Assignment Problem

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PRELIMINARY DRAFT

1 Cost minimization

A bakery+cafe company owns bakeries $\mathcal{X}$ and cafes $\mathcal{Y}$. Let $|\mathcal{X}| = |\mathcal{Y}| = n$. The bakeries have a supply of one bread each, and the cafes demand one bread each. The cost of transporting bread from bakery x_i to cafe y_j is $c(x_i, y_j)$. The bakery+cafe company's objective is find a bijective function (an assignment) $T : \mathcal{X} \to \mathcal{Y}$ that minimizes total cost,

$$\min_{T \text{ bijective}} \sum_{i=1}^{n} c(x_i, T(x_i)).$$

Existence of an optimal T is not an issue in this discrete problem, as we can check all the finitely many possible Ts.

1.1 Cyclical monotonicity

Definition 1. An assignment T is c-cyclically monotone if for any $k \leq n$, and any family $(x_1, y_1), \dots, (x_k, y_k) \in \{(x_i, y_i) : y_i = T(x_i)\}$ satisfies the inequality

$$\sum_{i=1}^{k} c(x_i, y_i) \leq \sum_{i=1}^{k} c(x_{i+1}, y_i),$$

with the convention that $x_{i+1} = x_1$.

Lemma 1. *An assignment T is optimal if and only if it is c-cyclically monotone.*

Proof. If T is optimal any rearrangement of assignments cannot decrease the cost. Since every cycle defined above is a rearrangement we conclude T is c-cyclically monotone.

The above definition of c-cyclical monotonicity covers all the rearrangements because we allow any cycle length k and any family from graph of T with that length. Hence, if T is c-cyclically monotone then it is optimal. □

Remark 1. The definition of c-cyclical monotonicity gives a structure to rearrangements. First choosing a cycle length, then choosing any subset of assignments with that length. Since this structure accounts for all the rearrangements, we can just focus on it for our analysis.

2 The Dual Problem

Consider a logistics company that can magically transport bread between bakeries and cafes. It offers to buy bread at bakeries and sell at cafes. Also assume that this contract is non-binding in the following sense: the bakery+cafe company has power to unilaterally deviate by refusing to trade at some bakery and cafe and transport it themselves[1] Taking this into account the logistics company ensures prices are such that this scenario won't occur[2].

Let $\psi : \mathcal{X} \to \mathbb{R}$ and $\phi : \mathcal{Y} \to \mathbb{R}$ price functions at bakeries and cafes, respectively. The logistics company's revenue-maximization problem is then

$$\begin{aligned} &\max_{\psi,\phi} \sum_{y_j=1}^{n} \phi(y_j) - \sum_{x_i=1}^{n} \psi(x_i), \\ &\text{s.t. } \phi(y_j) - \psi(x_i) \leq c(x_i, y_j) \qquad \forall\ x_i \in \mathcal{X},\ y_i \in \mathcal{Y}. \end{aligned}$$

The following Lemma is immediate as can be seen by summing up feasibility constraints in the dual.

Lemma 2.

$$\max_{\phi-\psi\leq c} \sum_{y_j=1}^{n} \phi(y_j) - \sum_{x_i=1}^{n} \psi(x_i) \leq \min_{T \text{ \textit{bijective}}} \sum_{i=1}^{n} c(x_i, T(x_i)).$$

However, it is not obvious that there exist prices such that the two objectives meet. The logistics company sets competitive prices between *all* pairs of bakeries and cafes, not just the ones between which bakery+cafe company transports in its optimal plan. Despite the restrictive constraints, we show that the optimal objectives match, which is called strong duality.

Theorem 1.

$$\max_{\phi-\psi\leq c} \sum_{y_j=1}^{n} \phi(y_j) - \sum_{x_i=1}^{n} \psi(x_i) = \min_{T \text{ \textit{bijective}}} \sum_{i=1}^{n} c(x_i, T(x_i)).$$

Proof. Due to Lemma 2, the strong duality can only hold at primal's optimal, fix such a T^*. Therefore, price difference between matched bakery-cafes must exactly equal cost of transport between them. Hence, to have any hope for strong duality we assume prices satisfy the following equations,

$$\phi(y_j) - \psi(T^{*-1}(y_j)) = c(T^{*-1}(y_j), y_j) \qquad \forall y_j \in \mathcal{Y}.$$

Now that we matched the objectives we need to make sure the constraints in the dual are satisfied. We restate the constraints using the above equations,

$$\psi(T^{*-1}(y_j)) - \psi(x_i) \leq c(x_i, y_j) - c(T^{*-1}(y_j), y_j) \qquad \forall x_i, y_j. \tag{1}$$

Thus, we eliminated ϕ in finding prices, and if we can find ψ satisfying above inequalities then we are done. To this end, consider the following *fully connected directed graph.*

The Graph. There $n+1$ nodes of which n correspond to bakeries $\mathcal{X}$. And the one remaining node a dummy that connects to every other node. The weight of the edge going from node x_i to node $x_{i'}$ denoted $w(x_i, x_{i'}) := c(x_i, T^*(x_{i'})) - c(x_{i'}, T^*(x_{i'}))$. The weight of any edge from the dummy is 0.

[1] If logistics company can make the bakery+cafe company to sign a binding contract it is enough for logistics company to guarantee total loss is less for cafe+bakery company is less than the optimal transport cost.

[2] This is without loss of generality due to revelation principle from mechanism design, but not important for our story here.

Claim 1. *Sum of edges on any cycle in the graph is non-negative.*

Proof. Consider any cycle of length $k:\ x_i, x_{i'}, \ldots, x_i$. Then $\sum_{x_i, x_{i'}, \ldots, x_i} w(x_i, x_{i'}) = \sum_{x_i, x_{i'}, \ldots, x_i} \Big(c(x_i, T^*(x_{i'})) - c(x_{i'}, T^*(x_{i'})) \Big)$. But these cycles are precisely what we used in Lemma 1. So we conclude the sum of edges in any cycle in the graph is non-negative. □

We define $\psi(x_i)$ as the minimum distance from dummy node to x_i. The above Claim implies that this is a well-defined function. If this constructed ψ satisfies Inequality 1, then we are done.

Pick any node $T^{*-1}(y_j)$. Since $\psi(T^{*-1}(y_j))$ is minimum distance from dummy, reaching here from an arbitrary node x_i which is $\psi(x_i) + w(x_i, T^{*-1}(y_j))$ must be greater, which is exactly Inequality 1. □

In constructing logistic company's optimal solution we exploited bakery+cafe company's non-existence of improving cycles at the optimum. This helped us consistently construct prices that satisfy all the constraints.