Chapter 2 Solutions (Tao Analysis I)

Chris McDonald

May 12, 2020

Section 2.2

Exercise 2.2.1

(Addition is associative). For any natural numbers a, b, c, we have $(a+b)+c = a+(b+c)$.

Proof. Induct on c, keeping a and b fixed. For the base case $c = 0$, we must show $(a+b)+0 = a+(b+0)$. Two applications of Lemma 2.2.2 gives $(a+b)+0 = a+b = a+(b+0)$, as desired. Suppose inductively that $(a+b)+c = a+(b+c)$. We need to show that $(a+b)+S(c) = a+(b+S(c))$. We have

$$\begin{aligned}
(a+b)+S(c) &= S((a+b)+c) && \text{(Lemma 2.2.3)}\\
&= S(a+(b+c)) && \text{(Inductive Hypothesis)}\\
&= a+S(b+c) && \text{(Lemma 2.2.3)}\\
&= a+(b+S(c)) && \text{(Lemma 2.2.3)}
\end{aligned}$$

This closes the induction.

□

Exercise 2.2.2

Let a be a positive number. Then there exists exactly one natural number b such that $b+1 = a$.

Proof. We are asked to prove $\forall a \in \mathbb{N} P(a)$, where

$$P(a) \iff [a \neq 0 \Rightarrow \exists! b \in \mathbb{N}(a = b+1)]$$

Induct on a.

For the base case $a = 0$, we have $P(0)$ is vacuously true since it's an implication with a false hypothesis, $0 \neq 0$.

Suppose inductively that P(a) is true. We need to show that if $a+1 \neq 0$, then there exists a unique natural number, call it c, such that $a+1 = c+1$. So suppose that $a + 1 \neq 0$. By the inductive hypothesis there is $b \in \mathbb{N}$ satisfying $a = b + 1$. Adding one to both sides gives $a + 1 = (b + 1) + 1$. Taking $c = b + 1$ gives existence. For uniqueness, one needs only recall Axiom 2.1.4 ensures the successor function is one to one. Thus $P(a + 1)$ must also be true.

This closes the induction.

$\square$

Exercise 2.2.3

Let $a, b, c \in \mathbb{N}$.

(a)

$a \geq a$

Proof. For any natural number a, $a = a + 0$. The desired result follows from Definition 2.2.11

$\square$

(b)

$a \geq b$ *and* $b \geq c \Rightarrow a \geq c$

Proof. Suppose that $a \geq b$ and $b \geq c$. Then by Definition 2.2.11 there exist natural numbers m and n such that $a = b + m$ and $b = c + n$. Thus $a = b + m = (c + n) + m = c + (m + n)$. Since $m + n \in \mathbb{N}$, Definition 2.2.11 and the previous sentence give $a \geq c$ as desired.

$\square$

(c)

$a \geq b$ *and* $b \geq a \Rightarrow a = b$

Proof. Suppose that $a \geq b$ and $b \geq a$. Then by Definition 2.2.11 there exist natural numbers m and n such that $a = b + m$ and $b = a + n$. Substitution for b in the first equation gives $a = a + n + m$. Together, Propositions 2.2.4-6 imply $0 = n + m$. Corollary 2.2.9 then implies that both $n = 0$ and $m = 0$. Substituting $m = 0$ into $a = b + m$ gives $a = b$ as desired.

$\square$

(d)

$a \geq b$ *if and only if* $a + c \geq b + c$

Proof. ($\Rightarrow$) Suppose $a \geq b$. Then by Definition 2.2.11 there exists a natural number n such that $a = b + n$. Adding c to both sides of this equation we see that $a + c = b + n + c = (b + c) + n$. Thus by Definition 2.2.11, $a + c \geq b + c$.

($\Leftarrow$) Suppose $a+c \geq b+c$. Then by Definition 2.2.11, there exists a natural number n such that $a+c = b+c+n$. The properties of addition in $\mathbb{N}$ derived to this point - including commutativity, associativity, and cancellation - imply that $a = b+n$. By Definition 2.2.11, we conclude $a \geq b$ as desired.

$\square$

(e & f)

Prove that the following are equivalent:

1. *$a < b$*
2. *There exists a positive natural number d such that $a + d = b$*
3. *$a + 1 \leq b$*

Proof. ($1 \Rightarrow 2$) Suppose $a < b$. By definition $a \leq b$ and $a \neq b$. By definition of $\leq$, there exists a natural number n such that $a + n = b$. Necessarily n is positive, since otherwise we have $a = b$, a contradiction.

($2 \Rightarrow 3$) Suppose there exists a positive natural number d such that $a+d = b$. By Lemma 2.2.10, there exists a unique natural number n such that $d = n+1$. Substituting for d in $a + d = b$, we have $a + n + 1 = b$. By Definition 2.2.11, $a + 1 \leq b$.

($3 \Rightarrow 1$) Suppose $a + 1 \leq b$. By Definition 2.2.11, there exists a natural number n such that $a+1+n = b$. Since $n+1$ is a natural number, Definition 2.2.11 gives $a \leq b$. Now suppose by contradiction that $a = b$. Then applying cancellation to $a+1+n = b$ we get $n+1 = 0$. But this contradicts Axiom 2.1.3. Thus $a \neq b$ and overall $a < b$ as desired.

$\square$

Exercise 2.2.4

(a)

$0 \leq b$ *for all natural numbers* b

Proof. Let $b \in \mathbb{N}$. Since $b = b + 0$, we have $0 \leq b$ by definition.

$\square$

(b)

If $a > b$ *then* $a + 1 > b$

Proof. Suppose $a > b$. By Proposition 2.2.12(f), there exists a positive natural number d such that $a = b + d$. Adding one to each side of this equation gives $a + 1 = b + d + 1$. By Axiom 2.1.3, we have that $d + 1$ must be positive. Thus, again by Proposition 2.2.12(f), we have $a + 1 > b$ as desired.

$\square$

(c)

If $a = b$ then $a + 1 > b$

Proof. Suppose $a = b$. Adding one to both sides of this equation gives $a + 1 = b + 1$. Since one is a positive natural number, we use Proposition 2.2.12(f) to conclude $a + 1 > b$ as desired.

□

Exercise 2.2.5

Prove the Strong Principle of Induction

Proof. Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true.

For each $n \in \mathbb{N}$, let $Q(n)$ denote the statement: $P(m)$ is true for all natural numbers m satisfying $m_0 \leq m < n$. We prove that $Q(n)$ is true for all natural numbers n by induction on n.

For the base case $n = 0$, we have that $Q(0)$ is true in a trivial manner. Indeed, since there are **no** natural numbers m such that $m_0 \leq m < 0$, it follows $Q(0)$ is vacuously true.

Suppose inductively that $Q(n)$ is true; that is, suppose $P(m)$ is true for $m_0 \leq m < n$. In the case that $n \geq m_0$ we have, by assumption, an implication that guarantees $P(n)$ is true here. So, $P(m)$ is true for $m_0 \leq m < n + 1$; that is, Q(n+1) is true as desired if $n \geq m_0$. In the case that $n < m_0$, we have $n + 1 \leq m_0$ by Proposition 2.2.12(e). This makes $Q(n + 1)$ vacuously true since there are no numbers m satisfying

$$n + 1 \leq m_0 \leq m < n + 1$$

. This closes the induction on n of $Q(n)$.

Hence, for arbitrary $n \in \mathbb{N}$, we have $Q(n + 1)$ is true - and consequently $P(n)$ is true. Therefore, $P(n)$ is true for all natural numbers n, as desired.

□

Exercise 2.2.6

Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m + 1)$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the principle of backwards induction.

Proof. Let $Q(n)$ denote the statement of backwards induction for an arbitrary natural number n. We show $Q(n)$ is true for all natural numbers by induction on n.

For the base case $n = 0$, suppose $P(0)$ is true. Then trivially $Q(0)$ is true, since the only natural number less than or equal to zero is zero itself.

Suppose inductively that $Q(n)$ is true. If $P(n + 1)$ is true, then by the definition of $P(m)$ we have that $P(n)$ is true. Then by the inductive hypothesis we have that $P(m)$ is true for all $m \leq n$. Overall, we have that the truth of $P(n+1)$ implies that $P(m)$ is true for all $m \leq n+1$.

This closes the induction.

□

Section 2.3

Exercise 2.3.1

(Multiplication is commutative). Let n, m be natural numbers. Then $n \times m = m \times n$.

Proof. We will prove this proposition in stages. Specifically, we will first prove preliminary claims analogous to ones proved in Section 2.2 for addition.

First, we claim that $n \times 0 = 0$ for every natural number n. We induct on n. The base case $n = 0$ corresponds to $0 \times 0 = 0$, and this is true from the definition of multiplication. Suppose inductively that $n \times 0 = 0$. We wish to show $(n + 1) \times 0 = 0$. Using the definition of multiplication and the inductive hypothesis we get $(n+1) \times 0 = n \times 0 + 0 = n \times 0 = 0$. This closes the induction and the first claim is proved.

Second, we claim that for all natural numbers n and m we have $n \times (m+1) = n \times m + n$. We induct on n. For the base case $n = 0$, we wish to show that $0 \times (m+1) = 0 \times m + 0$. But this follows just from the definition of multiplication. Suppose inductively that $n \times (m + 1) = n \times m + n$. We wish to show that $(n + 1) \times (m + 1) = (n + 1) \times m + (n + 1)$. We have

$$\begin{aligned}
(n + 1) \times (m + 1) &= n \times (m + 1) + (m + 1) && \text{(Definition of multiplication)} \\
&= n \times m + n + (m + 1) && \text{(Inductive Hypothesis)} \\
&= n \times m + m + n + 1 \\
&= (n + 1) \times m + (n + 1) && \text{(Definition of multiplication)}
\end{aligned}$$

as desired. This closes the induction and proves the second claim.

We now show that multiplication is commutative. Let n, m be natural numbers. We fix m and induct on n.

For the base case $n = 0$, we wish to show $0 \times n = n \times 0$. The left-hand side is equal to zero from the definition of multiplication, and the right-hand side is equal to zero by our first claim. This proves the base case.

Suppose inductively that $n \times m = m \times n$. We wish to show that $(n+1) \times m =$

$m \times (n+1)$. We have

$$\begin{aligned}
(n+1) \times m &= n \times m + m && \text{(Definition of multiplication)} \\
&= m \times n + m && \text{(Inductive Hypothesis)} \\
&= m \times (n+1) && \text{(Second Claim)}
\end{aligned}$$

as desired. This closes the induction and proves that multiplication is commutative in the natural number system.

$\square$

Exercise 2.3.2

Positive natural numbers have no zero divisors. Let n, m be natural numbers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.

Proof. Let n, m be natural numbers.

($\Leftarrow$) Suppose that at least one of n, m is equal to zero, that is, suppose that $n = 0$ or $m = 0$. Then $nm = 0$.

($\Rightarrow$) Suppose $nm = 0$. We wish to show that at least one of n, m is equal to zero. We suppose $n \neq 0$ and show that $m = 0$. By Lemma 2.2.10, there exists a natural number d such that $n = d + 1$. Substituting for n and using the distributive property proved earlier, $0 = nm = (d+1)m = dm + m$. By Corollary 2.2.9, $m = 0$ as desired.

$\square$

Exercise 2.3.3

(Multiplication is associative). For any natural numbers a, b, c, we have $(ab)c = a(bc)$.

Proof. Fix a, b and induct on c.

For the base case $c = 0$, we have $(ab)0 = 0 = a0 = a(b0)$.

Suppose inductively that $(ab)c = a(bc)$. We wish to show that $(ab)(c+1) = a(b(c+1))$. We have

$$\begin{aligned}
(ab)(c+1) &= (ab)c + (ab)1 && \text{(Distributive Property)} \\
&= a(bc) + a(b1) && \text{(Inductive Hypothesis)} \\
&= a(bc + b1) && \text{(Distributive Property)} \\
&= a(b(c+1)) && \text{(Distributive Property)}
\end{aligned}$$

as desired. This completes the induction and finishes the proof. $\square$

Exercise 2.3.4

Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Proof.

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) && \text{(Definition of Exponentiation)} \\
&= (a+b)a + (a+b)b && \text{(Distributive Property)} \\
&= aa + ba + ab + bb && \text{(Distributive Property)} \\
&= a^2 + ba + ab + b^2 && \text{(Definition of Exponentiation)} \\
&= a^2 + ab + ab + b^2 && \text{(Multiplication is Commutative)} \\
&= a^2 + 1(ab) + 1(ab) + b^2 && \text{(Def of mult. and Mult. associative)} \\
&= a^2 + (1+1)(ab) + b^2 && \text{(Distributive Property)} \\
&= a^2 + 2ab + b^2 && (1+1=2)
\end{aligned}$$

□

Exercise 2.3.5

(Euclidean algorithm). Let n be a natural number, and let q be a positive natural number. Then there exist natural numbers m, r such that $0 \le r < q$ and $n = mq + r$.

Proof. Fix q and induct on n.

For the base case $n = 0$, we wish to show that there exist natural numbers m, r such that $0 = mq + r$ and $0 \le r < q$. A choice of $m = 0$ and $r = 0$ will do.

Suppose inductively that the division algorithm is valid for a natural number n. We wish to show that there exist natural numbers m, r such that $n + 1 = mq + r$ and $0 \le r < q$. By the induction hypothesis, there exist natural numbers m_1, r_1 such that $n = m_1 q + r_1$ and $0 \le r_1 < q$. Hence $n + 1 = m_1 q + r_1 + 1$. By Proposition 2.2.12(e), $r_1 + 1 < q$ implies that $r_1 + 1 \le q$. Trichotomy then gives two exhaustive cases: $r_1 + 1 < q$ or $r_1 + 1 = q$. If $r_1 + 1 < q$, then it suffices to choose $m = m_1$ and $r = r_1 + 1$. If $r_1 + 1 = q$, then we have

$$\begin{aligned}
n + 1 &= m_1 q + r_1 + 1 \\
&= m_1 q + q \\
&= (m_1 + 1)q
\end{aligned}$$

and thus we may choose $m = m_1 + 1$ and $r = 0$, since then $0 \le r < q$ is also satisfied. This closes the induction.

□