.

Theorem 1. Let (A, ϕ) be a n -Lie algebra and τ a ϕ -trace, then (A, ϕ_τ) is a $(n+1)$ -Lie algebra. We shall say that (A, ϕ_τ) is induced by (A, ϕ) . In particular, let $(A, [\cdot, \cdot, \cdot])$ be a Lie algebra and $\tau : A \rightarrow \mathbb{K}$ be a trace map, the ternary bracket $[\cdot, \cdot, \cdot]$ given by: $[x, y, z] = \bigcirc_{x,y,z} \tau(x) [y, z]$ defines a 3-Lie algebra, we refer to A when considering the Lie algebra and A_τ when considering induced 3-Lie algebra.

3 Structure of 3-Lie Algebras Induced by Lie Algebras

Let $(A, [\cdot, \cdot])$ be a Lie algebra, τ a $[\cdot, \cdot]$ -trace and $(A, [\cdot, \cdot, \cdot]_\tau)$ the induced 3-Lie algebra.

Proposition 1. *If B is a subalgebra of $(A, [\cdot, \cdot])$ then B is also a subalgebra of $(A, [\cdot, \cdot, \cdot]_\tau)$.*

Proof. Let B be a subalgebra of $(A, [\cdot, \cdot])$ and $x, y, z \in B$:

$$[x, y, z]_\tau = \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y],$$

which is a linear combination of elements of B and then belongs to B . \square

Proposition 2. *Let J be an ideal of $(A, [\cdot, \cdot])$. Then J is an ideal of $(A, [\cdot, \cdot, \cdot]_\tau)$ if and only if:*

$$[A, A] \subseteq J \text{ or } J \subseteq \ker \tau.$$

Proof. Let J be an ideal of $(A, [\cdot, \cdot])$, and let $j \in J$ and $x, y \in A$, then we have:

$$[x, y, j]_\tau = \tau(x)[y, j] + \tau(y)[j, x] + \tau(j)[x, y].$$

We have that $\tau(x)[y, j] + \tau(y)[j, x] \in J$, then, to have $[x, y, j]_\tau \in J$ it is necessary and sufficient to have $\tau(j)[x, y] \in J$, which is equivalent to $\tau(j) = 0$ or $[x, y] \in J$. \square

Theorem 2. *Let $(A, [\cdot, \cdot])$ be a Lie algebra, τ a $[\cdot, \cdot]$ -trace and $(A, [\cdot, \cdot, \cdot]_\tau)$ the induced 3-Lie algebra. The 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_\tau)$ is solvable, more precisely $D^2(A_\tau) = 0$ i.e. $(D^1(A_\tau) = [A, A, A]_\tau, [\cdot, \cdot, \cdot]_\tau)$ is abelian.*

Proof. Let $x, y, z \in [A, A, A]_\tau$, $x = [x_1, x_2, x_3]_\tau$, $y = [y_1, y_2, y_3]_\tau$ and $z = [z_1, z_2, z_3]_\tau$, then:

$$\begin{aligned} [x, y, z]_\tau &= \tau([x_1, x_2, x_3]_\tau)[[y_1, y_2, y_3]_\tau, [z_1, z_2, z_3]_\tau] \\ &\quad + \tau([y_1, y_2, y_3]_\tau)[[z_1, z_2, z_3]_\tau, [x_1, x_2, x_3]_\tau] \\ &\quad + \tau([z_1, z_2, z_3]_\tau)[[x_1, x_2, x_3]_\tau, [y_1, y_2, y_3]_\tau] \\ &= 0. \end{aligned}$$

Because $\tau([\cdot, \cdot, \cdot]) = 0$. \square

Remark 1 ([8]). Let $(A, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra. If we fix $a \in A$, the bracket

$$[\cdot, \cdot]_a = [a, \cdot, \cdot]$$

is skew-symmetric and satisfies Jacobi identity. Indeed, we have, for $x, y, z \in A$:

$$\begin{aligned} [x, [y, z]_a]_a &= [a, x, [a, y, z]] \\ &= [[a, x, a], y, z] + [a, [a, x, y], z] + [a, y, [a, x, z]] \\ &= [a, [a, x, y], z] + [a, y, [a, x, z]] \\ &= [[x, y]_a, z]_a + [x, [y, z]_a]_a \end{aligned}$$

Proposition 3. *Let $(A, [\cdot, \cdot])$ be a Lie algebra, τ be a trace and $(A, [\cdot, \cdot, \cdot]_\tau)$ the induced algebra, let $(C^p(A))$ be the central descending series of $(A, [\cdot, \cdot])$, and $(C^p(A_\tau))$ be the central descending series of $(A, [\cdot, \cdot, \cdot]_\tau)$. then we have :*

$$C^p(A_\tau) \subset C^p(A), \forall p \in \mathbb{N}.$$

If there exists $i \in A$ such that $[i, x, y]_\tau = [x, y], \forall x, y \in A$ then:

$$C^p(A_\tau) = C^p(A), \forall p \in \mathbb{N}.$$

Proof. We proceed by induction over $p \in \mathbb{N}$. The case of $p = 0$ is trivial, for $p = 1$ we have:

$$\forall x = [a, b, c]_\tau \in C^1(A_\tau), x = \tau(a)[b, c] + \tau(b)[c, a] + \tau(c)[a, b],$$

which is a linear combination of elements of $C^1(A)$ and then is an elements of $C^1(A)$. Suppose now that there exists $i \in A$ such that $[i, x, y]_\tau = [x, y], \forall x, y \in A$, then for $x = [a, b] \in C^1(A)$, $x = [i, a, b]_\tau$ and then is an element of $C^1(A_\tau)$.

Now, we suppose this proposition is true for some $p \in \mathbb{N}$, and let $x \in C^{p+1}(A_\tau)$, then $x = [a, u, v]_\tau$ with $u, v \in A$ and $a \in C^p(A_\tau)$

$$x = [a, u, v]_\tau = \tau(u)[v, a] + \tau(v)[a, u] \quad (\tau(a) = 0)$$

which is an element of $C^{p+1}(A)$ because $a \in C^p(A_\tau) \subset C^p(A)$. If there exists $i \in A$ such that $[i, x, y]_\tau = [x, y], \forall x, y \in A$ then, if $x \in C^{p+1}(A)$ then $x = [a, u]$ with $a \in C^p(A)$ and $u \in A$ and we have:

$$x = [a, u] = [i, a, u]_\tau = [a, u, i]_\tau \in C^{p+1}(A_\tau).$$

□

Remark 2. It also results from the preceding proposition that $D^1(A_\tau) = [A, A, A]_\tau \subset D^1(A) = [A, A]$, and that if there exists $i \in A$ such that $[i, x, y]_\tau = [x, y], \forall x, y \in A$, then $D^1(A_\tau) = D^1(A)$. For the rest of the derived series, we have obviously the first inclusion by Theorem 2.

Theorem 3. *Let $(A, [\cdot, \cdot])$ be a Lie algebra, τ be a trace and $(A, [\cdot, \cdot, \cdot]_\tau)$ the induced algebra, then we have :*

$$(A, [\cdot, \cdot]) \text{ is nilpotent of class } p \implies (A, [\cdot, \cdot, \cdot]_\tau) \text{ is nilpotent of class at most } p.$$

Moreover, if there exists $i \in A$ such that $[i, x, y]_\tau = [x, y], \forall x, y \in A$ then:

$$(A, [\cdot, \cdot, \cdot]_\tau) \text{ is nilpotent of class } p \iff (A, [\cdot, \cdot, \cdot]_\tau) \text{ is nilpotent of class } p.$$

Proof. 1. Suppose that $(A, [\cdot, \cdot])$ is nilpotent of class $p \in \mathbb{N}$, then $C^p(A) = \{0\}$. By the preceding proposition, $C^p(A_\tau) \subseteq C^p(A) = \{0\}$, therefore $(A, [\cdot, \cdot, \cdot]_\tau)$ is nilpotent of class at most p .

2. We suppose now that $(A, [\cdot, \cdot, \cdot]_\tau)$ is nilpotent of class $p \in \mathbb{N}$, and that there exists $i \in A$ such that $[i, x, y]_\tau = [x, y]$, $\forall x, y \in A$, then $C^p(A_\tau) = \{0\}$. By the preceding proposition, $C^p(A) = C^p(A_\tau) = \{0\}$. Therefore $(A, [\cdot, \cdot, \cdot])$ is nilpotent, since $C^{p-1}(A) = C^{p-1}(A_\tau) \neq \{0\}$, $(A, [\cdot, \cdot, \cdot]_\tau)$ and $(A, [\cdot, \cdot, \cdot])$ have the same nilpotency class.

□

4 Lie and 3-Lie Algebras Cohomology

In this section, we study the connections between the Chevalley-Eilenberg cohomology for Lie algebras and the cohomology of 3-Lie algebras induced by Lie algebras.

Now, let us recall the main definitions of Lie algebras and n -Lie algebras cohomology, for reference and further details, see [5], [6], [9] and [13].

Definition 8. Let $(A, [\cdot, \cdot, \cdot])$ be a Lie algebra, ρ a representation of A in a vector space M . A M -valued p -cochain on A is a skew-symmetric p -linear map $\varphi : A^p \rightarrow M$, the set of M -valued p -cochain is denoted by $C^p(A, M)$.

The coboundary operator is the linear map $\delta^p : C^p(A, M) \rightarrow C^{p+1}(A, M)$ given by :

$$\begin{aligned} \delta^p \varphi(x_1, \dots, x_{p+1}) &= \sum_{j=1}^{p+1} (-1)^{j+1} \rho(x_j) \varphi(x_1, \dots, \hat{x}_j, \dots, x_{p+1}) \\ &\quad + \sum_{j=1}^{p+1} \sum_{k=j+1}^{p+1} (-1)^{j+k} \varphi([x_j, x_k], x_1, \dots, \hat{x}_j, \dots, \hat{x}_k, \dots, x_{p+1}). \end{aligned}$$

We will study two particular cases, the adjoint cohomology $M = A, \rho = ad$ and the scalar cohomology $M = \mathbb{K}, \rho = 0$.

Definition 9. Let $(A, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra, an A -valued p -cochain is a linear map $\psi : (\wedge^2 A)^{\otimes p-1} \wedge A \rightarrow A$.

Definition 10. The coboundary operator for the adjoint action is given by :

$$\begin{aligned} d^p \psi(x_1, \dots, x_{2p+1}) &= \sum_{j=1}^p \sum_{k=2j+1}^{2p+1} (-1)^j \psi(x_1, \dots, \hat{x}_{j-1}, \hat{x}_j, \dots, ad_{a_j} x_k, \dots, x_{2p+1}) \\ &\quad + \sum_{k=1}^p (-1)^{k-1} ad_{a_k} \psi(x_1, \dots, \hat{x}_{2k-1}, \hat{x}_{2k}, \dots, x_{2p+1}) \\ &\quad + (-1)^{p+1} [x_{2p-1}, \psi(x_1, \dots, x_{2p-2}, x_{2p}), x_{2p+1}] \\ &\quad + (-1)^{p+1} [\psi(x_1, \dots, x_{2p-1}, x_{2p}, x_{2p+1})], \end{aligned}$$

where $a_k = (x_{2k-1}, x_{2k})$.

Definition 11. Let $(A, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra, a \mathbb{K} -valued p -cochain is a linear map $\psi : (\wedge^2 A)^{\otimes p-1} \wedge A \rightarrow \mathbb{K}$.

Definition 12. The coboundary operator for the trivial action is given by :

$$d^p \psi(x_1, \dots, x_{2p+1}) = \sum_{j=1}^p \sum_{k=2j+1}^{2p+1} (-1)^j \psi(x_1, \dots, \hat{x}_{j-1}, \hat{x}_j, \dots, ad_{a_j} x_k, \dots, x_{2p+1}),$$

where $a_k = (x_{2k-1}, x_{2k})$.

The elements of $Z^p(A, M) = \ker \delta^p$ are called p -cocycles, those of $B^n(A, M) = \text{Im } \delta^{p-1}$ are called coboundaries. $H^p(A, M) = \frac{Z^p(A, M)}{B^n(A, M)}$ is the p -th cohomology group. We sometimes add in subscript the representation used in the cohomology complex, for example $Z_{ad}^p(A, A)$ denotes the set of p -cocycle for the adjoint cohomology and $Z_0^p(A, \mathbb{K})$ denotes the set of p -cocycle for the scalar cohomology.

In particular, the elements of $Z^1(A, A)$ are the derivations. Recall that a derivation of a n -Lie algebra is a linear map $f : A \rightarrow A$ satisfying:

$$f([x_1, \dots, x_n]) = \sum_{i=1}^n [x_1, \dots, f(x_i), \dots, x_n], \forall x_1, \dots, x_n \in A.$$

4.1 Derivations and 2-cocycles correspondence

Let $(A, [\cdot, \cdot, \cdot])$ be a Lie algebra, τ a $[\cdot, \cdot, \cdot]$ -trace and $(A, [\cdot, \cdot, \cdot]_\tau)$ the induced 3-Lie algebra, then we have the following correspondence between 1 and 2-cocycles of $(A, [\cdot, \cdot, \cdot])$ and those of $(A, [\cdot, \cdot, \cdot]_\tau)$.

Lemma 3. Let $f : A \rightarrow A$ be a Lie algebra derivation, then $\tau \circ f$ is a $[\cdot, \cdot, \cdot]$ -trace.

Proof. for all $x, y \in A$, we have :

$$\tau(f([x, y])) = \tau([f(x), y] + [x, f(y)]) = \tau([f(x), y]) + \tau([x, f(y)]) = 0.$$

□

Theorem 4. Let $f : A \rightarrow A$ be a derivation of the Lie algebra A , then f is a derivation of the induced 3-Lie algebra if and only if:

$$[x, y, z]_{\tau \circ f} = 0, \forall x, y, z \in A.$$

Proof. Let f be a derivation of A and $x, y, z \in A$:

$$\begin{aligned}
f([x, y, z]_\tau) &= \tau(x)f([y, z]) + \tau(y)f([z, x]) + \tau(z)f([x, y]) \\
&= \tau(x)[f(y), z] + \tau(y)[f(z), x] + \tau(z)[f(x), y] \\
&\quad + \tau(x)[y, f(z)] + \tau(y)[z, f(x)] + \tau(z)[x, f(y)] \\
&\quad + \tau(f(x))[y, z] + \tau(f(y))[z, x] + \tau(f(z))[x, y] \\
&\quad - \tau(f(x))[y, z] + \tau(f(y))[z, x] + \tau(f(z))[x, y] \\
&= [f(x), y, z]_\tau + [x, f(y), z]_\tau + [x, y, f(z)]_\tau - [x, y, z]_{\tau \circ f}. \square
\end{aligned}$$

Theorem 5. Let $\varphi \in Z_{ad}^2(A, A)$ and $\omega : A \rightarrow \mathbb{K}$ be a linear map satisfying :

1. $\tau(x)\omega(y) = \tau(y)\omega(x)$,
2. $\omega([x, y]) = 0$,
3. $\bigcirc_{x, y, z} \omega(x) \tau(\varphi(y, z)) = 0$.

Then $\psi(x, y, z) = \bigcirc_{x, y, z} \omega(x) \varphi(y, z)$ is a 2-cocycle of the induced 3-Lie algebra.

Proof. Let $\varphi \in Z_{ad}^2(A, A)$ and $\omega : A \rightarrow \mathbb{K}$ a linear map satisfying conditions 1, 2 and 3 above, and let $\psi(x, y, z) = \bigcirc_{x, y, z} \omega(x) \varphi(y, z)$, then we have:

$$\begin{aligned}
d^2\psi(x_1, x_2, y_1, y_2, z) &= \psi(x_1, x_2, [y_1, y_2, z]_\tau) - \psi([x_1, x_2, y_1]_\tau, y_2, z) \\
&\quad - \psi(y_1, [x_1, x_2, y_2]_\tau, z) - \psi(y_1, y_2, [x_1, x_2, z]_\tau) + [x_1, x_2, \psi(y_1, y_2, z)]_\tau \\
&\quad - [\psi(x_1, x_2, y_1), y_2, z]_\tau - [y_1, \psi(x_1, x_2, y_2), z]_\tau - [y_1, y_2, \psi(x_1, x_2, z)]_\tau \\
&= \tau(y_1) \psi(x_1, x_2, [y_2, z]) + \tau(y_2) \psi(x_1, x_2, [z, y_1]) + \tau(z) \psi(x_1, x_2, [y_1, y_2]) \\
&\quad - \tau(x_1) \psi(y_1, y_2, [x_2, z]) - \tau(x_2) \psi(y_1, y_2, [z, x_1]) - \tau(z) \psi(y_1, y_2, [x_1, x_2]) \\
&\quad - \tau(x_1) \psi([x_2, y_1] y_2, z) - \tau(x_2) \psi([y_1, x_1] y_2, z) - \tau(y_1) \psi([x_1, x_2] y_2, z) \\
&\quad - \tau(x_1) \psi(y_1, [x_2, y_2], z) - \tau(x_2) \psi(y_1, [y_2, x_1], z) - \tau(y_2) \psi(y_1, [x_1, x_2], z) \\
&\quad + \tau(x_1) [x_2, \psi(y_1, y_2, z)] + \tau(x_2) [\psi(y_1, y_2, z), x_1] + \tau(\psi(y_1, y_2, z)) [x_1, x_2] \\
&\quad - \tau(y_1) [y_2, \psi(x_1, x_2, z)] - \tau(y_2) [\psi(x_1, x_2, z), y_1] - \tau(\psi(x_1, x_2, z)) [y_1, y_2] \\
&\quad - \tau(\psi(x_1, x_2, y_1)) [y_2, z] - \tau(y_2) [z, \psi(x_1, x_2, y_1)] - \tau(z) [\psi(x_1, x_2, y_1), y_2] \\
&\quad - \tau(y_1) [\psi(x_1, x_2, y_2), z] - \tau(\psi(x_1, x_2, y_2)) [z, y_1] - \tau(z) [y_1, \psi(x_1, x_2, y_2)] \\
&= \tau(y_1) \left(\omega(x_1) \varphi(x_2, [y_2, z]) + \omega(x_2) \varphi([y_2, z], x_1) - \omega(y_2) \varphi(z, [x_1, x_2]) \right. \\
&\quad - \omega(z) \varphi([x_1, x_2], y_2) - \omega(x_1) [y_2, \varphi(x_2, z)] - \omega(x_2) [y_2, \varphi(z, x_1)] \\
&\quad - \omega(z) [y_2, \varphi(x_1, x_2)] - \omega(x_1) [\varphi(x_2, y_2), z] - \omega(x_2) [\varphi(y_2, x_1), z] \\
&\quad \left. - \omega(y) [\varphi(x_1, x_2), z] \right) \\
&\quad + \tau(y_2) \left(\omega(x_1) \varphi(x_2, [z, y_1]) + \omega(x_2) \varphi([z, y_1], x_1) - \omega(x_1) [\varphi(x_2, z), y_1] \right. \\
&\quad - \omega(x_2) [\varphi(z, x_1), y_1] - \omega(z) [\varphi(x_1, x_2), y_1] - \omega(x_1) [z, \varphi(x_2, y_1)] \\
&\quad - \omega(x_2) [z, \varphi(y_1, x_1)] - \omega(y_1) [z, \varphi(x_1, x_2)] - \omega(y_1) \varphi([x_1, x_2], z) \\
&\quad \left. - \omega(z) \varphi(y_1, [x_1, x_2]) \right)
\end{aligned}$$

$$\begin{aligned}
& + \tau(z) \left(\omega(x_1) \varphi(x_2, [y_1, y_2]) + \omega(x_2) \varphi([y_1, y_2], x_1) - \omega(y_1) \varphi(y_2, [x_1, x_2]) \right. \\
& - \omega(y_2) \varphi([x_1, x_2], y_1) - \omega(x_1) [\varphi(x_2, y_1), y_2] - \omega(x_2) [\varphi(y_1, x_1), y_2] \\
& - \omega(y_1) [\varphi(x_1, x_2), y_2] - \omega(x_1) [y_1, \varphi(x_2, y_2)] - \omega(x_2) [y_1, \varphi(y_2, x_1)] \\
& \left. - \omega(y_2) [y_1, \varphi(x_1, x_2)] \right) \\
& + \tau(x_1) \left(\omega(y_1) [x_2, \varphi(y_2, z)] + \omega(y_2) [x_2, \varphi(z, y_1)] + \omega(z) [x_2, \varphi(y_1, y_2)] \right. \\
& - \omega(y_1) \varphi(y_2, [x_2, z]) - \omega(y_2) \varphi([x_2, z], y_1) - \omega(y_2) \varphi(z, [x_2, y_1]) \\
& - \omega(z) \varphi([x_2, y_1], y_2) - \omega(y_1) \varphi([x_2, y_2], z) - \omega(z) \varphi(y_1, [x_2, y_2]) \left. \right) \\
& + \tau(x_2) \left(\omega(y_1) [\varphi(y_2, z), x_1] + \omega(y_2) [\varphi(z, y_1), x_1] + \omega(z) [\varphi(y_1, y_2), x_1] \right. \\
& - \omega(y_1) \varphi(y_2, [z, x_1]) - \omega(y_2) \varphi([z, x_1], y_1) - \omega(y_2) \varphi(z, [y_1, x_1]) \\
& - \omega(z) \varphi([y_1, x_1], y_2) - \omega(y_1) \varphi([y_2, x_1], z) - \omega(z) \varphi(y_1, [y_2, x_1]) \left. \right) \\
& + \left(\omega(y_1) \tau(\varphi(y_2, z)) + \omega(y_2) \tau(\varphi(z, y_1)) + \omega(z) \tau(\varphi(y_1, y_2)) \right) [x_1, x_2] \\
& - \left(\omega(x_1) \tau(\varphi(x_2, y_1)) + \omega(x_2) \tau(\varphi(y_1, x_1)) + \omega(y_1) \tau(\varphi(x_1, x_2)) \right) [y_2, z] \\
& - \left(\omega(x_1) \tau(\varphi(x_2, z)) + \omega(x_2) \tau(\varphi(z, x_1)) + \omega(z) \tau(\varphi(x_1, x_2)) \right) [y_1, y_2] \\
& - \left(\omega(x_1) \tau(\varphi(x_2, y_2)) + \omega(x_2) \tau(\varphi(y_2, x_1)) + \omega(y_2) \tau(\varphi(x_1, x_2)) \right) [z, y_1] \\
& = -\tau(y_1) \omega(x_1) \delta^2 \varphi(z, y_2, x_2) - \tau(y_1) \omega(x_2) \delta^2 \varphi(y_2, z, x_1) \\
& - \tau(y_2) \omega(x_1) \delta^2 \varphi(y_1, z, x_2) - \tau(y_2) \omega(x_2) \delta^2 \varphi(z, y_1, x_1) \\
& - \tau(z) \omega(x_1) \delta^2 \varphi(y_2, y_1, x_2) - \tau(z) \omega(x_2) \delta^2 \varphi(y_1, y_2, x_1) \\
& + \left(\omega(y_1) \tau(\varphi(y_2, z)) + \omega(y_2) \tau(\varphi(z, y_1)) + \omega(z) \tau(\varphi(y_1, y_2)) \right) [x_1, x_2] \\
& - \left(\omega(x_1) \tau(\varphi(x_2, y_1)) + \omega(x_2) \tau(\varphi(y_1, x_1)) + \omega(y_1) \tau(\varphi(x_1, x_2)) \right) [y_2, z] \\
& - \left(\omega(x_1) \tau(\varphi(x_2, z)) + \omega(x_2) \tau(\varphi(z, x_1)) + \omega(z) \tau(\varphi(x_1, x_2)) \right) [y_1, y_2] \\
& - \left(\omega(x_1) \tau(\varphi(x_2, y_2)) + \omega(x_2) \tau(\varphi(y_2, x_1)) + \omega(y_2) \tau(\varphi(x_1, x_2)) \right) [z, y_1].
\end{aligned}$$

Since

$$\bigcirc_{x,y,z} \omega(x) \tau(\varphi(y, z)) = 0, \forall x, y, z \in A,$$

we get

$$d^2 \psi = 0.$$

□

Theorem 6. *Every 1-cocycle for the scalar cohomology of $(A, [\cdot, \cdot])$ is a 1-cocycle for the scalar cohomology of the induced 3-Lie algebra.*

Proof. Let ω be a 1-cocycle for the scalar cohomology of $(A, [\cdot, \cdot])$, then

$$\forall x, y \in A, \delta^1 \omega(x, y) = \omega([x, y]) = 0,$$

which is equivalent to $[A, A] \subset \ker \omega$. By Remark 2 $[A, A, A]_\tau \subset [A, A]$ and then $[A, A, A]_\tau \subset \ker \omega$, that is

$$\forall x, y, z \in A, \omega([x, y, z]_\tau) = d^1 \omega(x, y, z) = 0,$$

which means that ω is a 1-cocycle for the scalar cohomology of $(A, [\cdot, \cdot, \cdot]_\tau)$. \square

Theorem 7. *Let $\varphi \in Z_0^2(A, \mathbb{K})$ and $\omega : A \rightarrow \mathbb{K}$ a linear map satisfying :*

1. $\tau(x)\omega(y) = \tau(y)\omega(x)$,
2. $\omega([x, y]) = 0$,
3. $\omega(y_2)(\tau(x_1)\varphi([x_1, z]x_2) + \tau(x_2)\varphi([z, y_1]x_1)) = 0$.

Then $\psi(x, y, z) = \bigcirc_{x, y, z} \omega(x)\varphi(y, z)$ is a 2-cocycle of the induced 3-Lie algebra.

Proof. Let $\varphi \in Z_0^2(A, \mathbb{K})$ and $\omega : A \rightarrow \mathbb{K}$ a linear map satisfying conditions 1, 2 and 3 above, and let $\psi(x, y, z) = \bigcirc_{x, y, z} \omega(x)\varphi(y, z)$, then we have:

$$\begin{aligned} d^2 \psi(x_1, x_2, y_1, y_2, z) &= \psi(x_1, x_2, [y_1, y_2, z]_\tau) - \psi([x_1, x_2, y_1]_\tau, y_2, z) \\ &\quad - \psi(y_1, [x_1, x_2, y_2]_\tau, z) - \psi(y_1, y_2, [x_1, x_2, z]_\tau) \\ &= \tau(y_1)\psi(x_1, x_2, [y_2, z]) + \tau(y_2)\psi(x_1, x_2, [z, y_1]) + \tau(z)\psi(x_1, x_2, [y_1, y_2]) \\ &\quad - \tau(x_1)\psi([x_2, y_1], y_2, z) - \tau(x_2)\psi([y_1, x_1], y_2, z) - \tau(y_1)\psi([x_1, x_2], y_2, z) \\ &\quad - \tau(x_1)\psi(y_1, [x_2, y_2], z) - \tau(x_2)\psi(y_1, [y_2, x_1], z) - \tau(y_2)\psi(y_1, [x_1, x_2], z) \\ &\quad - \tau(x_1)\psi(y_1, y_2, [x_2, z]) - \tau(x_2)\psi(y_1, y_2, [z, x_1]) - \tau(z)\psi(y_1, y_2, [x_1, x_2]) \\ &= \tau(y_1)(\omega(x_1)\varphi(x_2, [y_2, z]) + \omega(x_2)\varphi([y_2, z], x_1)) \\ &\quad + \tau(y_2)(\omega(x_1)\varphi(x_2, [y_1, z]) + \omega(x_2)\varphi([y_1, z], x_1)) \\ &\quad + \tau(z)(\omega(x_1)\varphi(x_2, [y_1, y_2]) + \omega(x_2)\varphi([y_1, y_2], x_1)) \\ &\quad - \tau(x_1)(\omega(y_2)\varphi(z, [x_2, y_1]) + \omega(z)\varphi([x_2, y_1], y_2)) \\ &\quad - \tau(x_2)(\omega(y_2)\varphi(z, [y_1, x_1]) + \omega(z)\varphi([y_1, x_1], y_2)) \\ &\quad - \tau(y_1)(\omega(y_2)\varphi(z, [x_1, x_2]) + \omega(z)\varphi([x_1, x_2], y_2)) \\ &\quad - \tau(x_1)(\omega(y_1)\varphi([x_2, y_2], z) + \omega(z)\varphi(y_1, [x_2, y_2])) \\ &\quad - \tau(x_2)(\omega(y_1)\varphi([y_2, x_1], z) + \omega(z)\varphi(y_1, [y_2, x_1])) \\ &\quad - \tau(y_2)(\omega(y_1)\varphi([x_1, x_2], z) + \omega(z)\varphi(y_1, [x_1, x_2])) \\ &\quad - \tau(x_1)(\omega(y_1)\varphi(y_2, [x_2, z]) + \omega(y_2)\varphi([x_2, z], y_1)) \\ &\quad - \tau(x_2)(\omega(y_1)\varphi(y_2, [z, x_1]) + \omega(y_2)\varphi([z, x_1], y_1)) \\ &\quad - \tau(z)(\omega(y_1)\varphi(y_2, [x_1, x_2]) + \omega(y_2)\varphi([x_1, x_2], y_1)) \end{aligned}$$

$$\begin{aligned}
&= \tau(x_1) \omega(y_1) \delta^2 \varphi(y_2, z, x_2) + \tau(x_1) \omega(y_2) \delta^2 \varphi(z, y_1, x_2) \\
&+ \tau(x_2) \omega(y_1) \delta^2 \varphi(z, y_2, x_1) + \tau(x_2) \omega(y_2) \delta^2 \varphi(x_1, y_1, z) \\
&+ \tau(x_1) \omega(z) \delta^2 \varphi(y_1, y_2, x_2) + \tau(x_2) \omega(z) \delta^2 \varphi(y_2, y_1, x_1) \\
&- 2\omega(y_2) (\tau(x_1) \varphi([y_1, z], x_2) + \tau(x_2) \varphi([z, y_1], x_1)).
\end{aligned}$$

Since

$$\omega(y_2) (\tau(x_1) \varphi([y_1, z], x_2) + \tau(x_2) \varphi([z, y_1], x_1)) = 0,$$

it follows that $d^2\psi = 0$. \square

Remark 3. Condition 1 in Theorems 5 and 7 are equivalent to $\omega = \lambda\tau, \lambda \in \mathbb{K}$, and therefore one may remove condition 2, which is redundant.

Lemma 4. Let $\alpha \in C^1(A, \mathbb{K})$. Then:

$$d^1\alpha(x, y, z) = \bigcirc_{x, y, z} \tau(x) \delta^1\alpha(y, z), \forall x, y, z \in A.$$

Proof. Let $\alpha \in C^1(A, \mathbb{K})$, $x, y, z \in A$, then we have:

$$d^1\alpha(x, y, z) = \alpha([x, y, z]) = \bigcirc_{x, y, z} \tau(x) \alpha([y, z]) = \bigcirc_{x, y, z} \tau(x) \delta^1\alpha(y, z)$$

\square

Proposition 4. Let $\varphi_1, \varphi_2 \in Z_0^2(A, \mathbb{K})$ satisfying conditions of Theorem 7. If φ_1, φ_2 are in the same cohomology class then ψ_1, ψ_2 defined by:

$$\psi_i(x, y, z) = \bigcirc_{x, y, z} \tau(x) \varphi_i(y, z), i = 1, 2$$

are in the same cohomology class.

Proof. Let $\varphi_1, \varphi_2 \in Z_0^2(A, \mathbb{K})$ be two cocycles in the same cohomology class, that is

$$\varphi_2 - \varphi_1 = \delta^1\alpha, \alpha \in C^1(A, \mathbb{K})$$

satisfying conditions of Theorem 7, and

$$\psi_i(x, y, z) = \bigcirc_{x, y, z} \tau(x) \varphi_i(y, z) : i = 1, 2,$$

then we have:

$$\begin{aligned}
\psi_2(x, y, z) - \psi_1(x, y, z) &= \bigcirc_{x, y, z} \tau(x) \varphi_2(y, z) - \bigcirc_{x, y, z} \tau(x) \varphi_1(y, z) \\
&= \bigcirc_{x, y, z} \tau(x) (\varphi_2 - \varphi_1)(y, z) \\
&= \bigcirc_{x, y, z} \tau(x) \delta^1\alpha(y, z) \\
&= d^1\alpha(x, y, z).
\end{aligned}$$

Which means that ψ_1 and ψ_2 are in the same cohomology class. \square

5 Central Extension of 3-Lie Algebras Induced by Lie Algebras

Definition 13. Let A, B, C be n -Lie algebras ($n \geq 2$). An extension of B by A is a short sequence:

$$A \xrightarrow{\lambda} C \xrightarrow{\mu} B,$$

such that λ is an injective homomorphism, μ is a surjective homomorphism, and $\text{Im } \lambda \subset \ker \mu$. We say also that C is an extension of B by A .

Definition 14. Let A, B be n -Lie algebras, and $A \xrightarrow{\lambda} C \xrightarrow{\mu} B$ be an extension of B by A .

- The extension is said to be trivial if there exists an ideal I of C such that $C = \ker \mu \oplus I$.
- It is said to be central if $\ker \mu \subset Z(C)$.

We may equivalently define central extensions by a 1-dimensional algebra (we will simply call it central extension) this way:

Definition 15. Let A be a n -Lie algebra, we call central extension of A the space $\bar{A} = A \oplus \mathbb{K}c$ equipped with the bracket:

$$\forall x_1, \dots, x_n \in A, [x_1, \dots, x_n]_c = [x_1, \dots, x_n] + \omega(x_1, \dots, x_n)c \text{ and } [x_1, \dots, x_{n-1}, c]_c = 0.$$

Where ω is a skew-symmetric n -linear form such that $[\cdot, \dots, \cdot]_c$ satisfies the fundamental identity (or Jacobi identity for $n = 2$).

Proposition 5 ([6]).

1. The bracket of a central extension satisfies the fundamental identity (resp. Jacobi identity) if and only if ω is a 2-cocycle for the scalar cohomology of n -Lie algebras (resp. Lie algebras).
2. Two central extensions of a n -Lie algebra (resp. Lie algebra) A given by two maps ω_1 and ω_2 are isomorphic if and only if $\omega_2 - \omega_1$ is a 2-coboundary for the scalar cohomology of n -Lie algebras (resp. Lie algebras).

Now, we look at the question of whether a central extension of a Lie algebra may give a central extension of the induced 3-Lie algebra (by some trace τ), the answer is given by the following theorem:

Theorem 8. Let $(A, [\cdot, \cdot])$ be a Lie algebra, τ be a trace and $(A, [\cdot, \cdot, \cdot]_\tau)$ be the induced 3-Lie algebra. If $(\bar{A}, [\cdot, \cdot, \cdot]_c)$ is a central extension of $(A, [\cdot, \cdot])$ where

$$\bar{A} = A \oplus \mathbb{K}c \text{ and } [x, y]_c = [x, y] + \omega(x, y)c,$$

and we extend τ to \bar{A} by assuming $\tau(c) = 0$ then $(\bar{A}, [\cdot, \cdot, \cdot]_{c, \tau})$ the 3-Lie algebra induced by $(\bar{A}, [\cdot, \cdot, \cdot]_c)$, is a central extension of $(A, [\cdot, \cdot, \cdot]_\tau)$.

Proof. Let $x, y, z \in A$:

$$\begin{aligned} [x, y, z]_{c, \tau} &= \tau(x) [y, z]_c + \tau(y) [z, x]_c + \tau(z) [x, y]_c \\ &= \tau(x) ([y, z] + \omega(y, z) c) + \tau(y) ([z, x] + \omega(z, x) c) + \tau(z) ([x, y] + \omega(x, y) c) \\ &= (\tau(x) [y, z] + \tau(y) [z, x] + \tau(z) [x, y]) \\ &\quad + (\tau(x) \omega(y, z) + \tau(y) \omega(z, x) + \tau(z) \omega(x, y)) c. \\ &= [x, y, z]_\tau + \omega_\tau(x, y, z) c \end{aligned}$$

The map $\omega_\tau(x, y, z) = \tau(x) \omega(y, z) + \tau(y) \omega(z, x) + \tau(z) \omega(x, y)$ is a skew-symmetric 3-linear form, and $[\cdot, \cdot, \cdot]_{c, \tau}$ satisfies the fundamental identity, we have also:

$$\begin{aligned} [x, y, c]_{c, \tau} &= \tau(x) [y, c]_c + \tau(y) [c, x]_c + \tau(c) [x, y]_c \\ &= 0. \quad \left([y, c]_c = [c, x]_c = 0 \text{ and } \tau(c) = 0. \right) \end{aligned}$$

Therefore $(\bar{A}, [\cdot, \cdot, \cdot]_{c, \tau})$ is a central extension of $(A, [\cdot, \cdot, \cdot]_\tau)$. \square

Example 1. Consider the 4-dimensional Lie algebra $(A, [\cdot, \cdot])$ with basis $\{e_1, e_2, e_3, e_4\}$ defined by:

$$[e_2, e_4] = e_3 ; [e_3, e_4] = e_3,$$

(remaining brackets are either obtained by skew-symmetry or zero), and let ω be a skew-symmetric bilinear form on A . ω is fully defined by the scalars

$$\omega_{ij} = \omega(e_i, e_j), 1 \leq i < j \leq 4.$$

By solving the equations for ω to be a 2-cocycle:

$$\delta^2 \omega(e_i, e_j, e_k) = 0, 1 \leq i < j < k \leq 4,$$

we get the conditions:

$$\omega_{13} = 0 \text{ and } \omega_{23} = 0.$$

Now, let α be a linear form on A , defined by $\alpha(e_i) = \alpha_i, 1 \leq i \leq 4$, we find that $\delta^1 \alpha(e_2, e_4) = \delta^1 \alpha(e_3, e_4) = \alpha_3$ and $\delta^1 \alpha(e_i, e_j) = 0$ for other values of i and j ($i < j$). Now consider the trace map τ such that $\tau(e_1) = 1$ and $\tau(e_i) = 0, i \neq 1$, and the 2-cocycles λ and μ defined by:

$$\lambda(e_1, e_2) = 1$$

and

$$\mu(e_2, e_4) = 1 ; \mu(e_3, e_4) = -1.$$

Central extensions of $(A, [\cdot, \cdot])$ by λ and μ are respectively given by $(\bar{A} = A \oplus \mathbb{K}c)$:

$$[e_1, e_2]_\lambda = c ; [e_2, e_4]_\lambda = e_3 ; [e_3, e_4]_\lambda = e_3$$

and

$$[e_2, e_4]_\mu = e_3 + c ; [e_3, e_4]_\mu = e_3 - c.$$

3-Lie algebras induced by $(A, [.,.])$ and by these central extensions are given by:

$$[e_1, e_2, e_4]_\tau = e_3 ; [e_1, e_3, e_4]_\tau = e_3,$$

$$[e_1, e_2, e_4]_{\tau, \lambda} = e_3 ; [e_1, e_3, e_4]_{\tau, \lambda} = e_3$$

and

$$[e_1, e_2, e_4]_{\tau, \mu} = e_3 + c ; [e_1, e_3, e_4]_{\tau, \mu} = e_3 - c.$$

We can see that, here, the central extension given by λ induces a trivial one, while the one given by μ induces a non-trivial one. This example shows also that the converse of Proposition 4 is, in general, not true.

6 3-Lie Algebras Induced by Lie Algebras in Low Dimensions

In this section, we give a list of all 3-Lie algebras induced by Lie algebras in dimension $d \leq 5$, based on the classifications given in [8] and [4]. For this, we shall use the following result:

Proposition 6. *Let $(A, [.,.,.])$ be a 3-Lie algebra, $(e_i)_{1 \leq i \leq d}$ a basis of A . If there exists e_{i_0} in this base, such that the multiplication table of $(A, [.,.,.])$ is given by:*

$$[e_{i_0}, e_j, e_k] = x_{jk}; j \neq i_0, k \neq i_0, k \neq j$$

with e_{i_0} and x_{jk} linearly independent, then $(A, [.,.,.])$ is induced by a Lie algebra

Proof. We define a bilinear skew-symmetric map $[.,.]$ on A and a form $\tau : A \rightarrow \mathbb{K}$ by:

$$[e_j, e_k] = x_{jk}, j \neq i_0, k \neq i_0, k \neq j \text{ and } [e_{i_0}, e_j] = 0$$

and

$$\tau(x) = \tau \left(\sum_{k=0}^d x_k e_k \right) = x_{i_0}$$

$[.,.]$ satisfies the Jacobi identity:

$$\begin{aligned} [e_j, [e_k, e_l]] &= [e_{i_0}, e_j, [e_{i_0}, e_k, e_l]] \\ &= [[e_{i_0}, e_j, e_{i_0}], e_k, e_l] + [e_{i_0}, [e_{i_0}, e_j, e_k], e_l] + [e_{i_0}, e_k, [e_{i_0}, e_j, e_l]] \\ &= [[e_j, e_k], e_l] + [e_k, [e_j, e_l]] \end{aligned}$$

The obtained Lie bracket $[.,.]$ and the trace τ given above indeed induce the ternary bracket considered above:

$$\begin{aligned}
[e_{i_0}, e_j, e_k]_{\tau} &= \tau(e_{i_0})[e_j, e_k] + \tau(e_j)[e_k, e_{i_0}] + \tau(e_k)[e_{i_0}, e_j] \\
&= \tau(e_{i_0})[e_j, e_k] \\
&= x_{jk} \\
&= [e_{i_0}, e_j, e_k]
\end{aligned}$$

for $i \neq i_0$:

$$[e_i, e_j, e_k]_{\tau} = \tau(e_i)[e_j, e_k] + \tau(e_j)[e_k, e_i] + \tau(e_k)[e_i, e_j] = 0 = [e_i, e_j, e_k]$$

□

Theorem 9 ([8] 3-Lie algebras of dimension less than or equal to 4). Any 3-Lie algebra A of dimension less than or equal to 4 is isomorphic to one of the following algebras: (omitted brackets are obtained by skew-symmetry, $(e_i)_{1 \leq i \leq \dim A}$ is a basis of A)

1. If $\dim A < 3$ then A is abelian.
2. If $\dim A = 3$, then we have 2 cases :
 - a. A is abelian.
 - b. $[e_1, e_2, e_3] = e_1$.
3. if $\dim A = 4$ then we have the following cases :
 - a. A is abelian.
 - b. $[e_2, e_3, e_4] = e_1$.
 - c. $[e_1, e_2, e_3] = e_1$.
 - d. $[e_1, e_2, e_4] = ae_3 + be_4; [e_1, e_2, e_3] = ce_3 + de_4$, with $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an invertible matrix. Two such algebras, defined by matrices C_1 and C_2 , are isomorphic if and only if there exists a scalar α and an invertible matrix B such that $C_2 = \alpha B C_1 B^{-1}$.
 - e. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = ae_2; [e_1, e_2, e_4] = be_3$ ($a, b \neq 0$).
 - f. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = ae_2; [e_1, e_2, e_4] = be_3; [e_1, e_2, e_3] = ce_4$ ($a, b, c \neq 0$).

Theorem 10 ([4] 5-dimensional 3-Lie algebras). Let \mathbb{K} be an algebraically closed field. Any 5-dimensional 3-Lie algebra A defined with respect to a basis $\{e_1, e_2, e_3, e_4, e_5\}$ is isomorphic to one of the algebras listed below, where A^1 denotes $[A, A, A]$:

1. If $\dim A^1 = 0$ then A is abelian.
2. If $\dim A^1 = 1$, let $A^1 = \langle e_1 \rangle$, then we have :
 - a. $A^1 \subseteq Z(A) : [e_2, e_3, e_4] = e_1$.
 - b. $A^1 \not\subseteq Z(A) : [e_1, e_2, e_3] = e_1$.
3. If $\dim A^1 = 2$, let $A^1 = \langle e_1, e_2 \rangle$, then we have :
 - a. $[e_2, e_3, e_4] = e_1; [e_3, e_4, e_5] = e_2$.

- b. $[e_2, e_3, e_4] = e_1; [e_2, e_4, e_5] = e_2; [e_1, e_4, e_5] = e_1.$
- c. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = e_2.$
- d. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = e_2; [e_2, e_4, e_5] = e_2; [e_1, e_4, e_5] = e_1.$
- e. $[e_2, e_3, e_4] = \alpha e_1 + e_2; [e_1, e_3, e_4] = e_2.$
- f. $[e_2, e_3, e_4] = \alpha e_1 + e_2; [e_1, e_3, e_4] = e_2; [e_2, e_4, e_5] = e_2; [e_1, e_4, e_5] = e_1.$
- g. $[e_1, e_3, e_4] = e_1; [e_2, e_3, e_4] = e_2.$

where $\alpha \in \mathbb{K} \setminus \{0\}$

4. If $\dim A^1 = 3$, let $A^1 = \langle e_1, e_2, e_3 \rangle$, then we have :

- a. $[e_2, e_3, e_4] = e_1; [e_2, e_4, e_5] = -e_2; [e_3, e_4, e_5] = e_3.$
- b. $[e_2, e_3, e_4] = e_1; [e_3, e_4, e_5] = e_3 + \alpha e_2; [e_2, e_4, e_5] = e_3; [e_1, e_4, e_5] = e_1.$
- c. $[e_2, e_3, e_4] = e_1; [e_3, e_4, e_5] = e_3; [e_2, e_4, e_5] = e_2; [e_1, e_4, e_5] = 2e_1.$
- d. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = e_2; [e_1, e_2, e_4] = e_3.$
- e. $[e_1, e_4, e_5] = e_1; [e_2, e_4, e_5] = e_3; [e_3, e_4, e_5] = \beta e_2 + (1 + \beta)e_3, \beta \in \mathbb{K} \setminus \{0, 1\}.$
- f. $[e_1, e_4, e_5] = e_1; [e_2, e_4, e_5] = e_2; [e_3, e_4, e_5] = e_3.$
- g. $[e_1, e_4, e_5] = e_2; [e_2, e_4, e_5] = e_3; [e_3, e_4, e_5] = se_1 + te_2 + ue_3.$ And 3-Lie algebras corresponding to this case with coefficients s, t, u and s', t', u' are isomorphic if and only if there exists a non-zero element $r \in K$ such that :

$$s = r^3 s'; t = r^2 t'; u = ru'$$

5. If $\dim A^1 = 4$, let $A^1 = \langle e_1, e_2, e_3, e_4 \rangle$, then we have :

- a. $[e_2, e_3, e_4] = e_1; [e_3, e_4, e_5] = e_2; [e_2, e_4, e_5] = e_3; [e_2, e_3, e_5] = e_4.$
- b. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = e_2; [e_1, e_2, e_4] = e_3; [e_1, e_2, e_3] = e_4.$

The 3-Lie algebras which are induced by Lie algebras are given by the following proposition:

Proposition 7. Let \mathbb{K} be an algebraically closed field of characteristic 0. According to Theorems 9 and 10, the 3-Lie algebras induced by Lie algebras of dimension $d \leq 5$ are:

- $d = 3$ Theorem 9: 2.
- $d = 4$ Theorem 9: 3.: $a, b, c, d, e.$
- $d = 5$ Theorem 10: 1. 2. 3. 4.

Proof. By applying Proposition 6, the algebras given in Theorem 9 2. and 3. a, b, c, d, e and Theorem 10 1., 2., 3. and 4. are all induced by Lie algebras, remaining algebras' derived algebras are not abelian and then they cannot be induced by Lie algebras (Theorem 2). \square

6.1 From Lie Algebras to 3-Lie Algebras

We list, below, all 3 and 4-dimensional Lie algebras and all 3-Lie algebras they may induce, 3-dimensional algebras are classified in [12] and 4-dimensional ones, par-

tially, in [7]. For every Lie algebra, we compute all the trace maps and the induced 3-Lie algebra using these trace maps.

Theorem 11 (3-dimensional Lie algebras [12]). *Let \mathfrak{g} be a Lie algebra and $\{e_1, e_2, e_3\}$ a basis of \mathfrak{g} , then \mathfrak{g} is isomorphic to one of the following algebras: (Remaining brackets are either obtained by skew-symmetry or zero)*

1. The abelian Lie algebra $[x, y] = 0, \forall x, y \in \mathfrak{g}$.
2. $L(3, -1) : [e_1, e_2] = e_2$.
3. $L(3, 1) : [e_1, e_2] = e_3$.
4. $L(3, 2, a) : [e_1, e_3] = e_1; [e_2, e_3] = ae_2; 0 < |a| \leq 1$.
5. $L(3, 3) : [e_1, e_3] = e_1; [e_2, e_3] = e_1 + e_2$.
6. $L(3, 4, a) : [e_1, e_3] = ae_1 - e_2; [e_2, e_3] = e_1 + ae_2; a \geq 0$.
7. $L(3, 5) : [e_1, e_2] = e_1; [e_1, e_3] = -2e_2; [e_2, e_3] = e_3$.
8. $L(3, 6) : [e_1, e_2] = e_3; [e_1, e_3] = -e_2; [e_2, e_3] = e_1$.

Remark 4. The classification given above is for the ground field $\mathbb{K} = \mathbb{R}$, if $\mathbb{K} = \mathbb{C}$ then $L(3, 2, \frac{x-i}{x+i})$ is isomorphic to $L(3, 4, x)$ and $L(3, 5)$ is isomorphic to $L(3, 6)$.

Theorem 12 (Solvable 4-dimensional Lie algebras [7]). *Let \mathfrak{g} be a solvable Lie algebra, and $\{e_1, e_2, e_3, e_4\}$ a basis of \mathfrak{g} , then \mathfrak{g} is isomorphic to one of the following algebras: (Remaining brackets are either obtained by skew-symmetry or zero)*

1. The abelian Lie algebra $[x, y] = 0, \forall x, y \in \mathfrak{g}$.
2. $M^2 : [e_1, e_4] = e_1; [e_2, e_4] = e_2; [e_3, e_4] = e_3$.
3. $M_a^3 : [e_1, e_4] = e_1; [e_2, e_4] = e_3; [e_3, e_4] = -ae_2 + (a+1)e_3$.
4. $M^4 : [e_2, e_4] = e_3; [e_3, e_4] = e_3$.
5. $M^5 : [e_2, e_4] = e_3$.
6. $M_{a,b}^6 : [e_1, e_4] = e_2; [e_2, e_4] = e_3; [e_3, e_4] = ae_1 + be_2 + e_3$.
7. $M_{a,b}^7 : [e_1, e_4] = e_2; [e_2, e_4] = e_3; [e_3, e_4] = ae_1 + be_2 (a = b \neq 0 \text{ or } a = 0 \text{ or } b = 0)$.
8. $M^8 : [e_1, e_2] = e_2; [e_3, e_4] = e_4$.
9. $M_a^9 : [e_1, e_4] = e_1 + ae_2; [e_2, e_4] = e_1; [e_1, e_3] = e_1; [e_2, e_3] = e_2 (X^2 - X - a \text{ has no root in } \mathbb{K})$.
10. $M^{11} : [e_1, e_4] = e_1; [e_3, e_4] = e_3; [e_1, e_3] = e_2$.
11. $M^{12} : [e_1, e_4] = e_1; [e_2, e_4] = e_2; [e_3, e_4] = e_3; [e_1, e_3] = e_2$.
12. $M_a^{13} : [e_1, e_4] = e_1 + ae_3; [e_2, e_4] = e_2; [e_3, e_4] = e_1; [e_1, e_3] = e_2$.
13. $M_a^{14} : [e_1, e_4] = ae_3; [e_3, e_4] = e_1; [e_1, e_3] = e_2. (M_a^{14} \text{ is isomorphic to } M_b^{14} \text{ if and only if } a = \alpha^2 b \text{ for some } \alpha \neq 0)$.

In the following, we will give all the traces τ on the Lie algebras listed above, we add to this list two non-solvable Lie algebras of dimension 4, and the induced 3-Lie algebras: (for a Lie algebra \mathfrak{g} $(e_i)_{1 \leq i \leq \dim \mathfrak{g}}$ is a basis of \mathfrak{g} , and for $x \in \mathfrak{g}$, $(x_i)_{1 \leq i \leq \dim \mathfrak{g}}$ are its coordinates in this basis).

Lie algebra	Trace	Induced 3-Lie algebra
Abelian Lie algebra	All linear forms	Abelian 3-Lie algebra
$L(3, -1)$	$\tau(x) = t_1x_1 + t_3x_3$	$[e_1, e_2, e_3] = t_3e_2$
$L(3, 1)$	$\tau(x) = t_1x_1 + t_2x_2$	Abelian 3-Lie algebra
$L(3, 2, a), L(3, 3)$ $L(3, 4, a)$	$\tau(x) = t_3x_3$	Abelian 3-Lie algebra
$L(3, 5), L(3, 6)$	$\tau(x) = 0$	Abelian 3-Lie algebra
M_a^2, M_a^3 $M_{a,b}^6; M_{a,b}^7$ ($a \neq 0$)	$\tau(x) = t_4x_4$	Abelian 3-Lie algebra
M_0^3	$\tau(x) = t_2x_2 + t_4x_4$	$[e_1, e_2, e_4] = -t_2e_1$
M^4	$\tau(x) = t_1x_1 + t_2x_2 + t_4x_4$	$[e_1, e_2, e_4] = t_1e_3$ $[e_1, e_3, e_4] = t_1e_3$ $[e_2, e_3, e_4] = t_2e_3$
M^5	$\tau(x) = t_1x_1 + t_2x_2 + t_4x_4$	$[e_1, e_2, e_4] = t_1e_3$
M_{0b}^6	$\tau(x) = t_1x_1 + t_4x_4$	$[e_1, e_2, e_4] = t_1e_3$ $[e_1, e_3, e_4] = t_1(be_2 + e_3)$
M_{0b}^7	$\tau(x) = t_1x_1 + t_4x_4$	$[e_1, e_2, e_4] = t_1e_3$ $[e_1, e_3, e_4] = t_1be_2$
M^8	$\tau(x) = t_1x_1 + t_3x_3$	$[e_1, e_2, e_3] = t_3e_2$ $[e_1, e_3, e_4] = t_1e_4$
M_a^9	$\tau(x) = t_3x_3 + t_4x_4$	$[e_1, e_3, e_4] = -t_3(e_1 + ae_2) + t_4e_1$ $[e_2, e_3, e_4] = t_3e_1 + t_4e_2$
M^{11}	$\tau(x) = t_4x_4$	$[e_1, e_3, e_4] = t_4e_2$ $[e_2, e_3, e_4] = t_4e_1$
M_a^{12}, M_a^{13} $M_a^{14}, a \neq 0$	$\tau(x) = t_4x_4$	$[e_1, e_3, e_4] = t_4e_2$
M_0^{13}	$\tau(x) = t_3x_3 + t_4x_4$	$[e_1, e_3, e_4] = -t_3e_1 + t_4e_2$ $[e_2, e_3, e_4] = -t_3e_2$
M_0^{14}	$\tau(x) = t_3x_3 + t_4x_4$	$[e_1, e_3, e_4] = t_4e_2$
$\mathfrak{gl}_2(\mathbb{K})$	$\tau(x) = t_4x_4$	$[e_1, e_2, e_4] = 2t_4e_2$ $[e_1, e_3, e_4] = -2t_4e_3$ $[e_2, e_3, e_4] = t_4e_1$
$E_3 \times \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$)	$\tau(x) = t_4x_4$	$[e_1, e_2, e_4] = t_4e_3$ $[e_1, e_3, e_4] = -t_4e_2$ $[e_2, e_3, e_4] = t_4e_1$

Where E_3 denotes the 3-dimensional Euclidean space equipped with the cross product.

7 Examples

7.1 Adjoint Representation 1-Cocycles and Coboundaries

Here, we give the set of 1-cocycles/coboundaries of 4 chosen Lie algebras ($\mathfrak{gl}_2(\mathbb{K})$ and M^4 , M^5 and M^8 in the classification above) in the classification above and 1-cocycles/coboundaries of the induced algebras using a chosen trace map for each one, the computations were done using the computer algebra software Mathematica.

Shortly explained, the computation goes this way:

Let $(A, [\cdot, \cdot, \cdot])$ be a Lie algebra of dimension n with a basis $B = \{e_1, \dots, e_n\}$, τ a trace and $(A, [\cdot, \cdot, \cdot]_\tau)$ the induced algebra. Denote the structure constants of $(A, [\cdot, \cdot, \cdot])$ with respect to this basis B by $(c_{ij}^k)_{1 \leq i, j, k \leq n}$ and by $(ct_{ijk}^q)_{1 \leq i, j, k, q \leq n}$ those of $(A, [\cdot, \cdot, \cdot]_\tau)$. The linear form τ is represented by the one-line matrix $T = (t_i)_{1 \leq i \leq n}$. A given linear map $f : A \rightarrow A$ (1-cochain) may be represented by a $n \times n$ matrix, $Z = (z_{ij})_{1 \leq i, j \leq n}$. In terms of structure constants, the condition for f (represented by the matrix Z) to be a cocycle writes for the Lie algebra:

$$\sum_{k=1}^n (c_{ij}^k z_{qk} - c_{kj}^q z_{ki} - c_{ik}^q z_{kj}) = 0, \forall i, j, q,$$

and for the induced ternary algebra

$$\sum_{p=1}^n (ct_{ijk}^p z_{qp} - ct_{pjk}^q z_{pi} - ct_{ipk}^q z_{pj} - ct_{ijp}^q z_{pk}) = 0, \forall i, j, k, q.$$

By solving these equations, we get a set of conditions, that we apply to Z to get the matrices listed in the tables below, under "Cocycle" and "Ternary cocycle" respectively.

Matrices listed under "Coboundary" and "Ternary coboundary" are obtained by putting in column j respectively $[y, e_j]$ or $[x, y, e_j]_\tau$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, and $x_{ij} = x_i y_j - x_j y_i$.

- $\mathfrak{gl}_2(\mathbb{K})$:

Cocycle	Coboundary	$\dim H^1$
$\begin{pmatrix} 0 & z_{12} & z_{13} & 0 \\ -2z_{13} & z_{22} & 0 & 0 \\ -2z_{12} & 0 & z_{22} & 0 \\ 0 & 0 & 0 & z_{44} \end{pmatrix}$	$\begin{pmatrix} 0 & -y_3 & y_2 & 0 \\ -2y_2 & 2y_1 & 0 & 0 \\ 2y_3 & 0 & -2y_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	1
Trace :	$\tau(x) = x_4$	
Ternary cocycle	Ternary coboundary	$\dim H_\tau^1$
$\begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ -2z_{13} & z_{22} & 0 & z_{24} \\ -2z_{12} & 0 & 2z_{11} - z_{22} & z_{34} \\ 0 & 0 & 0 & -z_{11} \end{pmatrix}$	$\begin{pmatrix} 0 & x_{34} & x_{42} & x_{23} \\ -2x_{42} & -2x_{14} & 0 & 2x_{12} \\ -2x_{34} & 0 & 2x_{14} & -2x_{13} \\ 0 & 0 & 0 & 0 \end{pmatrix}$	1

• M^4 :

Cocycle	Coboundary	$\dim H^1$
$\begin{pmatrix} z_{11} & z_{12} & 0 & z_{14} \\ z_{21} & z_{22} & 0 & z_{24} \\ -z_{21} & z_{32} & z_{22} + z_{32} & z_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -y_4 & -y_4 & y_2 + y_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	6
Trace : $\tau(x) = x_1 + x_2 + x_4$		
Ternary cocycle		
$\begin{pmatrix} z_{11} & z_{12} & 0 & z_{14} \\ z_{21} & z_{11} - z_{12} + z_{21} & 0 & z_{24} \\ z_{31} & z_{32} & z_{11} - z_{12} - z_{31} + z_{32} & z_{34} \\ z_{41} & z_{41} & 0 & -z_{11} - z_{21} \end{pmatrix}$		
Ternary coboundary		$\dim H^1_\tau$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_{24} + x_{34} & x_{34} - x_{14} & -x_{14} - x_{24} & x_{12} + x_{13} + x_{23} \\ 0 & 0 & 0 & 0 \end{pmatrix}$		6

• M^5 :

Cocycle	Coboundary	$\dim H^1$
$\begin{pmatrix} z_{11} & z_{12} & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ 0 & z_{42} & 0 & z_{33} - z_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -y_4 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	8
Trace : $\tau(x) = x_1$		
Ternary cocycle	Ternary coboundary	$\dim H^1_\tau$
$\begin{pmatrix} -z_{22} + z_{33} - z_{44} & z_{12} & 0 & z_{14} \\ z_{21} & z_{22} & 0 & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & 0 & z_{44} \end{pmatrix}$		9

• M^8 :

Cocycle	Coboundary	$\dim H^1$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z_{43} & z_{44} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -y_2 & y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y_4 & y_3 \end{pmatrix}$	0
Trace : $\tau(x) = x_1 + x_3$		
Ternary cocycle	Ternary coboundary	$\dim H^1_\tau$
$\begin{pmatrix} -z_{33} & 0 & z_{13} & 0 \\ z_{21} & z_{22} & z_{23} & 0 \\ z_{31} & 0 & z_{33} & 0 \\ z_{41} & 0 & z_{43} & z_{44} \end{pmatrix}$		4

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