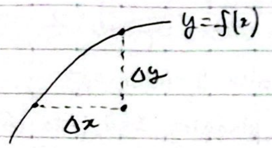
Date: August 4, 2025
Based on MIT OCW 18.01SC Video Lecture 2

☆ what we did last time:
- difinition of the derivative as the slope of a tangent line
- $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$
- $\frac{d}{dx}x^n = nx^{n-1}$ $(n = 1, 2, \dots)$

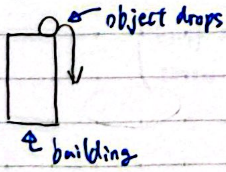
What is a derivative? (continuation of last time)
↳ Last time - geometric interpretations
Today: rate of change as an interpretation of the derivative



$\boxed{\frac{\Delta y}{\Delta x}} \longrightarrow \boxed{\frac{dy}{dx}}$

instantaneous rate
average change

Examples (from physics)
1. Q = charge, $\frac{dQ}{dt}$ = current
2. S = distance, $\frac{ds}{dt}$ = speed
   ☆ Pumpkin drop



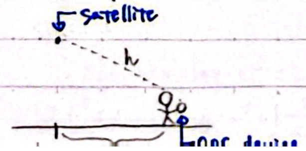
$h = 80 - 5t^2$ meters.
↳ $t = 0, h = 80$;
$t = 4, h = 0$

average speed: $\frac{\Delta h}{\Delta t} = \frac{0 - 80}{4 - 0}$
$= -20$ m/sec

final position
initial position

instaneous speed:
$\frac{d}{dt}h = 0 - 10t$ $\left(\because \frac{d}{dt}80 = 0, \frac{d}{dt}t^2 = 2t\right)$
↳ at final position: $t = 4$, $h' = -40$ m/s

3. T = temperature
   $\frac{dT}{dx}$ = temperature gredient
4. sensitivity of measurements
   ☆ GPS (from problem set 1)



h measured by radio waves/clock
L deduced from h
Error in h: Δh
$\Delta L \approx dL$ "easy"

Limits + continuity

1. Easy limits

ex.) $\lim_{x \to 4} \frac{x+3}{x^2+1} = \frac{4+3}{4^2+1} = \frac{7}{17}$

2. Derivatives are always harder:

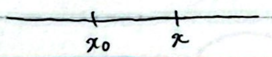
$\lim_{x \to x_0} \frac{f(x_0+\Delta x) - f(x_0)}{x - x_0}$

↳ $x = x_0$ gives $\frac{0}{0}$

→ need cancellation

$\lim_{x \to x_0^+} f(x)$ = right-hand limit

↳ $x \to x_0$ (in range $x > x_0$)

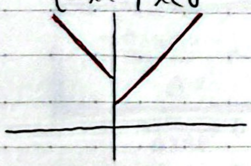


$\lim_{x \to x_0^-} f(x)$ = left-hand limit

↳ $x \to x_0$ (in range $x < x_0$)

Example

$$f(x) = \begin{cases} x+1, & x > 0 \\ -x+2, & x < 0 \end{cases}$$



$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x+1 = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} -x+2 = 2$

→ Did not need $x=0$ value

in this case $x=0$ value wasn't defined

※ if $f(x) = \begin{cases} x+1, & x \geq 0 \\ -x+2, & x < 0 \end{cases}$ ⇒ $x=0$ value = right-hand value ($x \to 0$)

Definition

f is continuous at x_0 means

$\lim_{x \to x_0} f(x) = f(x_0)$ ← "easy" limits

& $x \neq x_0$

↳ 1. $\lim_{x \to x_0} f(x)$ exists (from Right-hand limit = Left-hand limit)

2. $f(x_0)$ is defined

3. they are equal

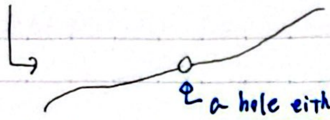
examples of discontinuity

1. Jump discontinuity

lim from left and right exist but are not equal

2. Removable discontinuity

lim from left and right are equal



a hole either...

- the function is undefined
- " defined up here

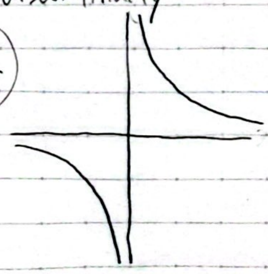
$\boxed{\text{example}}$ $g(x) = \frac{\sin x}{x}$, $h(x) = \frac{1-\cos x}{x}$

removable discontinuity at $x=0$

☆ $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1, \lim_{x\to 0} \frac{1-\cos x}{x} = 1}$

3. Infinite discontinuity

$y = \frac{1}{x}$



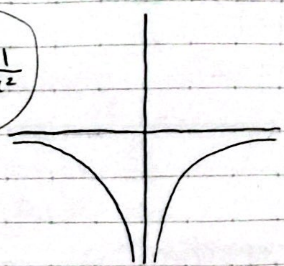
$\lim_{x\to 0^+} \frac{1}{x} = \infty$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\therefore$ ~~$\lim_{x\to 0} \frac{1}{x} = \infty$~~

odd function

$y = \frac{-1}{x^2}$



$\lim_{x\to 0} \frac{-1}{x^2} = -\infty$

↳ both $x \to 0^+$ and $x \to 0^-$ are $-\infty$

↳ even function

4. other (ugly) discontinuities

$y = \sin\frac{1}{x}$ as $x \to 0$: no left or right limit

Theorem (Differentiable implies continuous)

If f is differentiable at x_0, then f is continuous at x_0

↳ Proof $\lim_{x\to x_0} f(x) - f(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}(x-x_0) = f'(x_0)\cdot 0 = 0$

$x \neq x_0$