Date: August 22, 2025
Based on MIT OCW 18.01SC Video Lecture 9, 10, 11

UNIT 2 Applications of Differentiation

Linear Approximations

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)} \quad (x \approx x_0)$$

→ curve $y = f(x) \approx y = f(x) + f'(x_0)(x - x_0)$
tangent line

Same

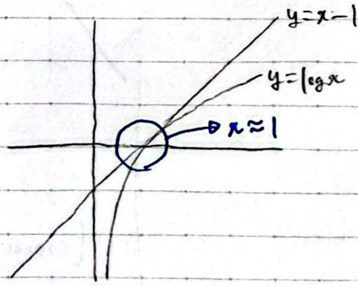
Example

$f(x) = \log x, \quad f'(x) = \frac{1}{x}$

$x_0 = 1, \ f(1) = \log 1 = 0, \ f'(1) = 1$

$\Rightarrow \log x \approx 0 + 1 \cdot (x - 1)$

$\therefore \log x \approx x - 1$



$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

$$\therefore \boxed{\frac{\Delta f}{\Delta x} \approx f'(x_0)} \quad (x \approx x_0)$$

& Δx small

$\Leftrightarrow \Delta f \approx f'(x_0) \Delta x$

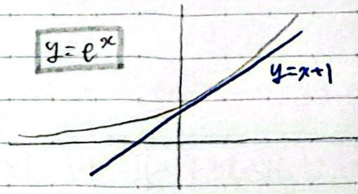
$\Leftrightarrow f(x) - f(x_0) \approx f(x_0)(x - x_0)$

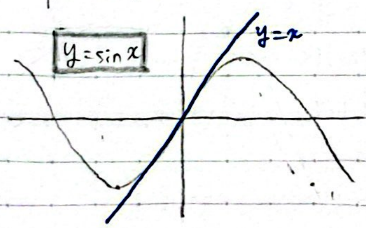
$\Leftrightarrow f(x_0) \approx f(x_0) + f'(x_0)(x - x_0)$

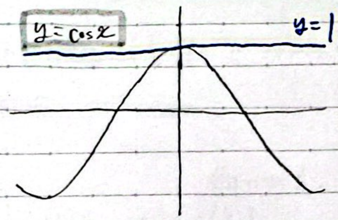
$(x_0 = 0)$ $\boxed{f(x) \approx f(0) + f'(0)x}$ $(x \approx 0)$

$$\Rightarrow \boxed{\begin{array}{l} \sin x \approx x \\ \cos x \approx 1 \\ e^x \approx 1 + x \end{array}} \quad (x \approx 0)$$

	f'	$f(0)$	$f'(0)$
$\sin x$	$\cos x$	0	1
$\cos x$	$-\sin x$	1	0
e^x	e^x	1	1







$$\boxed{\begin{array}{l} \log(1+x) \approx x \\ (1+x)^r \approx 1 + rx \end{array}} \quad (x \approx 0)$$

[connect to $\log u \approx u-1$, $u = 1+x$, $u-1 = x$]

	f'	$f(0)$	$f'(0)$
$\log(1+x)$	$1/(1+x)$	0	1
$(1+x)^r$	$1-(1+x)^{r-1}$	1	r

Ex. 2

$\log(1.1) \approx \frac{1}{10}$

$\Leftarrow \log(1+x) \approx x$, $x = \frac{1}{10}$

Ex. 3 Find linear approximation near $x=0$ $(x \approx 0)$ of $\frac{e^{-3x}}{\sqrt{1+x}}$

$$\frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-\frac{1}{2}}$$

$$\approx (1-3x)\left(1-\tfrac{1}{2}x\right)$$

$$= 1 - 3x - \tfrac{1}{2}x + \tfrac{3}{2}x^2$$

$$\approx 1 - \tfrac{7}{2}x$$

drop x^2 term(s) + x^3 and higher

Quadratic Approximations

$$\boxed{f(x) \approx f(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2}$$

Ex. 2 (compare to linear approximations)

Linear Approximations	Quadratic Approximations
$\log(1.1) \approx \frac{1}{10}$ $\uparrow \log(1+x) \approx x,$ $x = \frac{1}{10}$	$\log(1.1) \approx \frac{1}{10}$ $\uparrow \log(1.1) = \log\left(1+\frac{1}{10}\right)$ $\approx \frac{1}{10} - \frac{1}{2}\left(\frac{1}{10}\right)^2$ $= 0.95$

→ (use when linear approximation is not enough)

$$\boxed{f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2} \quad (x \approx 0)$$

$\boxed{\text{why } \frac{1}{2}f''(0)?}$

$f(x) = a + bx + cx^2$

$f'(x) = b + 2cx$

$f''(x) = 2c$

recover a, b, c

$\Rightarrow$

$f(0) = a$

$f'(0) = b$

$\frac{1}{2}f''(0) = c$

Ex.

$\sin x \approx x$

$\cos x \approx 1 - \frac{1}{2}x^2$

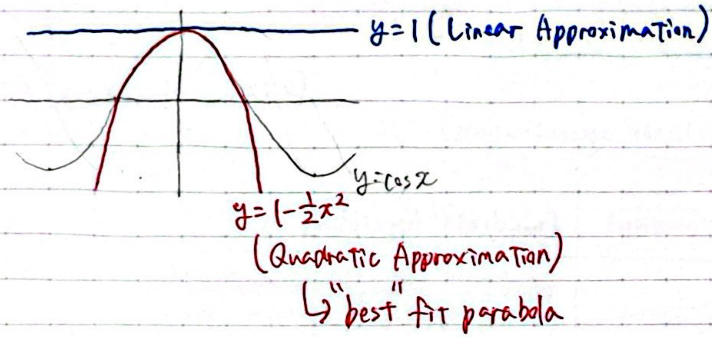
$e^x \approx 1 + x + \frac{1}{2}x^2$

$\log(1+x) \approx x - \frac{1}{2}x^2$

$(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$

	f''	$f''(0)$
$\sin x$	$-\sin x$	0
$\cos x$	$-\cos x$	-1
e^x	e^x	1

Geometric significance of the quadratic term



Ex.

$a_k = \left(1+\frac{1}{k}\right)^k \longrightarrow e \ (k \to \infty)$

$\log a_k = k\left(\log\left(1+\frac{1}{k}\right)\right)$

$\approx k\left(\frac{1}{k}\right) = 1$

$\uparrow$

$\log(1+x) \approx x \quad \left(x = \frac{1}{k} \approx 0\right)$ ($k \to \infty$)

rate of convergence

$\underbrace{\log a_k - 1}_{\text{how big is this?}} \longrightarrow 0$

$\Rightarrow$ use quadratic approximation

Ex. Find quadratic approximation for x near 0 to $e^{-3x}(1+x)^{-\frac{1}{2}}$

$$e^{-3x}(1+x)^{-\frac{1}{2}} \approx \left(1+(-3x)+\frac{(-3x)^2}{2}\right)\left(1-\frac{1}{2}x+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2\right)$$

$$\approx \underbrace{1}_{①}\underbrace{-3x}_{②}\underbrace{-\frac{1}{2}x}_{③}\underbrace{+\frac{3}{2}x^2}_{④}\underbrace{+\frac{9}{2}x^2}_{⑤}\underbrace{-\frac{3}{8}x^2}_{⑥}$$

$$= 1-\frac{7}{2}x+\frac{51}{8}x^2$$

Curve Sketching

Goal: Draw graph of f using f', f'' positive/negative

warning Don't abandon your precalculus skills and common sense

$f' > 0 \Rightarrow f$ is increasing
$f' < 0 \Rightarrow f$ is decreasing.

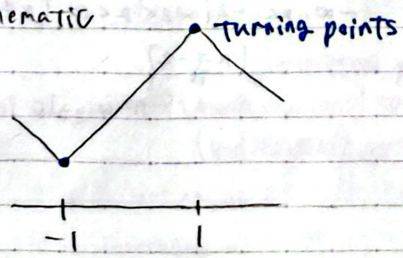
$f'' > 0 \Rightarrow f'$ is increasing
↳ f concave up
$f'' < 0 \Rightarrow f$ concave down

Ex. 1 $f(x) = 3x - x^3$

$f'(x) = 3 - 3x^2$
$= 3(1-x)(1+x)$
$\Rightarrow$ when $-1 < x < 1$: $f'(x) > 0$
f increasing
when $x > 1$, $x < -1$: $f'(x) < 0$
f decreasing

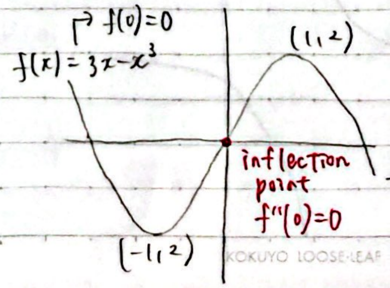
☆ Definition
If $f'(x_0) = 0$ we call x_0 a critical point.
$y_0 = f(x_0)$ is a critical value

$\Rightarrow$ schematic



$\boxed{\text{plot critical points/values}}$

$f(1) = 3 \cdot 1 - 1^2 = 2$
$f(-1) = 3 \cdot (-1) - (-1)^3 = -2$



function to be odd
$f(0) = 0$

$\rightarrow$

$\boxed{\text{Ends } (x \to \pm\infty)}$

$f(x) = 3x - x^3 \approx -x^3 \to -\infty \quad (x \to +\infty)$

$f(x) \to +\infty$ if $x \to -\infty$

$f''(x) = (3 - 3x^2)' = -6x$

$f''(x) < 0$ if $x > 0$ (concave down)

$f''(x) > 0$ if $x < 0$ (concave up)

Ex.) $f(x) = \frac{x+1}{x+2}$

$\Rightarrow f'(x) = \frac{1}{(x+2)^2}$

$\hookrightarrow f'(x) \neq 0$

$\Rightarrow$ no critical point

$\Rightarrow$ instead of this: plot points $x = -2$

$f((-2)^+) = \frac{-2+1}{(-2)^+ + 2} = \frac{-1}{0^+} = -\infty$

$f((-2)^-) = \frac{-2+1}{(-2)^- + 2} = \frac{-1}{0^-} = \infty$

$\boxed{\begin{matrix} a^+ : x > a \\ a^- : x < a \end{matrix}}$

$\boxed{\text{Ends}}$ $x \to \pm\infty$

$f(x) = \frac{x+1}{x+2} = \frac{1 + 1/x}{1 + 2/x} \to 1$

$\boxed{\text{Doublecheck}}$

$\frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$

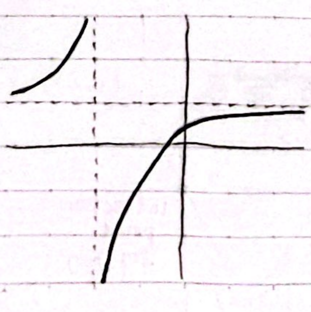
$f'(x) = \frac{1}{(x+2)^2} > 0 \Rightarrow$ ok

$\hookleftarrow$ f increasing on $-\infty < x < -2,\ -2 < x < \infty \quad (x \neq -2)$

$f''(x) = \frac{-2}{(x+2)^3} \quad (x \neq -2)$

$f''(x) < 0$ when $-2 < x < \infty$ (concave down) no wiggle in graph of f

$f''(x) > 0$ when $-\infty < x < -2$ (concave up)



General Strategy of sketching

1 Plot a) discontinuities especially infinite
 b) end points (or $x \to \pm\infty$)
 c) easy points (optional)

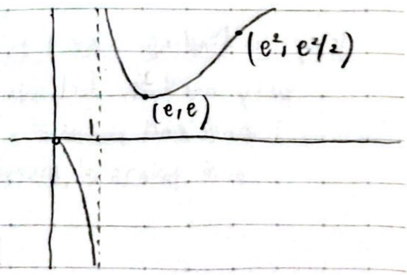
2 a) solve $f'(x)=0$
 b) Plot critical points and values

3 Decide whether f' is positive or negative — double check
 on each interval between critical points/discontinuities
 (※ consistent with 1, 2)

4 f'' is positive or negative
 $f''(x)=0 \Leftrightarrow x_0$ inflection point

5 Combine

Ex.) $f(x)=\frac{x}{\log x}$ $(x>0)$



1 a) $f(1^+)=\frac{1}{\log 1^+}=\frac{1}{0^+}=\infty$
 $f(1^-)=-\infty$

 b) $f(0^+)=\frac{0^+}{\log 0^+}=\frac{0^+}{-\infty}=0$
 $f(\infty)=\infty$ $\boxed{ex}$ $f(10^{10})=\frac{10^{10}}{\log 10^{10}}=\frac{10^{10}}{10\log 10}\gg 1$
 very big

2 $f'(x)=\frac{1\cdot\log x - x\cdot\frac{1}{x}}{(\log x)^2}$
 $=\frac{(\log x)-1}{(\log x)^2}$

 $f'(x)=0 \Rightarrow x=e$ — critical point
 $f(e)=e/\log e=e$

3 f is decreasing on $0<x<1$, $1<x<e$
 f is increasing on $e<x$
 ↳ $f'(x)<0$ on $0<x<1$, $1<x<e$
 $f'(x)>0$ on $e<x$
 $f'(0^+)=\frac{1}{\log 0^+}-\frac{1}{(\log 0^+)^2}$
 $=\frac{1}{-\infty}-\frac{1}{(\infty)^2}=0$

4 $f''(x)=-(\log x)^{-2}\frac{1}{x}$
 $+2(\log x)^{-3}\frac{1}{x}$
 $=\frac{2-\log x}{x(\log x)^3}$

 $\Rightarrow f''<0$ when $0<x<1$, $e^2<x$
 (concave down)
 $f''>0$ when $1<x<e^2$
 (concave up)