on Universality of Hyperbolicity

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Abstract

Interactive systems at their critical point exhibit divergence of correlation length, allowing information (pattern) to propagate (replicate) upto arbitrary range, giving rise to hierarchical scalefree structures everywhere around; One investigation is proposed how negative curvature universally underlies such phenomena.

Sir,

The ubiquity of negative curvature intrigued me very much. From your lecture on how trees and fractal-like structures all around, have natural hyperbolicity endowed within them; I got surprised due to this very non-trivial statement about something that's trivially spontaneous in nature. Being inclined into geometry and a student of physics, I personally delved into it as it seemed to me a very interesting direction.

I think there can be some fundamental results possible tying hyperbolicity from mathematics and critical phenomena from physics; underlying a very broad range of natural dynamics. In interactive evolution of complex systems, there occurs in general a critical point where correlations become scalefree and the correlation length diverges; and this lack of characteristic scale for the system is the signature of critical phenomena; which is very close to the point of phase transition in the system. The fractality discussed heavily in so many place, is the consequence of this scalefreeness or self-similarity of dynamical variable of that system near a critical point. What I am interested in now, is the geometric counterpart of such phenomena.

I have a number of propositions which I want to prove but am unable due to technical difficulties now. I am listing them, along with why I think so, and how I sketched them.

I have enclosed a number of informal discussions on why I think there must be such relations between critical phenomena and hyperbolicity.

1 A $b - ary\ tree$ can only be isometrically embedded in the upper-half space of curvature upper bounded by $-(lnb)^2$

Take $\mathbb{T}_b$ to be an infinite $b - ary$ tree with edges corresponding to integral distance[1], i.e.

$$d_g(n_{i,j}, n_{i+1,bj-k}) = 1 \ \forall k \in [0, b]$$

Therefore we can consider $(\mathbb{T}_b, d_g)$ to be a geodesic metric space.

Now consider the circles of radius r in it. The circumference $S_{tree} = b^r$ and the area $A_{tree} = \sum_{r' \leq r} b^{r'}$, grows as b^r which is very unlikely when considered any isometric embediing in any euclidean space, where circumference and area grows polynomially, not exponentially.

Recalling that in the hyperbolic upper-half space $S_{hy} = 2\pi sinh(\zeta r)$ and $A_{hy} = 2\pi(cosh(\zeta r) - 1)$, are the measure of periphery and area bounded by a circle of radius r, one sees it grows as $e^{\zeta r}$ for large r.

Relating the growth rate b^r for $(\mathbb{T}_b, d_g)$ and $e^{\zeta r}$ for $\mathbb{H}^2$ one finds $\zeta = lnb$ and $\zeta = \sqrt{|K|}$ where K is the scalar curvature, strictly negative for any hyperbolic space. Thus,

$$K = -(lnb)^2$$

. Now I think $\mathbb{T}_b$ can be isometrically put inside $\mathbb{H}^2$ of which curvature is upperbounded by $-(lnb)^2$

This can be an argument why it strictly requires negatively curved spaces to isometrically incorporate any tree-like structure[2]; but how do I more formally prove it?

2 Correlational structures in a system near critical point naturally carries hyperbolicity

[this is what I am most interested in, and sketched what I think, following the Ising model of magnetism at critical point and Gromov's 4 point condition]

L^d number of spins [3] at each vertex of a finite $\mathbb{Z}^d$ lattice with near neighbouring interaction. for $d \geq 2$ $\exists 0 T_c < \infty$ where statistical correlation between spins becomes a power-law, i.e.

$$G(x, y) = \langle S_x S_y \rangle \sim |x - y|^{-(d-2+\eta)}$$

with η some small positive number. I want to consider such a metric structure where for any two points, the more they will be correlated in real space, the closer they'll be in the defined metric space. Since correlation is positive-definite, symmetric and bilinear, (the necessary conditions for a function to regarded as a metric), one way to transform its monotonic increment to a monotonic decrement, is by defining:

$$d(x, y) := e^{-G(x,y)} = e^{-\langle S_x S_y \rangle} = e^{-C|x-y|^{-(d-2+\eta)}}$$

in order to capture - **the more the correlation, the less the distance** - effect. And one can directly verify that it satisfies the requirements of being a metric. We will consider cases for dimension $d \geq 2$, where phase transition occurs (there is no phase transition in 1D for this model) thus the exponent can be absorbed within a strictly positive factor, say $\gamma = (d - 2 + \eta)$ with the metric being:

$$d(x, y) = e^{-C|x-y|^{-\gamma}}$$

I can consider the spins to be in a graph where the edges are weighed by the corresponding correlation function between nodes, from which I define the distance above; and see what global

[1] with standard word metric

[2] The above argument I learnt from Hyperbolic Geometry of Complex Networks

[3] it has only two possible states; namely up and down orientation

characteristic of geometry I get [4]. From Gromov's 4 point condition, I choose randomly 4 vertices, the origin 0, and x, y, z actually denoting the nodes S_0, S_x, S_y, S_z and I call it a quadruplet $\square_{0xyz}$. Considering the following pari-sum order $[d(0,x)+d(y,z)] \geq [d(0,y)+d(x,z)] \geq [d(0,z)+d(x,y)]$ where the difference between largest pair-sums is thus: $[d(0,x)+d(y,z)]-[d(0,y)+d(x,z)]$ which by condition, requires to be upperbounded by a $2\delta \geq 0$, i.e.

$$if\ \exists\delta \geq 0\ such\ that\ \delta(\square_{0xyz}) := \frac{1}{2}[d(0,x)+d(y,z)-d(0,y)+d(x,z)] \leq \delta\ \forall x,y,z$$

Then I can say this graph to be hyperbolic, endowed with the defined metric [5].

But Now, with this definition of metric,

$$\delta(\square_{0xyz}) = \frac{1}{2}[e^{-|x|^{-\gamma}} + e^{-|y-z|^{-\gamma}} - e^{-|y|^{-\gamma}} - e^{-|x-z|^{-\gamma}}]$$

With noting, for a $L \times L$ lattice, $max\{|p|\} = \frac{L}{\sqrt{2}}$[6]; one trivial upperbound possible here would be:

$$\delta(\square_{0xyz}) \leq e^{-(\frac{L}{\sqrt{2}})^{-\gamma}}$$

with the argument that:

$$max\ \delta(\square_{0xyz}) = max\ \{\frac{1}{2}[e^{-|x|^{-\gamma}} + e^{-|y-z|^{-\gamma}} - e^{-|y|^{-\gamma}} - e^{-|x-z|^{-\gamma}}]\}$$

$$= max\ \{\frac{1}{2}[e^{-|x|^{-\gamma}} + e^{-|y-z|^{-\gamma}}]\} = max\ \{[e^{-|x|^{-\gamma}}]\} = e^{-(\frac{L}{\sqrt{2}})^{-\gamma}}$$

Where for an infinite square lattice, one has the $\delta = 1$. Now can I say, the correlational structure is $1-hyperbolic$ for this context?

Although this is a very rough sketch, some important aspects are discussed later.

3 Hierarchical scalefree networks carry hyperbolic metric structure

It has been anticipated and modelled how hyperbolic embedding is a good choice for incorporating and visualizing large real-world graphs, from collaboration network, hollywood actor's networks to World Wide Web and many others, in works like Treelike structures in large social and information networks, hyperbolic geometry of complex networks.

A crucial point in all of these is $power-law\ degree\ distribution$ of the nodes in these graphs, i.e. the proportion of $degree = k$ falls as

$$P(k) \sim k^{-\gamma}$$

with $2 \leq \gamma \leq 3$ I do not understand how it captures some underlying hyperbolicity, but I want to view it to be something of a simple phenomenon as it occurs ubiquitously. But I do not have any idea how to even approach this case, so I want to understand more.

4 Discussion:

But some more things require mentioning here:

[4] the $|x-y|$ can be thought as $d_g(x,y)$, the standard word metric for graph

[5] following the conventions from Treelike structures in large social and information networks

[6] p denotes distance from the points in the lattice, with origin at the centre

- The definition of the distance is **to keep correlated individuals close**, as per their strength of correlation. But the δ here derived is the trivial one, which the diameter of the graph itself; which isn't actually interesting. So, can I obtain any non-trivial upperbound for $\delta(\Box_{0xyz})$ and if so, what is that?

- The point of my motivation to relate correlation at criticality and hyperbolicity, is the following idea. Interactive evolution of complex systems can have sharp divergence in their correlation of thermodynamic variables near the point of phase transition, where system becomes self-similar w.r.t. its variables, and it is characterised by a number of critical exponents, one of which is the correlation function mentioned above. It is a hot topic in the physics of phase transition and critical phenomena how this self-similarity near critical point underlies the explanation of all fractal stuffs we find around, and why it is inevitable. I am listing down a number of cases of complexity where scalefree hierarchical structures prevail in nature.

  1. scalefree galaxy clustering [cosmological matter trajectory],
  2. lichtenberg figure [charge-flow trajectory in insulator],
  3. tree [transpiration trajectory] ,
  4. rivers [geological fluid trajectory],
  5. blood-vessels [biological fluid trajectory],
  6. neuronal structure [biological information trajectory]
  7. brain hierarchy at scales of cognitive-difficulty [trajectory of cognition]
  8. complexity in subjectivity [existential or immanent trajectory of a subject[7]]
  9. complexity in society [power trajectory [8]]

  Since, a critical phenomenon is identified by a power-law, and lack of characteristic spatiotemporal scale or divergence of correlation lengths; It occured to me whether they are very closely related to hyperbolic metric structures, or not, which I think, is very much.

- Another relation could be, that conformality of Poincar´e models of hyperbolicity preserves angles but distances are scaled as one goes away from origin; whereas the self-similarity of structures follow somewhat same property, i.e. the branching angle of a natural tree is same (stochastically, on an average) from bottom-to-top, where long-range correlation near criticality propagates information throughout the system (Information propagates by replicating patterns). If I define that **Information is Pattern** then it can be argued, that system repeats its pattern throughtout, but scales down as it goes along.

- Critical phenomena carry a notion of Universality, i.e. systems from very different microscopic details, are characterised by same cririical exponents, which might indicate why trees, lichtenberg figures, rivers, biological networks, social networks; all follow same outcome structures. But what about hyperbolicity then? Maybe that underlies the universality.

I really want to make proper sense of these occurances and how hyperbolicity emerges out from various phenomenon, simply and spontanously.
I am a 3rd year student of Integrated BS-MS with Physics Major from Indian Association for the Cultivation of Science, and an enthusiast of differential geometry, which I feel very imaginatively visualizable.

Now, I would like do a project under your guidance, on hyperbolic groups, and more intricate properties of hyperbolic geometry since it really fascinated me so I want to learn more, and it was your lecture that pushed me towards this direction.

I would be extremely thankful if you kindly allow this.

Thank You,

Pritam Sarkar.

[7]some works on networks of hippocampal neurons show that they are indeed scalefree near a critical point, where yhey tend spontaneously from arbitrary initial conditions

[8]hierarchical scalefreeness in large social graphs