Information geometry of magnetic systems

Pritam Sarkar

July 2022

1 Introduction:

The primary idea of classical many-body dynamics carry the concept of Phase Space of all possible position and momentum configurations of a system due to Willard Gibbs, who himself started making progress on geometric structures of the phase space. We find the abstract generalisation of this dynamic structure to the Hilbert Space in quantum mechanics as well; since in a many-body evolution the information of the state of a system must make sense in the most compact and efficient way possible. So rather than talking about state of all its constituents, we try to figure out the *distribution* of its possible configurations. Thus the statistics of possibilities govern our knowledge of systems with large degrees of freedom, be that of classical or quantum in nature.

One now generalises the statistical model itself as the set of probabilities it can have, i.e. the space of all probabilities now signify the system of consideration; and the evolution of one configuration to another now translates to traversal between two points in this set of probabilities. Since it's known that the probability itself is a measure over the sample space of the system, one can thus formalise this idea of moving from one probability to another by introducing a measure of distance between probabilities; so, all one needs is a metric structure in the space of probabilities and voilà! This set, endowed with the metric is denoted as Statistical Manifold $\mathcal{M}$, and will later be discussed how it has a natural Riemannian structure due to certain choice of metric over it. Therefore the *manifold and metric* at hand one can pretty much forget statistics and simply do geometry to obtain thermodynamic and non-thermodynamic information of the system which is all about this field of Information Geometry.

In this article the formalism of Riemannian Geometry and Information Geometry will be elementarily presented and will be shwon how scalar curvature, an intrinsic property of this statistical manifold encaptures significant information about phase transition of magnetic systems.

2 Ideas of Riemannian Geometry:

A manifold is locally glued Euclidean patches, i.e. spaces that locally looks $\mathbb{R}^n$. With $\mathcal{M}$ a manifold of dimension n, for any subset $U_i \in \mathcal{M}\ \exists \phi_i : U_i \to V_i \in \mathbb{R}^n$. These $\{\phi_i\}'s$ are the *chart maps* that takes you from the manifold to a subset of $\mathbb{R}^n$. In terms of coordinatised version $\psi_{ij} = \phi_j \circ \phi_i^{-1}$ are defined as *Trasition Maps*, and for smooth differentiable manifolds, transition maps are smooth.

Now the totality of the Tangent vectors at $p \in \mathcal{M}$ is called the Tangent Space $T_p\mathcal{M}$. Now a *Riemannian Metric* g on $\mathcal{M}$ is an assignment of a family of inner products in the tangent space, i.e. g associates to each $p \in \mathcal{M}$ a **positive definite symmetric bilinear form** on $T_p\mathcal{M}$ by $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$. With the metric at hand one then can define distances, geodesics (*straighest* path between two point in arbitrarily curved structure), and all other intrinsic properties of the manifold that are irrespective of coordinatisation. And the smoothness of $\mathcal{M}$ is carried in to the fact that $p \in \mathcal{M} \to g_p(X_p, Y_p) \in \mathbb{R}$ is smooth.

In this notation the, for a curve $\gamma : [a, b] \to \mathcal{M}$ length of distance between them is given as:

$$L(\gamma) = \int_a^b ||\dot{\gamma}(\tau)||d\tau = \int_a^b \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}d\tau \tag{1}$$

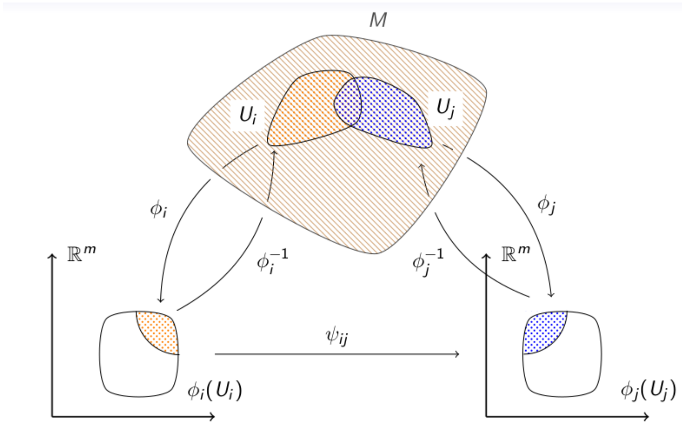

Figure 1: transition maps

$$d(a,b) = \underset{all\gamma's}{L(\gamma)} \tag{2}$$

Although Riemannian Geometry itself is a vast subject, we will basically need these things for further proceeding into Informtation geometry

- the idea of a *metric structure*
- the abstraction of distance within any mathematical object
- With g_{ij} at hand one can *derive* all intrinsic properties of the manifold like, christoffel connections, riemann curvature tensor, ricci tensor and any other curvature scalars, although the calculations might get utterly tedious.

3 Statistical Manifold and Information metric:

In statistical mechanical context the probability distribution of states of a systes naturally carries the structure of Gibbs measure

$$p(x,\theta) = q(x)\ exp\Big(-\sum_i \theta^i H_i - W(\theta)\Big) \tag{3}$$

Where x runs over configurations, $\{\theta_i\}$ are the parameters, $\theta^i H_i$ are the terms in theromdynamic Hamiltonian βH, $W(\theta) = ln\ Z(\theta)$, i.e. $Z(\theta)$ is our partition function and $q(x) = p(x, \theta = 0)$. This generalises the statistical distribution of any classical system.
Consider now $\psi_\theta(x) = \sqrt{p(x,\theta)}$ inside a *real* Hilbert space $\mathcal{H}$ endowed with some symmetric metric g_{ab}. Due to the normalisation condition

$$g_{ab}\ \psi^a_\theta(x)\ \psi^b_\theta(x) = ||\psi_\theta(x)||^2 = \int^x p(x,\theta) = 1 \tag{4}$$

We find that our system reswides inside the unit sphere $\mathcal{S} := \{\psi_\theta(x) : g_{ab}\ \psi^a_\theta(x)\ \psi^b_\theta(x) = 1\}$. With the probability distributions conditioned on the set of parameters $\{\theta_i\}$ we have restriction. With $\mathcal{P}$ the space of parameters, we can see for each θ_i there is a $\psi_\theta(x) \in \mathcal{S}$ and the map $f : \mathcal{P} \to \mathcal{H}$ further restricts within $\mathcal{S}$. In particular the $Image(f) = \mathcal{M} \subset \mathcal{S}$ is our **Statistical Manifold** [1].

Given the manifold representation of our statistical system, one seeks the choice of metric for its endowment. There are already vast kinds of information metrics present in the literature due to recent works of Fisher, Rao, G. Ruppeiner, H. Janysczek, D.C.Brody on information geometry encapturing geometric essence of statistical evolution. Among many forms we will choose for our convinience, the *Entropy Derivarive Metric*

$$g_{rs}(\theta) = -\partial_r \partial_s F \tag{5}$$

with F the free energy and $\partial_r = \frac{\partial}{\partial \theta^i}$. The essence it captures is that 2nd derivative of the free energy (positive since the system runs towards a global *minima*) directs the system towards spontaneity, whereas information metrics measure the dissimilarities between probabilities, which gives the negative sign. Noting the usual $F(\theta) = -kTlnZ(\theta)$ we can use for our purpose, the definition

$$I_{rs} := \partial_r \partial_s \; ln \; Z(\theta) \tag{6}$$

With the factors already absorbed within definitons.
With the geometric structures, that are the statistical manifold and information metric at hand we can go on characterising the system of our interest in the following section. Note that, from the local coordinate reprsentation of the metric itself we can work out pretty much any intrinsic characteristics of the manifold.

4 Mean Field Magnetic System:

For information geometric treatment of magnetic system, for simplicity of calculation yet with conceptual robustness, we will consider a mean field treatment of a model of magnetic system. In the mean field model interactions between spins are replaced nby a field obytained by averaging over all spons. Consider a system of N spjins $\{\sigma_i\}$ with the following Hamiltonian:

$$\mathcal{H} = -\frac{J}{N-1} \sum_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{7}$$

where q is the number of near neighbours, which for a 2D system is 4, J is the exchange interaction strength between spins. Considering variables (α, β) with $\alpha = \frac{h}{kT}$ and $\beta = \frac{J}{kT}$ and using mean field scheme mentioned in Baxter's Exacty solvable models in statistical mechanics [2], we find in the thermodybamic limit

$$ln \; Z(\alpha, \beta) \approx \frac{1}{2} ln\Big(\frac{4}{1-M^2}\Big) - \frac{1}{2} q\beta M^2 \tag{8}$$

where the magnetisation M is given by,

$$M = tanh(q\beta M + \alpha) \tag{9}$$

With the approximation for $tanh(x)$ for $x \to 0$ considering only first order terms, i.e. $M \approx \Big(\frac{\alpha}{(1-q\beta)^2}\Big)$ and replacing it inside (8) we obtain:

$$\ln Z(\alpha, \beta) = \frac{1}{2} ln4 - \frac{1}{2} ln\Big(1 - \Big(\frac{\alpha}{1-q\beta}\Big)^2\Big) - 2\beta\Big(\frac{\alpha}{1-q\beta}\Big) \tag{10}$$

Using the formula in the literature Scalar Curvature due to metric structure of the form $I_{rs} := \partial_r \partial_s \; ln \; Z(\theta)$

$$\mathcal{R} = -\frac{1}{2 \; det(I)} \begin{vmatrix} lnZ_{,11} & lnZ_{,12} & lnZ_{,22} \\ lnZ_{,112} & lnZ_{,112} & lnZ_{,122} \\ lnZ_{,112} & lnZ_{,122} & lnZ_{,222} \end{vmatrix} \tag{11}$$

Where $lnZ_{,ijk} = \partial_i \partial_j \partial_k lnZ$. And usage of this formula straightforwardly gives us a simple form of scalar curvature. The calculation has been done in Mathematica which can be found here:

$$\boxed{\mathcal{R} = \frac{q^2}{(q\beta - 1)^3}} \tag{12}$$

From where we can see this scalar curvature is negative for $\beta < \beta_c$, and positive for $\beta > \beta_c$, and clearly **diverges at the critical point**

$$\beta_c = \frac{J}{kT_c} = \frac{1}{4} \implies T_c = \frac{4J}{k} \tag{13}$$

Which we can clearly identify from standard mean field consideration of Ising models. Thus only first order approximation captures the phase transition information in the scalar curvature pretty well. Now the assymtotic behaviour of the scalar curvature near β_c, considering $q = 4$ for 2D magnetic systems

$$\mathcal{R} = \frac{16}{(\frac{\beta}{\beta_c} - 1)^3} = 16\Big(\frac{\beta - \beta_c}{\beta_c}\Big)^{-3} \tag{14}$$

Says something very fundamental behind the structure of phase transition from informational point of view, i.e. from one phase to another, the statistical manifold must deform through an infinitely curved structure of itself so that those two phase remain contactless in terms of information-exchange. The essence of singularity of scalar curvature as a characterisation of phase transition and critical point of very strikingly beautiful about Information Geometry.

5 Discussion on Mean-Field Schemes:

Although the mean field scheme presented in Baxter works finely along with this formalism, there are intrinsic discrepancies with regular mean field scheme where one considers $\sigma_i = \langle\sigma_i\rangle + \delta(\sigma_i)$ and ignores the second orders of fluctions, i.e. $\delta(\sigma_i)\delta(\sigma_j) \to 0$. from which we obtain a partition function of the kind:

$$Z = exp(-\frac{1}{2}\beta qNJM^2)\{2cosh(\beta\{JqM + h\})\}^N \tag{15}$$

Which when considering the $\frac{lnZ}{N} \to lnZ$ we obtain:

$$lnZ = -4\beta JM^2 + ln2 + ln\{cosh(\beta\{JqM + h\})\} \tag{16}$$

Therefore the Informatiom matrix takes the form:

$$I_{rs} = \begin{pmatrix} \partial_\beta^2\, lnZ & \partial_\beta\partial_h\, lnZ \\ \partial_\beta\partial_h\, lnZ & \partial_\beta^2\, lnZ \end{pmatrix} = \begin{pmatrix} M^2sech^2(\beta\{JqM + h\})) & Msech^2(\beta\{JqM + h\})) \\ Msech^2(\beta\{JqM + h\})) & sech^2(\beta\{JqM + h\})) \end{pmatrix} \implies det(I) = 0 \tag{17}$$

Which clearly states, according to the formula (11),

$$\mathcal{R} = -\infty \tag{18}$$

which isn't good. This clearly indicates there are subtle discrepancies with the mean field schemes along with information geometry, where one scheme works out and other just blows down to negative infinity.

6 Conclusion:

Overall in this article an expository presentation has been given for information geometric treatment of Ising like 2D magnetic system under the mean field approximation. Further on there are more subtleties with finite size issues of magnetic systems and there are number of literature concerned about this. For the moment the crucial idea conveyed here is that, geometric abstraction of statistical systems capture the overall essence in rather novel and beautiful way, unlike mere statistical calculations.

References

[1] Geometrical aspects of statistical mechanics - Dorje Brody and Nicolas Rivier

[2] Exactly solvable models in Statistical Mechanics - R.Baxter - pg - 53