Existence of Optimal Transport

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PRELIMINARY DRAFT

Notation: $P(\mathcal{X})$: set of Borel measures on $\mathcal{X}$.

1 Preliminaries

Theorem 1 (Prokhorov)**.** *If $\mathcal{X}$ is Polish space, then a set $\mathcal{P} \in P(\mathcal{X})$ is precompact for the weak topology if and only if it tight, i.e. for every $\epsilon > 0$ there exists compact $K_\epsilon \in \mathcal{X}$ such that $\mu(\mathcal{X} \backslash K_\epsilon) \leq \epsilon$ for every $\mu \in \mathcal{P}$.*

Proof. Tight $\implies$ precompact.
Let $\{\mu_n\}$ be a sequence, we need to find a subsequence that converges. Tightness implies for each $k \in \mathbb{N}$, there exists compact K_k such that $\mu(\mathcal{X} \setminus K_k) \leq 1/k$ for all $\mu \in \mathcal{P}$. Enlarging if necessary we consider $K_1 \subset K_2 \subset K_3 \subset \ldots$.

Define restricted measures $\nu_n^{(1)}(A) := \mu_n(K_1 \cap \mathcal{X})$. Each $\nu_n^{(1)}$ is supported on compact K_1 and total mass is bounded, so $\{\nu_n^{(1)}\}$ has a convergent subsequence. Hence there exists a subsequence $\{\mu_{n_j^{(1)}}\}$ such that its restriction to K_1 converges.

Let $\tilde{\mu}_1 = \mu_{n_1^{(1)}}$. Now, consider K_2 and the sequence $\{\mu_{n_j^{(1)}}\}$, there is a subsequence of it, call it $\{\mu_{n_j^{(2)}}\}$, such that its restriction to K_2 converges. We set $\tilde{\mu}_2 = \mu_{n_2^{(2)}}$ and proceed inductively to construct the diagonal subsequence $\{\tilde{\mu}_d\}$.

Fix some k and we claim that the diagonal subsequence $\{\tilde{\mu}_d\}$ converges on K_k. This is because $\{\tilde{\mu}_d\}_{d \geq k} \subset \{\mu_{n_j^{(k)}}\}$ which we know converges on K_k.

Fix k. For any d, we have

$$\begin{aligned} \int f \; d\tilde{\mu}_d &= \int_{K_k} f \; d\tilde{\mu}_d + \int_{\mathcal{X} \backslash K_k} f \; d\tilde{\mu}_d \\ &\leq \int_{K_k} f \; d\tilde{\mu}_d + \|f_\infty\|/k. \end{aligned}$$

Letting $k \to \infty$, since the first term above converges we conclude $\int f \; d\tilde{\mu}_d$ converges for any $f \in C_b(\mathcal{X})$.

Define $\Lambda(f) = \lim_{d \to \infty} \int f \; d\tilde{\mu}_d$, then using Reisz representation theorem, we know there exists $\mu \in P(\mathcal{X})$ such that $\Lambda(f) = \int f \; d\mu$. Therefore $\{\tilde{\mu_d}\} \to \mu$.

Precompact $\implies$ tight.
If not tight, then there exists an ϵ_0 such that for every compact $K \subset \mathcal{X}$ there is a measure μ with $\mu(\mathcal{X} \setminus K) > \epsilon_0$. Consider $K_1 \subset K_2 \subset K_3 \subset \ldots$, then there exists $\{\mu_n\}$ such that $\mu_n(\mathcal{X} \setminus K_n) > \epsilon_0$ for all n. Since $K_1 \subset K_2 \subset K_3 \subset \ldots$, for any fixed m and for all $n \geq m$ we have $\mu_n(\mathcal{X} \setminus K_m) \geq \mu_n(\mathcal{X} \setminus K_n) > \epsilon_0$.

If precompact, there exists a converging subsequence $\{\mu_{n_j}\}$. Say it converges to $\tilde{\mu}$. Also, note from earlier that, for a fixed m and sufficiently large j we have $\mu_{n_j}(\mathcal{X} \setminus K_m) > \epsilon$.

Fix m and construct f_m continuous such that it is 0 on K_m (plus some small extension) and 1 otherwise. Then for a large enough j, we have $\int f_m \, d\mu_{n_j} \geq \mu_{n_j}(\mathcal{X} \setminus K_m) > \epsilon_0$. Since $\mu_{n_j} \to \tilde{\mu}$, we have $\int f_m \, d\tilde{\mu} \geq \epsilon_0$. From structure of f_m we conclude $\tilde{\mu}(\mathcal{X} \setminus K_m) \geq \epsilon_0$ for all m.

We know $\cap_{m=1}^{\infty}(\mathcal{X} \setminus K_m) = \emptyset$. By continuity of measures from above, $\tilde{\mu}(\cap_{m=1}^{\infty}(\mathcal{X} \setminus K_m)) = \lim_{m\to\infty} \tilde{\mu}(\mathcal{X} \setminus K_m) \geq \epsilon_0$. But LHS equals 0, hence a contradiction. $\square$

2 Existence

Lemma 1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Polish spaces, and $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous cost function. Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ be an upper semicontinuous function such that $c \geq h$. Let $(\pi_k)_{k\in\mathbb{N}}$ be a sequence of probability measures on $\mathcal{X} \times \mathcal{Y}$, converging weakly to some $\pi \in P(\mathcal{X} \times \mathcal{Y})$, in such a way that $h \in L^1(\pi_k)$, $h \in L^1(\pi)$, and*

$$\int_{\mathcal{X}\times\mathcal{Y}} h \, d\pi_k \xrightarrow[k\to\infty]{} \int_{\mathcal{X}\times\mathcal{Y}} h \, d\pi.$$

Then

$$\int_{\mathcal{X}\times\mathcal{Y}} c \, d\pi \leq \liminf_{k\to\infty} \int_{\mathcal{X}\times\mathcal{Y}} c \, d\pi_k.$$

In particular, if c is nonnegative, then $F : \pi \to \int c \, d\pi$ is lower semicontinuous on $P(\mathcal{X} \times \mathcal{Y})$, equipped with the topology of weak convergence.