Date: August 4, 2025
Based on MIT OCW 18.01 SC Video Lecture 3

Derivative formulas
- Specific formulas
- General formulas

Specific formulas

$$\boxed{\frac{d}{dx}\sin x = \cos x,\quad \frac{d}{dx}\cos x = -\sin x}$$

$$\frac{d}{dx}\sin x = \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

Sum formula for sine function
$\sin(a+b) = \sin a\cos b + \cos a\sin b$

$$= \frac{\sin x\cos\Delta x + \cos x\sin\Delta x - \sin x}{\Delta x}$$

$$= \boxed{\sin\left(\frac{\cos\Delta x - 1}{\Delta x}\right)} + \boxed{\cos x\left(\frac{\sin\Delta x}{\Delta x}\right)}$$

$\xrightarrow{\Delta x \to 0} 0$ $\qquad$ $\xrightarrow{\Delta x \to 0} 1$

$\xrightarrow{\Delta x \to 0} \cos x$

$$\frac{d}{dx}\cos x = \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$

$$= \frac{\cos x\cos\Delta x - \sin x\sin\Delta x - \cos x}{\Delta x}$$

$$= \boxed{\cos\left(\frac{\cos\Delta x - 1}{\Delta x}\right)} + \boxed{(-\sin x)\left(\frac{\sin\Delta x}{\Delta x}\right)}$$

$\xrightarrow{\Delta x \to 0} -\sin x$

No. 8

Date

Remarks

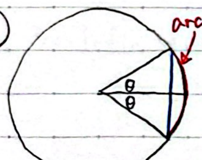
$$\left.\frac{d}{dx}\cos x\right|_{x=0} = \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = 0 \quad \text{(A)}$$

※ θ : in radians

$$\left.\frac{d}{dx}\sin x\right|_{x=0} = \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{(B)}$$

→ Geometric proof

(B)



$$\frac{2\sin\theta}{2\theta} = \frac{\sin\theta}{\theta} \xrightarrow[\theta \to 0]{} 1$$

↑ bow string (2 sin θ)

↑ bow (2θ)

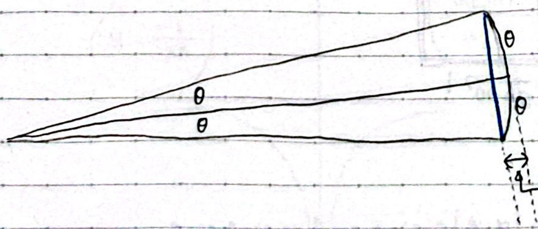
↑ θ get smaller

→ the curved piece looks more like a straight one

Principle

Short pieces of curves are nearly straight =

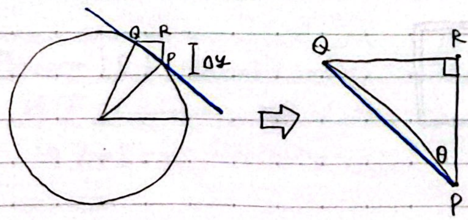
(A)



$$\frac{1-\cos\theta}{\theta} = -\left(\frac{1-\cos\theta}{\theta}\right) \to 0$$

as $\theta \to 0$

Geometric proof of $\frac{d}{dx}\sin\theta = \cos\theta$ (all θ)



$\widehat{PQ} \approx PQ$

$PQ \approx \Delta\theta$

$PQ \perp OP$; PR vertial

$\angle QPR \approx \theta$

$\therefore \frac{\Delta y}{\Delta\theta} \approx \cos\theta$

$\lim_{\Delta\theta \to 0} \frac{\Delta y}{\Delta\theta} = \cos\theta$

$y = \sin\theta$

vertial position of circular motion

General Derivative Rules

Product rule:

$$(uv)' = u'v + uv'$$ ← change at a time

Quotient rule:

$$\left(\frac{v}{u}\right)' = \frac{u'v - uv'}{v^2} \quad (v \neq 0)$$

Date: August 5, 2025

Based on MIT OCW 18.01 SC Video Lecture 4

$$\frac{d}{dt}(cu) = c\frac{du}{dt}$$

$$\frac{d}{dt}(u+v) = \frac{du}{dt} + \frac{dv}{dt}$$

$$(u+v)' = u' + v'$$

Product rule

$$\boxed{(uv)' = u'v + uv'}$$

ex.) $\frac{d}{dx}(x^n \sin(x)) = nx^{n-1}\sin(x) + x^n \cos(x)$

proof $\Delta(uv)$

$= u(x+\Delta x)v(x+\Delta x) - u(x)v(x)$

$= (u(x+\Delta x) - u(x))v(x+\Delta x)$ ← $u(x+\Delta x)v(x+\Delta x) - u(x)v(x+\Delta x)$

$\quad + u(x)(v(x+\Delta x) - v(x))$ ← $u(x)v(x+\Delta x) - u(x)v(x)$ Cancel

$= (\Delta u)v(x+\Delta x) + u(x)\Delta v$

$$\frac{\Delta(uv)}{\Delta x} = \frac{\Delta u}{\Delta x}v(x+\Delta x) + u\frac{\Delta u}{\Delta x}$$

$\Delta x \to 0 \downarrow$

$$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

Date

Quotient rule

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

proof
$$\Delta\left(\frac{v}{u}\right) = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}$$
$$= \frac{uv + (\Delta u)v - uv + u\Delta v}{(v+\Delta v)v}$$
$$= \frac{(\Delta u)v - u\Delta v}{(v+\Delta v)v}$$
$$\frac{\Delta\left(\frac{v}{u}\right)}{\Delta x} = \frac{\frac{\Delta u}{\Delta x}v - u\frac{\Delta v}{\Delta x}}{(v+\Delta v)v}$$
$\downarrow \Delta x \to 0$
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\cdot v - u\cdot\frac{dv}{dx}}{v\cdot v}$$

ex.) $u=1$
$$\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{-1\cdot v'}{v^2} = -v^{-2}v'$$

Subexample) $u=1$, $v=x^n$, $n=0, \pm1, \pm2, \pm3, \ldots$
$$\frac{d}{dx}x^{(-n)} = \frac{d}{dx}\left(\frac{1}{x^n}\right)' = -x^{-2n}nx^{n-1}$$
$$= (-n)x^{(-n)-1}$$

Composition rule

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta t}$$ cancel

$\downarrow \Delta t \to 0$
$$\boxed{\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}}$$ Chain rule

↑ Differentiation of a composition is a product

ex.) $(\sin t)^{10} = \sin^{10}(t) \longrightarrow \underset{\text{inside}}{x=\sin(t)},\ \underset{\text{outside}}{y=x^{10}}$
$$\frac{d}{dt}(\sin t)^{10} = 10x^9\cdot\cos t = 10(\sin(t))^9\cos(t) = 10\sin^9(t)\cos t$$

Higher Derivatives

$u = u(x)$, $u' \to u'' = (u')'$

ex.) $u = \sin(x)$

$u' = \cos(x)$

$u'' = -\sin(x)$

$u''' = (u'')' = -\cos(x)$ ← third derivative

$u^{(4)} = (u''')' = \sin(x)$

Other notation: "operator", applied to a function

$$u' = \frac{du}{dx} = \left(\frac{d}{dx}\right)u = Du$$

$$u'' = \frac{d}{dx}\frac{du}{dx} = \frac{d}{dx}\frac{d}{dx}u = \left(\frac{d}{dx}\right)^2 u = \frac{d^2}{(dx)^2}u = \frac{d^2u}{\underbrace{dx^2}_{\text{not } d(x^2)}} = D^2u$$

$$u''' = \frac{d^3u}{dx^3} = D^3u \text{ etc.}$$

ex.) $D^n x^n = ?$ $(n = 1, 2, 3, \dots)$

$Dx^n = nx^{n-1}$

$D^2x^n = n(n-1)x^{n-2}$

$D^3x^n = n(n-1)(n-2)x^{n-3}$

$\vdots$

$D^{n-1}x^n = n(n-1)\cdots 2x^1$ ← constant function

$D^n x^n = \underbrace{(n(n-1)\cdots 2 \cdot 1)}_{n\text{-factorial}} \cdot 1$

$n!$

$\boxed{D^n x^n = n!}$

$* \; D^{n+1}x^n = 0$