Lecture 1

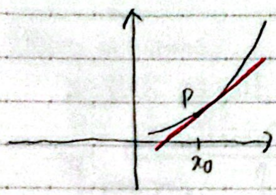
Date : August 3, 2025
Based on MIT OCW 18.01SC Video Lecture 1

UNIT 1 Differentiation

A. What is a derivative?
- the geometric interpretation
- physical interpretation
- importance of derivatives to all measurements (science, engineering, economics, political science, etc.)

B. How to differentiate any F'N you know

Geometric interpretation of derivatives



Target: To find the tangent line to $y=f(x)$ at $P=(x_0, y_0)$
→ the task: To figure out how to find the line analytically

☆ what we learned in high school:

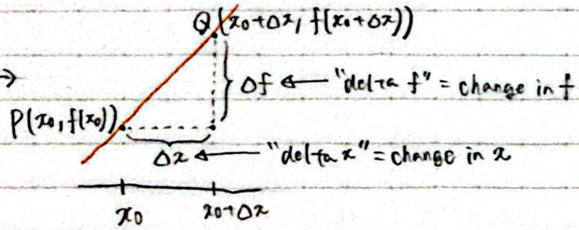
a tangent line has
- an equation
- any line through a point has the equation $y-y$
  ↳ has the equation $y-y_0=m(x-x_0)$

2 pieces of information to work out what the line is:
- point P
  ↳ given x, $y=f(x_0)$
- slope
  ↳ number $m=f'(x_0)$

Definition

$f'(x_0)$, the derivative of f at x_0, is the slope of the tangent line to $y=f(x)$ at P

Tangent line = limit of secant lines PQ as $Q \to P$ (P fixed)



→ Slope of the tangent line

$$\boxed{m} = \lim_{\Delta x \to 0} \boxed{\frac{\Delta f}{\Delta x}}$$

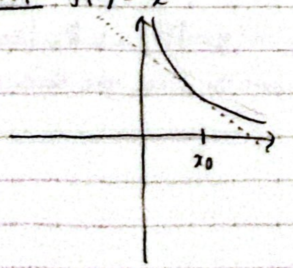
↳ Slope of secant

→ m

$$\boxed{\boxed{f'(x_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}$$

↳ difference quotient

Ex.1 $f(x) = \frac{1}{x}$



$$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0 + \Delta x} - \frac{1}{x_0}}{\Delta x}$$

$$= \frac{1}{\Delta x}\left(\frac{x_0 - (x_0 + \Delta x)}{(x_0 + \Delta x)x_0}\right)$$

$$= \frac{1}{\Delta x}\left(\frac{-\Delta x}{(x_0 + \Delta x)x_0}\right)$$

$$= \frac{-1}{(x_0 + \Delta x)x_0}$$

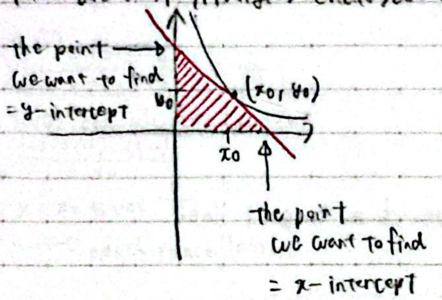
$$\xrightarrow[\Delta x \to 0]{} \frac{-1}{x_0^2}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{x} \Rightarrow f'(x_0) = \frac{-1}{x_0^2}}$$

↳ < 0 : negative slope

$x_0 \to \infty$: less & less steep

* Find areas of triangles enclosed by axes and tangent to $y = \frac{1}{x}$



$$(\ast)\quad y - y_0 = \frac{-1}{x_0^2}(x - x_0)$$

Find x-intercept $(y=0)$ ← $\underline{\underline{\text{not}}}$ $y = \frac{1}{x}$

$$0 - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0)$$
$$= \frac{-x}{x_0^2} + \frac{1}{x_0}$$
$$\Rightarrow \frac{x}{x_0^2} = \frac{2}{x_0}$$
$$\Rightarrow x = 2x_0$$

Shortcut to y-intercept:

use symmetry $y = 2y_0$

↳ symmetry explanation:

$y = \frac{1}{x} \Leftrightarrow xy = 1 \Leftrightarrow x = \frac{1}{y}$

(Can also y-intercept by plugging $x = 0$ into $(\ast)$)

More notations

$y = f(x)$, $\Delta y = \Delta f$

$$\underbrace{f'}_{\text{Newton's}} = \underbrace{\frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f = \frac{d}{dx}y}_{\text{Leipniz}} \quad (\text{omits } \Delta)$$

Ex. 2 $f(x) = x^n$, $n = 1, 2, 3, \dots$ ← (x is fixed, Δx moves)

task: to find $\frac{d}{dx}x^n$

$$\frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$
$$= \frac{1}{\Delta x}\{(x+\Delta x)^n - x^n\}$$
$$= \frac{1}{\Delta x}\left(x^n + nx^{n-1}\Delta x + O((\Delta x)^2) - x^n\right)$$
$$= \frac{1}{\Delta x}\left(nx^{n-1}\Delta x + O((\Delta x)^2)\right)$$
$$= nx^{n-1} + O(\Delta x)$$
$$\xrightarrow{\Delta x \to 0} nx^{n-1}$$

binomical theorem

$(x+\Delta x)^n = (x+\Delta x)\cdots(x+\Delta x)$

$= x^n + nx^{n-1}\Delta x + \text{junk}$

($O((\Delta x)^2)$ terms of order so with $(\Delta x)^2, (\Delta x)^3, \dots$)

$$\Rightarrow \boxed{\frac{d}{dx}x^n = nx^{n-1}}$$ → extends to polynomials!

$$\frac{d}{dx}(x^3 + 50x^{10}) = 3x^2 + 50x^9$$