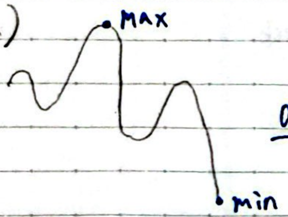
Date : August 29, 2025

Based on MIT OCW 18.01SC Video Lecture 11, 12

Maximum and Minimum problems

Ex.)



done!

⇒ Easy to find max and min with sketch!

Goal : short cuts!

Key to finding max + min

only need to look up at...

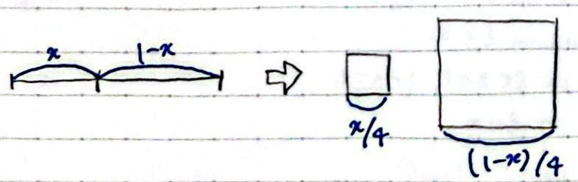
- critical points
- end points
- points of discontinuity

Ex. Wire length 1

↳ cut into 2 pieces, each pieces encloses a square

2 steps :

1. Draw diagram
2. Name variables



→

→ (Area)

$A = \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2$

$\Rightarrow A' = \frac{x}{8} - \frac{(1-x)}{8}$

$\Rightarrow$ when $A' = 0$:

$x = 1-x \Leftrightarrow 2x = 1 \Leftrightarrow x = \frac{1}{2}$

↑ critical point / value

$A\left(\frac{1}{2}\right) = \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 = \frac{1}{32}$

↳ Not done!

Check end points $0 < x < 1$

$A(0^+) = 0^+ + \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

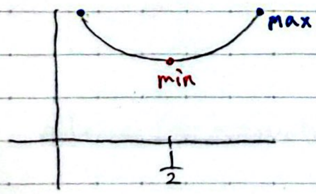
$A(1^-) = \left(\frac{1}{4}\right)^2 + 0 = \frac{1}{16}$

Least area enclosed: $\frac{1}{32}$ when $x = \frac{1}{2}$ (equal square)

What is the Minimum? $\Rightarrow 1/32$
(minimum value)

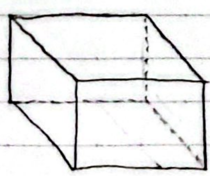
where is the Minimum? $\Rightarrow x = 1/2$ $\left(\frac{1}{2}, \frac{1}{32}\right)$
(minimum point)

Largest area enclosed: $\frac{1}{16}$ (in the limit $\begin{matrix} x \to 0^+ \\ x \to 1^- \end{matrix}$)



Ex. 2 Find the box without a top with least surface area for a fixed volume

$\Rightarrow$ Diagram, variables



$V = x^2 y$ ← constraint

$A = \underline{\underline{x^2}} + 4xy$

bottom

→

→

$y = \frac{V}{x^2}$

$\Rightarrow A = x^2 + 4x\left(\frac{V}{x^2}\right)$

$= x^2 + 4V/x$

$\Rightarrow A' = 2x - \frac{4V}{x^2}$

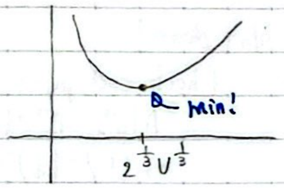
$\Rightarrow$ when $A' = 0$: critical point

$2x = \frac{4V}{x^2} \Rightarrow 2x^3 = 4V \Rightarrow x = 2^{\frac{1}{3}}V^{\frac{1}{3}}$

Ends, $0 < x < \infty$

$A(0^+) = x^2 + \frac{4V}{x}\Big|_{x=0^+} = \infty$

$A(\infty) = \infty$



☆ Alternative to checking ends: 2nd derivative test

$A'' = 2 + \frac{8V}{x^2} > 0$

Concave up $\Rightarrow$ critical point is a min point

$\Rightarrow$ minimum:

$x = 2^{\frac{1}{3}}V^{\frac{1}{3}}$

$y = \frac{V}{(2^{\frac{1}{3}}V^{\frac{1}{3}})^2} = 2^{-\frac{2}{3}}V^{\frac{1}{3}}$

$A = (2^{\frac{1}{3}}V^{\frac{1}{3}})^2 + \frac{4V}{2^{\frac{1}{3}}V^{\frac{1}{3}}}$

$= 3 \cdot 2^{\frac{1}{3}} \cdot V^{\frac{2}{3}}$

More meaningful answers: dimention was variable

↳ $A/V^{\frac{2}{3}} = 3 \cdot 2^{\frac{1}{3}}$

$x/y = \frac{2^{\frac{1}{3}}V^{\frac{1}{3}}}{2^{-\frac{2}{3}}V^{\frac{1}{3}}} = \boxed{2}$ best answer

optimal shape

Ex. 2 by implicit differentiation

$V = x^2y$, $A = x^2 + 4xy$

goal: min of A with V constant

→

$\rightarrow$

$\frac{d}{dx}(V = x^2 y) \Rightarrow 0 = 2xy + x^2 y'$

$\Rightarrow y' = \frac{-2xy}{x^2}$

$\Rightarrow -\frac{2y}{x}$

$\frac{dA}{dx} = 2x + 4y + 4xy'$

$= 2x + 4y + 4x\left(\frac{-2y}{x}\right)$

$\Rightarrow$ when $\frac{dA}{dx} = 0$:

$2x + 4y - 8y = 0$

$\Rightarrow 2x = 4y$

$\Rightarrow \boxed{x/y = 2}$ Faster and Nicer

disadvantage:

did not check whether this critical point in A max / min / neither