No. 12

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Based on MIT OCW 18.01SC Video Lecture 5

Implicit differentiation

Example

$$\frac{d}{dx}x^a = ax^{a-1}$$

So far: $a = 0, \pm 1, \pm 2, \cdots$
Today: $a = m/n$, m and n integers

$y = x^{m/n}$ (1)
$\therefore y^n = x^m$ (2)
Apply $\frac{d}{dx}$ to equation (2)

$$\frac{d}{dx}y^n = \frac{d}{dx}x^m$$

$$\left(\frac{d}{dy}y^n\right)\frac{dy}{dx} = mx^{m-1}$$

$$\therefore ny^{n-1}\frac{dy}{dx} = mx^{m-1}$$

$$\therefore \frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}}$$

$$= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}}$$

$$= ax^{\boxed{m-1-(n-1)\frac{m}{n}}}$$

$$= ax^{\boxed{a-1}}$$

$$m-1-(n-1)\frac{m}{n} = m-1-m+\frac{m}{n}$$

$$= -1+\frac{m}{n}$$

$$= a-1$$

Example 2

$x^2+y^2=1$

$\therefore\ y^2=1-x^2,\ y=\pm\sqrt{1-x^2}$ (now consider positive branch)

explicit differentiation:

$y=(1-x^2)^{\frac{1}{2}}$

$\therefore\ y'=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x)$ ← $a=\frac{1}{2},\ a-1=-\frac{1}{2}$

$=\frac{-x}{\sqrt{1-x^2}}$

implicit differentiation:

$\frac{d}{dx}(x^2+y^2=1)$

$2x+2yy'=0$

$\therefore\ y'=\frac{-2x}{2y}=-x/y\underset{y=\sqrt{1-x^2}}{=}\frac{-x}{\sqrt{1-x^2}}$

Example 3

$y^4+xy^2-2=0$

explicit:

$y^2=\frac{-x\pm\sqrt{x^2-4(-2)}}{2}$

$\therefore\ y=\pm\sqrt{\frac{-x\pm\sqrt{x^2+8}}{2}}$ (*)

implicit:

$4y^3y'+y^2+x(2yy')-0=0$

$\therefore\ (4y^3+2xy)y'=-y^2$

$\therefore\ y'=\frac{-y^2}{4y^3+2xy}$

Ex. at $x=1,\ y=1$ = at $(1,1)$ along the curve

slope $=\frac{-1^2}{4\cdot1^3+2\cdot1\cdot1}=\frac{-1}{6}$

But at, say, $x=2$

we're stuck using (*) to find y

Inverse function

Example

$y=\sqrt{x},\ x>0,\ y^2=x$

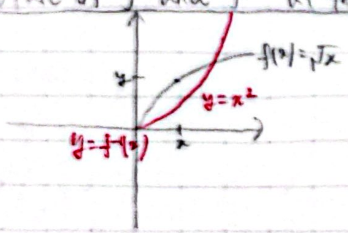
$f(x)=\sqrt{x},\ g(y)=x,\ g(y)=y^2$

$\Rightarrow$ In general:

$y=f(x),\ g(y)=x$

$g(f(x))=x,\ g=f^{-1},\ f=g^{-1}$

Picture of f and f^{-1} at the same graph



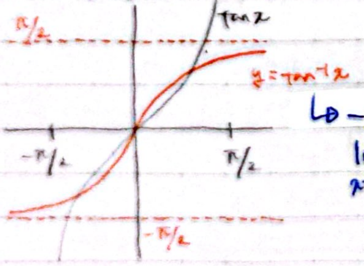
from $g(y)=x$ to $g(x)=?$:

Exchange x and y

Implicit differentiation allows us to find the derivative of any inverse function provided we know the derivative of the function

Example $y=\tan^{-1}x$ [$\arctan x$]

→ use $\tan y = x$



↳ $-\infty < x < \infty$

$\lim_{x\to\infty} \tan^{-1}x = \frac{\pi}{2}$

Recall

$$\frac{d}{dy}\tan y = \frac{d}{dy}\frac{\sin y}{\cos y} = \cdots = \frac{1}{\cos^2 y} = \sec^2 y$$

$\Rightarrow \dfrac{d}{dx}(\tan y = x)$

$\therefore \left(\dfrac{d}{dy}\tan y\right)\dfrac{dy}{dx} = 1$

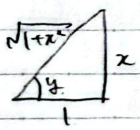
$\therefore \dfrac{1}{\cos^2 y}\cdot y' = 1$

$\therefore y' = \cos^2 y$

$\therefore \dfrac{d}{dx}\tan^{-1} x = \cos^2 y\,(\tan^{-1} x)$ ← correct

but way too complicated

$\Rightarrow \tan y = x \ (y = \tan^{-1} x)$



$\cos y = \dfrac{1}{\sqrt{1+x^2}}$

$\therefore \cos^2 y = \dfrac{1}{1+x^2}$

$$\boxed{\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}}$$