Information Geometry of Magnetic Systems.

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The Big Picture :-

- Geometry is the Perception of Structure.
- The Abstraction from SpatioTemporal Geometry to Parametric Geometry.
- Giving Geometric Meaning to Statistics.
- What essence do these Geometric Objects carry ??

Why Bother at all, about Geometry?

- For Long, Many-Body Evolution has been dealt within Statistics, which contains only algebraic-differential structures.

- Information is Pattern - and so is within an interactive system.

- The Geometry is already underneath.

Riemannian Geometry :-

- **Theorema Egregium (1827 - Gauss) :**

→ There are intrinsic properties of a geometric object, like curvature, torsion, geodesics, irrespective of how you coordinatise it.

- **Riemann (1853) :-**

→ For Smooth Geometric objects, there is a "metric tensor" that tells you everything about it.

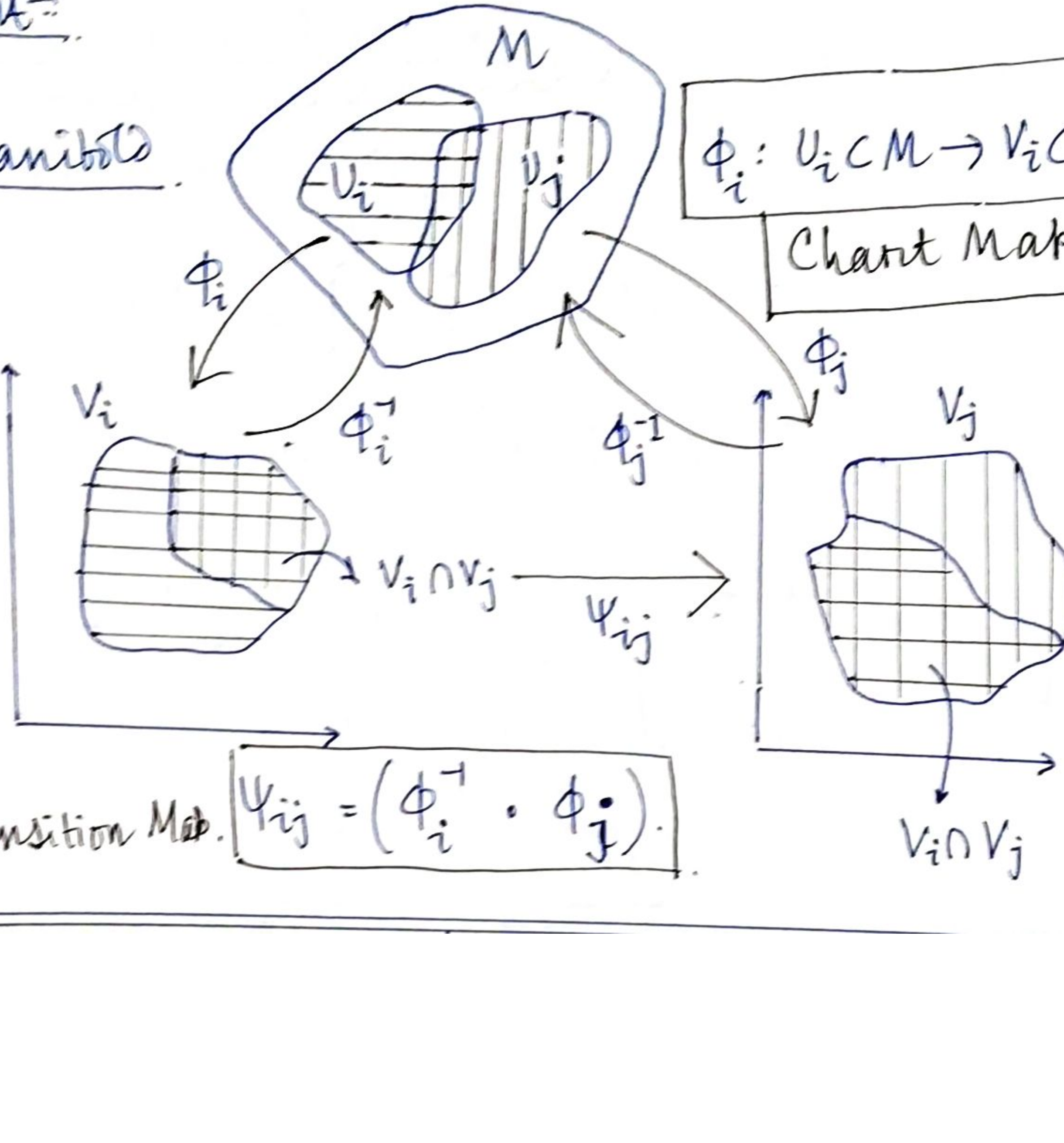
$$\boxed{g_{ab}(p) := \left(\hat{e}_a\big|_p \cdot \hat{e}_b\big|_p\right).}$$

Cont^n

Manifold

$$\boxed{\phi_i : U_i \subset M \to V_i \subset \mathbb{R}^{\dim(M)}}$$

Chart Map



Transition Map. $\boxed{\Psi_{ij} = \left(\phi_i^{-1} \cdot \phi_j\right)}$.

Cont^{d}:

$\boxed{*}$ Smoothness of M, must be carried on the smoothness of ψ_{ij}.

$\boxed{\cdot}$ A metric : is
$d(a,b)$

i. Positive definite

ii. Symmetric

iii. Bilinear.

⊙ One needs a metric over manifolds.

$\boxed{\cdot}$ Riemann (1853): $g_{ab}(p) : T_pM \times T_pM \to \mathbb{R}$

$$ds^2 = g_{ab}\, dx^a\, dx^b .$$

$\boxed{\cdot}$ A manifold, Equipped with a smooth (g_{ab}) is a Riemannian Manifold.

What about distance?

For a curve $\gamma : [a, b] \to M$

$$L(\gamma) = \int_a^b \| \dot{\gamma}(\tau) \| \, d\tau$$

$$= \int_a^b \sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j} \, d\tau$$

Geodesic: $\gamma_{GD} : [a, b] \to M$ such that

$$L(\gamma_{GD}) = \inf_{\text{all } \gamma} \int_a^b \sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j} \, d\tau$$

Distance: $\boxed{d(a, b) = L(\gamma_{GD})}$

☐ the Whole Idea of Differential Geometry is that — once you have g_{ab}, you have every information of the Manifold.

Formula for Curvature Tensor / Scalar:

$$\Gamma^{i}{}_{jk} = \left[\partial_j g^{i}{}_{k} + \partial_k g^{i}{}_{j} - \partial^{i} g_{jk} \right]$$

$$R^{l}{}_{ijk} = \left[\partial_j \Gamma^{l}{}_{ik} - \partial_i \Gamma^{l}{}_{jk} \right] + \left[\Gamma^{p}{}_{ik} \Gamma^{l}{}_{jp} - \Gamma^{p}{}_{jk} \Gamma^{l}{}_{ip} \right].$$

$$R_{ik} = R^{l}{}_{ilk}.$$

the Ricci Scalar:

$$\boxed{R = g^{ik} R_{ik}.}$$

Statistical Manifold :— ($\mathcal{M}$).

Gibbs Measure :— $p(x,\theta) = q(x) \exp\left(-\sum_i \theta^i H_i - W(\theta)\right)$.

Consider a REAL Hilbert Space :—
[Equipped with some metric g_{ab}]. $\psi_\theta(x) = \sqrt{p(x,\theta)} \in \mathcal{H}$.

- Normalization : $\|\psi_\theta(x)\|^2 = 1 = \left(g_{ab}\, \psi_\theta^a\, \psi_\theta^b\right)$

$\Rightarrow$ The system resides within the unit Sphere $\mathcal{S}$ in $\mathcal{H}$.

- With $\mathcal{P} := \{\text{parameters}\}$ the Parameter Space and
$f : \mathcal{P} \to \mathcal{H}$. $\boxed{\mathcal{M} = \text{Image}(f) \cap \mathcal{S}}$ Statistical Manifold

- Now the Statistical System is a point in M.
- The probabilistic evolution of the system, is trajectories in M.
- But We Still Need a Metric Structure to Give it a Riemannian Signature.
- There are Fisher-Rao Information metric, Entropy Derivative metric, Kullback-Leibler Divergence (Symmetrized), Jensen metric — As Dissimilarity measures of probability distribution functions.
- Entropy Derivative Metric :-

$$\partial_i \equiv \frac{\partial}{\partial\theta^i}$$

$$\boxed{g_{ij} = -\partial_i \partial_j F}$$

F = Free Energy.

Application to Mean Field Magnetic System :-

$$\mathcal{H} = -\frac{J}{(N-1)} \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

In the Thermodynamic Limit :- $\left[\text{with} \begin{cases} \alpha = \frac{h}{kT} \\ \beta = \frac{J}{kT} \end{cases} \right]$

$$\ln Z(\alpha, \beta) \sim \frac{1}{2} \ln \left[\frac{4}{1-m^2} \right] - 4\beta m^2 \quad [m = \text{magnetization}].$$

with: $m = \tanh(q\beta m + \alpha)$.

$$\sim (q\beta m + \alpha) \quad \left[\tanh(x) \cong x + \left(-\frac{x^3}{3}\right) \forall x \to 0 \right]$$

$$\Rightarrow \ln Z(\alpha, \beta) \sim \frac{1}{2} \ln 4 + \frac{1}{2}(1-q\beta) \frac{\alpha^2}{(1-q\beta)^2} + \frac{1}{4} \frac{\alpha^4}{(1-q\beta)^4}.$$

$$= \left[\ln 2 + \frac{\alpha^2}{2(1-q\beta)^2} + \frac{\alpha^4}{4(1-q\beta)^4} \right].$$

It's a 2D model : (parametrically & spatially).

The Information metric :

$$[g_{ij}] = \begin{pmatrix} \dfrac{1}{1-q\beta} + \dfrac{3\alpha^2}{(-1+q\beta)^4} & \dfrac{q\alpha(-4\alpha^2+(-1+q\beta)^3)}{(-1+q\beta)^5} \\[2ex] \dfrac{q\alpha(-4\alpha^2+(-1+q\beta)^3)}{(-1+q\beta)^5} & -\dfrac{q^2\alpha^2(-5\alpha^2+(-1+q\beta)^3)}{(-1+q\beta)^6} \end{pmatrix}$$

$$\boxed{\det[g] = -\frac{q^2\alpha^6}{(-1+q\beta)^{10}}}$$

A Formula for the Scalar Curvature:

$$
R = -\frac{1}{2\det[g]} \begin{vmatrix} \ln Z_{,11} & \ln Z_{,12} & \ln Z_{,22} \\ \ln Z_{,111} & \ln Z_{,112} & \ln Z_{,122} \\ \ln Z_{,112} & \ln Z_{,122} & \ln Z_{,222} \end{vmatrix}
$$

where: $\ln Z_{,123} \equiv \left[\partial_1 \partial_2 \partial_3 \ln Z\right]$.

With this, the Scalar Curvature takes the form:-

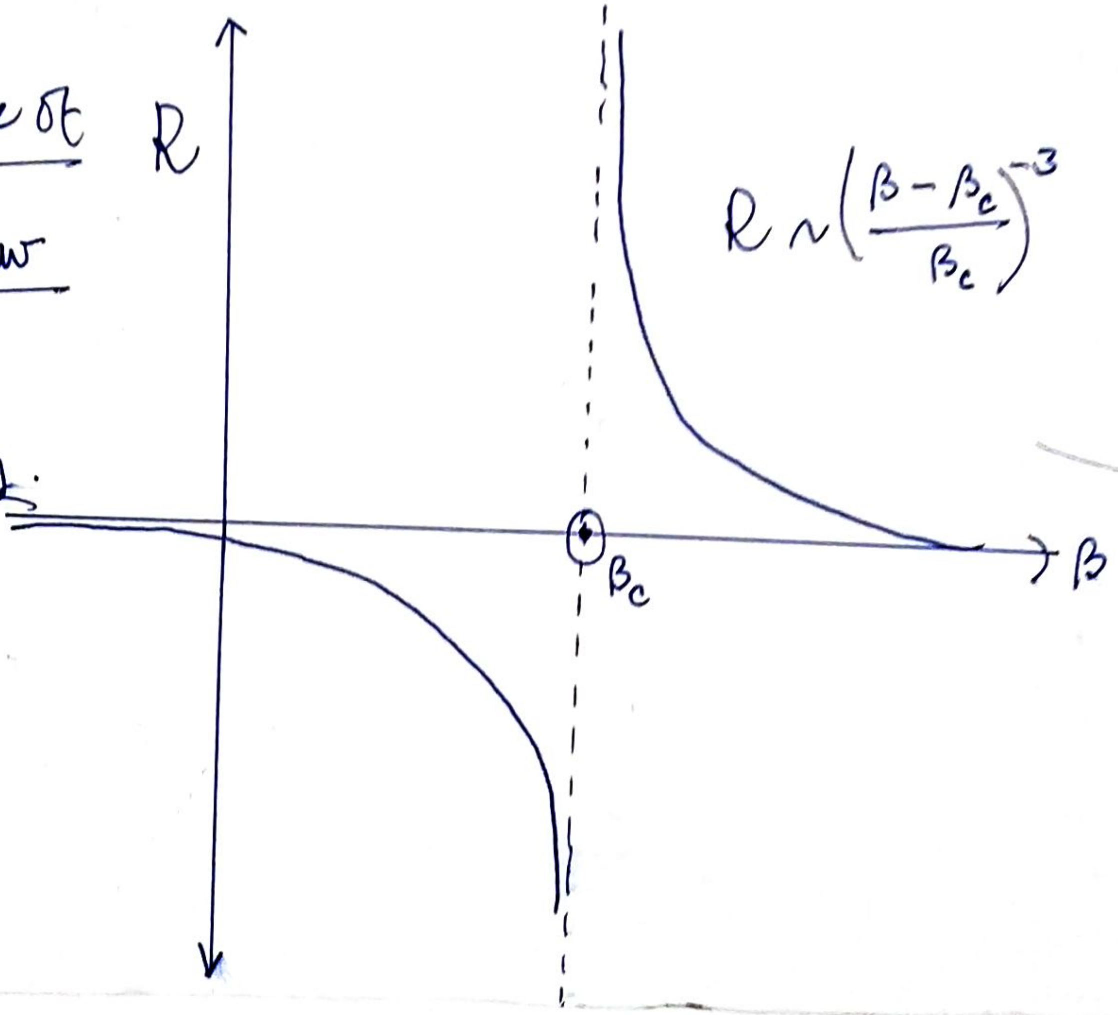
$$
\boxed{R = \frac{q^2}{(q\beta - 1)^3}}. \qquad \boxed{\beta_c = \frac{1}{q}}
$$

$$
\Rightarrow \frac{J}{k T_c} = \frac{1}{4} \Rightarrow \boxed{T_c = \frac{4J}{k}}
$$

- Scalar Curvature Diverges @ Critical Point.
- $R \sim \left(\frac{\beta - \beta_c}{\beta_c}\right)^{-3}$
- Changes sign from $\beta < \beta_c$ to $\beta > \beta_c$.

■ The Scalar Curvature Contains rich Information of Phase Transition.

Emergence of Power Law @ Criticality.



Trouble with Mean-Field Schemes :-

- Our Standard Approach of $\sigma_i \to \langle \sigma_i \rangle + \delta\sigma_i$ and ignoring $(\delta\sigma_i \delta\sigma_j) \to 0$ terms, gives us:

$$\ln Z = -\frac{qN\beta J m^2}{2} + N\ln 2 + N\ln\left[\cosh(\beta q m + \alpha)\right].$$

considering Intrinsic properties: $\frac{\ln Z}{N} \to \ln Z$.

$$[g_{ab}] = \begin{pmatrix} m^2 \operatorname{sech}^2(\beta q m + \alpha) & m \operatorname{sech}^2(\beta q m + \alpha) \\ m \operatorname{sech}^2(\beta q m + \alpha) & \operatorname{sech}^2(\beta q m + \alpha) \end{pmatrix}.$$

$$\Rightarrow \det[g] = 0 \Rightarrow \boxed{R = -\infty.}$$

- Subtle Discrepancies Between Mean Field Schemes.

Thank You For YOUR KIND ATTENTION

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