Introduction to Wavelets

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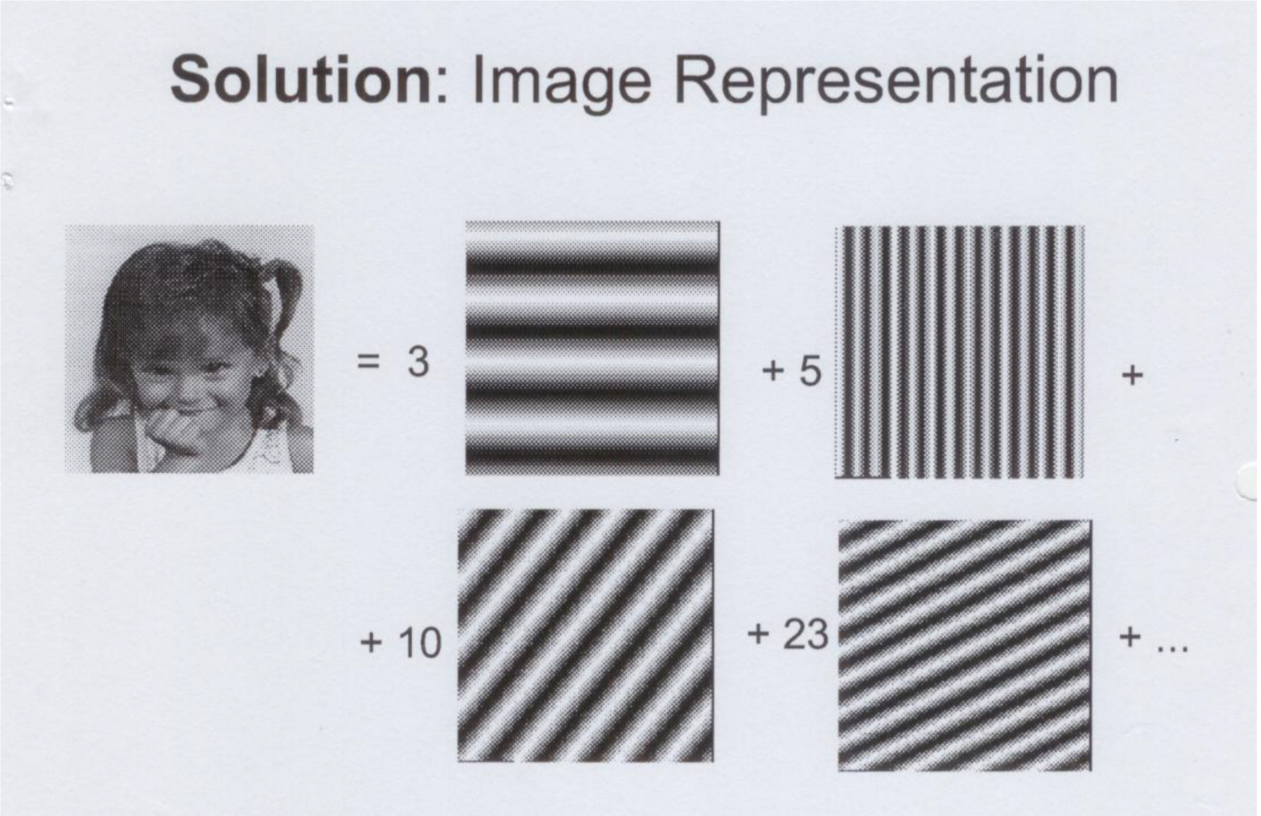
List of topics

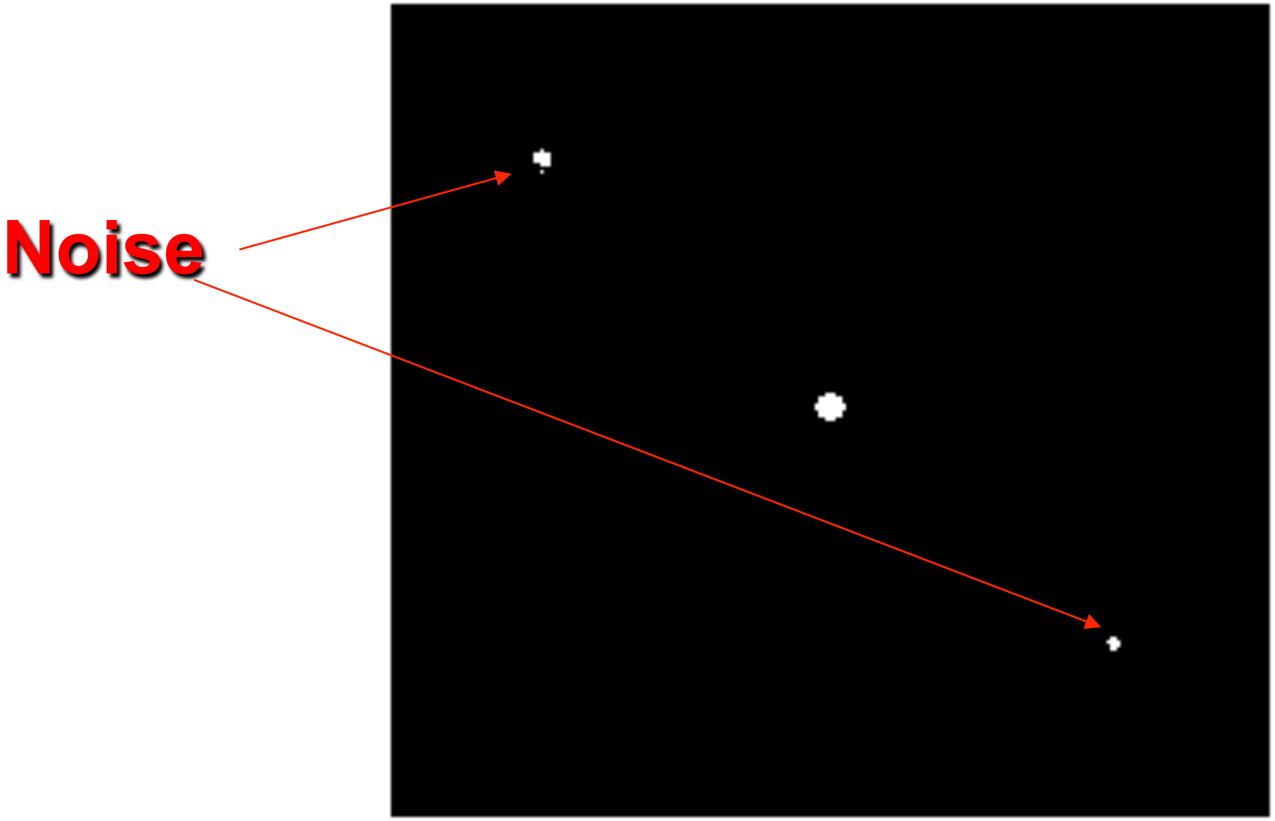
- Why transform?
- Why wavelets?
- Wavelets like basis components.
- Wavelets examples.
- Fast wavelet transform .
- Wavelets like filter.
- Wavelets advantages.

Why transform?

Image representation

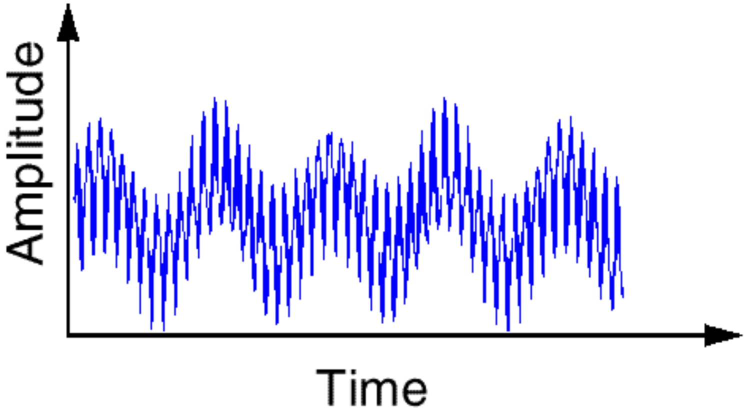


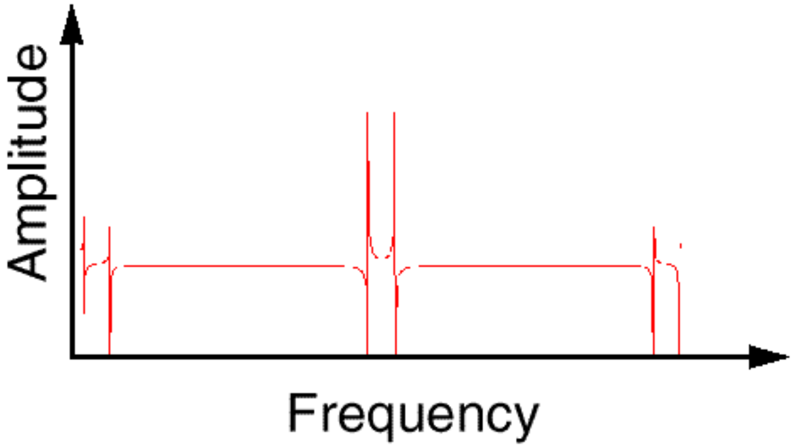
Reminder – Noise in Fourier spectrum



Fourier Analysis

- Breaks down a signal into constituent sinusoids of different frequencies





In other words: Transform the view of the signal from time-base to frequency-base.

What’s wrong with Fourier?

- By using Fourier Transform , we loose the time information : WHEN did a particular event take place ?
- FT can not locate drift, trends, abrupt changes, beginning and ends of events, etc.
- Calculating use complex numbers.

Time and Space definition

- Time – for one dimension waves we start point shifting from <u>source</u> to <u>end</u> in time scale .
- Space – for image point shifting is two dimensional .
- Here they are <u>synonyms</u> .

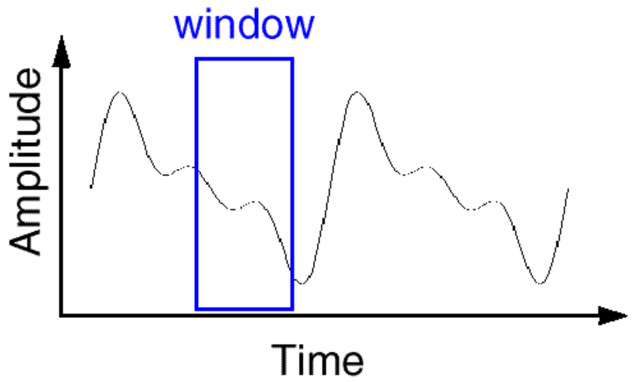
Kronneker function

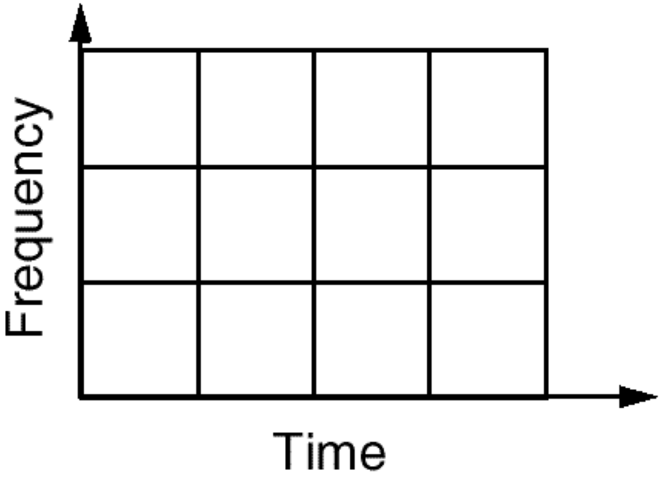
- $$\psi_k(t) = \delta_k(t) = \left\{ \frac{1, k = t}{0, k \neq t} \right.$$

Can exactly show the time of appearance but have not information about frequency and shape of signal.

Short Time Fourier Analysis

- In order to analyze small section of a signal, Denis Gabor (1946), developed a technique, based on the FT and using *windowing* : STFT





STFT (or: Gabor Transform)

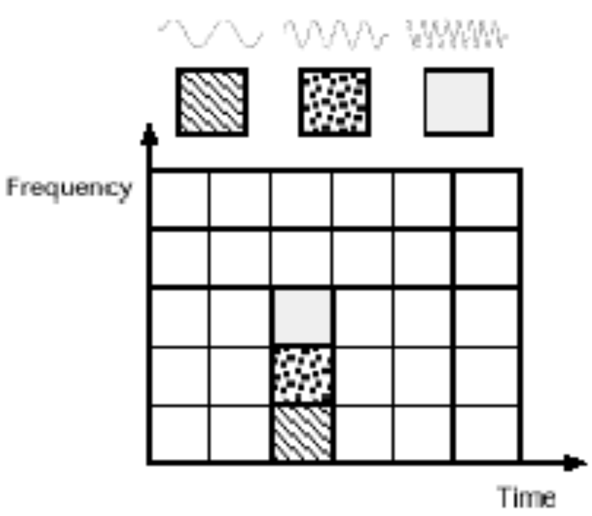
- A compromise between time-based and frequency-based views of a signal.
- both time and frequency are represented in limited precision.
- The precision is determined by the size of the window.
- Once you choose a particular size for the time window - it will be the same for all frequencies.

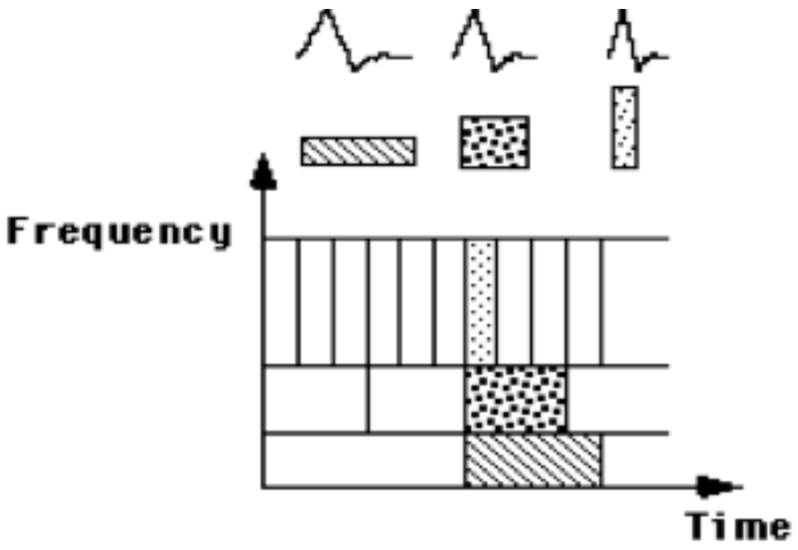
What's **wrong** with Gabor?

- Many signals require a more flexible approach - so we can vary the window size to determine more accurately <u>either time or frequency.</u>

Wavelet Analysis

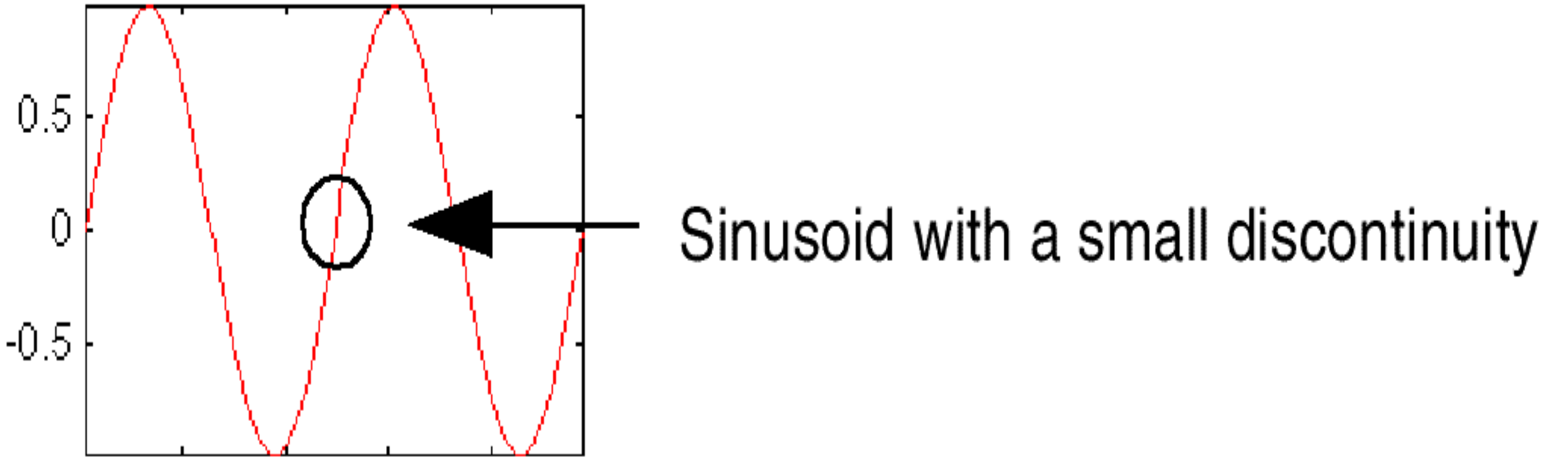
- Windowing technique with variable size window:
- Long time intervals - Low frequency
- Shorter intervals - High frequency





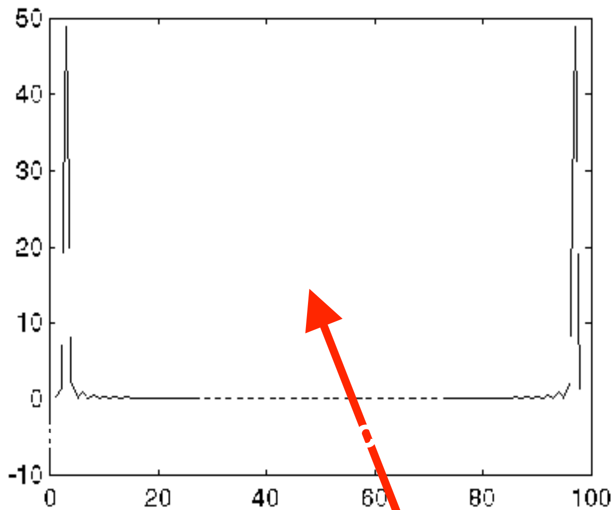
The main advantage:
Local Analysis

- To analyze a localized area of a larger signal.
- **For example**:

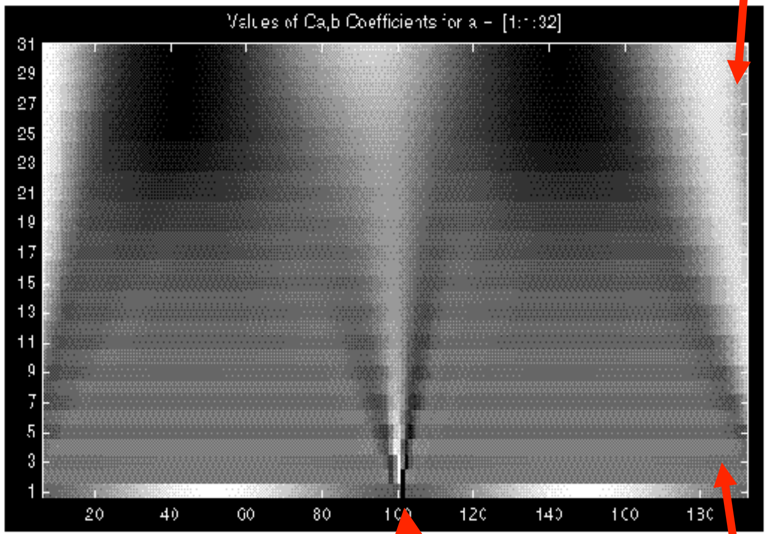


Local Analysis (Cont’d)

low
frequency



Fourier Coefficients



Wavelet Coefficients

High
frequency

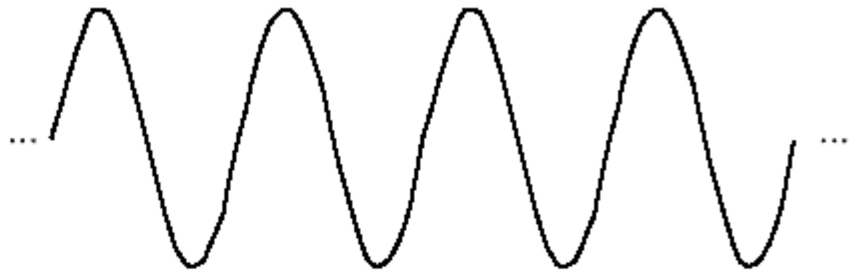
NOTHING!

exact location
in time of the discontinuity.

more

What is Wavelet Analysis ?

- And…what is a wavelet…?



Sine Wave

Wavelet (db10)

- A wavelet is a waveform of effectively limited duration that has an average value of zero.

Wavelet's properties

- Short time localized waves with zero integral value.
- Possibility of time shifting.
- Flexibility.

The Continuous Wavelet Transform (CWT)

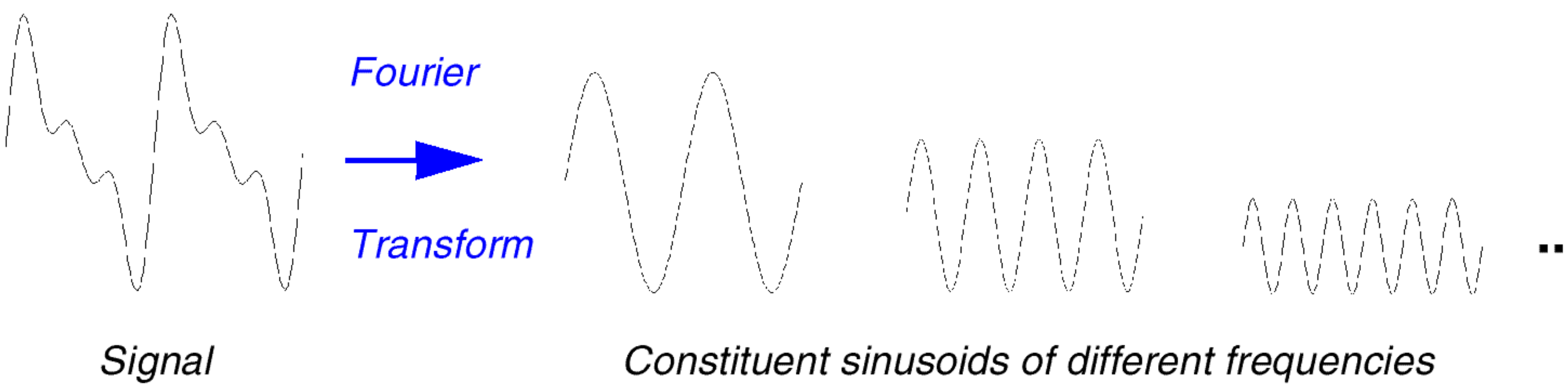
- A mathematical representation of the Fourier transform:

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

- Meaning: the sum over all time of the signal *f(t)* multiplied by a complex exponential, and the result is the Fourier coefficients F(w) .

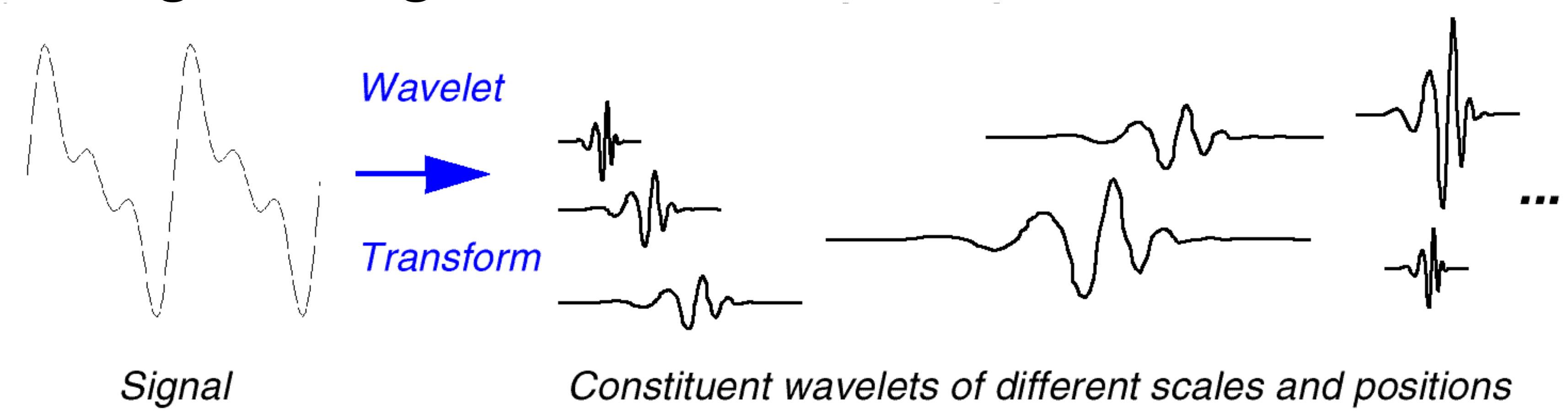
Wavelet Transform (Cont’d)

- Those coefficients, when multiplied by a sinusoid of appropriate frequency w, yield the constituent sinusoidal component of the original signal:



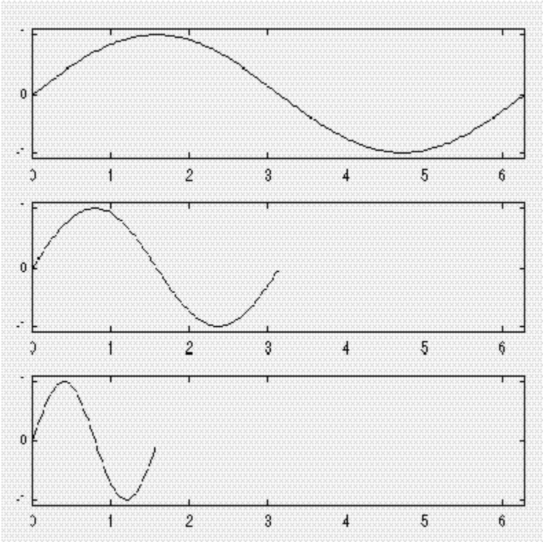
Wavelet Transform

- And the result of the CWT are Wavelet coefficients .
- Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelet of the original signal:



Scaling

- Wavelet analysis produces a time-scale view of the signal.
- *Scaling* means stretching or compressing of the signal.
- scale factor (a) for sine waves:



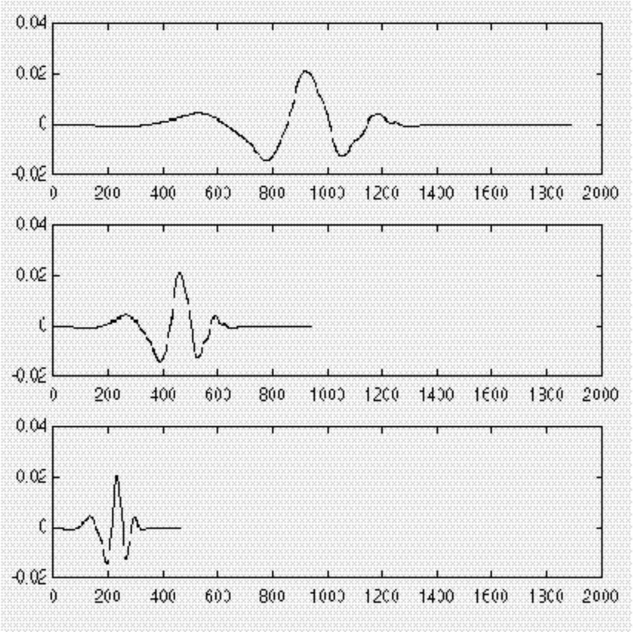
$$f(t) = \sin(t) \quad ; \quad a = 1$$

$$f(t) = \sin(2t) \quad ; \quad a = \tfrac{1}{2}$$

$$f(t) = \sin(4t) \quad ; \quad a = \tfrac{1}{4}$$

Scaling (Cont'd)

- Scale factor works exactly the same with wavelets:



$$f(t) = \Psi(t) \quad ; \quad a = 1$$

$$f(t) = \Psi(2t) \quad ; \quad a = 1/2$$

$$f(t) = \Psi(4t) \quad ; \quad a = 1/4$$

Wavelet function

$$\Psi_{a,b}(x) = \frac{1}{\sqrt{a}}\Psi\left(\frac{x-b}{a}\right)$$

- b – shift coefficient
- a – scale coefficient

$$\Psi_{a,b_x,b_y}(x,y) = \frac{1}{|a|}\Psi\left(\frac{x-b_x}{a}, \frac{y-b_y}{a}\right)$$

- 2D function

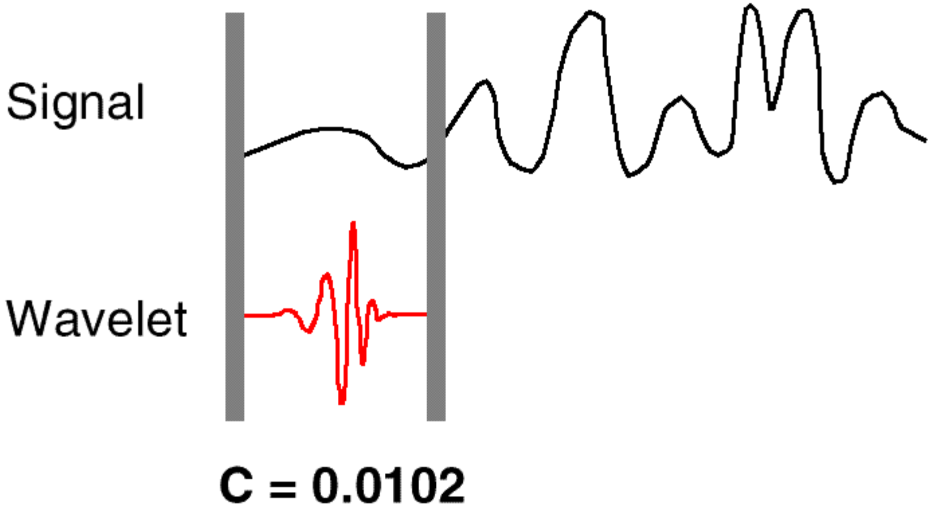
CWT

- Reminder: The *CWT* Is the sum over all time of the signal, multiplied by scaled and shifted versions of the wavelet function

Step 1:
Take a Wavelet and compare it to a section at the start of the original signal

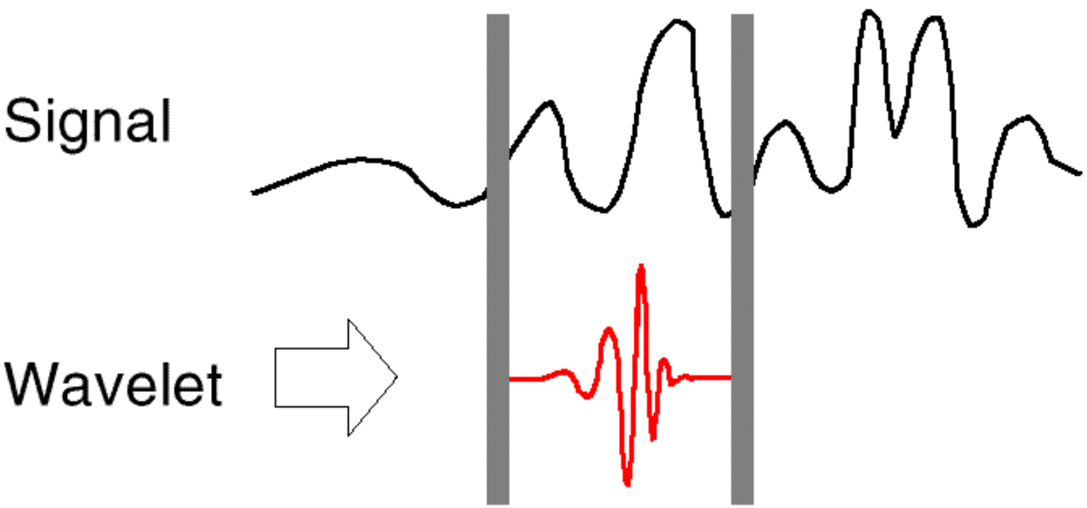
CWT

Step 2:
Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity.



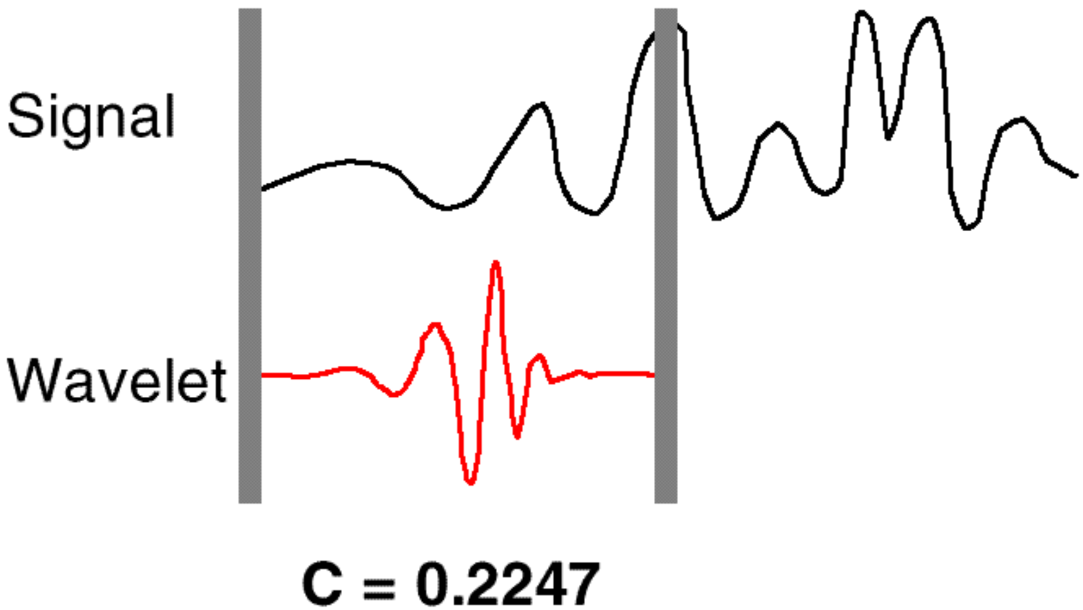
CWT

- Step 3: Shift the wavelet to the right and repeat steps 1-2 until you' ve covered the whole signal



CWT

- <u>Step 4:</u> Scale (stretch) the wavelet and repeat steps 1-3



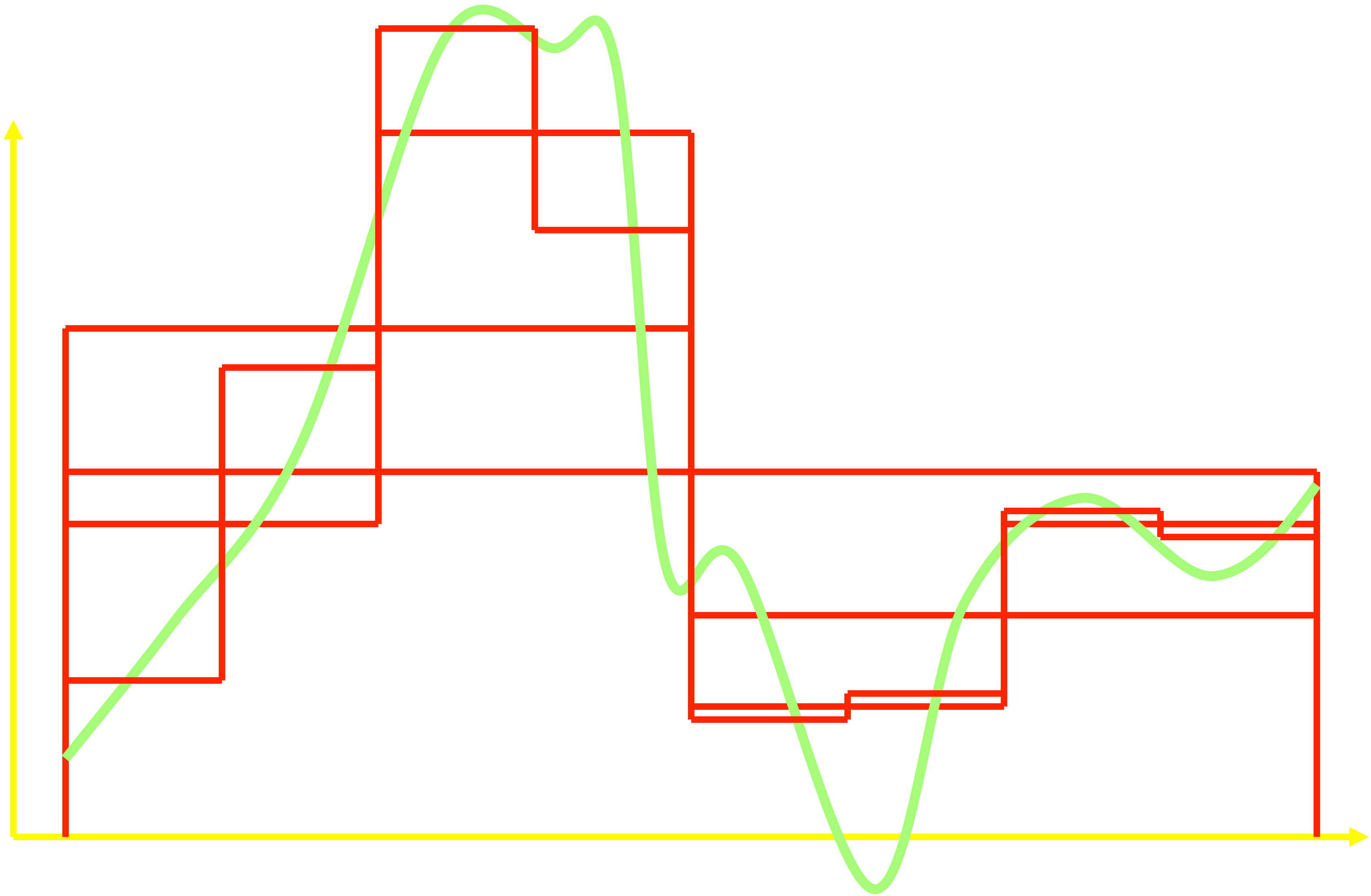
Wavelets examples

Dyadic transform

- For easier calculation we can to discrete continuous signal.
- We have a grid of discrete values that called dyadic grid .
- Important that wavelet functions compact (e.g. no overcalculatings) .

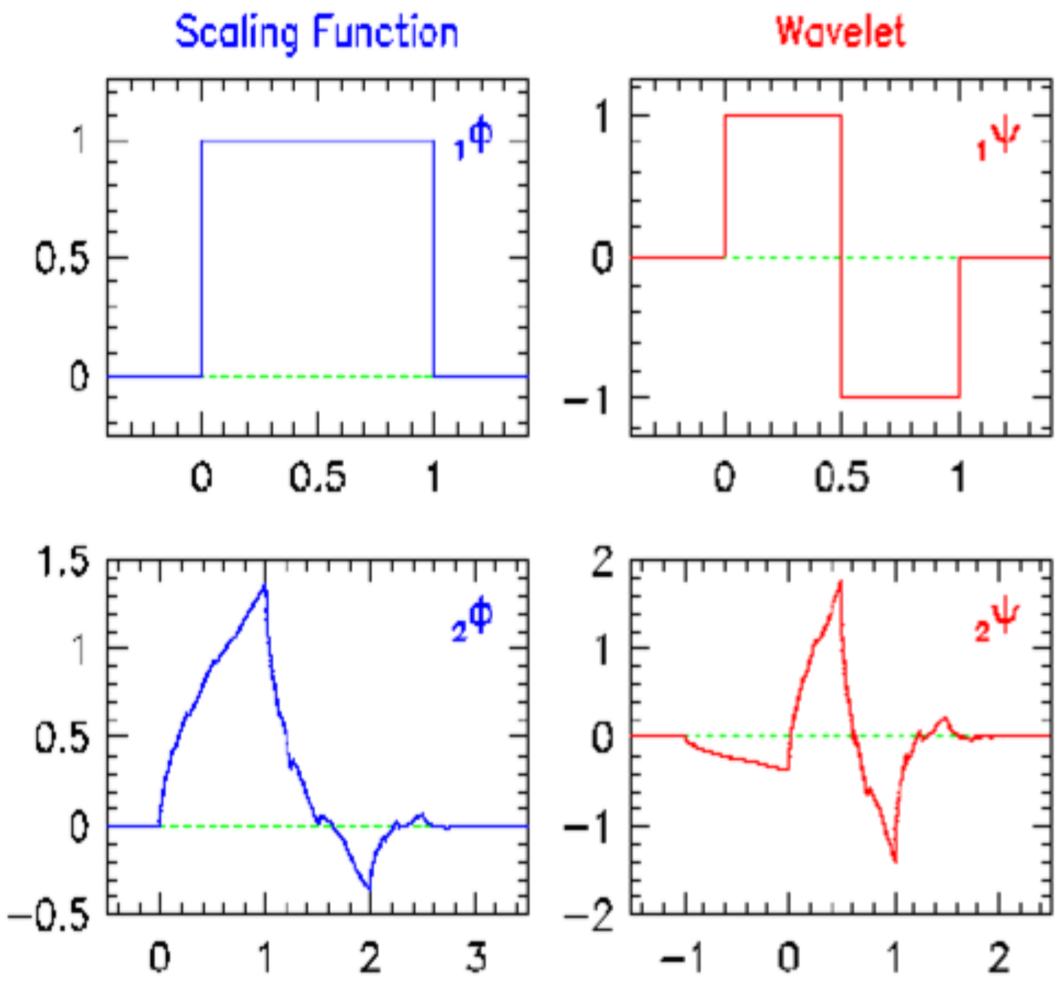
$$\boxed{\begin{aligned} a &= 2^j \\ b &= k2^j \end{aligned}}$$

Haar transform



Wavelet functions examples

- Haar function
- Daubechies function



Properties of Daubechies wavelets

I. Daubechies, *Comm. Pure Appl. Math.* 41 (1988) 909.

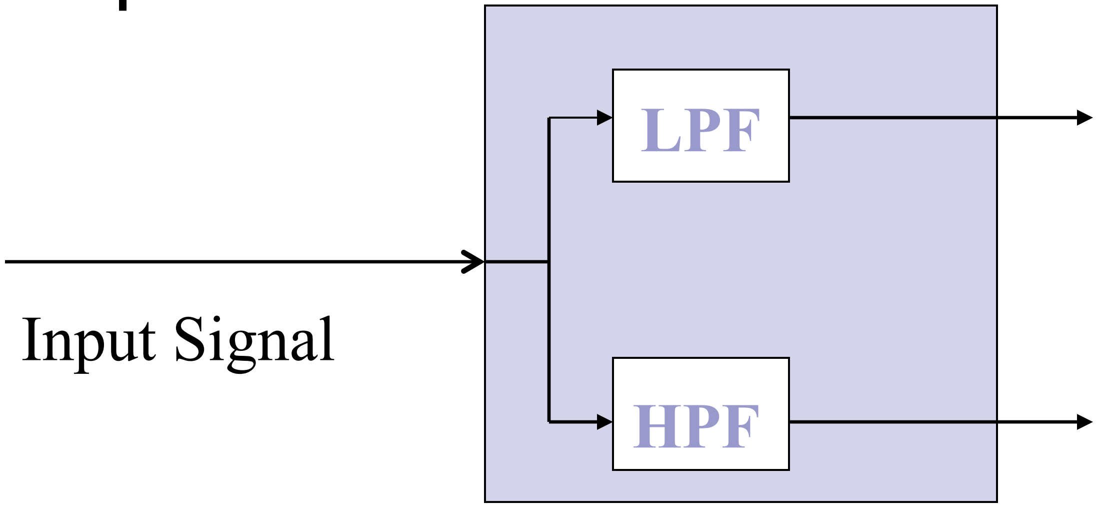
- Compact support
  - finite number of filter parameters / fast implementations
  - high compressibility
  - fine scale amplitudes are very small in regions where the function is smooth / sensitive recognition of structures
- Identical forward / backward filter parameters
  - fast, exact reconstruction
  - very asymmetric

Mallat* Filter Scheme

- Mallat was the first to implement this scheme, using a well known filter design called "two channel sub band coder", yielding a *'Fast Wavelet Transform'*

Approximations and Details:

- Approximations: High-scale, low-frequency components of the signal
- Details: low-scale, high-frequency components

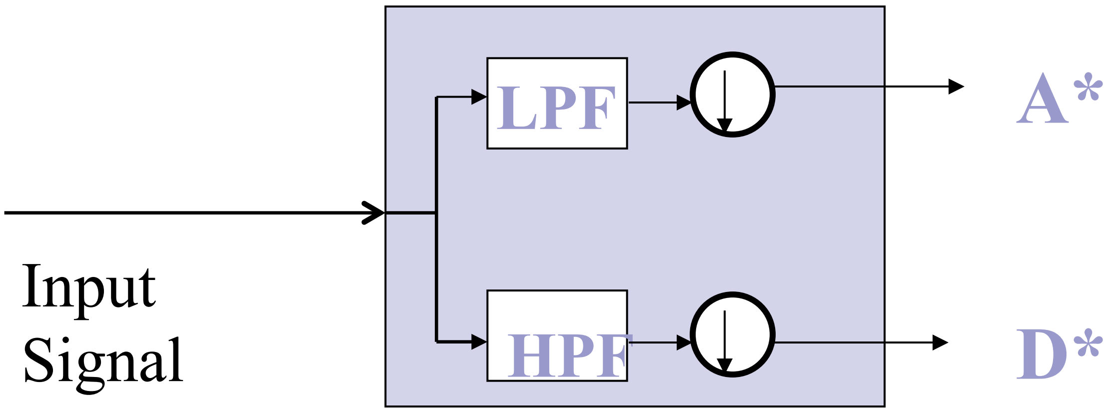


Decimation

- The former process produces <u>twice the data</u> it began with: N input samples produce N approximations coefficients and N detail coefficients.
- To correct this, we *Down sample* (or: *Decimate)* the filter output by two, by simply throwing away every second coefficient.

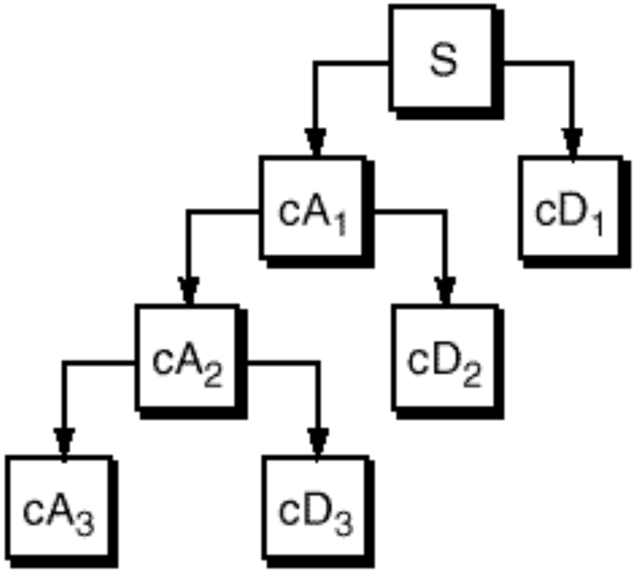
Decimation (cont'd)

So, a complete one stage block looks like:



Multi-level Decomposition

- Iterating the decomposition process, breaks the input signal into many lower-resolution components: *Wavelet decomposition tree*:



Orthogonality

- For 2 vectors
$$\langle v, w \rangle = \sum_n v_n w_n{}^* = 0$$

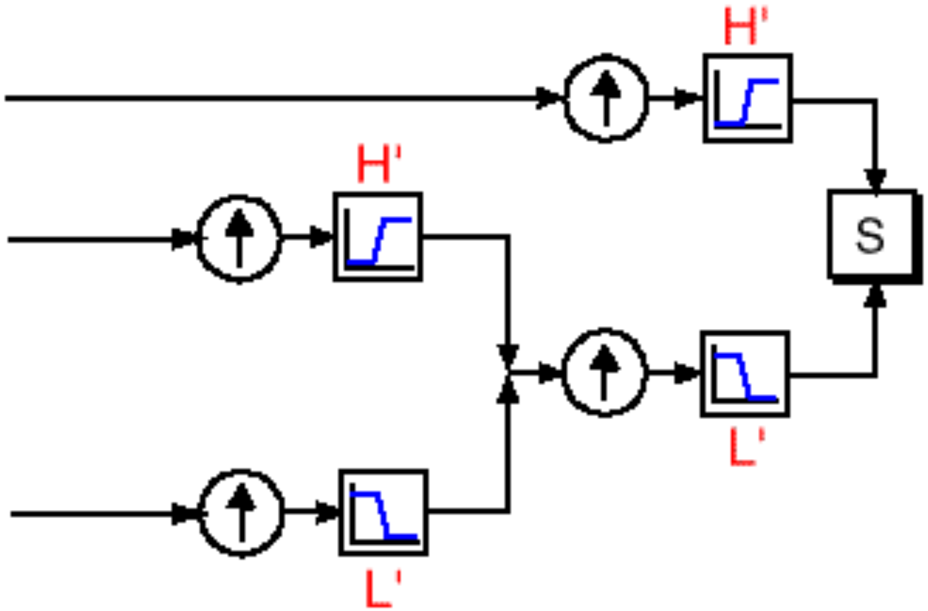
- For 2 functions
$$\langle f(t), g(t) \rangle = \int_a^b f(t) g^*(t) dt = 0$$

Why wavelets have orthogonal base ?

- It easier calculation.
- When we decompose some image and calculating zero level decomposition we have accurate values .
- Scalar multiplication with other base function equals zero.

Wavelet reconstruction

- Reconstruction (or synthesis) is the process in which we assemble all components back



Up sampling (or interpolation) is done by zero inserting between every two coefficients

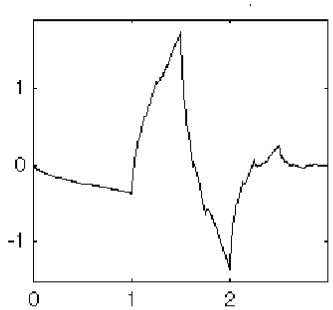
Wavelets like filters

Relationship of Filters to Wavelet Shape

- Choosing the correct filter is most important.
- The choice of the filter determines the shape of the wavelet we use to perform the analysis.

Example

- A low-pass reconstruction filter (L’) for the db2 wavelet:



The filter coefficients (obtained by Matlab *dbaux* command:

0.3415   0.5915   0.1585   -0.0915

reversing the order of this vector and multiply every second coefficient by -1 we get the high-pass filter H’ :

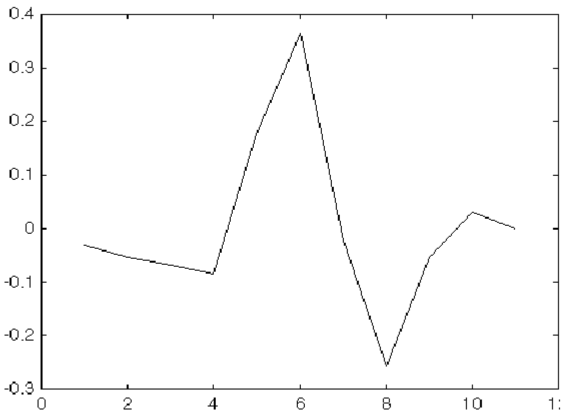
-0.0915   -0.1585   0.5915   -0.3415

Example (Cont'd)

- Now we up-sample the H' coefficient vector:

-0.0915 0 -0.1585 0 0.5915 0 -0.3415 0

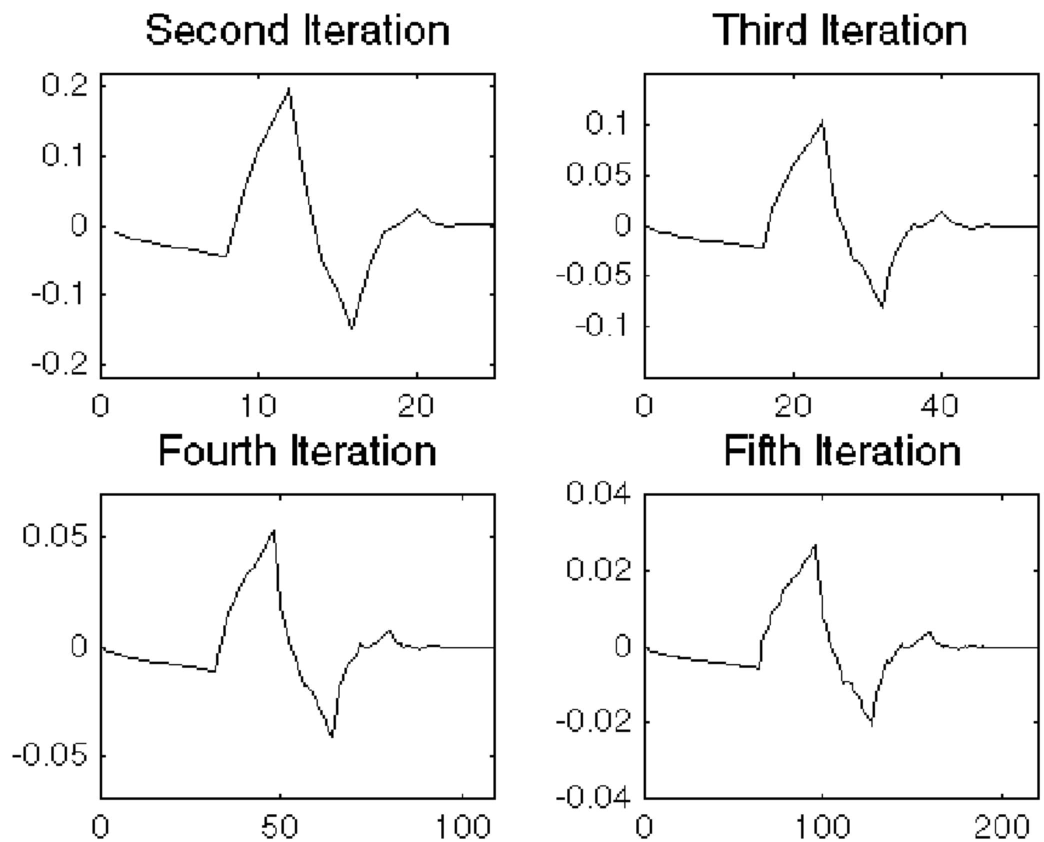
- and Convolving the up-sampled vector with the original low-pass filter we get:



Example (Cont’d)

- Now iterate this process several more times, repeatedly up-sampling and convolving the resultant vector

with the original low-pass filter, a *pattern* begins to emerge:



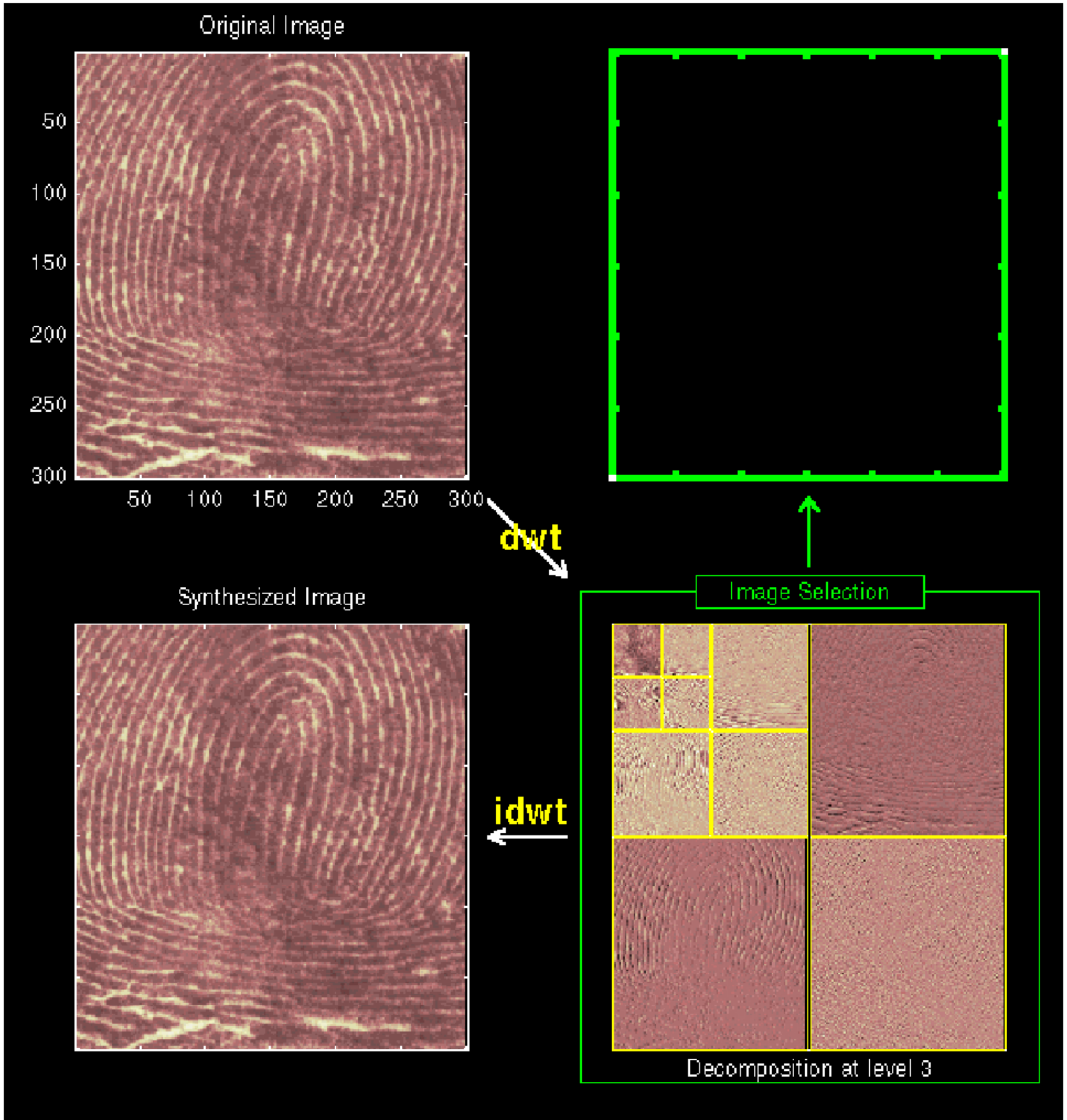
Example: Conclusion

- The curve begins to look more like the *db2* wavelet: the wavelet shape is determined entirely by the coefficient Of the reconstruction filter
- You can’t choose an arbitrary wavelet waveform if you want to be able to reconstruct the original signal accurately!

Compression Example

- A two dimensional (image) compression, using 2D wavelets analysis.
- The image is a Fingerprint.
- FBI uses a wavelet technique to compress its fingerprints database.

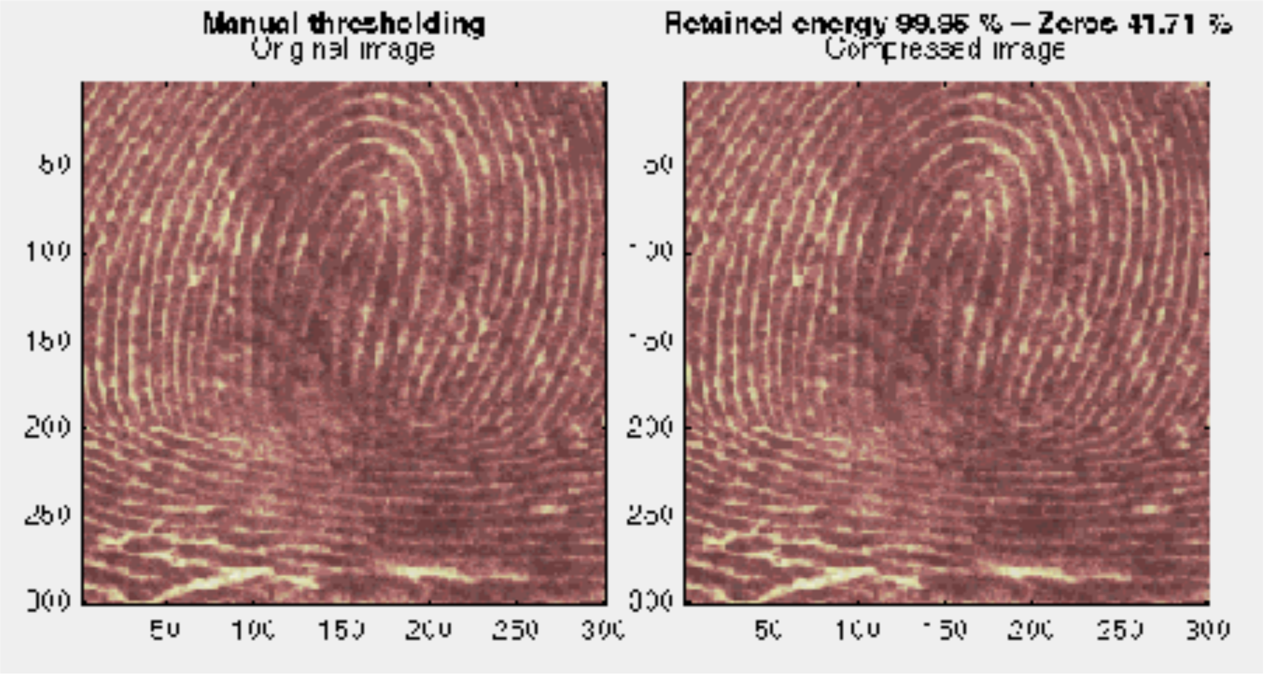
Fingerprint compression



Wavelet:
Haar
Level:3

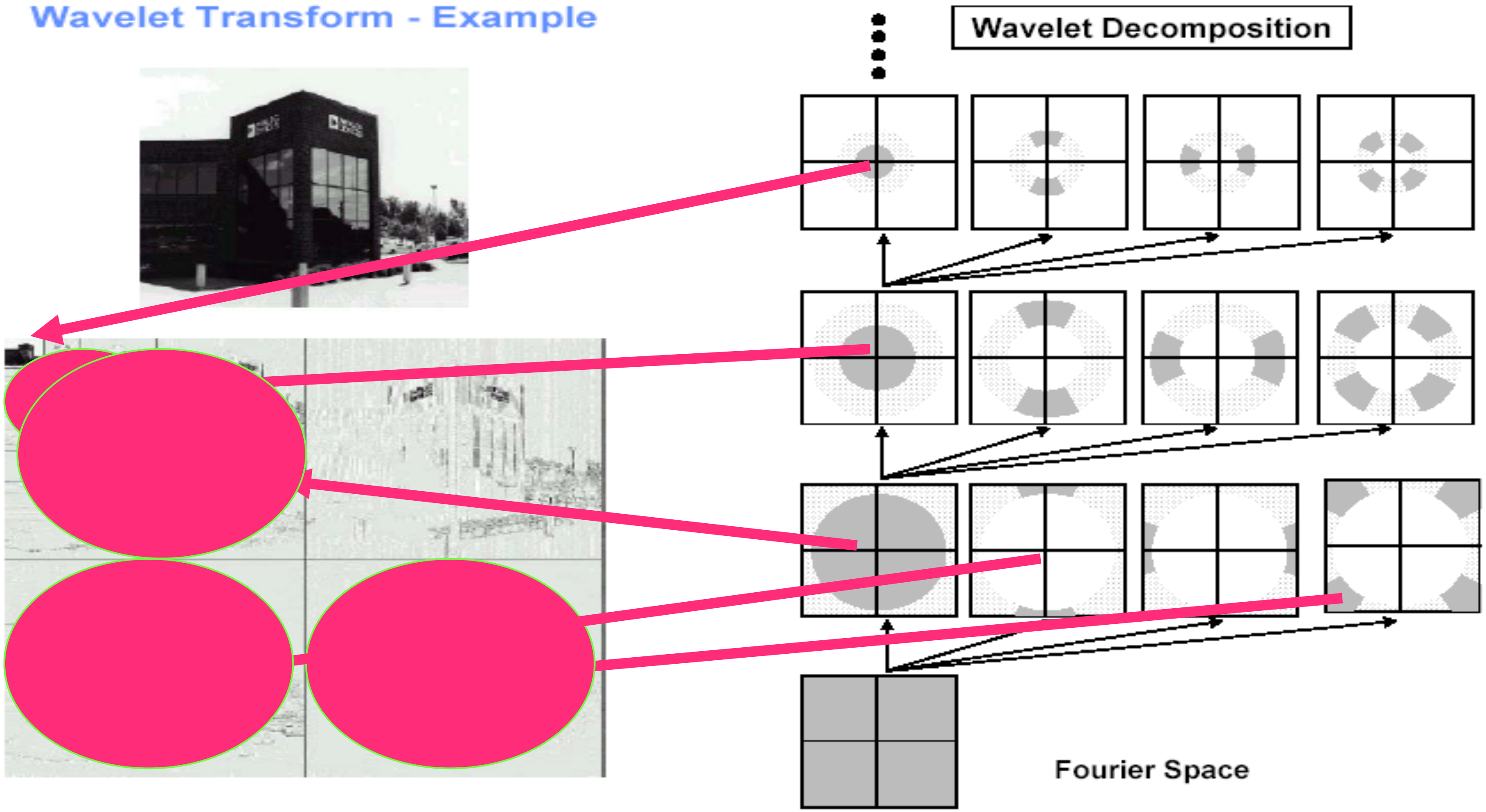
Results (1)

Original Image Compressed Image



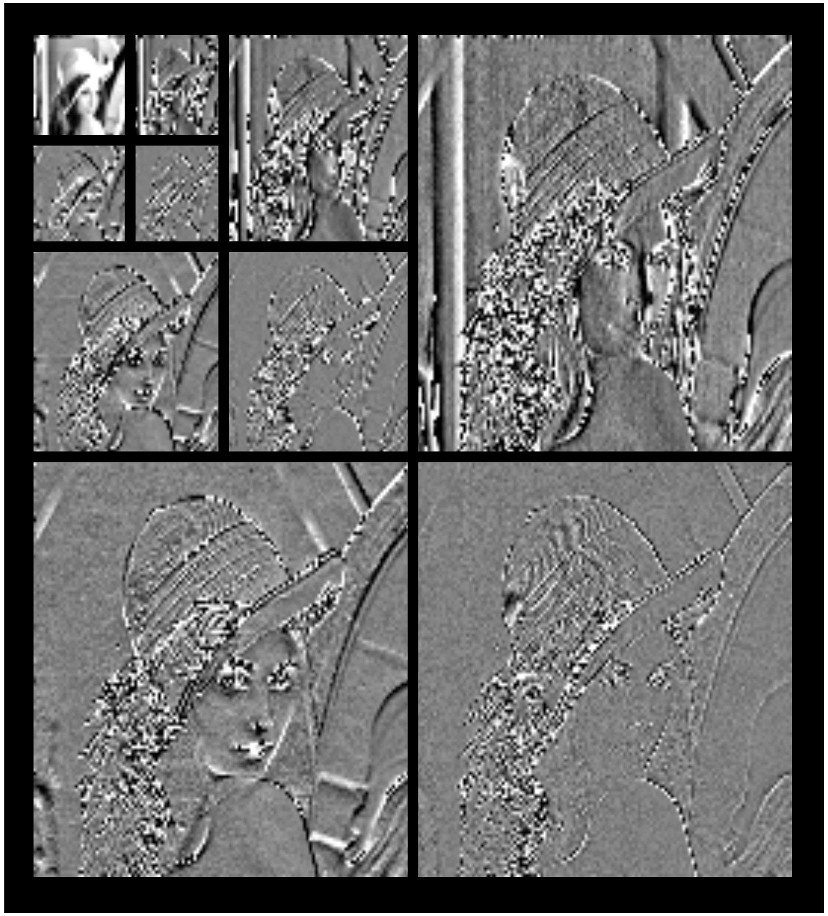
Threshold: 3.5
Zeros: 42%
Retained energy: 99.95%

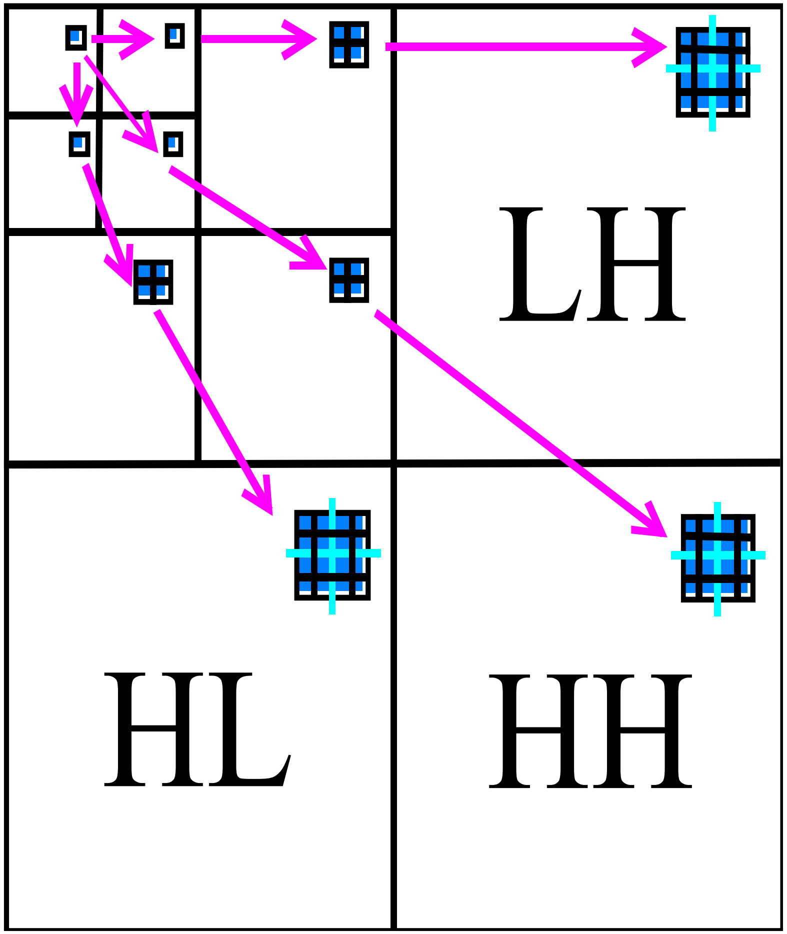
Wavelet Decomposition

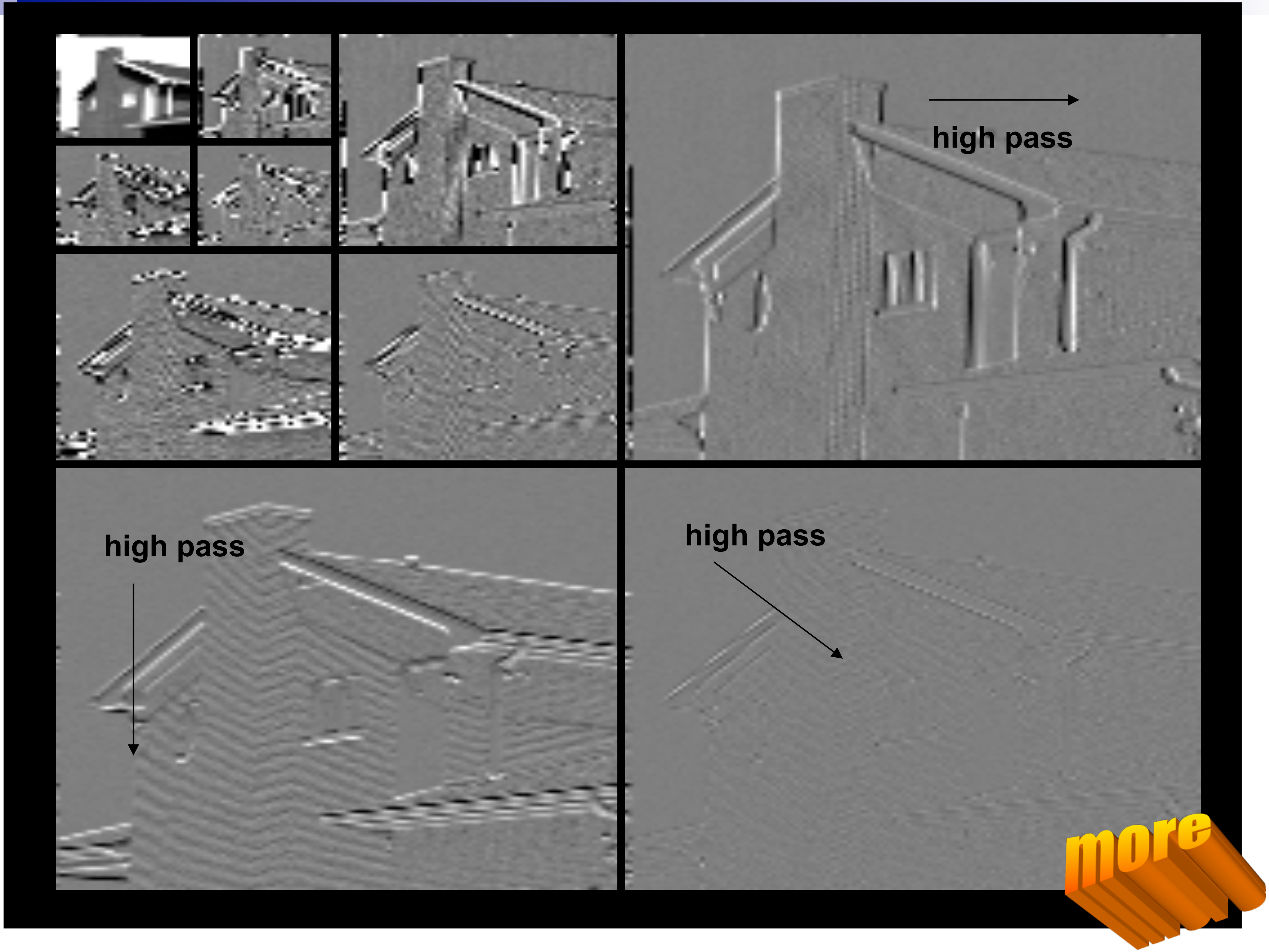


Wavelet Decomposition- Another Example

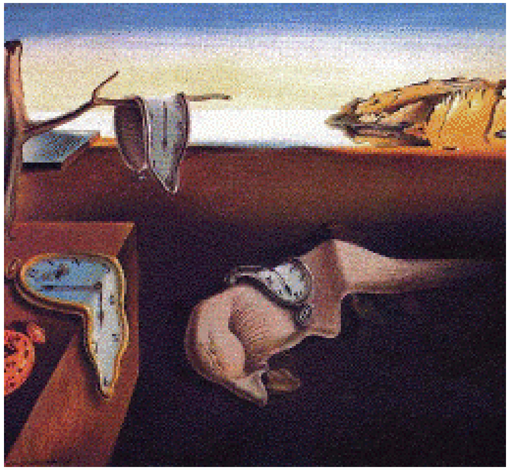
LENNA






high pass
high pass
high pass
more

Coding Example



Original @ 8bpp

DWT

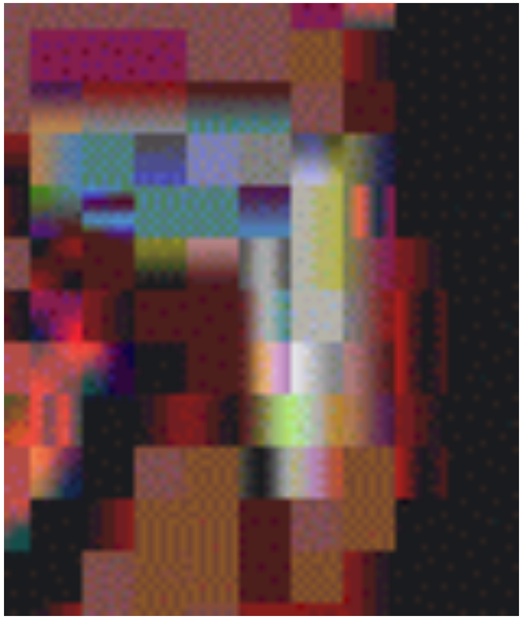
@0.5bpp

DCT
@0.5 bpp

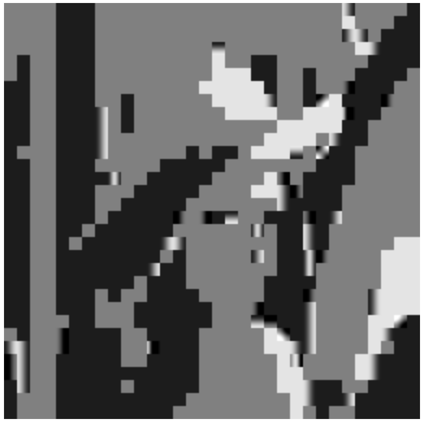
Zoom on Details

DWT

DCT



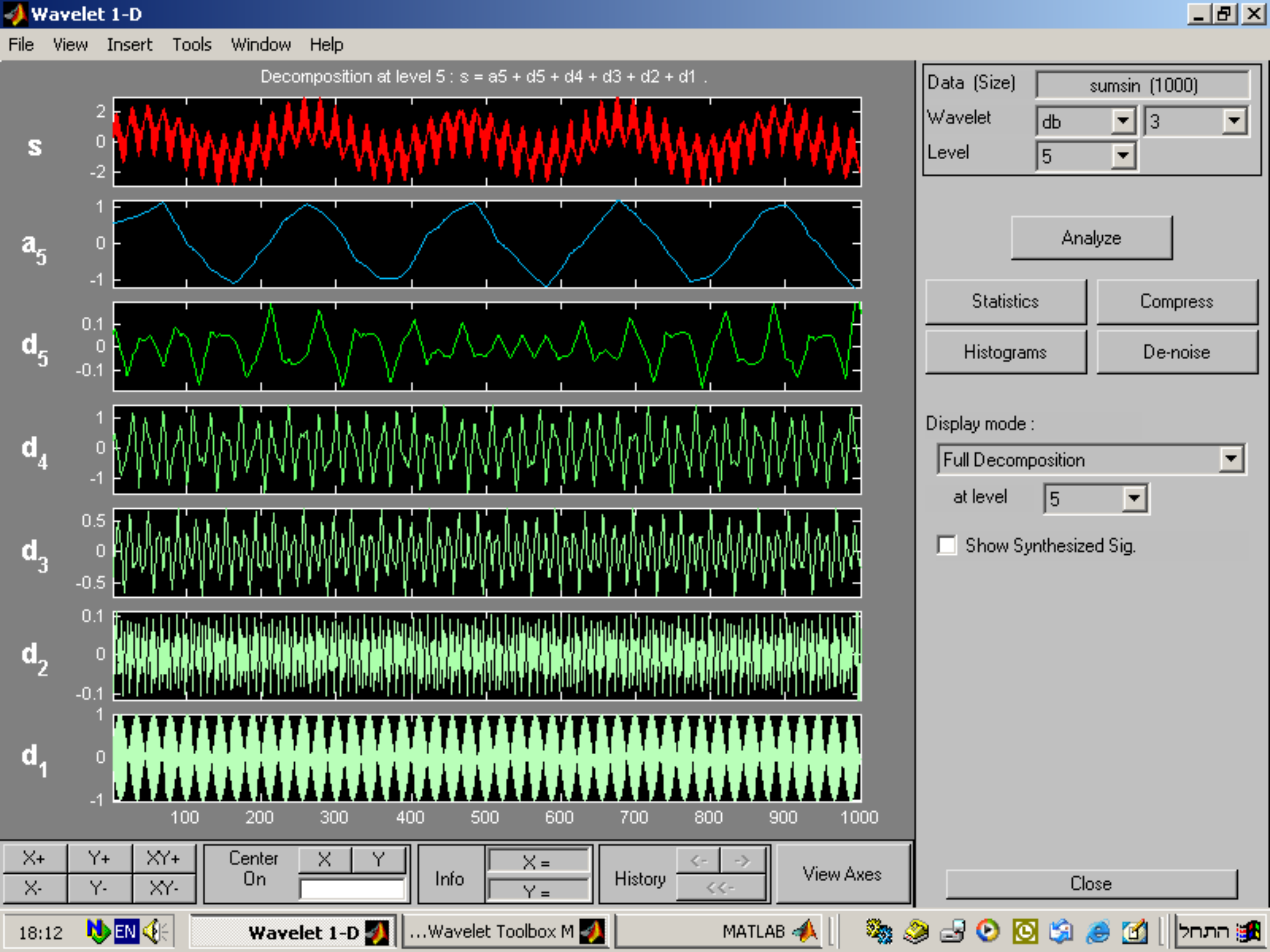
Another Example

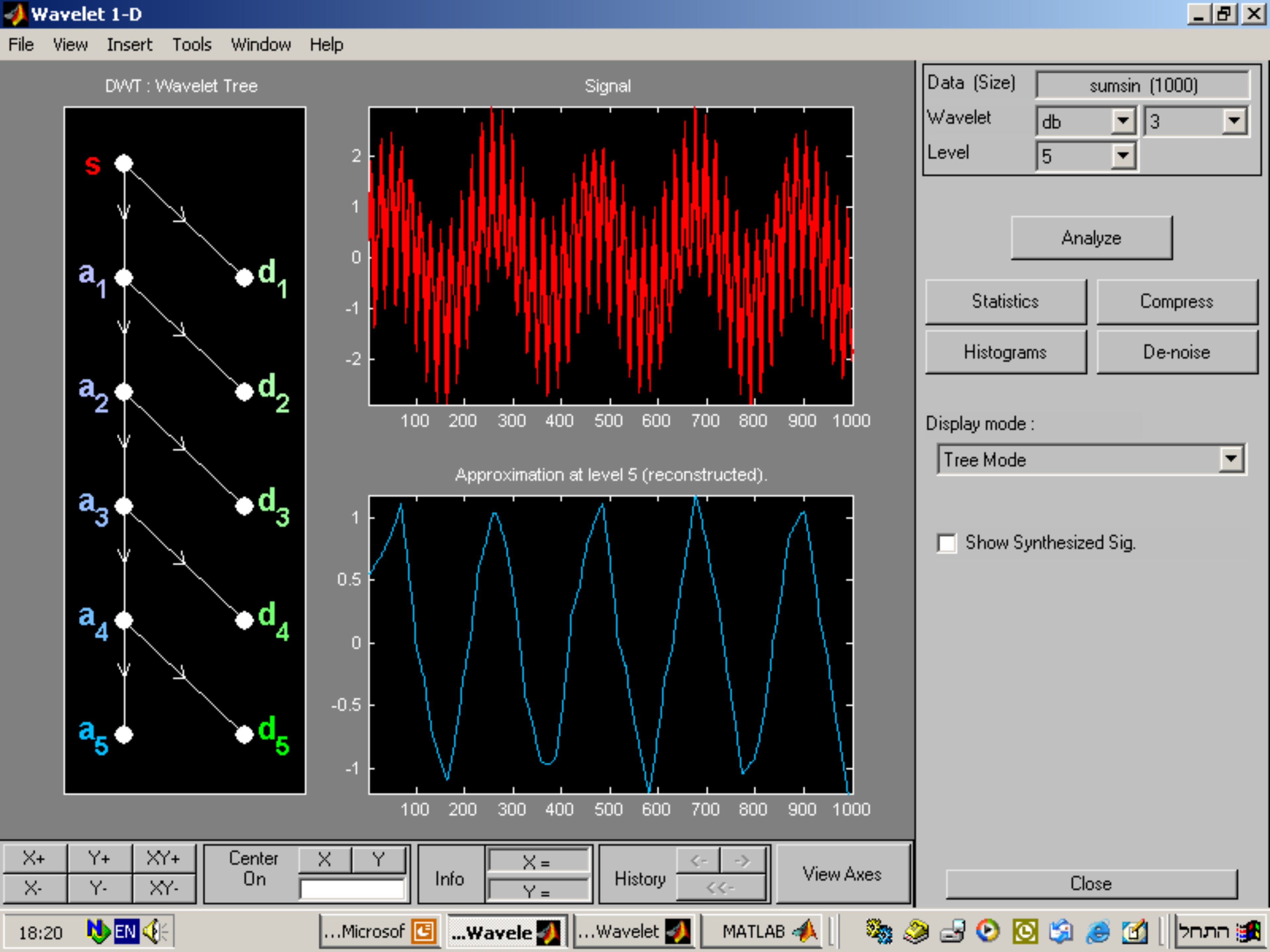
	0.15bpp	0.18bpp	0.2bpp
DCT			
DWT			

Where do we use Wavelets?

- **Everywhere around us are signals that can be analyzed**
- **<u>For example:</u>**
  - seismic tremors
  - human speech
  - engine vibrations
  - medical images
  - financial data
  - Music

Wavelet analysis is a new and promising set of tools for analyzing these signals


Wavelet 1-D
File View Insert Tools Window Help
Decomposition at level 5 : s = a5 + d5 + d4 + d3 + d2 + d1 .
s
a5
d5
d4
d3
d2
d1
100 200 300 400 500 600 700 800 900 1000
Data (Size)
sumsin (1000)
Wavelet
db
3
Level
5
Analyze
Statistics
Compress
Histograms
De-noise
Display mode :
Full Decomposition
at level
5
Show Synthesized Sig.
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
Close
18:12
EN
Wavelet 1-D
...Wavelet Toolbox M
MATLAB



Wavelet 1-D
File
View
Insert
Tools
Window
Help
DWT : Wavelet Tree
s
a1
a2
a3
a4
a5
d1
d2
d3
d4
d5
Signal
2
1
0
-1
-2
100 200 300 400 500 600 700 800 900 1000
Approximation at level 5 (reconstructed).
1
0.5
0
-0.5
-1
100 200 300 400 500 600 700 800 900 1000
Data (Size)
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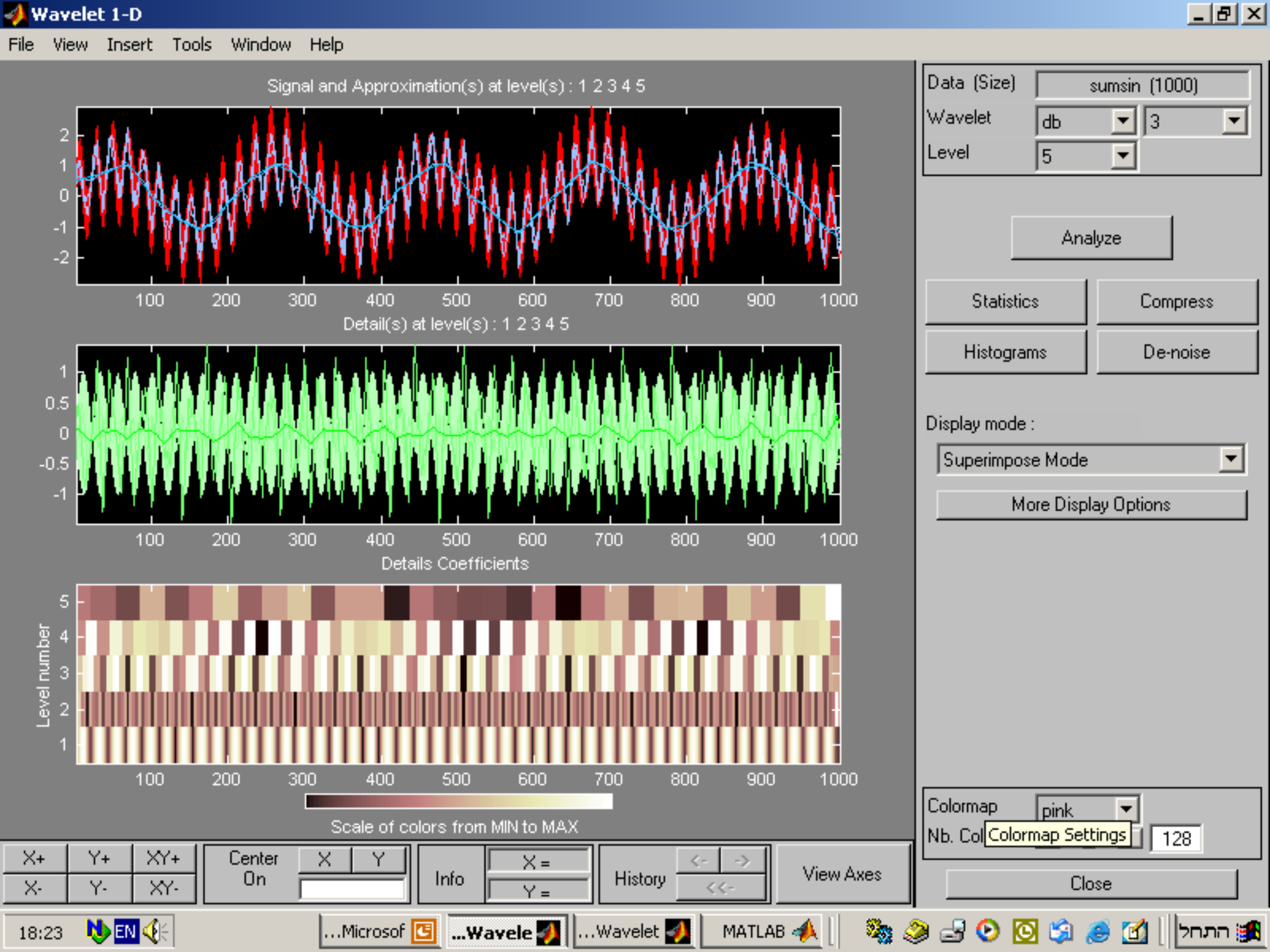
Wavelet 1-D
File View Insert Tools Window Help
Signal and Approximation(s) at level(s) : 1 2 3 4 5
Detail(s) at level(s) : 1 2 3 4 5
Details Coefficients
Level number
Scale of colors from MIN to MAX
Data (Size)
sumsin (1000)
Wavelet
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3
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Compress
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De-noise
Display mode :
Superimpose Mode
More Display Options
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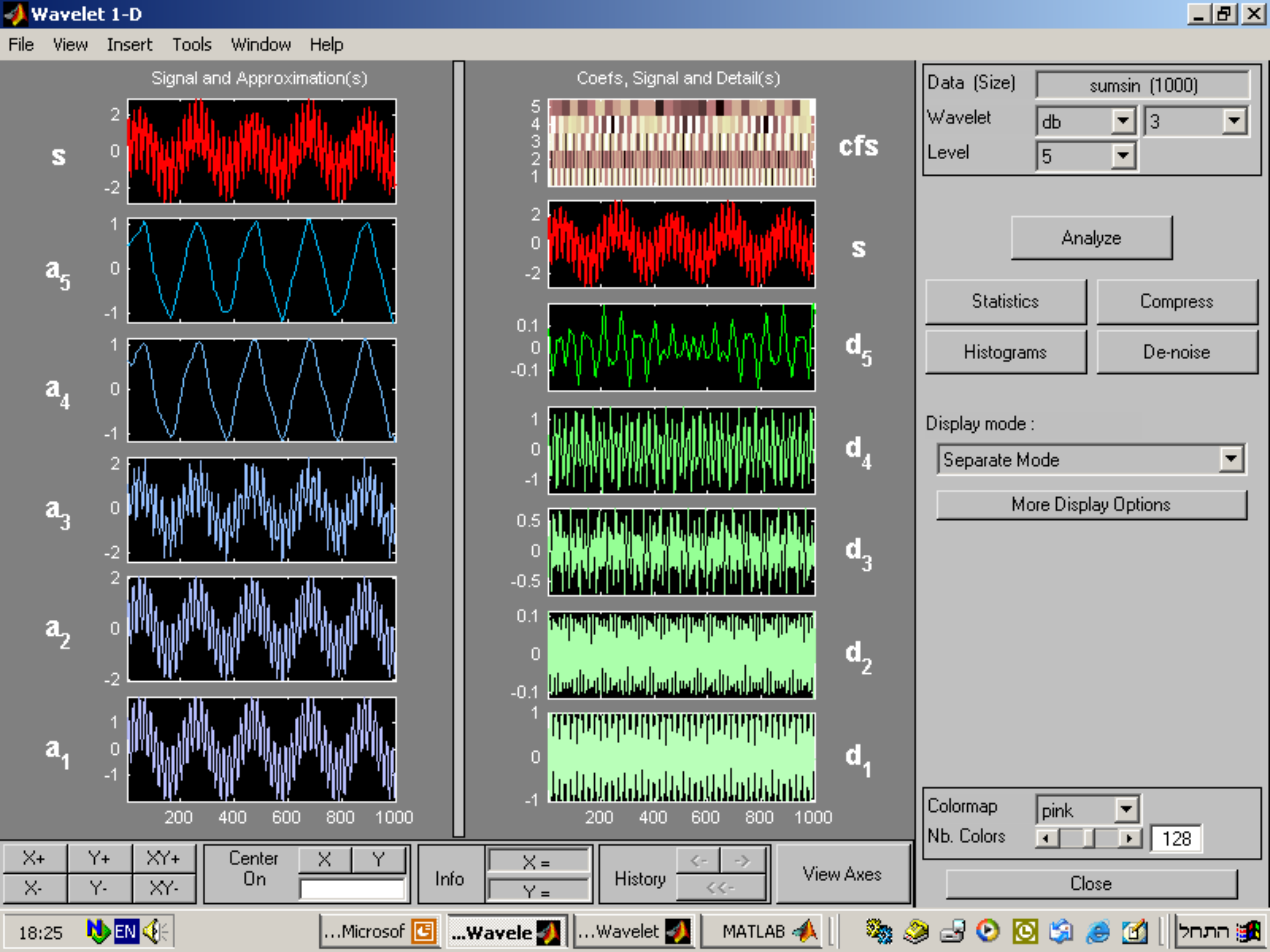
MATLAB


Wavelet 1-D
File View Insert Tools Window Help
Signal and Approximation(s)
Coefs, Signal and Detail(s)
s
a5
a4
a3
a2
a1
cfs
s
d5
d4
d3
d2
d1
200 400 600 800 1000
Data (Size)
sumsin (1000)
Wavelet
db
3
Level
5
Analyze
Statistics
Compress
Histograms
De-noise
Display mode :
Separate Mode
More Display Options
Colormap
pink
Nb. Colors
128
Close
X+
Y+
XY+
X-
Y-
XY-
Center On
X
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X =
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History
View Axes
18:25
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...Microsof
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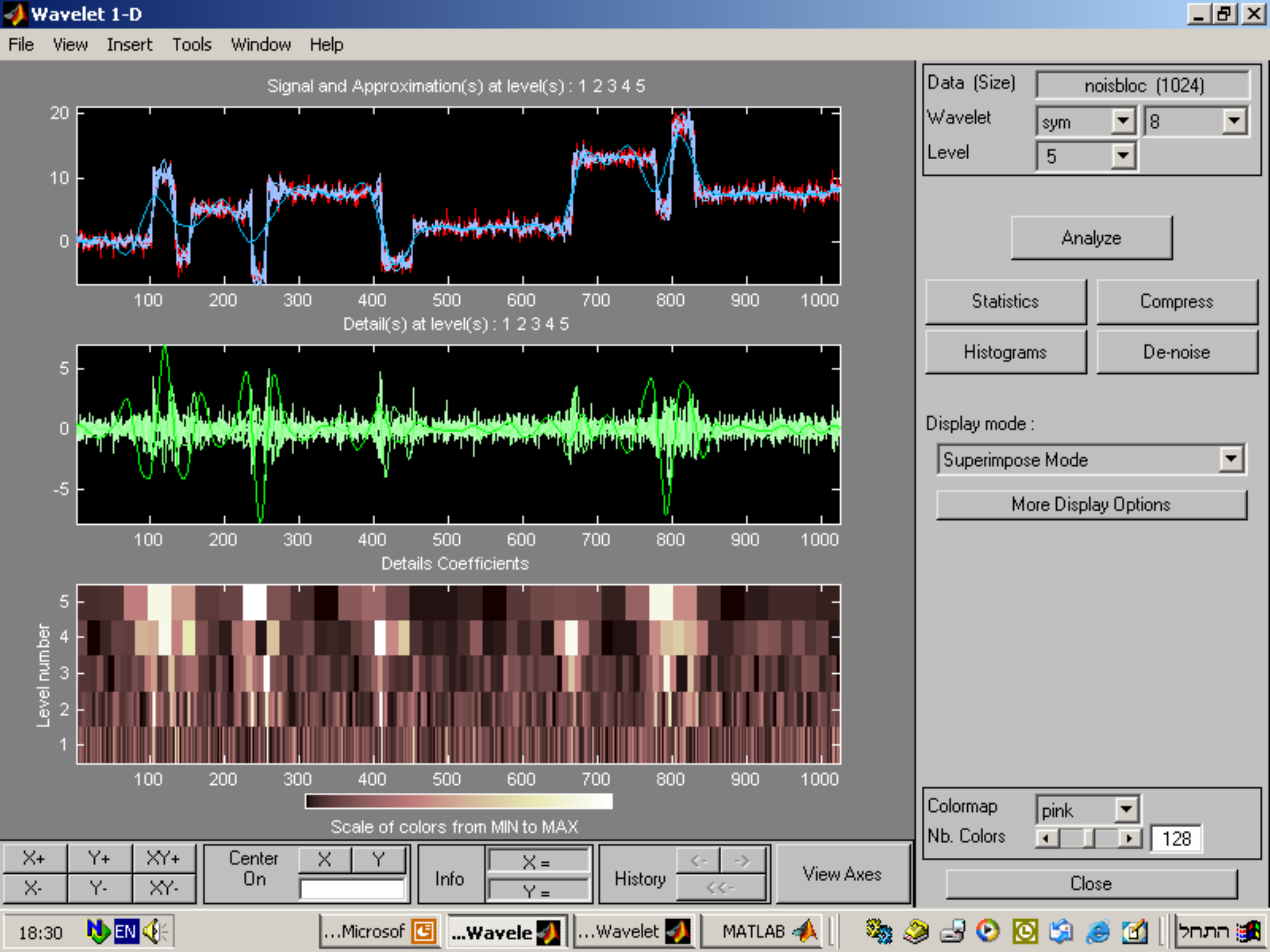
...Wavelet
MATLAB
התחל


Wavelet 1-D
File View Insert Tools Window Help
Signal and Approximation(s) at level(s) : 1 2 3 4 5
Detail(s) at level(s) : 1 2 3 4 5
Details Coefficients
Level number
Scale of colors from MIN to MAX
Data (Size)
noisbloc (1024)
Wavelet
sym
8
Level
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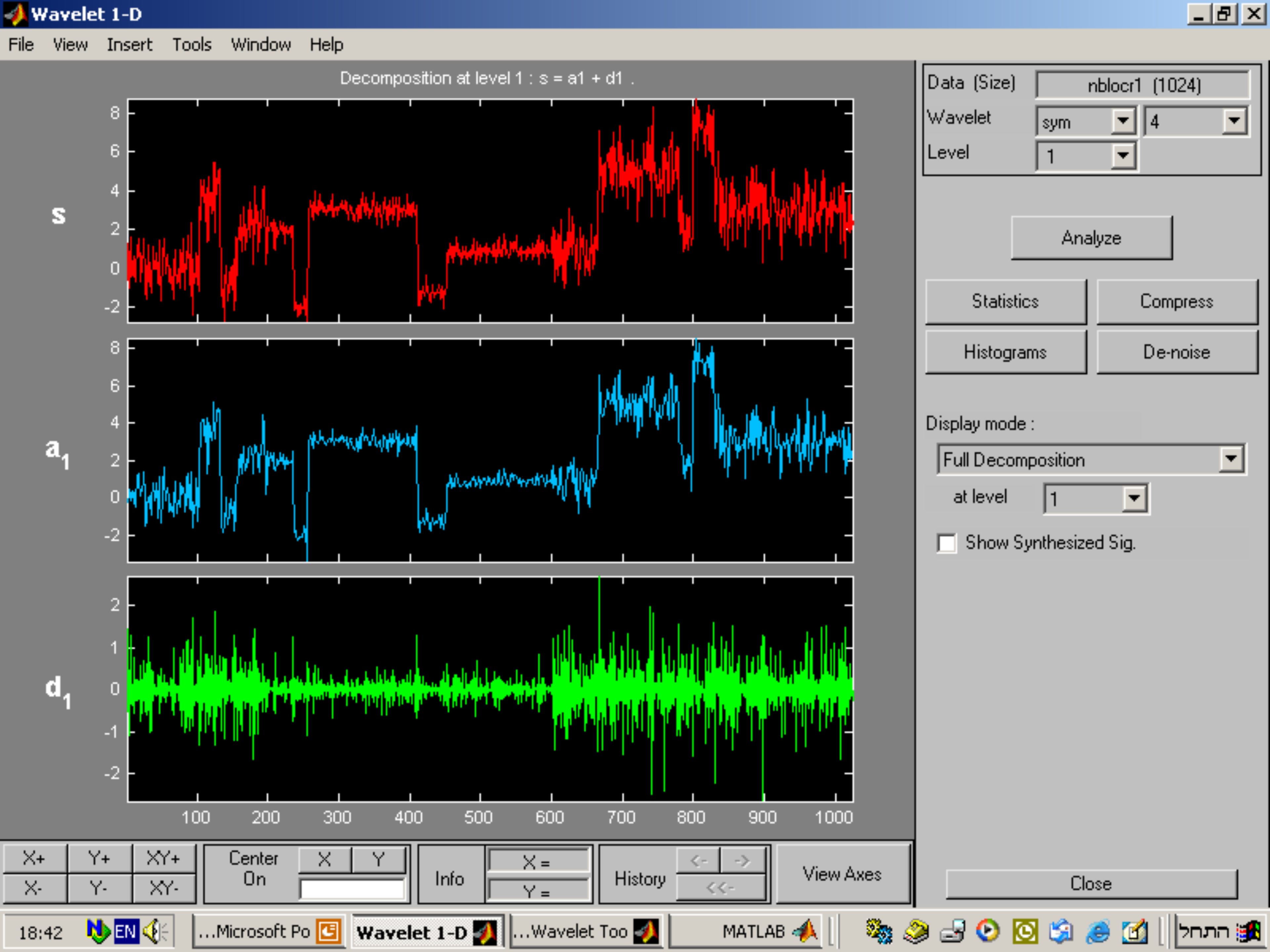
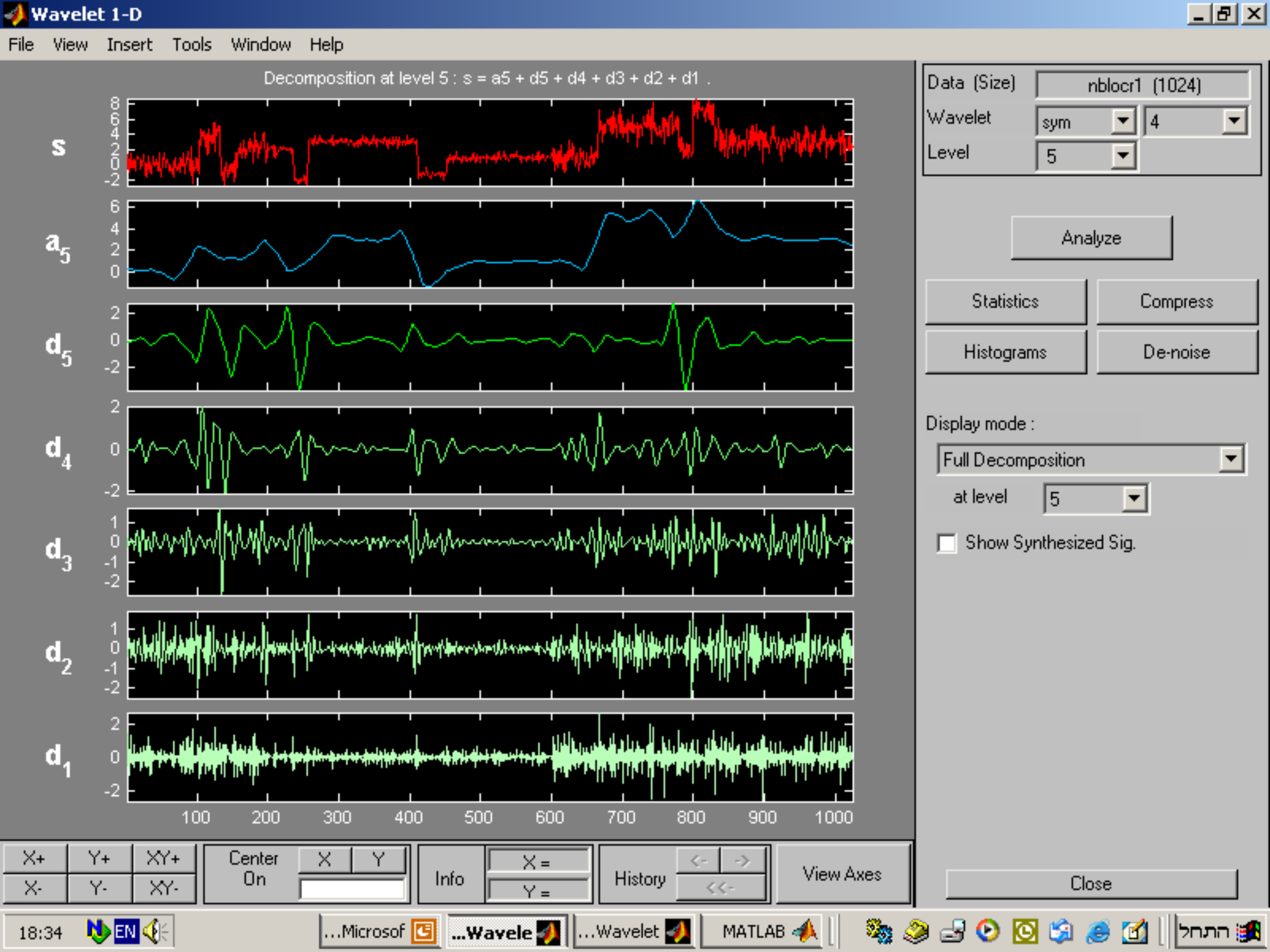
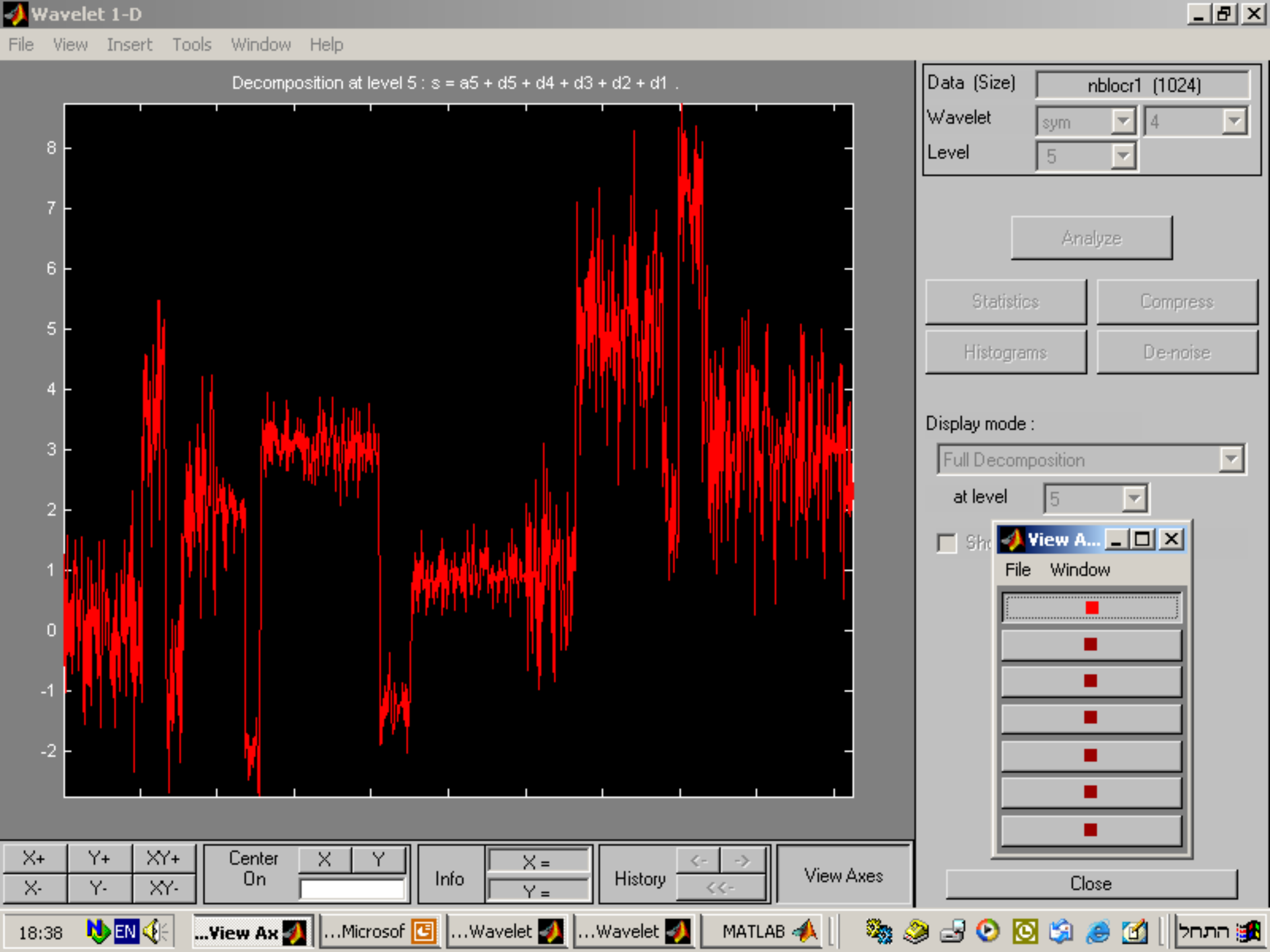
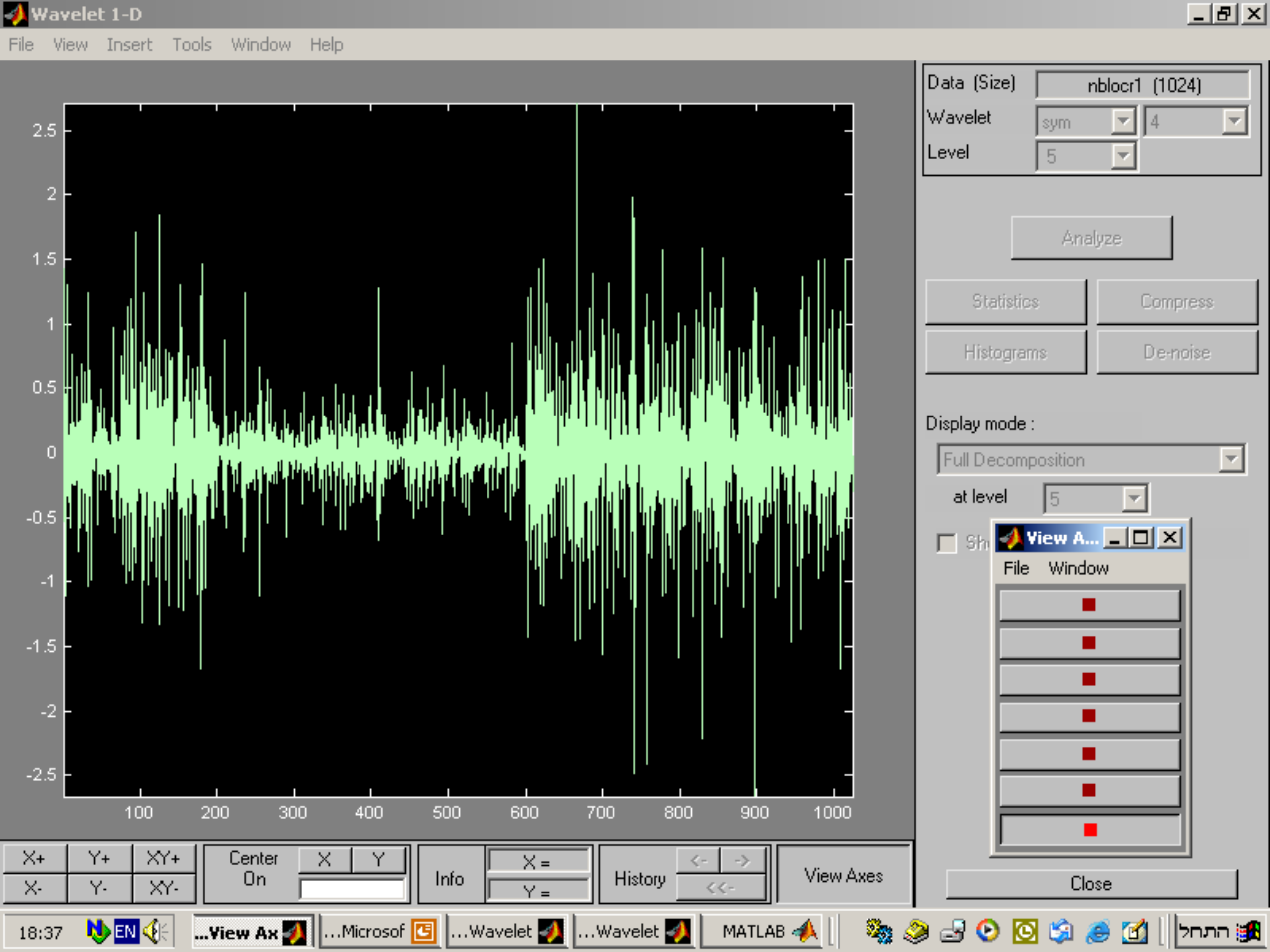
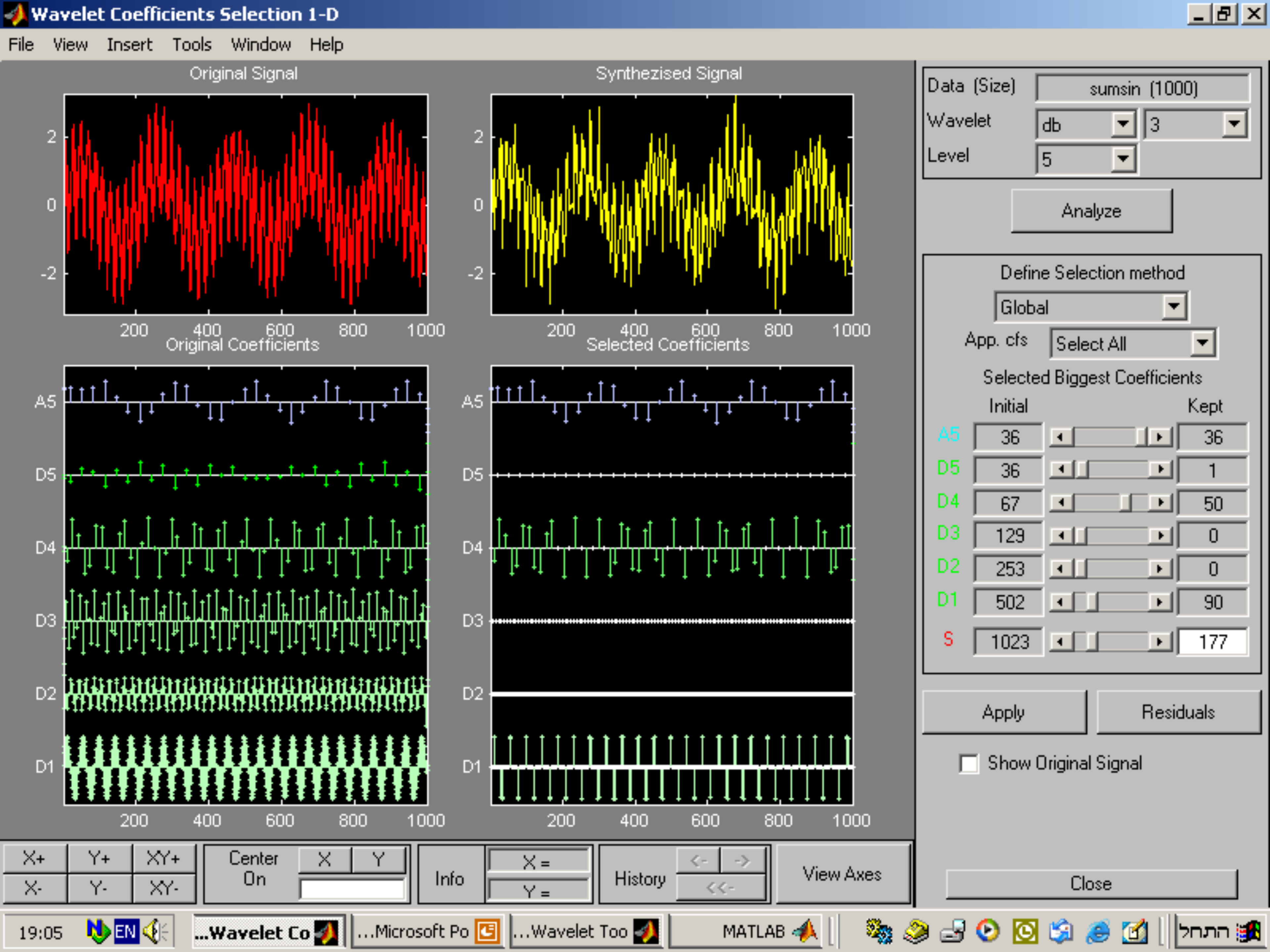
Analyze
Statistics
Compress
Histograms
De-noise
Display mode :
Superimpose Mode
More Display Options
Colormap
pink
Nb. Colors
128
Close
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes
18:30
EN
...Microsof
...Wavele
...Wavelet
MATLAB
התחל

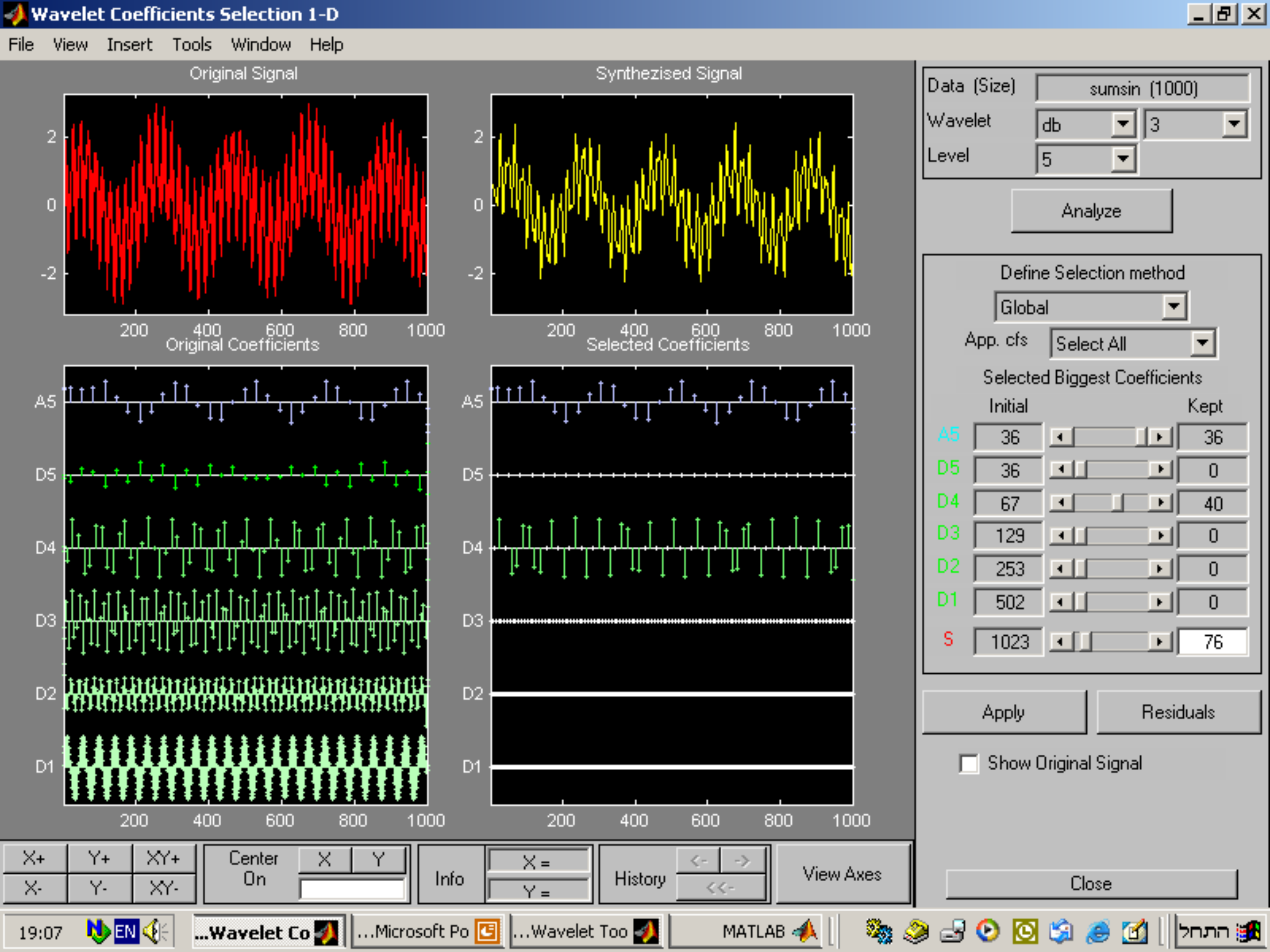

Wavelet 1-D
File View Insert Tools Window Help
Decomposition at level 1 : s = a1 + d1 .
s
a_1
d_1
100 200 300 400 500 600 700 800 900 1000
Data (Size)
nblocr1 (1024)
Wavelet
sym
4
Level
1
Analyze
Statistics
Compress
Histograms
De-noise
Display mode :
Full Decomposition
at level
1
Show Synthesized Sig.
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
Close
18:42
EN
...Microsoft Po
Wavelet 1-D
...Wavelet Too
MATLAB
התחל

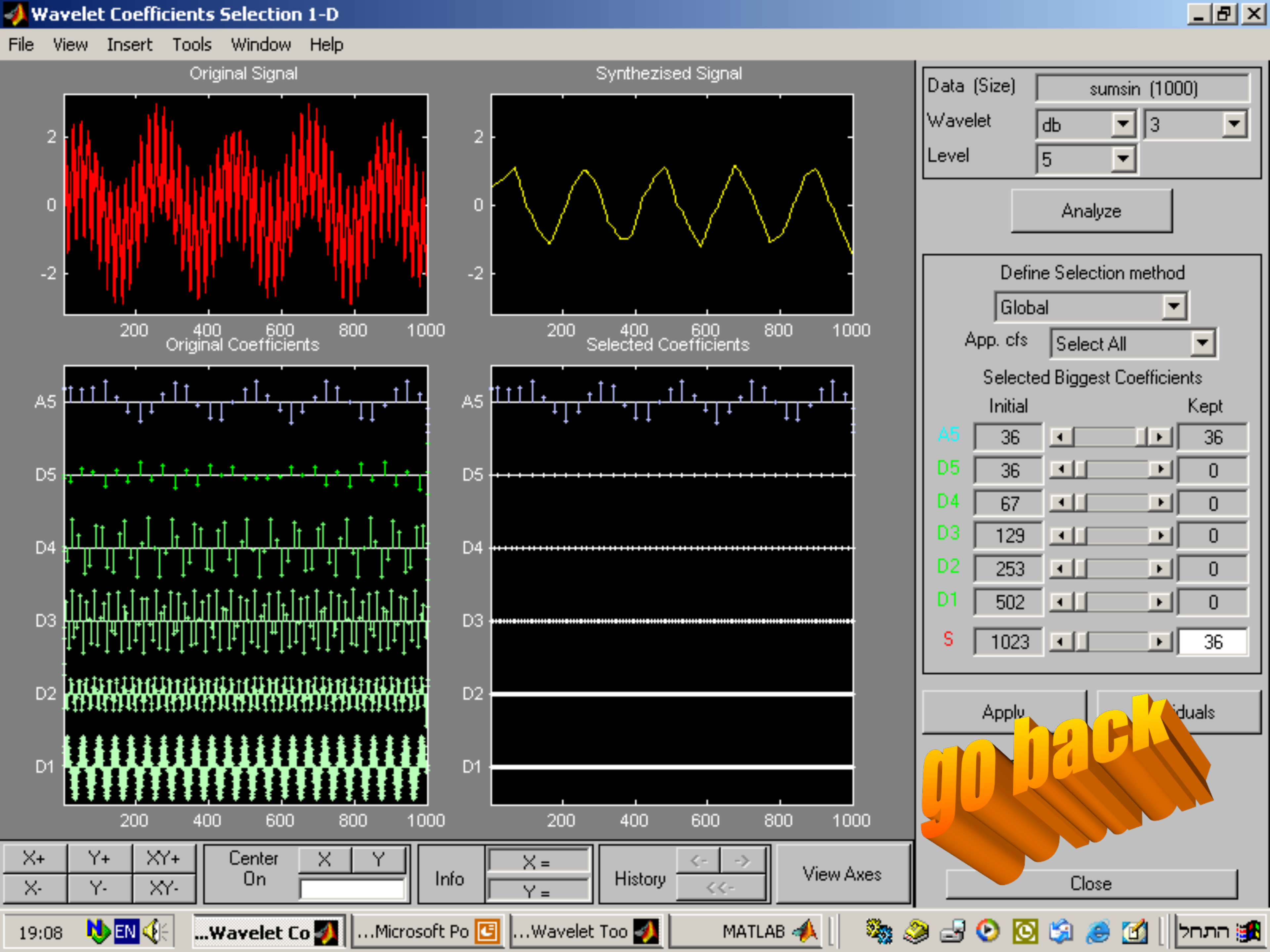

Wavelet 1-D
File View Insert Tools Window Help
Decomposition at level 5 : s = a5 + d5 + d4 + d3 + d2 + d1 .
s
a_5
d_5
d_4
d_3
d_2
d_1
100 200 300 400 500 600 700 800 900 1000
Data (Size)
nblocr1 (1024)
Wavelet
sym
4
Level
5
Analyze
Statistics
Compress
Histograms
De-noise
Display mode :
Full Decomposition
at level
5
Show Synthesized Sig.
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
Close
18:34
EN
...Microsof
...Wavele
...Wavelet
MATLAB
התחל

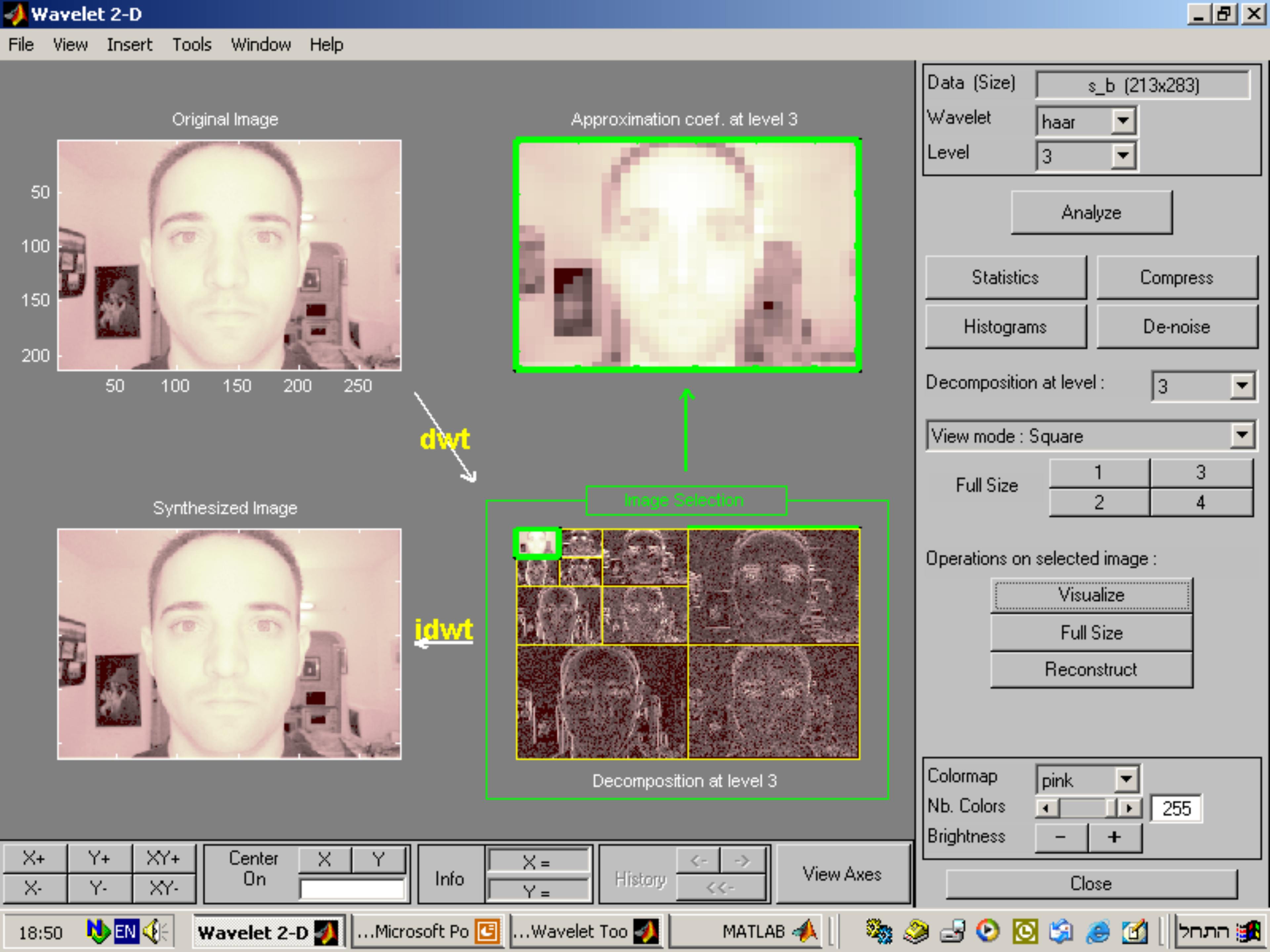

Wavelet 1-D
File View Insert Tools Window Help
Decomposition at level 5 : s = a5 + d5 + d4 + d3 + d2 + d1 .
8
7
6
5
4
3
2
1
0
-1
-2
Data (Size)
nblocr1 (1024)
Wavelet
sym
4
Level
5
Analyze
Statistics
Compress
Histograms
De-noise
Display mode :
Full Decomposition
at level
5
View A...
File Window
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes
Close
18:38
EN
...View Ax
...Microsof
...Wavelet
...Wavelet
MATLAB

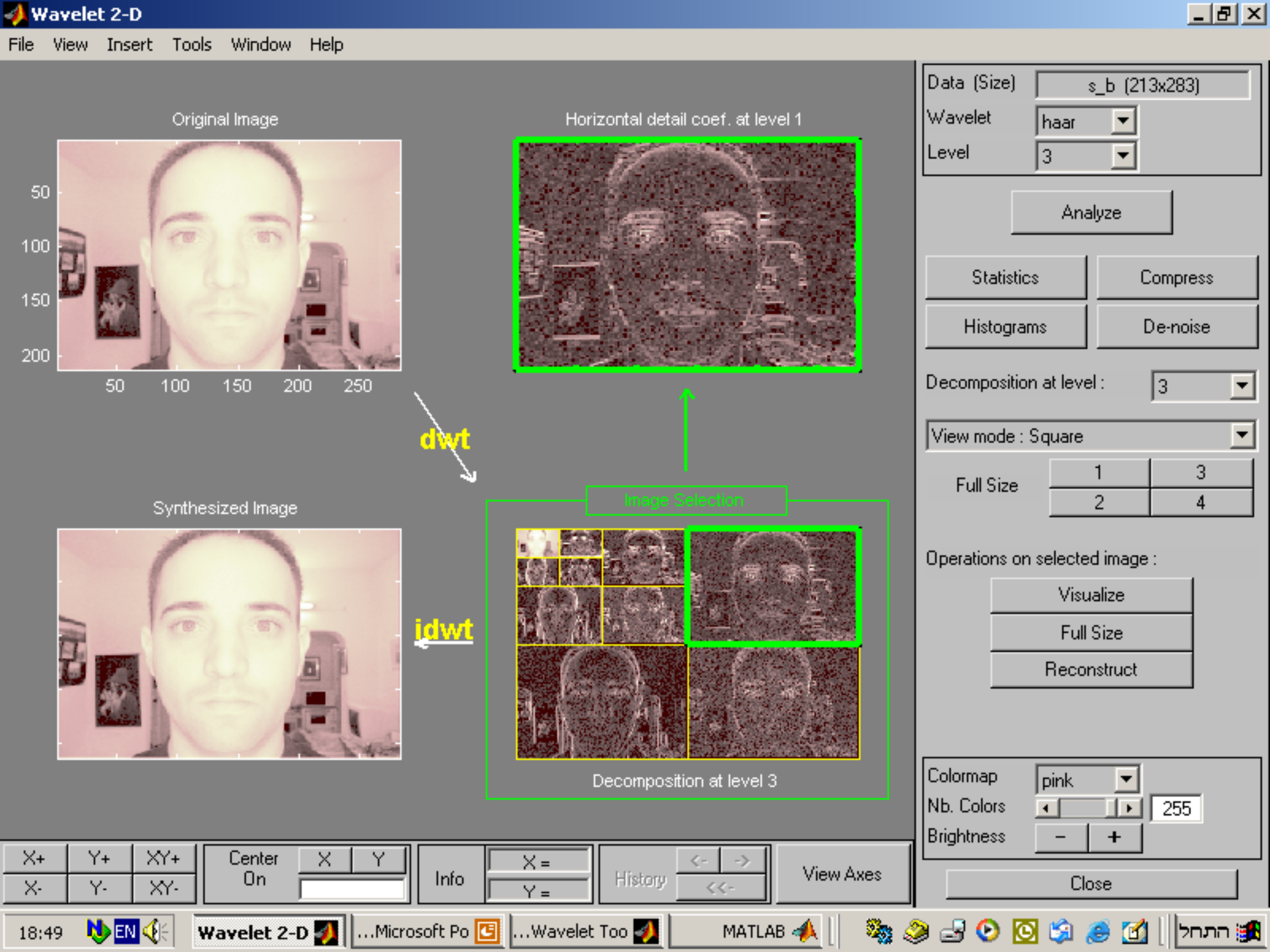

Wavelet 1-D
File View Insert Tools Window Help
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
100 200 300 400 500 600 700 800 900 1000
Data (Size)
nblocr1 (1024)
Wavelet
sym
4
Level
5
Analyze
Statistics
Compress
Histograms
De-noise
Display mode :
Full Decomposition
at level
5
View A...
File Window
Close
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
18:37
EN
...View Ax
...Microsof
...Wavelet
...Wavelet
MATLAB
התחל

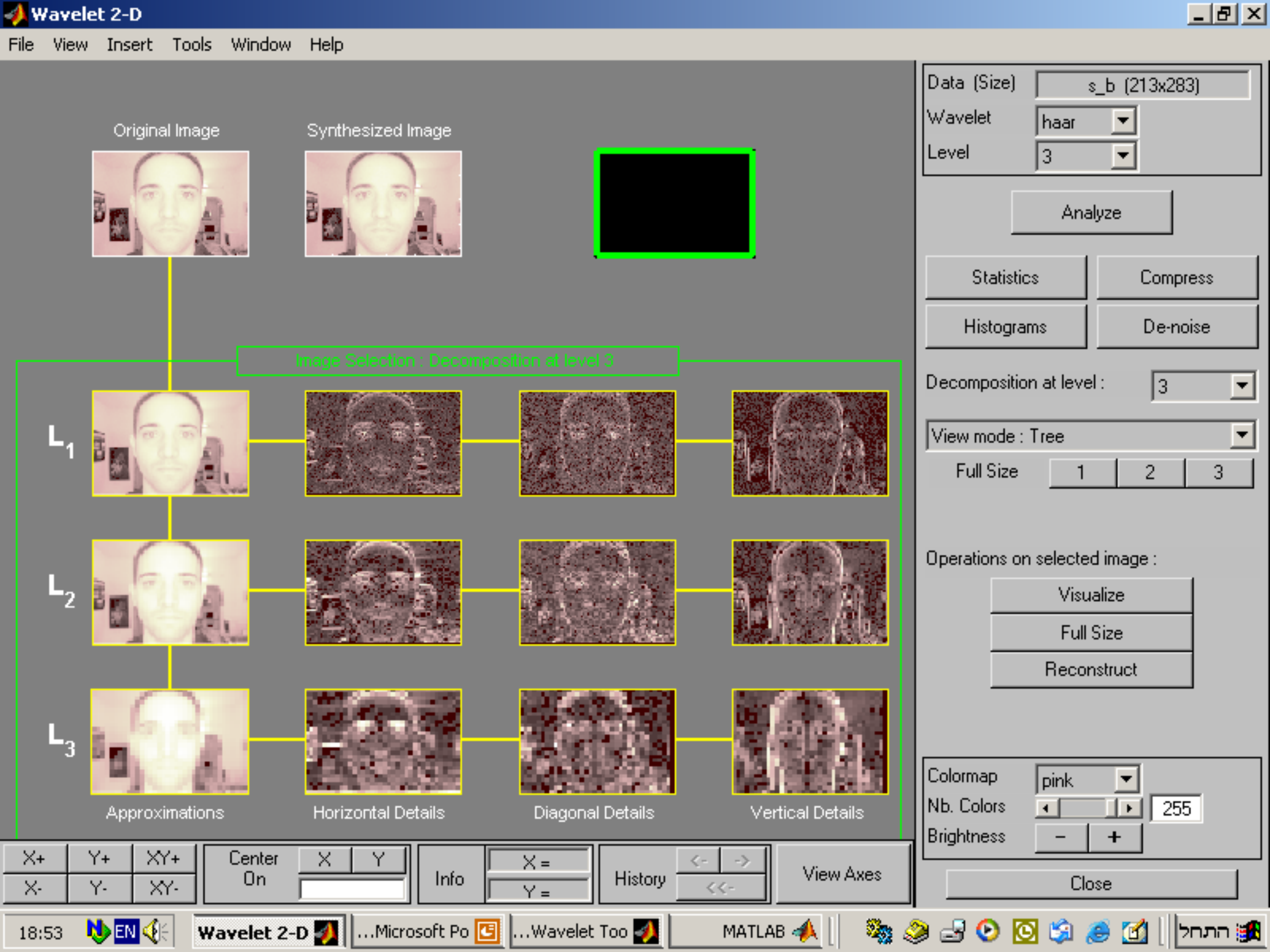

Wavelet Coefficients Selection 1-D
File View Insert Tools Window Help
Original Signal
Synthezised Signal
Original Coefficients
Selected Coefficients
A5
D5
D4
D3
D2
D1
200 400 600 800 1000
Data (Size)
sumsin (1000)
Wavelet
db
3
Level
5
Analyze
Define Selection method
Global
App. cfs
Select All
Selected Biggest Coefficients
Initial
Kept
A5 36 36
D5 36 1
D4 67 50
D3 129 0
D2 253 0
D1 502 90
S 1023 177
Apply
Residuals
Show Original Signal
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
Close
19:05
EN
...Wavelet Co
...Microsoft Po
...Wavelet Too
MATLAB

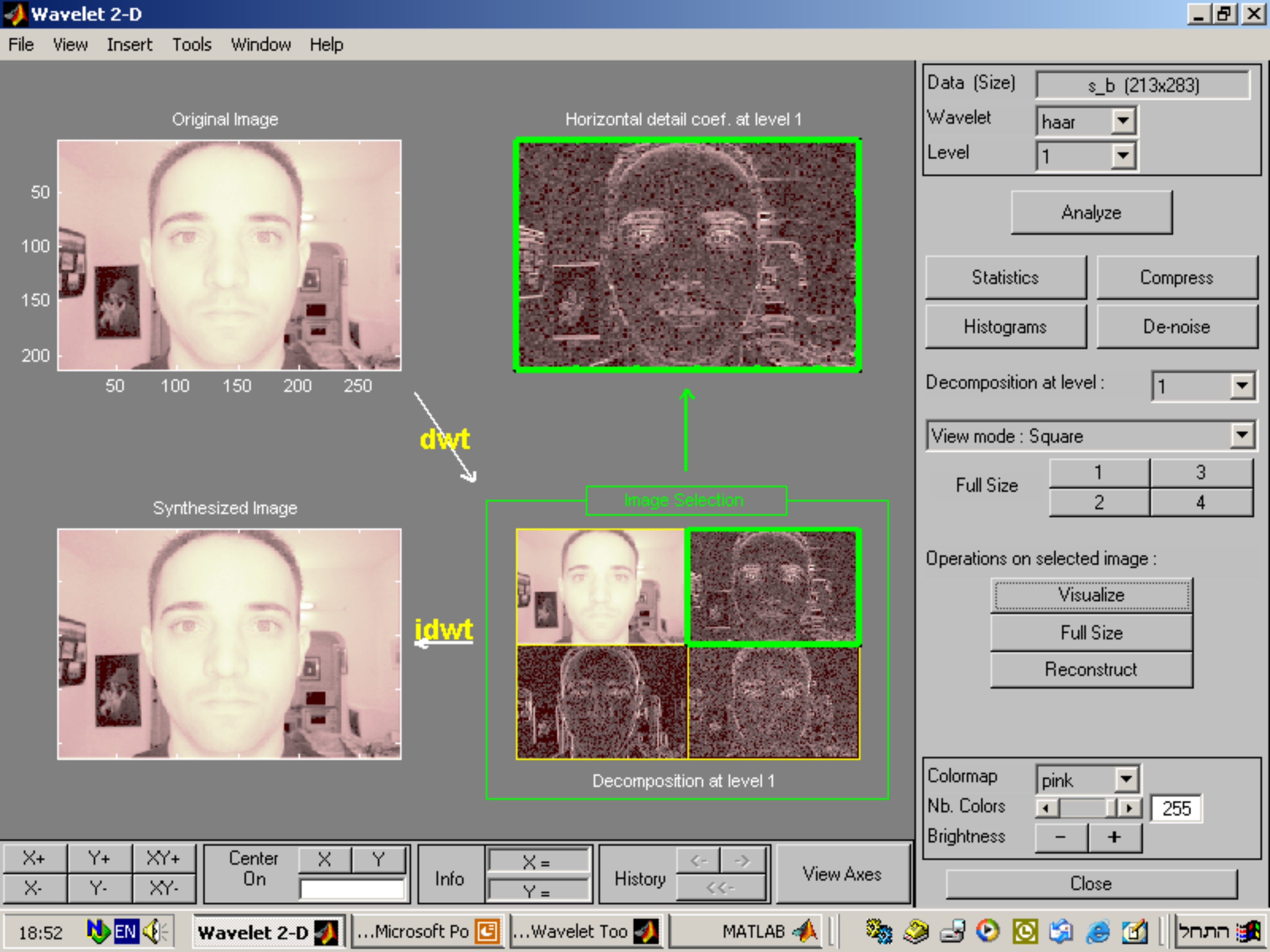

Wavelet Coefficients Selection 1-D
File View Insert Tools Window Help
Original Signal
Synthezised Signal
Original Coefficients
Selected Coefficients
A5
D5
D4
D3
D2
D1
Data (Size)
sumsin (1000)
Wavelet
db
3
Level
5
Analyze
Define Selection method
Global
App. cfs
Select All
Selected Biggest Coefficients
Initial
Kept
A5 36 36
D5 36 0
D4 67 40
D3 129 0
D2 253 0
D1 502 0
S 1023 76
Apply
Residuals
Show Original Signal
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
Close
19:07
EN
...Wavelet Co
...Microsoft Po
...Wavelet Too
MATLAB
התחל

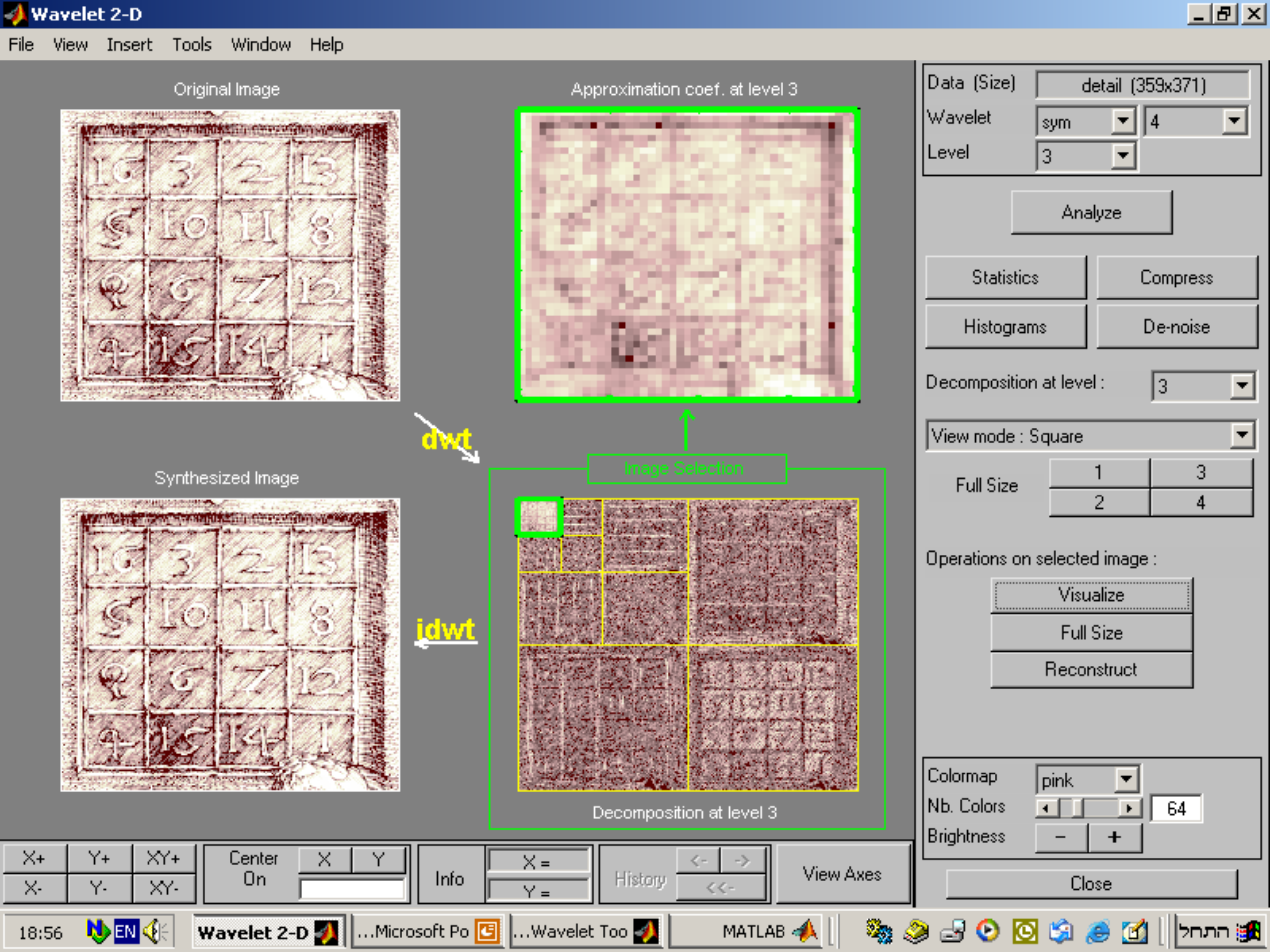

Wavelet Coefficients Selection 1-D
File View Insert Tools Window Help
Original Signal
Synthezised Signal
Original Coefficients
Selected Coefficients
A5
D5
D4
D3
D2
D1
Data (Size) sumsin (1000)
Wavelet db 3
Level 5
Analyze
Define Selection method
Global
App. cfs Select All
Selected Biggest Coefficients
Initial Kept
A5 36 36
D5 36 0
D4 67 0
D3 129 0
D2 253 0
D1 502 0
S 1023 36
Apply
go back
Close
X+ Y+ XY+
X- Y- XY-
Center On X Y
Info X = Y =
History
View Axes
19:08
...Wavelet Co
...Microsoft Po
...Wavelet Too
MATLAB

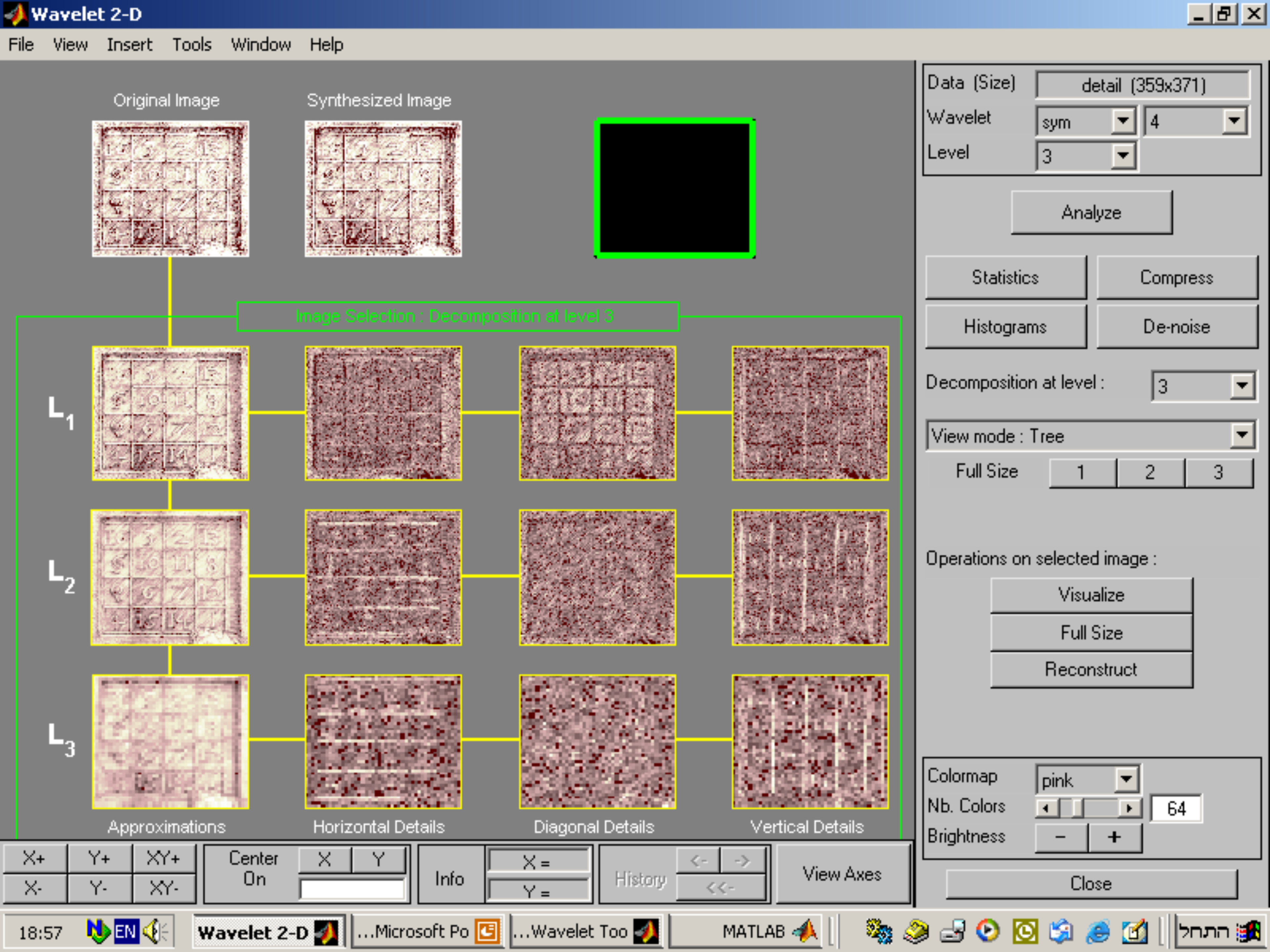

Wavelet 2-D
File View Insert Tools Window Help
Original Image
Approximation coef. at level 3
Data (Size)
s_b (213x283)
Wavelet
haar
Level
3
Analyze
Statistics
Compress
Histograms
De-noise
Decomposition at level :
3
View mode : Square
Full Size
1
2
3
4
Operations on selected image :
Visualize
Full Size
Reconstruct
dwt
idwt
Synthesized Image
Image Selection
Decomposition at level 3
Colormap
pink
Nb. Colors
255
Brightness
Close
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
18:50
Wavelet 2-D
...Microsoft Po
...Wavelet Too
MATLAB

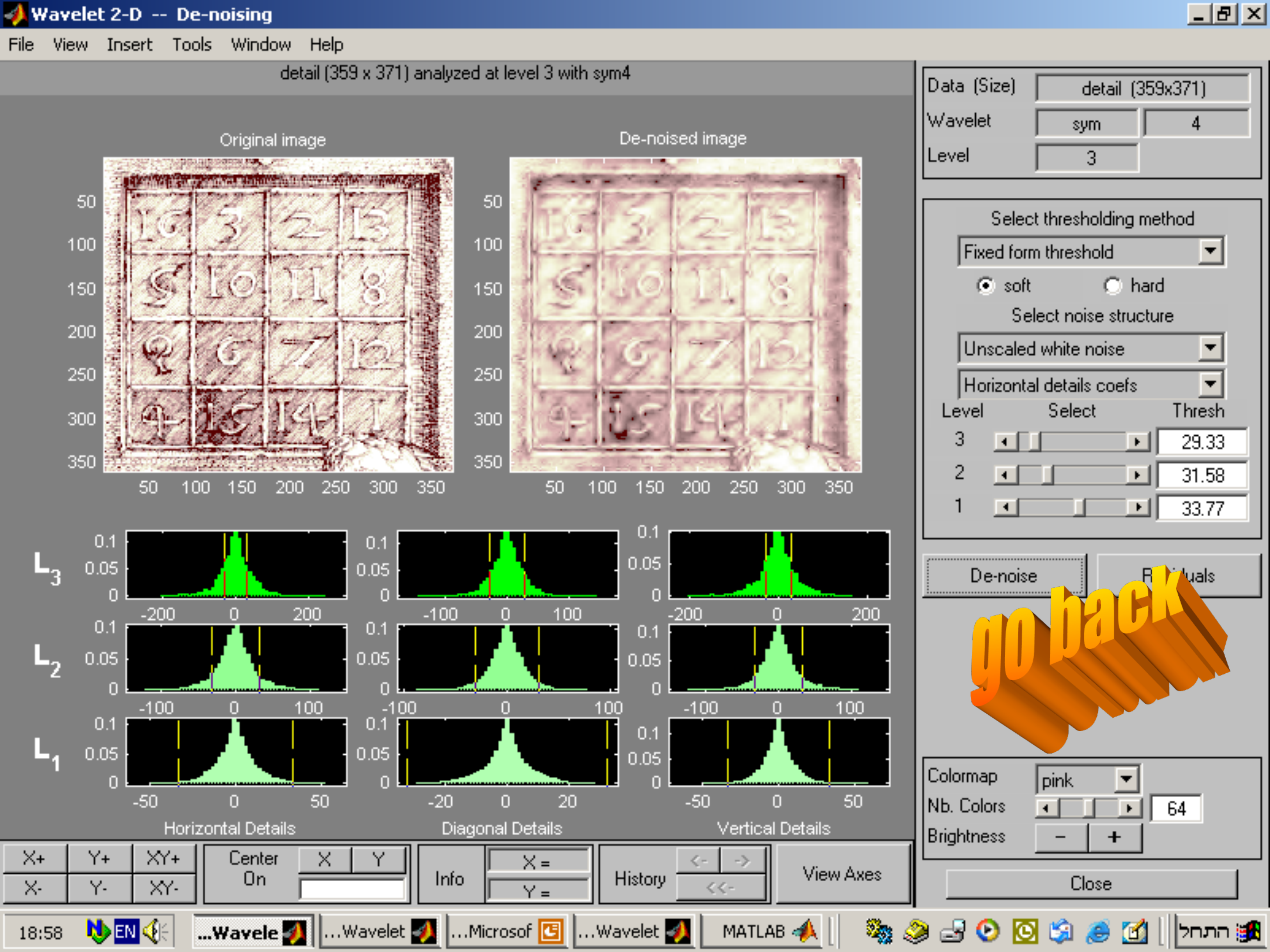

Wavelet 2-D
File View Insert Tools Window Help
Original Image
50
100
150
200
50 100 150 200 250
Horizontal detail coef. at level 1
dwt
Synthesized Image
Image Selection
idwt
Decomposition at level 3
Data (Size)
s_b (213x283)
Wavelet
haar
Level
3
Analyze
Statistics
Compress
Histograms
De-noise
Decomposition at level :
3
View mode : Square
Full Size
1
3
2
4
Operations on selected image :
Visualize
Full Size
Reconstruct
Colormap
pink
Nb. Colors
255
Brightness
-
+
Close
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
<- ->
<<-
View Axes
18:49
EN
Wavelet 2-D
...Microsoft Po
...Wavelet Too
MATLAB


Wavelet 2-D
File
View
Insert
Tools
Window
Help
Original Image
Synthesized Image
Image Selection : Decomposition at level 3
L1
L2
L3
Approximations
Horizontal Details
Diagonal Details
Vertical Details
Data (Size)
s_b (213x283)
Wavelet
haar
Level
3
Analyze
Statistics
Compress
Histograms
De-noise
Decomposition at level :
3
View mode : Tree
Full Size
1
2
3
Operations on selected image :
Visualize
Full Size
Reconstruct
Colormap
pink
Nb. Colors
255
Brightness
Close
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes
18:53
EN
Wavelet 2-D
...Microsoft Po
...Wavelet Too
MATLAB


Wavelet 2-D
File View Insert Tools Window Help
Original Image
Horizontal detail coef. at level 1
dwt
idwt
Synthesized Image
Image Selection
Decomposition at level 1
Data (Size)
s_b (213x283)
Wavelet
haar
Level
1
Analyze
Statistics
Compress
Histograms
De-noise
Decomposition at level :
View mode : Square
Full Size
Operations on selected image :
Visualize
Full Size
Reconstruct
Colormap
pink
Nb. Colors
255
Brightness
Close
Center On
Info
History
View Axes
18:52
Wavelet 2-D
...Microsoft Po
...Wavelet Too
MATLAB


Wavelet 2-D
File View Insert Tools Window Help
Original Image
Approximation coef. at level 3
Data (Size)
detail (359x371)
Wavelet
sym
4
Level
3
Analyze
Statistics
Compress
Histograms
De-noise
Decomposition at level :
3
View mode : Square
Full Size
1
3
2
4
Operations on selected image :
Visualize
Full Size
Reconstruct
dwt
idwt
Synthesized Image
Image Selection
Decomposition at level 3
Colormap
pink
Nb. Colors
64
Brightness
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes
Close
18:56
EN
Wavelet 2-D
...Microsoft Po
...Wavelet Too
MATLAB


Wavelet 2-D
File View Insert Tools Window Help
Original Image
Synthesized Image
Image Selection : Decomposition at level 3
L1
L2
L3
Approximations
Horizontal Details
Diagonal Details
Vertical Details
Data (Size)
detail (359x371)
Wavelet
sym
4
Level
3
Analyze
Statistics
Compress
Histograms
De-noise
Decomposition at level :
3
View mode : Tree
Full Size
1
2
3
Operations on selected image :
Visualize
Full Size
Reconstruct
Colormap
pink
Nb. Colors
64
Brightness
−
+
Close
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes
18:57
EN
Wavelet 2-D
...Microsoft Po
...Wavelet Too
MATLAB
התחל


Wavelet 2-D -- De-noising
File View Insert Tools Window Help
detail (359 x 371) analyzed at level 3 with sym4
Original image
De-noised image
Data (Size)
detail (359x371)
Wavelet
sym
4
Level
3
Select thresholding method
Fixed form threshold
soft
hard
Select noise structure
Unscaled white noise
Horizontal details coefs
Level
Select
Thresh
29.33
31.58
33.77
De-noise
Residuals
go back
Colormap
pink
Nb. Colors
64
Brightness
Close
Horizontal Details
Diagonal Details
Vertical Details
X+ Y+ XY+
X- Y- XY-
Center On
X Y
Info
X =
Y =
History
View Axes
18:58
EN
...Wavele
...Wavelet
...Microsof
...Wavelet
MATLAB