Unknown Coefficients in Inequalities

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1 Introduction

In our previous article we looked at an curious inequality from the recent New Zealand Squad Selection Test:

(2024 NZSST 3 P7) *Given that $0 \le a, b, c \le 1$, what is the maximum possible value of the following expression?*

$$\frac{a}{bc+1} + \frac{b}{ac+1} + \frac{c}{ab+1}$$

where we proved

$$\frac{a}{bc+1} \le \frac{2a}{a+b+c}$$

and 2 other similar ones. Summing them up would give the desired maximum of 2. In this article, we will explain how the technique of "unknown coefficients" could be used to solve more complicated and advanced problems.

2 Examples

2.1 A Household Classic

a, b, c are positive real numbers such that $abc = 1$, prove that

$$\frac{1}{1+2a} + \frac{1}{1+2b} + \frac{1}{1+2c} \ge 1.$$

This is a textbook problem most will encounter (or not) when studying olympiad inequalities. Firstly, let k be a real number such that

$$\frac{1}{1+2a} \ge \frac{a^k}{a^k + b^k + c^k}.$$

This is equivalent to

$$a^k + b^k + c^k \ge a^k + 2a^{k+1}$$
$$\Leftrightarrow b^k + c^k \ge 2a^{k+1}.$$

Note that by AM-GM and $abc = 1$ we have

$$b^k + c^k \ge 2(bc)^{\frac{k}{2}} = 2a^{-\frac{k}{2}}.$$

Hence we just have to find a k such that

$$a^{-\frac{k}{2}} = a^{k+1}$$

where evidently $k = -\dfrac{2}{3}$. Hence we have

$$\frac{1}{1+2a} \geq \frac{a^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}}.$$

Similarly, we have

$$\frac{1}{1+2b} \geq \frac{b^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}}$$
$$\frac{1}{1+2c} \geq \frac{c^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}}$$

Summing them up gives

$$\frac{1}{1+2a} + \frac{1}{1+2b} + \frac{1}{1+2c} \geq \frac{a^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}} + \frac{b^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}} + \frac{c^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}} = 1$$

which effectively solves our problem.

2.2 Greetings from the IMO

(2001 IMO P2) *Prove that for all positive real a, b, c,*

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

Again, we shall find a real k such that

$$\frac{a}{\sqrt{a^2 + 8bc}} \geq \frac{a^k}{a^k + b^k + c^k}$$

This is equivalent to

$$\frac{a^2}{a^2 + 8bc} \geq \frac{a^{2k}}{(a^k + b^k + c^k)^2}$$

$$\Leftrightarrow \quad (a^k + b^k + c^k)^2 \geq a^{2k-2}(a^2 + 8bc)$$

$$\Leftrightarrow \quad (b^2k + +2b^k c^k + c^2 k) + 2a^k(b^k + c^k) \geq 8a^{2k-2}bc$$

$$\Leftrightarrow \quad (b^k + c^k)^2 + 2a^k(b^k + c^k) \geq 8a^{2k-2}bc$$

Using AM-GM we have

$$(b^k + c^k) \geq 2(bc)^{\frac{k}{2}},$$

Hence the above is equivalent to

$$4(bc)^k + 4a^k b^{\frac{k}{2}} c^{\frac{k}{2}} \geq 8a^{2k-2}bc$$

$$\Leftrightarrow (bc)^k + a^k b^{\frac{k}{2}} c^{\frac{k}{2}} \geq 2a^{2k-2}bc$$

Note that by AM-GM

$$(bc)^k + a^k b^{\frac{k}{2}} c^{\frac{k}{2}} \geq 2a^{\frac{k}{2}} b^{\frac{3k}{4}} c^{\frac{3k}{4}}$$

Which means we only have to find a real k such that

$$a^{\frac{k}{2}} b^{\frac{3k}{4}} c^{\frac{3k}{4}} = a^{2k-2}bc$$

and evidently $k = \dfrac{4}{3}$. Therefore,

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} + \frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} + \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} = 1$$

2.3 A Date with the Devil

(2023 China Centennial Schools MO Day 1 P1) *Prove that for all positive real x, y, z satisfying $xyz = 1$,*

$$\frac{1}{1+x+y^2} + \frac{1}{1+y+z^2} + \frac{1}{1+z+x^2} \leq 1.$$

2.3.1 Motivation

This seemingly simple problem is in fact *not* so simple, but it does give rise to (in my opinion) a pretty elegant solution. First let's look at the following problem which motivates our solution.

(2015 Austrian MO Day 2 P4) *Let x, y, z be positive real numbers with $x + y + z \geq 3$. Prove that*

$$\frac{1}{x+y+z^2} + \frac{1}{y+z+x^2} + \frac{1}{z+x+y^2} \leq 1$$

First observe that by Cauchy-Schwarz Inequality we have

$$(x+y+z^2)(x+y+1) \geq (x+y+z)^2$$

Which means

$$\frac{1}{x+y+z^2}+\frac{1}{y+z+x^2}+\frac{1}{z+x+y^2} \le \frac{(x+y+1)+(y+z+1)+(z+x+1)}{(x+y+z)^2}$$

Now note that

$$\begin{aligned}
&\frac{(x+y+1)+(y+z+1)+(z+x+1)}{(x+y+z)^2} \\
&= \frac{2(x+y+z)+3}{(x+y+z)^2} \\
&\le \frac{2(x+y+z)+(x+y+z)}{(x+y+z)^2} \\
&= \frac{3}{x+y+z} \\
&\le 1 && (x+y+z \ge 3)
\end{aligned}$$

You can also check out this video on *Youtube*, where I explained the same problem. Now we shall try to implement this idea into our solution while using unknown coefficients introduced above.

2.3.2 Solution

Let k be a real number, by Cauchy-Schwarz Inequality we have

$$(x+y^2+1)(x^{2k-1}+y^{2k-2}+z^{2k}) \ge (x^k+y^k+z^k)^2$$

This implies

$$\begin{aligned}
&\frac{1}{1+x+y^2}+\frac{1}{1+y+z^2}+\frac{1}{1+z+x^2} \\
&\le \frac{(x^{2k-1}+y^{2k-2}+z^{2k})}{(x^k+y^k+z^k)^2}+\frac{(y^{2k-1}+z^{2k-2}+x^{2k})}{(x^k+y^k+z^k)^2}+\frac{(z^{2k-1}+x^{2k-2}+y^{2k})}{(x^k+y^k+z^k)^2} \\
&= \frac{(x^{2k}+y^{2k}+z^{2k})+(x^{2k-1}+y^{2k-1}+z^{2k-1})+(x^{2k-2}+y^{2k-2}+z^{2k-2})}{(x^k+y^k+z^k)^2}
\end{aligned}$$

Hence we just have to find a real k such that

$$(x^{2k}+y^{2k}+z^{2k})+(x^{2k-1}+y^{2k-1}+z^{2k-1})+(x^{2k-2}+y^{2k-2}+z^{2k-2}) \le (x^k+y^k+z^k)^2$$

Expanding and conquering gives us

$$(x^{2k-1}+y^{2k-1}+z^{2k-1})+(x^{2k-2}+y^{2k-2}+z^{2k-2}) \le 2(x^ky^k+y^kz^k+x^kz^k)$$

Using $xyz = 1, (xy = z^{-1}, yz = x^{-1}, xz = y^{-1})$ we have

$$2(x^k y^k + y^k z^k + x^k z^k) = 2(x^{-k} + y^{-k} + z^{-k})$$

Evidently, if we simply let $2k-1 = 2k-2 = -k$, one would get no solutions. Hence we have to try something different. Now observe by AM-GM and $xyz = 1$

$$x^{-k} + y^{-k} + z^{-k} = \frac{x^{-k} + y^{-k}}{2} + \frac{y^{-k} + z^{-k}}{2} + \frac{z^{-k} + x^{-k}}{2}$$

$$\geq (xy)^{-\frac{k}{2}} + (yz)^{-\frac{k}{2}} + (xz)^{-\frac{k}{2}}$$

$$= x^{\frac{k}{2}} + y^{\frac{k}{2}} + z^{\frac{k}{2}}$$

This implies

$$2(x^k y^k + y^k z^k + x^k z^k) \geq (x^{-k} + y^{-k} + z^{-k}) + (x^{\frac{k}{2}} + y^{\frac{k}{2}} + z^{\frac{k}{2}})$$

Which means we just have to find a real k such that

$$(x^{-k} + y^{-k} + z^{-k}) + (x^{\frac{k}{2}} + y^{\frac{k}{2}} + z^{\frac{k}{2}}) \geq (x^{2k-1} + y^{2k-1} + z^{2k-1}) + (x^{2k-2} + y^{2k-2} + z^{2k-2})$$

Now, if we let $\frac{k}{2} = 2k-1, -k = 2k-2$, we get $k = \frac{2}{3}$, which effectively solves our problem. Finally, we have

$$\frac{1}{1+x+y^2} + \frac{1}{1+y+z^2} + \frac{1}{1+z+x^2}$$

$$\leq \frac{(x^{\frac{1}{3}} + y^{-\frac{2}{3}} + z^{\frac{4}{3}})}{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})^2} + \frac{(y^{\frac{1}{3}} + z^{-\frac{2}{3}} + x^{\frac{4}{3}})}{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})^2} + \frac{(z^{\frac{1}{3}} + x^{-\frac{2}{3}} + y^{\frac{4}{3}})}{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})^2}$$

$$= \frac{(x^{\frac{1}{3}} + y^{-\frac{2}{3}} + z^{\frac{4}{3}}) + (y^{\frac{1}{3}} + z^{-\frac{2}{3}} + x^{\frac{4}{3}}) + (z^{\frac{1}{3}} + x^{-\frac{2}{3}} + y^{\frac{4}{3}})}{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})^2}$$

$$\leq \frac{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})^2}{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})^2}$$

$$= 1$$

3 Practice Problems

Problem 1 (IMO 1996 A1) Suppose $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ca}{ca + a^5 + c^5} \leq 1$$

Problem 2 (China) Suppose a, b, c are positive real numbers. Prove that

$$(1) \quad \sqrt[3]{\frac{a}{b+c}} + \sqrt[3]{\frac{b}{c+a}} + \sqrt[3]{\frac{c}{a+b}} > \frac{3}{2}$$

$$(2) \quad \sqrt[3]{\frac{a^2}{(b+c)^2}} + \sqrt[3]{\frac{b^2}{(c+a)^2}} + \sqrt[3]{\frac{c^2}{(a+b)^2}} \geq \frac{3}{\sqrt[3]{4}}$$

Problem 3 (Generalization of 2001 IMO P2) Suppose a, b, c are positive real numbers. Let $\lambda \geq 8$ be a real number. Prove that

$$\frac{a}{\sqrt{a^2 + \lambda bc}} + \frac{b}{\sqrt{b^2 + \lambda ca}} + \frac{c}{\sqrt{c^2 + \lambda ab}} \geq \frac{3}{\sqrt{1+\lambda}}$$

4 Afterword

I hope you have enjoyed this article. If you have any questions or I have made any mistakes (I am after all just a maths enthusiast), feel free to email to primusmathematica1729@gmail.com. Check us out on Youtube, and stay tuned at *Prime Pursuit* for more articles and monthly problems!