No. 16

Date

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Based on MIT OCW 18.01SC Video Lecture 6

Exponential and Logarithms

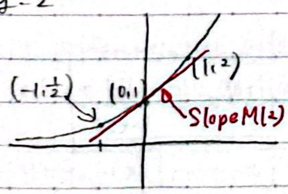
$a>0$ (base) $\Rightarrow a^0=1,\ a^1=a,\ a^2=a\cdot a$, etc.

$$\boxed{\begin{aligned} a^{x_1+x_2} &= a^{x_1}a^{x_2} \\ (a^{x_1})^{x_2} &= a^{x_1 x_2} \end{aligned}}$$

$a^{m/n} = \sqrt[n]{a^m}$ ← definition

$\Rightarrow a^x$ defined for all x by "filling in" by continuity

Ex, $y=2^x$



Goal $\dfrac{d}{dx}a^x = ?$

$a^{x+\Delta x} = a^x a^{\Delta x}$

$$\lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x} = \lim_{\Delta x\to 0} a^x \frac{a^{\Delta x}-1}{\Delta x}$$

$$\therefore \frac{d}{dx}a^x = a^x \lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}$$

x is fixed → constant times

$\Rightarrow$ to define $M(a)$:

$$M(a) = \lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}$$

$$\Rightarrow \frac{d}{dx}a^x = M(a)a^x$$

Plug in $x=0$:

$$\left.\frac{d}{dx} a^x \right|_{x=0} = M(a) a^0 = M(a)$$

in the graph of p.16

$\Rightarrow M(a)$ is $\boxed{\text{the slope}}$ of a^x at $x=0$

What is $M(a)$? Beg the question

Define base e as the unique number so that $M(e)=1$

$\hookrightarrow \boxed{\frac{d}{dx} e^x = e^x}$

$$\left.\frac{d}{dx} e^x \right|_{x=0} = 1$$

$\hookrightarrow$ Slope 1 at $x=0$

Why e exists :

$f(x) = 2^x$, $f'(0) = M(2)$

$\Downarrow$ stretch by k

$f(kx) = 2^{kx} = (2^k)^x = b^x \quad (b = 2^k)$

$$\frac{d}{dx} b^x = \frac{d}{dx} f(kx) = k f'(kx)$$

$$\left.\frac{d}{dx} b^x \right|_{x=0} = k f'(0) = k M(2)$$

$b = e$ when $k = 1/M(2)$

Natural log

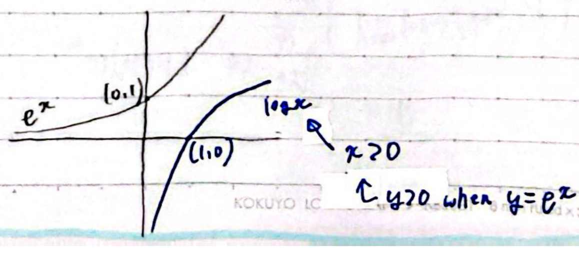
$w = \log x$

$y = e^x \Leftrightarrow \log y = x$

$\hookrightarrow$ inverse function

$\log(x_1 x_2) = \log x_1 + \log x_2$

$\log 1 = 0$; $\log e = 1$



To find $\frac{d}{dx}\log x$:

use implicit differentiation

$$w = \log x \Rightarrow e^w = x$$

$$\frac{d}{dx} e^w = \frac{d}{dx} x = 1$$

$$\therefore \left(\frac{d}{dw} e^w\right)\left(\frac{dw}{dx}\right) = 1$$

$$\therefore e^w \frac{dw}{dx} = 1 \Rightarrow \frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x}$$

$$\Rightarrow \boxed{\frac{d}{dx}\log x = \frac{1}{x}}$$

To differentiate any exponential: two method

Method 1

$$\frac{d}{dx} a^x = ?$$

use base e: $a^x = (e^{\log a})^x = e^{x\log a}$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x\log a}$$

$$= (\log a) e^{x\log a}$$

$$\Rightarrow \boxed{\frac{d}{dx} a^x = (\log a) a^x}$$

$M(a) = \log a$

Ex. $\frac{d}{dx} 2^x = (\log 2) 2^x$

$\frac{d}{dx} 10^x = (\log 10) 10^x$

Method 2 Logarithmic differentiation

$$\frac{d}{dx} u ??,$$

$$\frac{d}{dx} \log u = \left(\frac{d \log u}{du}\right) \cdot \left(\frac{d}{dx}\right) = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\Rightarrow (\log u)' = u'/u$$

$$\frac{d}{dx} a^x = ?$$

$u = a^x$

$\therefore \log u = x \log a$

$\therefore (\log u)' = \log a$

$\Rightarrow u'/u = (\log u)' = \log a$

$\therefore u' = u \log a$

$\Rightarrow \frac{d}{dx} a^x = (\log a) a^x$

Ex 2 (moving exponents)

$V = x^x$

$\Rightarrow \log V = x \log x$

$\Rightarrow (\log V)' = \log x + x \cdot \frac{1}{x}$

$V'/V = 1 + \log x$

$\Rightarrow V' = V(1 + \log x)$

$\Rightarrow \frac{d}{dx} x^x = x^x (1 + \log x)$

Ex 3 $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

$$\log\left(\left(1+\frac{1}{n}\right)^n\right) = n \log\left(1 + \frac{1}{n}\right)$$

$\Delta x = \frac{1}{n} \to 0$

$$= \frac{1}{\Delta x} \log(1 + \Delta x)$$

$$\downarrow \Delta x \to 0$$

$$\frac{d}{dx} \log x \Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$= e^{\left[\lim_{n \to \infty} \log\left\{\left(1+\frac{1}{n}\right)^n\right\}\right]}$$

$$= e^1 = e$$

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Derivatives of Hyperbolic Sine and Cosine

Hyperbolic sine (sinsh):

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine (cosh):

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\hookrightarrow \frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \cosh(x)$$

$$\Rightarrow \boxed{\frac{d}{dx}\sinh(x) = \cosh(x)}$$

Likewise, $\boxed{\frac{d}{dx}\cosh(x) = \sinh(x)}$

Important Identity:

$$\boxed{\cosh^2(x) - \sinh^2(x) = 1}$$

Proof:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x - e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left(e^{2x} + 2e^x e^{-x} + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$= \frac{1}{4}(2+2)$$

$$= 1$$

☆ why are those function called "hyperbolic"?

Let $u = \cosh(x)$ and $v = \sinh(x)$, then $u^2 - v^2 = 1$
which is the equation of a hyperbola.
Regular trig functions are "circular" functions.
If $u = \cos(x)$ and $v = \sin(x)$, $u^2 + v^2 = 1$
which is the equation of a circle.