CONSTRUCTING 3-LIE ALGEBRAS

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Abstract. 3-Lie algebras are constructed by Lie algebras, derivations and linear functions, associative commutative algebras, whose involutions and derivations. Then the 3-Lie algebras are obtained from group algebras $F[G]$. An infinite dimensional simple 3-Lie algebra $(A, [, ,]_{\omega,\delta_0})$ and a non-simple 3-Lie algebra $(A, [, ,]_{\omega_1,\delta})$ are constructed by Laurent polynomials $A = F[t, t^{-1}]$ and its involutions ω and ω_1 and derivations δ and δ_0. At last of the paper, we summarize the methods of constructing n-Lie algebras for $n \geq 3$ and provide a problem.

1. Introduction

To construct the generalized Hamiltonian dynamics, Nambu [1, 2] first proposed the notion of 3-bracket. Thus Nambu dynamics is described by the phase flow given by Nambu-Hamilton equations of motion which involves two Hamiltonians. The notion of n-Lie algebra was introduced by Filippov in 1985 ([3]). It is a natural generalization of the concept of a Lie algebra to the case where the fundamental multiplication is n-ary, $n \geq 2$ (when $n = 2$ the definition agrees with the usual definition of a Lie algebra). 3-Lie algebras have close relationships with many important fields in mathematics and mathematical physics (cf. [1, 2, 4, 5, 6, 7, 8]). For example, the metric 3-Lie algebras are used to describe a world volume of multiple $M2$-branes [4, 7] (BLG).

Since the multiple multiplication, the structure of n-Lie algebras is more complicated than that of Lie algebras. Especially, the realizations of n-Lie algebras is hardness.

Filippov in [9] constructed n-Lie algebra structure on commutative associative algebras by commuting derivations $\{D_1, \cdots, D_n\}$. Let A be a commutative associative algebra, then A is an n-Lie algebra with the multiplication:

$$[x_1, \cdots, x_n] = \det \begin{pmatrix} D_1(x_1) & \cdots & D_1(x_n) \\ \cdots & \cdots & \cdots \\ D_n(x_1) & \cdots & D_n(x_n) \end{pmatrix}, \tag{1.1}$$

which is called a *Jcobian algebra* defined by $\{D_1, \cdots, D_n\}$.

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Filippov and Pozhidaev in papers [9, 11, 12] defined *monomial n-Lie algebra* $A_G(f,t)$ with the multiplication

$$[e_{a_1},\cdots,e_{a_n}] = f(a_1,\cdots,a_n)e_{a_1+\cdots+a_n+t}, \tag{1.2}$$

where G is an Abelian group, A_G is a vector space with a basis $\{e_g | g \in G\}$, $t \in G$, $f : G^n \to F$.

Dzhumadildaev in [10] constructed an $(n+1)$-Lie algebra from a strong n-Lie-Poisson algebra $(L,\cdot,\omega)$ by endowing a skew-symmetric $(n+1)$-multiplication

$$\bar{\omega} = Id_L \wedge \omega, \tag{1.3}$$

where Id_L is the identity map of L.

Bai and collaborators in [13, 14] constructed n-Lie algebras by two-dimensional extensions of metric n-Lie algebras. And obtain $(n+1)$-Lie algebras by n-Lie algebras and linear functions $f \in L^*$, which satisfies $f(L^1) = 0$.

In this paper, we pay our main attention to construct 3-Lie algebras. The paper is organized as follows. Section 2 introduces some basic notions of n-Lie algebras. Section 3 provides a construction of 3-Lie algebras from a commutative associative algebras, involutions and derivations. Section 4 constructs 3-Lie algebras from group algebras. Section 5 studies 3-Lie algebras constructed by Laurent polynomials.

2. Fundamental notions of n-Lie algebra

An n-Lie algebra L (cf.[3]) is a vector space endowed with an n-ary multilinear skew-symmetric multiplication satisfying the n-Jacobi identity: $\forall x_1,\cdots,x_n,y_2,\cdots,y_n \in L$

$$[[x_1,\cdots,x_n],y_2,\cdots,y_n] = \sum_{i=1}^{n}[x_1,\cdots,[x_i,y_2,\cdots,y_n],\cdots,x_n]. \tag{2.1}$$

The n-ary skew-symmetry of the operation $[x_1,\cdots,x_n]$ means that

$$[x_1,\cdots,x_n] = sgn(\sigma)[x_{\sigma(1)},\cdots,x_{\sigma(n)}], \ \ \forall x_1,\cdots,x_n \in L, \tag{2.2}$$

for any permutation $\sigma \in S_n$.

A subspace B of L is called a *subalgebra* if $[B,\ldots,B] \subseteq B$. In particular, the subalgebra generated by the vectors $[x_1,\cdots,x_n]$ for any $x_1,\cdots,x_n \in L$ is called the *derived algebra* of L, which is denoted by L^1. If $L^1 = 0$, then L is called an *abelian n-Lie algebra*.

A derivation of an n-Lie algebra is a linear transformation D of L into itself satisfying

$$D([x_1,\cdots,x_n]) = \sum_{i=1}^{n}[x_1,\cdots,D(x_i),\cdots,x_n], \tag{2.3}$$

for $x_1,\cdots,x_n \in L$. Let $DerL$ be the set of all derivations of L. Then $DerL$ is a subalgebra of the general Lie algebra $gl(L)$ and is called *the derivation algebra* of L.

The map ad $(x_1, \cdots, x_{n-1}) : L \to L$ given by

$$ad(x_1, \cdots, x_{n-1})(x_n) = [x_1, \cdots, x_{n-1}, x_n], \forall x_n \in L,$$

is referred to as *a left multiplication* defined by elements $x_1, \cdots, x_{n-1} \in L$. It follows from identity (2.1) that $ad(x_1, \cdots, x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $DerL$, which is denoted by $ad(L)$. Every derivation in $ad(L)$ is by definition an inner derivation.

An *ideal* of an n-Lie algebra L is a subspace I such that $[I, L, \cdots, L] \subseteq I$. If $L^1 \neq 0$ and L has no ideals except for 0 and itself, then L is called a *simple n-Lie algebra*.

3. 3-Lie algebras constructed by commutative associative algebras

R.Bai and collaborators (cf [14]) constructed 3-Lie algebra $(L, [, ,]_\alpha)$ by a Lie algebra $(L, [,])$ and linear function $\alpha \in L^*$, where $\alpha([L, L]) = 0$, and $[, ,]_\alpha$ defined as follows

$$[a, b, c]_\alpha = \alpha(c)[a, b] + \alpha(a)[b, c] + \alpha(b)[c, a], \ for \ a, b, c \in L.$$

And it is proved that the 3-Lie algebra $(L, [, ,]_\alpha)$ is a two step 3-solvable 3-Lie algebra, that is, $L^{(2)} = [L^1, L^1, L^1]_\alpha = 0$, where $L^1 = [L, L, L]_\alpha$.

In this section, we try to construct 3-Lie algebras by commutative associative algebras and their derivations, involutions and linear functions.

Let A be a a commutative associative algebra over a field F. A derivation Δ of A is a linear mapping of A satisfying $\Delta(xy) = \Delta(x)y + x\Delta(y)$ for every $x, y \in A$. If a linear mapping $\omega : A \to A$ satisfying for every $a, b \in A$, $\omega(ab) = \omega(a)\omega(b)$ and $\omega^2(a) = a$, then ω is called *an involution* of A. If $chF \neq 2$, then for every $c \in A$,

$$\omega(c + \omega(c) = c + \omega(c), \ \omega(c - \omega(c)) = -(c - \omega(c)), \text{ and } c = \tfrac{1}{2}(c + \omega(c)) + \tfrac{1}{2}(c - \omega(c)).$$

Therefore,

$$A = A_1 \dot{+} A_{-1}, \ \text{where } A_1 = \{a |\ a \in A, \omega(a) = a\ \}, \ A_{-1} = \{b |\ b \in A, \omega(b) = -b\ \}.$$

First we give the following result.

Lemma 3.1. *Let A be a commutative associative algebra, ω be an involution of A and $\Delta \in DerA$. Then $(A, [,]_\Delta)$ and $(A, [,]_\omega)$ are Lie algebras, where*

$$[a, b]_\Delta = (Id_A \wedge \Delta)(a, b) = a\Delta(b) - b\Delta(a) \ a, b \in A; \tag{3.1}$$

$$[a, b]_\omega = (\omega \wedge Id_A)(a, b) = \omega(a)b - \omega(b)a, \ a, b \in A. \tag{3.2}$$

If Δ and ω satisfy $\Delta\omega + \omega\Delta = 0$, then $(A, [,]_{\omega,\Delta})$ is a Lie algebra, where

$$[a, b]_{\omega,\Delta} = ((Id_A - \omega) \wedge \Delta)(a, b) = (a - \omega(a))\Delta(b) - (b - \omega(b))\Delta(a), \ a, b \in A. \tag{3.3}$$

Proof. Eq. (3.1) is well know result. By the direct computation, Eqs. (3.2) and (3.3) satisfy Eq (2.1), respectively. □

Theorem 3.2. *Let A be a commutative associative algebra, $\Delta \in Der(A)$ and ω be an involution of A. If $\alpha,\ \beta, \gamma \in A^*$ satisfy*

$$\alpha(ab) = 0,\ \beta(a\Delta(b) - b\Delta(a)) = 0, \gamma(a\Delta(b) - b\Delta(a)) = \gamma(\omega(a)\Delta(b) - \omega(b)\Delta(a)), \tag{3.4}$$

then $(A, [, ,]_{\alpha,f})$ and $(A, [, ,]_{\beta,\Delta})$ are 3-Lie algebras, where for arbitrary $a, b, c \in A$,

$$[a, b, c]_{\alpha,\omega} = (\alpha \wedge Id_A \wedge \omega)(a, b, c) = \alpha(a)[b, c]_\omega + \alpha(b)[c, a]_\omega + \alpha(c)[a, b]_\omega, \tag{3.5}$$

$$[a, b, c]_{\beta,\Delta} = (\beta \wedge Id_A \wedge \Delta)(a, b, c) = \beta(a)[b, c]_\Delta + \beta(b)[c, a]_\Delta + \beta(c)[a, b]_\Delta. \tag{3.6}$$

If Δ and ω satisfy $\Delta\omega + \omega\Delta = 0$, then $(A, [,]_{\gamma,\omega,\Delta})$ is a 3-Lie algebra, where

$$\begin{aligned}[a, b, c]_{\gamma,\omega,\Delta} &= (\gamma \wedge (Id_A - \omega) \wedge \Delta)(a, b, c) = \gamma(a)((b - \omega(b))\Delta(c) - (c - \omega(c))\Delta(b)) \\ &+\gamma(b)((c - \omega(c))\Delta(a) - (a - \omega(a))\Delta(c)) + \gamma(c)((a - \omega(a))\Delta(b) - (b - \omega(b))\Delta(a)).\end{aligned} \tag{3.7}$$

Proof. Let $\alpha \in A^*$. Then for every $a, b \in A$,

$$\alpha(\omega(a)b - \omega(b)a) = \begin{cases} 0, & \text{if } a, b \in A_1;\ \text{or}\ \ a, b \in A_{-1}, \\ 2\alpha(ab), & \text{if } a \in A_1, b \in A_{-1}, \\ -2\alpha(ab), & \text{if } b \in A_1, a \in A_{-1}. \end{cases}$$

Therefore, if $\alpha(ab) = 0$, then $\alpha(\omega(a)b - \omega(b)a) = 0$. From Lemma 3.1, Eq. (3.1), and Theorem 3.1 in [14], the result holds. □

Theorem 3.3. *Let A be a commutative associative algebra over a field F, Δ be a derivation of A and $\omega : A \to A$ be an involution of A satisfying*

$$f\Delta + \Delta\omega = 0.$$

Then A is a 3-Lie algebra in the multiplication $[, ,]_{\omega,\Delta} : A \otimes A \otimes A \to A,\ \ \forall\ a, b, c \in A$,

$$[a, b, c]_{\omega,\Delta} = \omega \wedge Id_A \wedge \Delta(a, b, c) = \begin{vmatrix} \omega(a) & \omega(b) & \omega(c) \\ a & b & c \\ \Delta(a) & \Delta(b) & \Delta(c) \end{vmatrix} \tag{3.8}$$

$$= \omega(a)(b\Delta(c)-c\Delta(b))+\omega(b)(c\Delta(a)-a\Delta(c))+\omega(c)(a\Delta(b)-b\Delta(a)).$$

Proof. It is clear that $[, ,]_{\omega,\Delta}$ is a 3-ary linear skew-symmetric multiplication on A. Now we prove that $[, ,]_{\omega,\Delta}$ satisfies Eq.(2.1). Since A is commutative and ω is an involution of A which satisfies $\omega\Delta + \Delta\omega = 0$, by Eq.(3.8), $\forall a, b, c, d, e \in A$

$$\omega([a,b,c]_{\omega,\Delta}) = \begin{vmatrix} \omega(a) & \omega(b) & \omega(c) \\ a & b & c \\ \Delta\omega(a) & \Delta\omega(b) & \Delta\omega(c) \end{vmatrix},$$

$$\Delta([a,b,c]_{\omega,\Delta}) = \begin{vmatrix} \Delta\omega(a) & \Delta\omega(b) & \Delta\omega(c) \\ a & b & c \\ \Delta(a) & \Delta(b) & \Delta(c) \end{vmatrix} + \begin{vmatrix} \omega(a) & \omega(b) & \omega(c) \\ a & b & c \\ \Delta^2(a) & \Delta^2(b) & \Delta^2(c) \end{vmatrix},$$

$$[[a,b,c]_{\omega,\Delta}, d, e]_{\omega,\Delta} = \begin{vmatrix} \omega([a,b,c]_{\omega,\Delta}) & \omega(d) & \omega(e) \\ [a,b,c]_{\omega,\Delta} & d & e \\ \Delta([a,b,c]_{\omega,\Delta}) & \Delta(d) & \Delta(e) \end{vmatrix}$$

$$= \begin{vmatrix} \omega(a) & \omega(b) & \omega(c) \\ a & b & c \\ \Delta\omega(a) & \Delta\omega(b) & \Delta\omega(c) \end{vmatrix} \begin{vmatrix} d & e \\ \Delta(b) & \Delta(c) \end{vmatrix} - \begin{vmatrix} \omega(a) & \omega(b) & \omega(c) \\ a & b & c \\ \Delta(a) & \Delta(b) & \Delta(c) \end{vmatrix} \begin{vmatrix} \omega(d) & \omega(e) \\ \Delta(d) & \Delta(e) \end{vmatrix}$$

$$+ \begin{vmatrix} \Delta\omega(a) & \Delta\omega(b) & \Delta\omega(c) \\ a & b & c \\ \Delta(a) & \Delta(b) & \Delta(c) \end{vmatrix} \begin{vmatrix} \omega(d) & \omega(e) \\ d & e \end{vmatrix} + \begin{vmatrix} \omega(a) & \omega(b) & \omega(c) \\ a & b & c \\ \Delta^2(a) & \Delta^2(b) & \Delta^2(c) \end{vmatrix} \begin{vmatrix} \omega(d) & \omega(e) \\ d & e \end{vmatrix}.$$

Then we have

$$[[a,d,e]_{\omega,\Delta}, b, c]_{\omega,\Delta} + [[b,d,e]_{\omega,\Delta}, c, a]_{\omega,\Delta} + [[c,d,e]_{\omega,\Delta}, a, b]_{\omega,\Delta}$$

$$= \circlearrowleft_{a,b,c} \Delta\omega(a) \begin{vmatrix} b & c \\ \Delta(b) & \Delta(c) \end{vmatrix} \begin{vmatrix} \omega(d) & \omega(e) \\ d & e \end{vmatrix} + \circlearrowleft_{a,b,c} a \begin{vmatrix} \omega(b) & \omega(c) \\ \Delta(b) & \Delta(c) \end{vmatrix} \begin{vmatrix} \omega(d) & \omega(e) \\ \Delta(d) & \Delta(e) \end{vmatrix}$$

$$+ \circlearrowleft_{a,b,c} \Delta\omega(a) \begin{vmatrix} \omega(b) & \omega(c) \\ b & c \end{vmatrix} \begin{vmatrix} d & e \\ \Delta(d) & \Delta(e) \end{vmatrix} + \circlearrowleft_{a,b,c} \Delta^2(a) \begin{vmatrix} \omega(b) & \omega(c) \\ b & c \end{vmatrix} \begin{vmatrix} \omega(d) & \omega(e) \\ d & e \end{vmatrix}$$

$$= [[a,b,c]_{\omega,\Delta}, d, e]_{\omega,\Delta},$$

where $\circlearrowleft_{a,b,c}$ is the circulation of a, b, c, for example

$$\circlearrowleft_{a,b,c} \Delta\omega(a) \begin{vmatrix} b & c \\ \Delta(b) & \Delta(c) \end{vmatrix} = \Delta\omega(a) \begin{vmatrix} b & c \\ \Delta(b) & \Delta(c) \end{vmatrix} + \Delta\omega(b) \begin{vmatrix} c & a \\ \Delta(c) & \Delta(a) \end{vmatrix} + \Delta\omega(c) \begin{vmatrix} a & b \\ \Delta(a) & \Delta(b) \end{vmatrix}.$$

Therefore, $(A, [, ,]_{\omega,\Delta})$ is a 3-Lie algebra in the multiplication (3.8). $\square$

Corollary 3.4. *Let A be a commutative associative algebra, $\Delta \in Der(A)$ and ω be an involution of A satisfying $\Delta\omega + \omega\Delta = 0$. Then 3-Lie algebra $(A, [, ,]_{\Delta,\omega})$ with the multiplication (3.8) can be decomposed into the direct sum of abelian subalgebras A_1 and A_{-1}, that is, $A = A_1 \dot{+} A_{-1}$, and*

$$[A_1, A_1, A_1]_{\omega,\Delta} = [A_{-1}, A_{-1}, A_{-1}]_{\omega,\Delta} = 0.$$

And $\Delta(A_1) \subseteq A_{-1}$, $\Delta(A_{-1}) \subseteq A_1$.

Proof. For every $a_i \in A_1$, $\omega(a_i) = a_i$ for $i = 1, 2, 3$, $\omega\Delta(a_1) = -\Delta\omega(a_1) = -\Delta(a_1)$, then

$$[a_1,a_2,a_3]_{\omega,\Delta}=\begin{vmatrix}\omega(a_1) & \omega(a_2) & \omega(a_3)\\ a_1 & a_2 & a_3\\ \Delta(a_1) & \Delta(a_2) & \Delta(a_3)\end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3\\ a_1 & a_2 & a_3\\ \Delta(a_1) & \Delta(a_2) & \Delta(a_3)\end{vmatrix}=0.$$

Thus A_1 is an abelian subalgebra, and $\Delta(A_1)\subseteq A_{-1}$. $\square$

Theorem 3.5. *Let $(A,[,,]_{\omega,\Delta})$ be the 3-Lie algebra with the multiplication (3.8). If I is an ideal of the associative algebra A which satisfies $\omega(I)\subseteq I$ and $\Delta(I)\subseteq I$. Then I is an ideal of the 3-Lie algebra $(A,\ [,,]_{\omega,\Delta})$.*

Proof. Let I be an ideal of the associative algebra A satisfying $\omega(I)\subseteq I$ and $\Delta(I)\subseteq I$. Then for every $a\in I$, and $\forall b,c\in A$, by Eq. (3.8)

$$[a,b,c]_{\omega,\Delta}=\omega(a)(b\Delta(c)-c\Delta(b))+\omega(b)(a\Delta(c)-c\Delta(a))+\omega(c)(a\Delta(b)-b\Delta(a))\in I.$$

Therefore, I is an ideal of the 3-Lie algebra $(A,\ [,,]_{\omega,\Delta})$. $\square$

Theorem 3.6. *Let A_1 and A_2 be commutative associative algebras, ω_i be an involution of A_i and Δ_i be a derivation of A_i which satisfies $\omega_i\Delta_i+\Delta_i\omega_i=0$ for $i=1,2$. If $\sigma: A_1\rightarrow A_2$ is an associative algebra isomorphism satisfying $\sigma\omega_1=\omega_2\sigma$, and $\sigma\Delta_1=\Delta_2\sigma$. Then σ is a 3-Lie algebra isomorphism from $(A_1,[,,]_{\omega_1,\Delta_1})$ onto $(A_2,[,,]_{\omega_2,\Delta_2})$.*

Proof. For every $a,b,c\in A_1$, by Eq.(3.8),

$\sigma([a,b,c]_{\omega_1,\Delta_1})=\sigma(\omega_1(a)(b\Delta_1(c)-c\Delta_1(b))+\omega_1(b)(c\Delta_1(a)-a\Delta_1(c))+\omega_1(c)(a\Delta_1(b)-b\Delta_1(a)))$

$$=\omega_2(\sigma(a))(\sigma(b)\Delta_2\sigma(c)-\sigma(c)\Delta_2(\sigma(b)))+\omega_2(\sigma(b))(\sigma(c)\Delta_2(\sigma(a))-\sigma(a)\Delta_2(\sigma(c)))$$
$$+\omega_2(\sigma(c))(\sigma(a)\Delta_2(\sigma(b))-\sigma(b)\Delta_2(\sigma(a)))$$
$$=[\sigma(a),\sigma(b),\sigma(c)]_{\omega_2,\Delta_2}.$$

It follows the result. $\square$

Corollary 3.7. *If A is a nilpotent commutative associative algebra, then $(A,[,,]_{\omega,\Delta})$ is a nilpotent 3-Lie algebra.*

Proof. Suppose $A^m=0$ for some positive integer m. Then for every $a_1,\cdots,a_m\in A$, $a_1\cdots a_m=0$. Thanks to Eq. (3.8), for every $a,b\in A$, $ad^m(a,b)(A)\subseteq A^m=0$. Therefore, $(A,[,,]_{\omega,\Delta})$ is a nilpotent 3-Lie algebra. $\square$

4. 3-Lie algebras constructed by group algebras

Let $(G,+)$ be an additive Abelian group, and $F[G]$ be the group algebra, that is, $F[G]$ is a commutative associative algebra with a basis $\{e_g\ |g\in G\}$, and for every $x=\sum\limits_{g\in G}\lambda_g e_g$,

$y=\sum\limits_{h\in G}\mu_h e_h\in F[G]$,

$$x+y=\sum_{g\in G}(\lambda_g+\mu_g)e_g,\ xy=(\sum_{g\in G}\lambda_g e_g)(\sum_{h\in G}\mu_h e_h)=\sum_{g,h\in G}\lambda_g\mu_h e_{g+h}.$$

Define linear mapping $\omega: F[G]\rightarrow F[G]$,

$$\omega(x)=\omega(\sum_{g\in G}\lambda_g e_g)=\sum_{g\in G}\lambda_g e_{-g},\ \forall x=\sum_{g\in G}\lambda_g e_g\in F[G]. \tag{4.1}$$

Then ω is a linear isomorphism of $F[G]$ and for $x=\sum\limits_{g\in G}\lambda_g e_g$, $y=\sum\limits_{h\in G}\mu_h e_h\in F[G]$,

$$\omega^2(x)=\omega(\sum_{g\in G}\lambda_g e_{-g})=\sum_{g\in G}\lambda_g e_g=x,$$

$$\omega(xy)=\omega(\sum_{g\in G}\lambda_g e_g\sum_{h\in G}\mu_h e_h)=\omega(\sum_{g,h\in G}\lambda_g\mu_h e_{g+h})=\sum_{g,h\in G}\lambda_g\mu_h e_{-g-h}$$

$$=\sum_{g\in G}\lambda_g e_{-g}\sum_{hG}\mu_h e_{-h}=\omega(\sum_{g\in G}\lambda_g e_g)\omega(\sum_{h\in G}\mu_h e_h)=\omega(x)\omega(y).$$

Therefore, ω is an involution of the commutative associative algebra $F[G]$.

Denote F^+ the addition group of F. For every $\alpha\in Hom(G,F^+)$, then α satisfies $\alpha(g+h)=\alpha(g)+\alpha(h)$, $\forall g,h\in G$. Define linear mapping $\alpha^*: F[G]\rightarrow F[G]$ as follows:

$$\alpha^*(\sum_{g\in G}\lambda_g e_g)=\sum_{g\in G}\lambda_g\alpha(g)e_g,\ \forall\sum_{g\in G}\lambda_g e_g\in F[G]. \tag{4.2}$$

Lemma 4.1. *Let $(G,+)$ be an abelian group, $\omega: F[G]\rightarrow F[G]$ be defined as Eq.(4.1). Then for every $\alpha\in Hom(G,F^+)$, α^* defined as Eq. (4.2) is a derivation of the algebra $F[G]$, and satisfies $\omega\alpha^*+\alpha^*\omega=0$.*

Proof. By Eqs. (4.1) and (4.2), for arbitrary $x=\sum\limits_{g\in G}\lambda_g e_g$, $y=\sum\limits_{h\in G}\mu_h e_h\in F[G]$,

$$\alpha^*((\sum_{g\in G}\lambda_g e_g)(\sum_{h\in G}\mu_h e_h))=\alpha^*(\sum_{g,h\in G}\lambda_g\mu_h e_{g+h})=\sum_{g,h\in G}\lambda_g\mu_h(\alpha(g)+\alpha(h))e_{g+h}$$

$$=\sum_{g,h\in G}\lambda_g\mu_h\alpha(g)e_{g+h}+\sum_{g,h\in G}\lambda_g\mu_h\alpha(h)e_{g+h}=\alpha^*(\sum_{g\in G}\lambda_g e_g)(\sum_{h\in G}\mu_h e_h)+(\sum_{g\in G}\lambda_g e_g)\alpha^*(\sum_{h\in G}\mu_h e_h),$$

$$(\omega\alpha^*+\alpha^*\omega)(\sum_{g\in G}\lambda_g e_g)=\omega(\sum_{g\in G}\lambda_g\alpha(g)e_g)+\alpha^*(\sum_{g\in G}\lambda_g e_{-g})$$

$$=\sum_{g\in G}\lambda_g\alpha(g)e_{-g}+\sum_{g\in G}\lambda_g\alpha(g^{-1})e_{-g}=\sum_{g\in G}\lambda_g\alpha(g)e_{-g}-\sum_{g\in G}\lambda_g\alpha(g)e_{-g}=0.$$

It follows the result. $\square$

Theorem 4.2. *Let G be an abelian group, $\omega: F[G]\rightarrow F[G]$ be defined as Eq.(4.1), $\alpha\in Hom(G, F^{+})$, α^* be defined as Eq. (4.2). Then $(F[G], [, ,]_{\omega,\alpha^*})$ is a 3-Lie algebra, where for arbitrary $\sum\limits_{g\in G}\lambda_g e_g,\ \sum\limits_{h\in G}\mu_h e_h,\ \sum\limits_{q\in G}\nu_q e_q\in F[G]$,*

$$[\sum_{g\in G}\lambda_g e_g, \sum_{h\in G}\mu_h e_h, \sum_{q\in G}\nu_q e_q]_{\omega,\alpha^*} = \sum_{g,h,q\in G}\lambda_g\mu_h\nu_q(\alpha(q-h)e_{h+q-g} +\alpha(g-q)e_{g+q-h}+\alpha(h-g)e_{g+h-q}). \tag{4.3}$$

Proof. By Lemma 4.1 and Theorem 3.3, for arbitrary $\sum\limits_{g\in G}\lambda_g e_g,\ \sum\limits_{h\in G}\mu_h e_h,\ \sum\limits_{q\in G}\nu_q e_q\in F[G]$,

$$[\sum_{g\in G}\lambda_g e_g, \sum_{h\in G}\mu_h e_h, \sum_{q\in G}\nu_q e_q]_{\omega,\alpha^*} = \begin{vmatrix} \sum\limits_{g\in G}\lambda_g e_{-g} & \sum\limits_{h\in G}\mu_h e_{-h} & \sum\limits_{q\in G}\nu_q e_{-q}\\ \sum\limits_{g\in G}\lambda_g e_{g} & \sum\limits_{h\in G}\mu_h e_{h} & \sum\limits_{q\in G}\nu_q e_{q}\\ \sum\limits_{g\in G}\lambda_g \alpha(g)e_{g} & \sum\limits_{h\in G}\mu_h\alpha(h) e_{h} & \sum\limits_{q\in G}\nu_q\alpha(q) e_{q}\end{vmatrix}$$

$$= \sum_{g,h,q\in G}\lambda_g\mu_h\nu_q\{(\alpha(q)-\alpha(h))e_{h+q-g}+(\alpha(g)-\alpha(q))e_{g+q-h}+(\alpha(h)-\alpha(g))e_{g+h-q}\}$$
$$= \sum_{g,h,q\in G}\lambda_g\mu_h\nu_q(\alpha(q-h)e_{h+q-g}+\alpha(g-q)e_{g+q-h}+\alpha(h-g)e_{g+h-q}).$$

It follows the result. □

By the above discussions, the products of basis vectors $\{e_g \mid g\in G\}$ of 3-Lie algebra $(F[G], [, ,]_{\omega,\alpha^*})$ are as follows: for arbitrary $g, h, w\in G$,

$$[e_g, e_h, e_w]_{\omega,\alpha^*} = \begin{vmatrix} e_{-g} & e_{-h} & e_{-w}\\ e_g & e_h & e_w\\ \alpha(g)e_g & \alpha(h)e_h & \alpha(w)e_w\end{vmatrix} = \alpha(w-h)e_{h+w-g}+\alpha(g-w)e_{g+w-h}+\alpha(h-g)e_{g+h-w}. \tag{4.4}$$

For $\alpha\in Hom(G, F^{+})$, define mapping

$$\phi_\alpha: F[G]\rightarrow F,\ \phi_\alpha(x)=\sum_{g\in G}\lambda_g\alpha(g),\ \forall\ x=\sum_{g\in G}\lambda_g e_g\in F[G]. \tag{4.5}$$

Denote $I_0=\{x|\ x=\sum\limits_{g\in G}\lambda_g e_g\in F[G],\ \phi_\alpha(x)=\sum\limits_{g\in G}\lambda_g\alpha(g)=0\}$. Then I_0 is a subspace of $F[G]$. And we have the following result.

Theorem 4.3. *For $\alpha\in Hom(G, F^{+})$, if $\alpha\neq 0$, then I_0 is a maximal ideal of 3-Lie algebra $(F[G], [, ,]_{\omega,\alpha^*})$. Therefore, $(F[G], [, ,]_{\omega,\alpha^*})$ is a non-simple 3-Lie algebra.*

Proof. For arbitrary $g, h, w \in G$, by Eq. (4.4),

$$\phi_\alpha([e_g, e_h, e_w]_{\omega,\alpha^*}) = \phi_\alpha(\alpha(w-h)e_{h+w-g} + \alpha(g-w)e_{g+w-h} + \alpha(h-g)e_{g+h-w})$$

$$= (\alpha(w)-\alpha(h))(\alpha(h)+\alpha(w)-\alpha(g))+(\alpha(g)-\alpha(w))(\alpha(g)+\alpha(w)-\alpha(h))$$

$$+(\alpha(h)-\alpha(g))(\alpha(g)+\alpha(h)-\alpha(w))= 0.$$

It follows that the derived algebra of $(F[G], [, ,]_{\omega,\alpha^*})$ is contained in I_0. Therefore, I_0 is an ideal of the 3-Lie algebra $(F[G], [, ,]_{\omega,\alpha^*})$.

Since $\alpha \neq 0$, without loss of generality suppose $\alpha(d) = 1$ for some non-zero element d of G. Then for every $x \in F[G]$, $x = \phi_\alpha(x)e_d + (x - \phi_\alpha(x)e_d)$. Since

$$\phi_\alpha(x - \phi_\alpha(x)e_d) = \phi_\alpha(x) - \phi_\alpha(x) = 0,$$

we have $F[G] = Fe_d \dot{+} I_0$ as the direct sum of subspaces. Therefore, I_0 is a maximal ideal of 3-Lie algebra $(F[G], [, ,]_{\omega,\alpha^*})$. □

Example 4.1 Let $G = \{A = (a_{ij}) | a_{ij} \in F, 1 \leq i \leq m, 1 \leq j \leq n\}$ be the set of all $(m \times n)$-matrices over a field F. Then G is an abelian group in the addition: $\forall A = (a_{ij}), B = (b_{ij}) \in G$, $A + B = (a_{ij} + b_{ij})$. Define $\alpha : G \to F^+$ and $\omega : F[G] \to F[G]$ as follows

$$\alpha(A) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}, \qquad \omega(e_A) = e_{-A}.$$

Then $\alpha \in Hom(G, F^+)$ and ω is an involution of $F[G]$. By Theorem 4.2, $F[G]$ is an mn-dimensional 3-Lie algebra with the multiplication: $\forall A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \in G$,

$$[e_A, e_B, e_C]_{\omega,\alpha^*} = \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}-b_{ij})e_{B+C-A}+\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}-c_{ij})e_{C+A-B}+\sum_{i=1}^{m}\sum_{j=1}^{n}(b_{ij}-a_{ij})e_{A+B-C}.$$

Example 4.2 Let $G = \{A = (a_{ij}) | a_{ij} \in F, 1 \leq i, j \leq n\}$ be the group of all $(n \times n)$-matrices over a field F with the addition: $\forall A = (a_{ij}), B = (b_{ij}) \in G$, $A + B = (a_{ij} + b_{ij})$. Let $\beta \in Hom(G, F^+)$, $\beta(A) = tr(A) = \sum\limits_{i=1}^{n} a_{ii}$, $\omega : F[G] \to F[G]$, $\omega(e_A) = e_{-A}$. Then by Theorem 4.2, $F[G]$ is an n^2-dimensional 3-Lie algebra in the multiplication: $\forall A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \in G$,

$$[e_A, e_B, e_C]_{\omega,\beta^*} = tr(C - B)e_{B+C-A} + tr(A - C)e_{A+C-B} + tr(B - A)e_{A+B-C}.$$

Example 4.3 Let $G = Z_p^+$ be the addition group of the prime field Z_p, $chF_p = p$. Then the multiplication of the group algebra $Z_p[G]$ is

$$e_{\bar{r}}e_{\bar{s}} = e_{\overline{r+s}}, \ \forall \bar{s}, \bar{r} \in G.$$

Define

$$\alpha : G \to Z_p^+,\ \alpha(\bar{r}) = \bar{r},\ \forall \bar{r} \in Z_p;$$

$$\omega : Z_p[G] \to Z_p[G],\ \omega(e_{\bar{r}}) = e_{\overline{-r}},\ \bar{r} \in Z_p.$$

By Theorem 4.2 , $(Z_p[G], [, ,]_{\omega,\alpha^*})$ is a p- dimensional 3-Lie algebra with the multiplication as follows

$$[e_{\bar{r}}, e_{\bar{s}}, e_{\bar{k}}] = \begin{vmatrix} e_{\overline{-r}} & e_{\overline{-s}} & e_{\overline{-k}} \\ e_{\bar{r}} & e_{\bar{s}} & e_{\bar{t}} \\ \bar{r}e_{\bar{r}} & \bar{s}e_{\bar{s}} & \bar{k}e_{\bar{k}} \end{vmatrix} = \overline{k-s}e_{\overline{s+k-r}} + \overline{r-k}e_{\overline{k+r-s}} + \overline{s-r}e_{\overline{r+s-k}}.$$

5. 3-Lie algebras constructed from Laurent polynomials

In this section we study 3-Lie algebras constructed by Laurent polynomials. In the following, denote $A = F[t^{-1}, t]$, the set of Laurent polynomials over a field F.

We know that the derivation algebra $DerA = \{t^s \frac{d}{dt} |\ s \in Z\}$ with the product

$$[t^m\delta, t^n\delta] = (n-m)t^{m+n}\delta,\ m, n \in Z, \tag{5.1}$$

where $\delta = t\frac{d}{dt}$, Z is the set of all integer numbers.

Let $\omega : A \to A$ be an algebra homomorphism and satisfy $\omega^2 = Id_A$. Since $\omega(1) = 1$ and $\omega(t)\omega(t^{-1}) = 1$, we have $\omega(t) = \lambda t^r$, $\lambda \in F$ and $\lambda \neq 0$. Then $\omega(t^m) = \lambda^m t^{rm}$,

$$t^m = \omega^2(t^m) = \omega(\omega(t^m)) = \omega((\omega(t))^m) = \omega(\lambda^m t^{rm}) = \lambda^{m+rm}t^{r^2m}.$$

We obtain $r = -1,\ \lambda \neq 0$; or $r = 1,\ \lambda = \pm 1$. Therefore, we get the following result.

Lemma 5.1. *Let $\omega : A \to A$ be a linear map. If $chF \neq 2$, then ω is an involution of A if and only if ω satisfies*

$$\omega(t^m) = \varepsilon^m t^m, \varepsilon = \pm 1,\ \forall m \in Z;\ \text{or}\ \omega(t^m) = \lambda^m t^{-m},\ \lambda \in F,\ \lambda \neq 0,\ \forall\ m \in Z. \tag{5.2}$$

Proof. The result follows from the above discussions. □

Lemma 5.2. *Let ω be an involution of A, $\delta = t^l\frac{d}{dt} \in DerF[t^{-1}, t]$ and $chF \neq 2$. Then $\omega\delta + \delta\omega = 0$ if and only if ω, δ satisfy the following one possibilities*

(*i*) $\omega(t^m) = (-1)^m t^m,\ \forall m \in Z,\ \delta = t^{2k}\frac{d}{dt}, k \in Z.$

(*ii*) $\omega(t^m) = \lambda^m t^{-m},\ \lambda \in F,\ \lambda \neq 0,\ \forall\ m \in Z,\ \delta = t\frac{d}{dt}.$

(*ii*) *If $chF = 2$,* $\omega(t^m) = t^{-m},\ \forall\ m \in Z,\ \delta = t\frac{d}{dt}.$

Proof. The result follows from Lemma 5.1 and the direct computation. □

Theorem 5.3. *Let $\delta = t\frac{d}{dt}$, $chF \neq 2$, $\omega_\lambda : A \to A$,*

$$\omega_\lambda(t^m) = \lambda^m t^{-m},\ \lambda \in F,\ \lambda \neq 0,\ \forall m \in Z.$$

Then $(A,\ [,,]_{\omega_\lambda,\delta})$ *is a* 3*-Lie algebra in the multiplication:* $\forall\ t^l, t^m, t^n \in L$,

$$[t^l,t^m,t^n]_{\omega_\lambda,\delta} = \begin{vmatrix} \lambda^l t^{-l} & \lambda^m t^{-m} & \lambda^n t^{-n} \\ t^l & t^m & t^n \\ lt^l & mt^m & nt^n \end{vmatrix} \tag{5.3}$$

$$= \lambda^l(n-m)t^{m+n-l} + \lambda^m(l-n)t^{n+l-m} + \lambda^n(m-l)t^{l+m-n}.$$

Proof. The result follows from Theorem 3.3 and Lemma 5.2. □

Corollary 5.4. *Let* $\lambda = 1$ *in Theorem 5.3. Then the multiplication of the* 3*-Lie algebra* $(A, [,,]_{\omega_1,\delta})$ *is as follows:* $\forall\ t^l, t^m, t^n \in A$,

$$[t^l,t^m,t^n]_{\omega_1,\delta} = \begin{vmatrix} t^{-l} & t^{-m} & t^{-n} \\ t^l & t^m & t^n \\ lt^l & mt^m & nt^n \end{vmatrix} \tag{5.4}$$

$$= (n-m)t^{m+n-l} + (l-n)t^{n+l-m} + (m-l)t^{l+m-n}.$$

And $$\omega([t^l,t^m,t^n]_{\omega_1,\delta}) = -[\omega(t^l), \omega(t^m), \omega(t^n)]_{\omega_1,\delta}.$$

Theorem 5.5. *Let* $chF \neq 2$*. Then the* 3*-Lie algebra* $(A, [,,]_{\omega_\lambda,\delta})$ *with the multiplication (5.3) for some* $\lambda \neq 0$ *is isomorphic to the* 3*-Lie algebra* $(A, [,,]_{\omega_1,\delta})$ *with the multiplication (5.4), where* $\delta = t\frac{d}{dt}$.

Proof. Denote $\omega_\lambda : A \to A$, $\omega_\lambda(t^m) = \lambda^m t^{-m}, \lambda \in F, \lambda \neq 0$. Then we have $\omega_\lambda\delta + \delta\omega_\lambda = 0$.

Define $\sigma : A \to A$, $\sigma(t^m) = \lambda^{\frac{m}{2}} t^m$, $\forall\ m \in Z$. Then

$$\sigma(t^m t^n) = \sigma(t^m)\sigma(t^n),\ \ \delta\sigma(t^m) = \delta(\lambda^{\frac{m}{2}} t^m) = m\lambda^{\frac{m}{2}} t^m = \sigma\delta(t^m),$$
$$\omega_1\sigma(t^m) = \omega_1(\lambda^{\frac{m}{2}} t^m) = \lambda^{\frac{m}{2}} t^{-m} = \lambda^{\frac{-m}{2}}(\lambda^m t^{-m}) = \sigma\omega_\lambda(t^m).$$

$$\delta\sigma(t^m) = \delta(\lambda^{\frac{m}{2}} t^m) = m\lambda^{\frac{m}{2}} t^m = \sigma\delta(t^m).$$

Follows from Theorem 3.7, the result holds. □

Theorem 5.6. *If* $chF = p > 2$*, then for every integer* $k \in Z$ *and* $k \neq 0$,

$$I_k = \{\ (t^{kp} + t^{-kp})h(t) |\ \forall h(t) \in A\ \},\ \ J_k = \{\ (t^{kp} - t^{-kp})h(t) |\ \forall h(t) \in A\ \}$$

are non-zero proper ideals of the 3*-Lie algebra* $(A, [,,]_{\omega_1,\delta})$.

Proof. Since for every integer $k \in Z$ and $k \neq 0$, I_k, J_k are ideals of the associative algebra $A = F[t^{-1}, t]$, and satisfy

$$\omega(I_k) \subseteq I_k,\ \omega(J_k) \subseteq J_k,\ \Delta(I_k) \subseteq I_k,\ \Delta(J_k) \subseteq J_k.$$

By Theorem 3.5, I_k and J_k are proper ideals of the 3-Lie algebra. □

By the above discussions, A_1 and A_{-1} are two abelian subalgebras of $(A, [,,]_{\omega_1,\delta})$, and $A = A_1 \dot{+} A_{-1}$, where

$$A_1 = \{\ p(t)|\ p(t) \in A, \omega(p(t)) = p(t)\ \} = \{\ \sum_{i=r}^{s} a_i(t^i + t^{-i}),\ a_i \in F, r, s \in Z\ \},$$

$$A_{-1} = \{\ p(t)|\ p(t) \in A, \omega(p(t)) = -p(t)\ \} = \{\ \sum_{i=r}^{s} a_i(t^i - t^{-i}),\ a_i \in F, r, s \in Z\ \}.$$

If $B = F[t_1^{-1}, \cdots, t_k^{-1}, t_1, \cdots, t_k]$ is the commutative associative algebra of k variable Laurent polynomials over a field F of characteristic zero. Then for every $1 \le j \le k$, $\delta_j = t_j \frac{\partial}{\partial t_j}$ are derivations of B, where for every $p(t_1, \cdots, t_k) = \sum\limits_{i_1 \cdots i_k} a_{i_1 \cdots i_k} t_1^{i_1} \cdots t_k^{i_k} \in B$, $\delta_j(p(t_1, \cdots, t_k)) = \sum\limits_{i_1 \cdots i_k} i_j a_{i_1 \cdots i_k} t_1^{i_1} \cdots t_k^{i_k}$.

Theorem 5.7. *Let $B = F[t_1^{-1}, \cdots, t_k^{-1}, t_1, \cdots, t_k]$ with $chF = 0$, $\Delta_j = t_j \frac{\partial}{\partial t_j}$ be a derivation of B. For an algebra homomorphism $\omega : B \to B$, ω satisfies $\Delta_j \omega + \omega \Delta_j = 0$ and $\omega^2 = id_B$ if and only if*

$$\omega(t_1^{r_1} \cdots t_j^{r_j} \cdots t_k^{r_k}) = (\lambda_1^{r_1} \cdots \lambda_j^{r_j} \cdots \lambda_k^{r_k}) t_1^{-r_1} \cdots t_j^{-r_j} \cdots t_k^{-r_k}, \tag{5.5}$$

where $\lambda_s \in F,\ \lambda_s \neq 0,\ r_s \in Z,\ 1 \le s \le k$. Therefore, $(B,\ [,,]_{\omega,\delta_j})$ is a 3-Lie algebra in the multiplication: $\forall\ t_1^{r_1} \cdots t_j^{r_j} \cdots t_k^{r_k},\ t_1^{i_1} \cdots t_j^{i_j} \cdots t_k^{i_k},\ t_1^{n_1} \cdots t_j^{n_j} \cdots t_k^{n_k} \in B$,

$$\begin{aligned}
&[t_1^{r_1} \cdots t_j^{r_j} \cdots t_k^{r_k},\ t_1^{i_1} \cdots t_j^{i_j} \cdots t_k^{i_k},\ t_1^{n_1} \cdots t_j^{n_j} \cdots t_k^{n_k}]_{\omega,\Delta_j} \\
&= (\lambda_1^{r_1} \cdots \lambda_j^{r_j} \cdots \lambda_k^{r_k})(n_j - i_j) t_1^{i_1+n_1-r_1} \cdots t_j^{i_j+n_j-r_j} \cdots t_k^{i_k+n_k-r_k} \\
&+ (\lambda_1^{i_1} \cdots \lambda_j^{i_j} \cdots \lambda_k^{i_k})(r_j - n_j) t_1^{r_1+n_1-i_1} \cdots t_j^{r_j+n_j-i_j} \cdots t_k^{r_k+n_k-i_k} \\
&+ (\lambda_1^{n_1} \cdots \lambda_j^{n_j} \cdots \lambda_k^{n_k})(i_j - r_j) t_1^{i_1-n_1+r_1} \cdots t_j^{i_j-n_j+r_j} \cdots t_k^{i_k-n_k+r_k}.
\end{aligned} \tag{5.6}$$

Proof. The proof is completely similar to Theorem 5.3. □

In the following we study the 3-Lie algebra (A, ω, δ_{2k}), where the derivation

$$\delta_{2k} = t^{2k} \frac{d}{dt} \in DerA,\ k \in Z.$$

From Lemma 5.2, if $chF \neq 2$, for $\delta_{2k} = t^{2k} \frac{d}{dt} \in Der(F[t^{-1}, t]),\ k \in Z$, then the involution $\omega : A \to A$ satisfies

$$\delta_{2k} \omega + \omega \delta_{2k} = 0$$

if and only if ω is defined as $\omega(t^m) = (-1)^m t^m,\ \forall m \in Z$

Therefore, we have the following result.

Theorem 5.8. *If $chF \neq 2$, then A is a 3-Lie algebra in the multiplication $[,,]_{\omega,\delta_{2k}}$: for arbitrary $t^l, t^m, t^n \in A$,*

$$[t^l, t^m, t^n]_{\omega,\delta_{2k}} = \begin{vmatrix} (-1)^l t^l & (-1)^m t^m & (-1)^n t^n \\ t^l & t^m & t^n \\ l t^{2k+l-1} & m t^{m+2k-1} & n t^{2k+n-1} \end{vmatrix} \tag{5.7}$$

$$= \{(-1)^l (n-m) + (-1)^m (l-n) + (-1)^n (m-l)\} t^{2k+l+m+n-1}.$$

Proof. The result follows from Lemma 5.1. Lemma 5.2 and Theorem 3.3. □

Define linear functions α, β, $\gamma : A \to F$:

$$\alpha(t^m) = (-1)^m,\ \beta(t^m) = 1,\ \gamma(t^m) = m,\ \forall t^m \in A,$$

that is, for every $p(t) = \sum_{i=m}^{n} a_i t^i \in A$,

$$\alpha(p(t)) = \sum_{i=m}^{n} (-1)^i a_i,\ \beta(p(t)) = \sum_{i=m}^{n} a_i,\ \gamma(p(t)) = \sum_{i=m}^{n} i a_i.$$

Then Eq.(5.7) can be written as: $\forall\ t^l, t^m, t^n \in A$,

$$[t^l, t^m, t^n]_{\omega,\delta_{2k}} = (\alpha \wedge \beta \wedge \gamma)(t^l, t^m, t^n) t^{l+m+n+2k-1},$$

where $(\alpha \wedge \beta \wedge \gamma)(t^l, t^m, t^n) = \begin{vmatrix} \alpha(t^l) & \alpha(t^m) & \alpha(t^n) \\ \beta(t^l) & \beta(t^m) & \beta(t^n) \\ \gamma(t^l) & \gamma(t^m) & \gamma(t^n) \end{vmatrix} = \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix}.$

Remark References [11, 12, 9] studied n-Lie algebras $A(G, f, t)$, where G is an additive Abelian group, $f : G^n \to F$. By the above discussions, the 3-Lie algebra $(A, [,,]_{\omega,\delta_{2k}})$ is isomorphic to the 3-Lie algebra $A(Z, f, 2k+1)$ in [12], where Z is the set of all integers, $f(l, m, n) = \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix}, \forall\ l, m, n \in Z.$

Theorem 5.9. *Let $A = F[t^{-1}, t]$ be the Laurent polynomials over the field of complex numbers. Then for any integer k, $k \neq 0$, 3-Lie algebra $(A, [,,]_{\omega,\delta_{2k}})$ with the multiplication (5.7) is isomorphic to the 3-Lie algebra $(A, [,,]_{\omega,\delta_0})$, where $\delta_0 = \frac{d}{dt}$, and for every*

$t^l, t^m, t^n \in A$,

$$[t^l, t^m, t^n]_{\omega,\delta_0} = \{(-1)^l(n-m) + (-1)^m(l-n) + (-1)^n(m-l)\}t^{l+m+n-1}. \tag{5.8}$$

Proof. If $k = 2s$, define linear mapping $\sigma : (A, [, ,]_{\omega,\delta_0}) \to (A, [, ,]_{\omega,\delta_k})$, $\sigma(t^m) = t^{m-k}$, $\sigma(1) = 1$, for every $t^m \in A$. Then for every $t^l, t^m, t^n \in A$,

$$\sigma([t^l, t^m, t^n]_{\omega,\delta_0}) = \{(-1)^l(n-m) + (-1)^m(l-n) + (-1)^n(m-l)\}t^{l+m+n-k-1},$$

$$\begin{aligned}[\sigma(t^l), \sigma(t^m), \sigma(t^n)]_{\omega,\delta_{2k}} &= [t^{l-k}, t^{m-k}, t^{n-k}]_{\omega,\delta_{2k}} \\ &= \{(-1)^{l-k}(n-m) + (-1)^{m-k}(l-n) + (-1)^{n-k}(m-l)\}t^{l+m+n-k-1} \\ &= \{(-1)^l(n-m) + (-1)^m(l-n) + (-1)^n(m-l)\}t^{l+m+n-k-1}.\end{aligned}$$

If $k = 2s+1$, define linear mapping $\sigma : A \to A$, $\sigma(t^m) = it^{m-k}$, $\sigma(1) = 1$, where $i^2 = -1$. Then for every $t^l, t^m, t^n \in A$,

$$\sigma([t^l, t^m, t^n]_{\omega,\delta_0}) = i\ \{(-1)^l(n-m) + (-1)^m(l-n) + (-1)^n(m-l)\}t^{l+m+n-k-1},$$

$$\begin{aligned}[\sigma(t^l), \sigma(t^m), \sigma(t^n)]_{\omega,\delta_{2k}} &= [it^{l-k}, it^{m-k}, it^{n-k}]_{\omega,\delta_{2k}} \\ &= -i\{(-1)^{l-k}(n-m) + (-1)^{m-k}(l-n) + (-1)^{n-k}(m-l)\}t^{l+m+n-k-1} \\ &= i\ \{(-1)^l(n-m) + (-1)^m(l-n) + (-1)^n(m-l)\}t^{l+m+n-k-1} \\ &= \sigma([t^l, t^m, t^n]_{\omega,\delta_0}).\end{aligned}$$

The result holds. □

Theorem 5.10. *Let $A = F[t^{-1}, t]$ over the field F of complex numbers. Then 3-Lie algebra $(A, [, ,]_{\omega,\delta_0})$ is a simple 3-Lie algebra.*

Proof. By Eq.(5.8), for every $t^l, t^m, t^n \in A$,

$$\begin{aligned}[t^l, t^m, t^{-m+1}]_{\omega,\delta_0} &= \{(-1)^l(-m+1-m) + (-1)^m(l+m-1) + (-1)^{-m+1}(m-l)\}t^l \\ &= \begin{cases} 0, & if\ l = m\ or\ l = -m+1, \\ \{(-1)^{l+1}2m + (-1)^m 2l + (-1)^l + (-1)^{m+1}\}t^l \neq 0, & others.\end{cases}\end{aligned} \tag{5.9}$$

If $m = 0, n = 1$, then we have

$$[t^l, 1, t^1]_{\omega,\delta_0} = \{(-1)^l + (l-1) + l)\}t^l = \begin{cases} 0, & if\ l = 1,\ 0, \\ \{2l + (-1)^l - 1\}t^l, & others.\end{cases}$$

Let I be a non-zero ideal of the 3-Lie algebra $(A, [, ,]_{\omega,\delta_0})$. For every non-zero vector $p(t) = \sum\limits_{i=r}^{s} a_i t^i \in I$, where $a_s \neq 0, a_r \neq 0$, and m is a positive integer such that $m > s$ and $-m+1 < r$. Thanks to Eq.(5.8) and the Vandermonde determinant, we have $t^l \in I$ if $a_l \neq 0$ for $r \leq l \leq s$. We conclude that there is an integer l such that $t^l \in I$.

Now we prove $I = A$. If $t^m \in I$, then by Eq.(5.9), for every $l \neq m$ and $l \neq -m+1$, we have $t^l \in I$. Therefore, we can choose j satisfying $j \neq \pm m$, $j \neq -m+1$ (and then $-m+1 \neq -j+1$) such that $t^j \in I$. Again by Eq.(5.8), we have $t^m \in I$ and $t^{-m+1} \in I$.

Summarizing above discussions, we get $I = A$. Therefore, $(A, [, ,]_{\omega,\delta_0})$ is a simple 3-Lie algebra. □

Theorem 5.11. *If $chF = p > 2$, then the 3-Lie algebra $(A, [, ,]_{\omega,\delta_0})$ in Theorem 5.8 is a non-simple 3-Lie algebra.*

Proof. Suppose $I_k = \{(t^{kp} + t^{-kp})h(t) |\ \forall h(t) \in A\}$, $J_k = \{(t^{kp} - t^{-kp})h(t) |\ \forall h(t) \in A\}$, $k \neq 0$. Then $\omega(I_k) \subseteq I_k$, $\delta_0(I_k) \subseteq I_k$, $\omega(J_k) \subseteq J_k$ and $\delta_0(J_k) \subseteq J_k$. Thanks to Theorem 3.4, I_k and J_k are non-zero proper ideals of the 3-Lie algebra $(A, [, ,]_{\omega,\delta_0})$. Therefore, the result holds. □

By the above discussions, if $chF = p > 2$, then $J_1 = \{(t^p - t^{-p})h(t) |\ \forall h(t) \in A\}$ is an ideal of the 3-Lie algebra $(A, [, ,]_{\omega,\delta_0})$, and satisfies $\omega(J_1) \subseteq J_1$ and $\delta_0(J_1) \subseteq J_1$. Then we get the quotient 3-Lie algebra of $(A, [, ,]_{\omega,\delta_0})$ relating to the ideal J_1, which is denoted by $(\bar{A}, [, ,]_{\omega,\delta_0})$. The multiplication of $\bar{A} = A/J_1$ in the basis $\bar{t}^{-p+1}, \cdots \bar{t}^{-1}, \bar{1}, \bar{t}, \cdots, \bar{t}^p$ as follows

$$[\bar{t}^{\,l}, \bar{t}^{\,m}, \bar{t}^{\,n}]_{\omega,\delta_0} = \{(-1)^l(n-m) + (-1)^m(l-n) + (-1)^n(m-l)\}\bar{t}^{\,l+m+n-1}, \tag{5.10}$$

where $\bar{t}^p = \bar{t}^{-p}$.

Theorem 5.12. *The 3-Lie algebra $(\bar{A}, [, ,]_{\omega,\delta_0})$ is a simple 3-Lie algebra, where $[, ,]_{\omega,\delta_0}$ is defined as Eq. (5.10) and $\dim \bar{A} = 2p$.*

Proof. Let $\bar{I}$ be a nonzero ideal of the 3-Lie algebra $(\bar{A}, [, ,]_{\omega,\delta_0})$. Suppose $h(\bar{t}) = \sum\limits_{i=1-p}^{p} a_i \bar{t}^i \in \bar{I}$ and $h(\bar{t}) \neq 0$.

Case I. If $h(\bar{t}) = \sum\limits_{i=1-p}^{p} a_i \bar{t}^i = \bar{t}^p$. For every l satisfying $1-p < l < p$, since

$$[\bar{t}^l, \bar{t}^p, \bar{t}^{1-p}] = \{(-1)^l - 2l + 1\}\bar{t}^l \in \bar{I},\ (-1)^l - 2l + 1 \neq 0,\ l \neq 1-p,$$

we get $\bar{t}^l \in \bar{I}$ for $1-p < l \leq p$.

Thanks to $p \geq 3$, $\bar{t}^2 \in \bar{I}$, then $-2\bar{t}^{p-1} = [\bar{t}^2, \bar{t}^{-1}, \bar{t}^{1-p}] \in \bar{I}$. It follows $\bar{I} = \bar{A}$.

Case II. If $h(\bar{t}) = \sum\limits_{i=1-p}^{p} a_i \bar{t}^i \in \bar{I}$ satisfies $a_p \neq 0$, and there is an integer k satisfying $1-p \leq k < p$ and $a_k \neq 0$. Without loss of generality, we suppose $a_p = 1$. By Eq.(5.10)

$$[h(\bar{t}), \bar{t}^{p-1}, \bar{t}^{2-p}] = \sum_{i=1-p}^{p} a_i(3(-1)^i + 2i - 1)\bar{t}^i \in I.$$

Since $3(-1)^i + 2i - 1 = 0$ if and only if $i = p - 1$ or $i = 2 - p$, and $3(-1)^i + 2i - 1 = 3(-1)^j + 2j - 1$ if and only if $i = j$ for $i \neq p-1, i \neq 2-p, j \neq p-1, j \neq 2-p$, we obtain $\bar{t}^p \in I$ (using Vandermonde determinant). Follows the discussions of the Case I, $\bar{I} = \bar{A}$.

Case III. If $h(\bar{t}) = \sum\limits_{i=1-p}^{p} a_i \bar{t}^i = \sum\limits_{i=1-p}^{s} a_i \bar{t}^i$, where $s < p$, $a_s = 1$ (that is, $a_p = a_{p-1} = \cdots = a_{s+1} = 0$) and there is an integer k satisfying $1 - p \leq k < s$ such that $a_k \neq 0$. Then $s > 1 - p$ and

$$[h(\bar{t}), \bar{t}^p, \bar{t}^{1-s}] = \sum_{i=1-p}^{s} a_i \{(-1)^i(1-s) - i - s + 1 + i(-1)^s\} \bar{t}^{i+p-s} = \sum_{j=1-s}^{p} b_j \bar{t}^j \in \bar{I},$$

where $b_i = a_i((-1)^i(1-s) - i - s + 1 + i(-1)^s)$. We obtain

$b_p = (-1)^s(1-s) - s - s + 1 + s(-1)^s = -2s + (-1)^s + 1 \neq 0$ since $1 - p < s < p$.

Follows from Case II, $\bar{I} = \bar{A}$.

Case IV. If $h(\bar{t}) = \sum\limits_{i=1-p}^{p} a_i \bar{t}^i = \bar{t}^l$, where $l < p$. If $l < p - 1$, then

$[\bar{t}^l, \bar{t}^p, \bar{t}^{l+1}] = \{(-1)^l(2l+1) + 1\} \bar{t}^p \in \bar{I}$, and $(-1)^l(2l+1) + 1 \neq 0$, we obtain $\bar{t}^p \in \bar{I}$.

If $l = p - 1$, then $[\bar{t}^{p-1}, \bar{t}^p, \bar{t}^{-p+2}] = 4\bar{t}^p \in \bar{I}$. Therefore, $\bar{t}^p \in \bar{I}$.

Summarizing above discussions, $\bar{I} = \bar{A}$. It follows the result. □

6. CONCLUSIONS AND DISCUSSIONS

Since the multiple multiplication, constructions of n-Lie algebras is a continuously difficult problem in the structure theory of n-Lie algebras, for $n \geq 3$.

In [3, 10, 11, 12], n-Lie algebras are realized by associative commutative algebras and its arbitrary n pairwise commuting derivations, and linear functions.

Papadopoulos in ([7]) constructed 3-Lie algebras by Dirac γ-matrices. Let A be spanned by the four-dimensional γ-matrices (γ^μ) and let $\gamma^5 = \gamma^1 \ldots \gamma^4$. Then the product

$$[x, y, z] = [[x, y]\gamma^5, z], \quad \forall a, b, c \in A. \tag{6.1}$$

defines a 3-Lie algebra which is isomorphic to the unique simple 3-Lie algebra ([3]).

In [17] 3-Lie algebras are constructed from metric Lie algebras. Let $(\mathfrak{g}, B)$ be a metric Lie algebra over a field $\mathbb{F}$, that is, B is a nondegenerate symmetric bilinear form on $\mathfrak{g}$ satisfying $B([x, y], z) = -B(y, [x, z])$ for every $x, y, z \in \mathfrak{g}$. Suppose $\{x_1, \cdots, x_m\}$ is a basis of $\mathfrak{g}$ and $[x_i, x_j] = \sum\limits_{k=1}^{m} a_{ij}^k x_k,\ 1 \leq i, j \leq m$. Set

$$\mathfrak{g}_0 = \mathfrak{g} \oplus \mathbb{F}x^0 \oplus \mathbb{F}x^{-1} \quad \text{(the direct sum of vector space)}. \tag{6.2}$$

Then there is a 3-Lie algebra structure on $\mathfrak{g}_0$ given by

$$[x_0, x_i, x_j] = [x_i, x_j],\ 1 \leq i, j \leq m;\ [x^{-1}, x_i, x_j] = 0,\ 0 \leq i, j \leq m; \tag{6.3}$$

$$[x_i, x_j, x_k] = \sum_{s=1}^{m} a_{ij}^{s} B(x_s, x_k) x^{-1},\ 1 \leq i, j, k \leq m. \tag{6.4}$$

And A is a metric 3-Lie algebra in the multiplication (6.4) and (6.5).

In [14], 3-Lie algebras are realized by Lie algebras and linear functions. Let $(L, [,])$ be a Lie algebra, $f \in L^*$ satisfying $f([x, y]) = 0$ for every $x, y \in L$. Then L is a 3-Lie algebra in the multiplication

$$[x, y, z]_f = f(x)[y, z] + f(y)[z, x] + f(z)[x, y], \forall x, y, z \in L. \tag{6.5}$$

And it is proved in [14] that every m-dimensional 3-Lie algebras can be obtained by the multiplication (6.2) and (6.6) for $m \leq m$.

Awata, Li and et al in [18] constructed a 2-step solvable 3-Lie algebra from $(n \times n)$-matrices. Let $\mathfrak{g} = gl(m, \mathbb{F})$ be the general linear Lie algebra. Then there is a 3-Lie algebra structure on $\mathfrak{g}$ defined by

$$[A, B, C] = (trA)[B, C] + (trB)[C, A] + (trC)[A, B],\ \ \forall A, B, C \in \mathfrak{g}. \tag{6.6}$$

In this paper we construct 3-Lie algebras by associative commutative algebras and their derivations and involutions. From Example 4.1 and 4.2 we can obtain 3-Lie algebras from any $\alpha \in Hom(G^+, F^+)$ which is not isomorphic to the 3-Lie algebra obtain by Eq.(6.7), where G is the set of all $(n \times n)$-matrices over a field F.

So we may provide a problem that is how can we realize 3-Lie algebras by Lie algebras, and associative commutative algebras with multilinear functions and general linear mappings.

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