

Date: August 22, 2025

Based on MIT OCW 18.01SC Video Lecture 9

## UNIT 2 Applications of Differentiation

## Linear Approximations

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad (x \approx x_0)$$

↳ curve  $y = f(x) \approx y = f(x_0) + f'(x_0)(x - x_0)$   
tangent line

Example

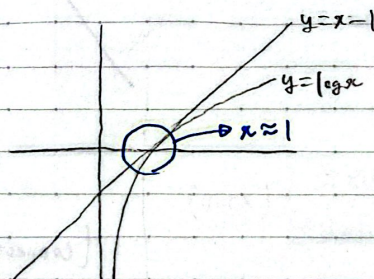
$$f(x) = \log x, \quad f'(x) = \frac{1}{x}$$

Same

$$x_0 = 1, \quad f(1) = \log 1 = 0, \quad f'(1) = 1$$

$$\Rightarrow \log x \approx 0 + 1 \cdot (x - 1)$$

$$\therefore \log x \approx x - 1$$



$$\lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

$$\therefore \frac{\Delta f}{\Delta x} \approx f'(x_0) \quad (x \approx x_0)$$

if  $\Delta x$  small

$$\Leftrightarrow \Delta f \approx f'(x_0) \Delta x$$

$$\Leftrightarrow f(x) - f(x_0) \approx f'(x_0)(x - x_0)$$

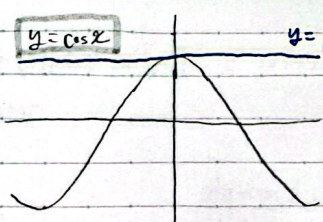
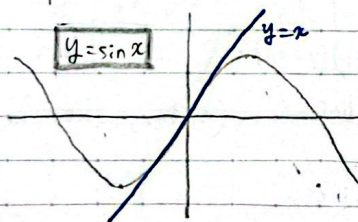
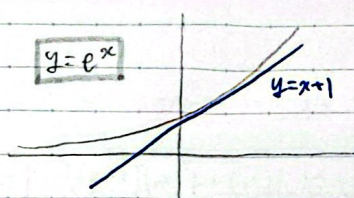
$$\Leftrightarrow f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$



$$(x_0 = 0) \quad f(x) \approx f(0) + f'(0)x \quad (x \approx 0)$$

$$\Rightarrow \begin{cases} \sin x \approx x \\ \cos x \approx 1 \\ e^x \approx 1+x \end{cases} \quad (x \approx 0)$$

	$f'$	$f(0)$	$f'(0)$
$\rightarrow \sin x$	$\cos x$	0	1
$\cos x$	$-\sin x$	1	0
$e^x$	$e^x$	1	1



$$\begin{cases} \log(1+x) \approx x \\ (1+x)^r \approx 1+rx \end{cases} \quad (x \approx 0)$$

[connect to  $\log u \approx u-1$ ,  $u = 1+x$ ,  $u-1 = x$ ]

	$f'$	$f(0)$	$f'(0)$
$\rightarrow \log(1+x)$	$1/(1+x)$	0	1
$(1+x)^r$	$r(1+x)^{r-1}$	1	r

Ex. 2

$$\log(1.1) \approx \frac{1}{10}$$

$$\uparrow \log(1+x) \approx x, \quad x = \frac{1}{10}$$

Ex. 3 Find linear approximation near  $x=0$  ( $x \approx 0$ ) of  $\frac{e^{-3x}}{\sqrt{1+x}}$

$$\frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x} (1+x)^{-\frac{1}{2}}$$

$$\approx (1-3x)(1-\frac{1}{2}x)$$

$$= 1 - 3x - \frac{1}{2}x + \frac{3}{2}x^2$$

$$\approx 1 - \frac{7}{2}x$$

drop  $x^2$  terms +  $x^3$  and higher

Quadratic Approximations

$$f(x) \approx f(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

Ex. 2 (compare to linear approximations)

Linear Approximations

$$\begin{aligned}\log(1.1) &\approx \frac{1}{10} \\ \uparrow \log(1+x) &\approx x, \\ x &= \frac{1}{10}\end{aligned}$$

Quadratic Approximations

$$\begin{aligned}\log(1.1) &\approx \frac{1}{10} \\ \uparrow \log(1.1) &= \log\left(1 + \frac{1}{10}\right) \\ &\approx \frac{1}{10} - \frac{1}{2}\left(\frac{1}{10}\right)^2 \\ &= 0.95\end{aligned}$$