Optimality

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Outline

Maximizing Profit

A factory produces milk chocolate ($10 per box) and dark chocolate ($30 per box). The daily demands are 500 and 200 boxes for milk and dark chocolate, respectively. The factory produces 600 boxes of chocolate per day.

What is the optimum production plan?

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- How can they convince the factory that this profit is indeed optimal (i.e., maximum)?
- They need to show two things:
    1. Existential statement: there exists a plan achieving profit \$10 000
    2. Universal statement: all plans give profit at most \$10 000

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- It is indeed a valid plan:
  - no more than 500 boxes of milk chocolate
  - no more than 200 boxes of dark chocolate
  - no more than 600 boxes
- Profit: $400 \times 10 + 200 \times 30 = 10\,000$

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Universal Part: All Plans are Not Better

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- $M \leq 500$, $D \leq 200$, $M + D \leq 600$
- Want to show that $10M + 30D \leq 10\,000$
- Sum up the inequalities

$$\begin{aligned} 10M + 10D &\leq 6\,000 \\ 20D &\leq 4\,000 \end{aligned}$$

Summary

- A proof of the fact that some value α is optimal usually consists of two parts:
    1. Existential statement: there exists a solution achieving the value α
    2. Universal statement: all solutions achieve the value not greater than α
- In this lesson, we'll see several proofs of optimality, following the same pattern

Outline

Warm-up: Producing Chocolate

Subset without x and $100 - x$

Rooks on a Chessboard

Knights on a Chessboard

Bishops on a Chessboard

Subset without x and $2x$

Problem

What is the maximum number of two-digit integers $(10, 11, \ldots, 99)$ that one can select, if it is not allowed to select simultaneously x and y such that $x + y = 100$?

Solving the Problem

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- 40 pairs: $(10, 90), (11, 89), \ldots, (49, 51)$
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- If one takes 10, then one cannot take 90, and vice versa
- Similarly, 11 and 89 cannot be taken simultaneously
- 40 pairs: $(10, 90), (11, 89), \ldots, (49, 51)$
- 10 numbers without pairs: $50, 91, 92, \ldots, 99$
- Definitely no more than $40 + 10 = 50$ numbers
- Optimal solution: $50, 51, \ldots, 99$

Once Again, Formally

Theorem

The maximum number of two-digit numbers s.t. no two of them sum up to 100 is 50.

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Proof

- Solution of size 50: $50, 51, \ldots, 99$ (the sum of any two is greater than 100)
- Any solution should exclude at least one number from each of the 40 pairs. Hence, at most $90 - 40 = 50$ numbers $\square$

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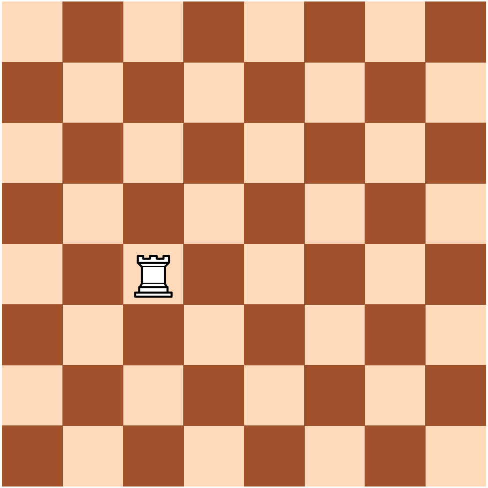
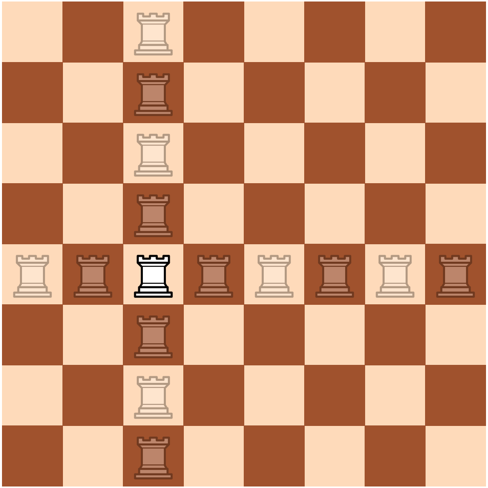
Chess Rook



A chess rook moves vertically and horizontally

Chess Rook



A chess rook moves vertically and horizon-tally

Maximum Number of Rooks

Problem

What is the maximum number of rooks on a chessboard s.t. no two attack each other?

The Pigeonhole Principle

If n pigeons are placed into m boxes and $m < n$, then there is a box containing more than one pigeon

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Solving the Problem

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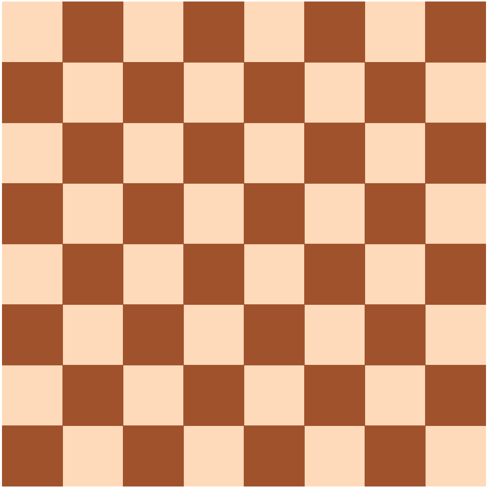
- There should be at most one rook in each row
- Hence, the total number of rooks is at most 8
- In other words, if the number of rooks is greater than the number of rows, then, by the pigeon hole principle, there is a row containing at least two rooks (and these two rooks attack each other)

Solution



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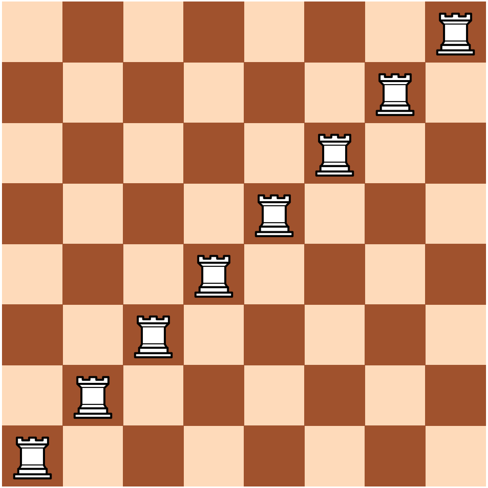

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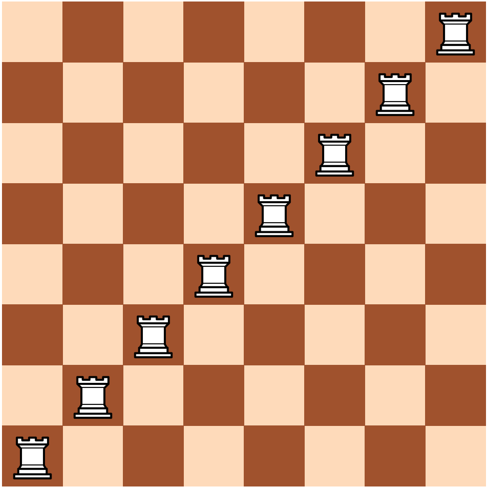

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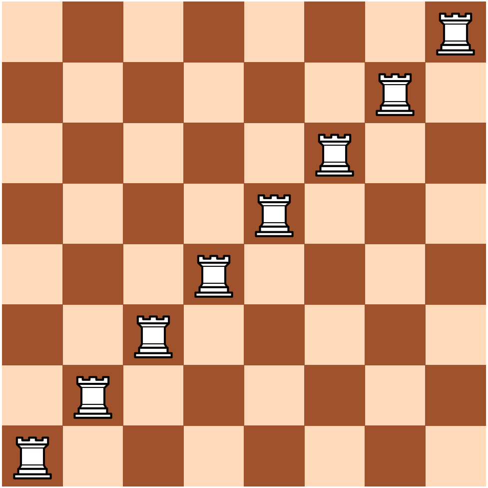

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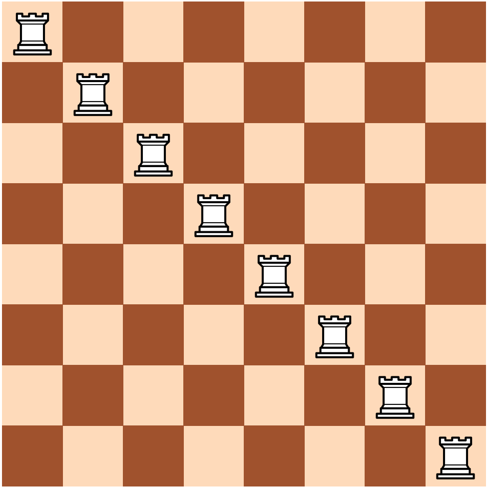

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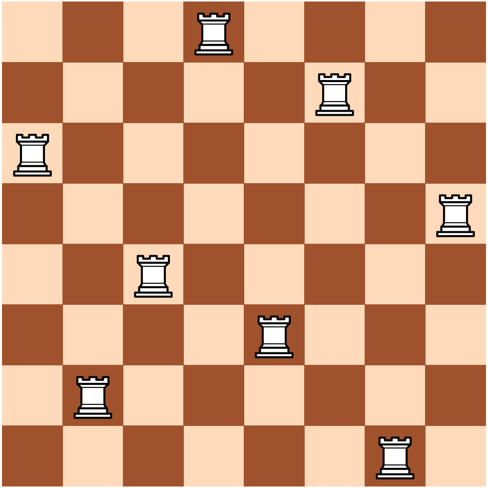

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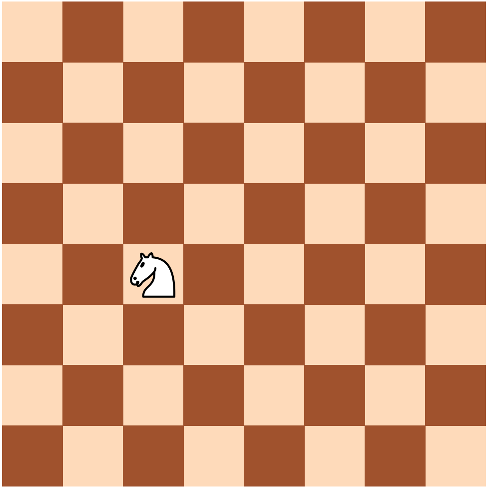
Chess Knight



A chess knight moves in an L-shape

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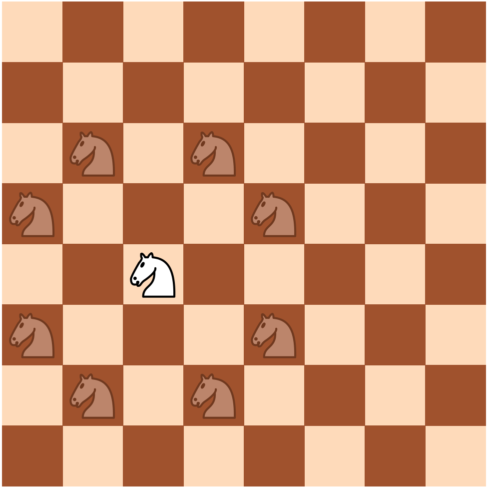

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Maximum Number of Knights

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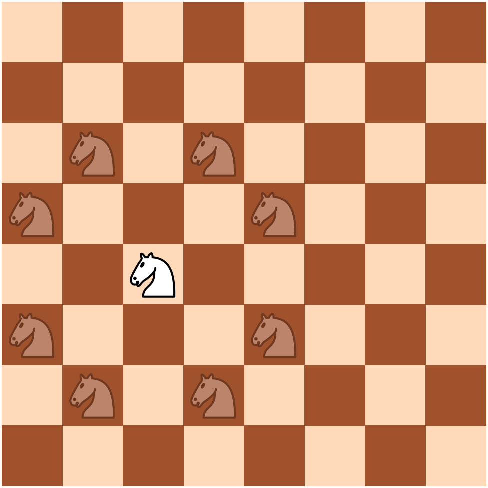
Speculating



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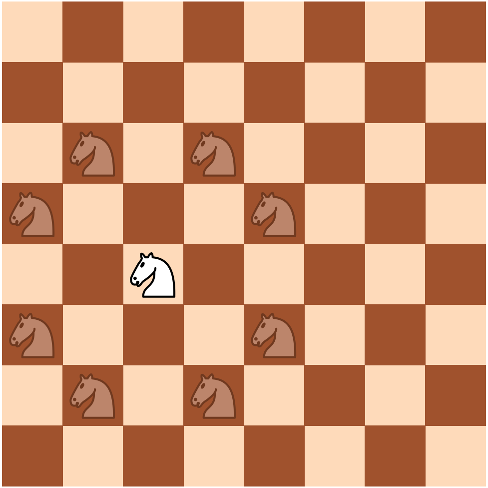

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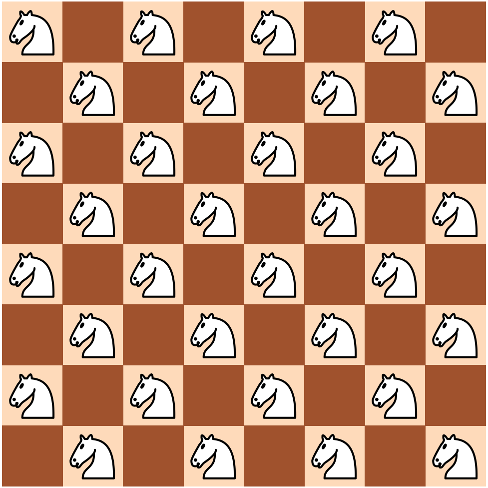

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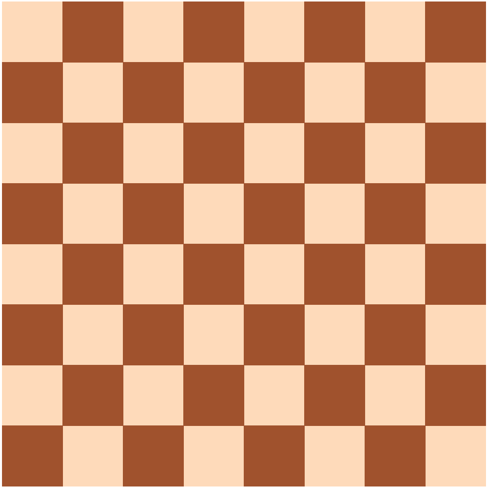

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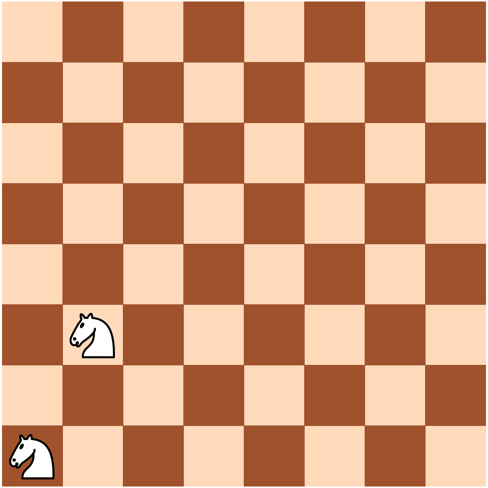

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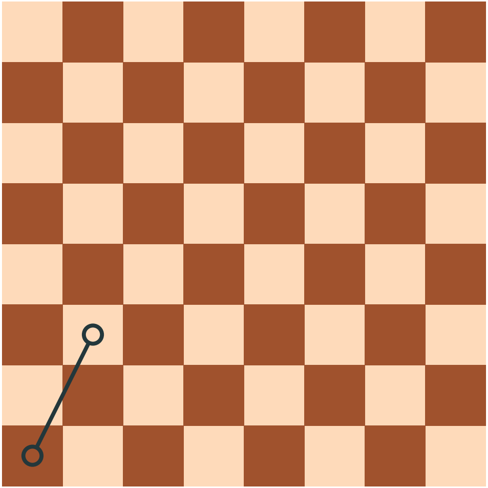

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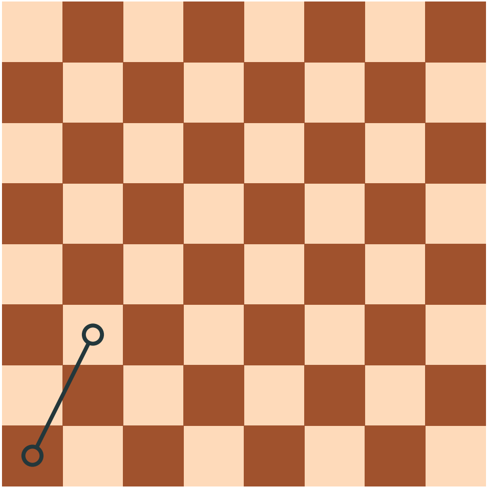

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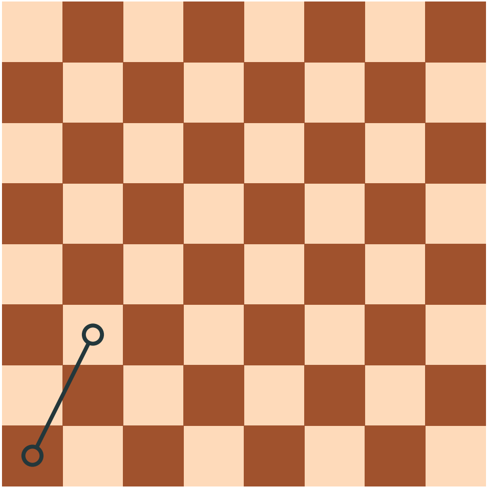

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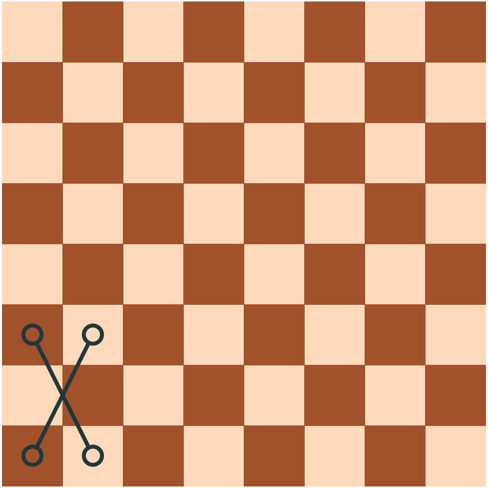

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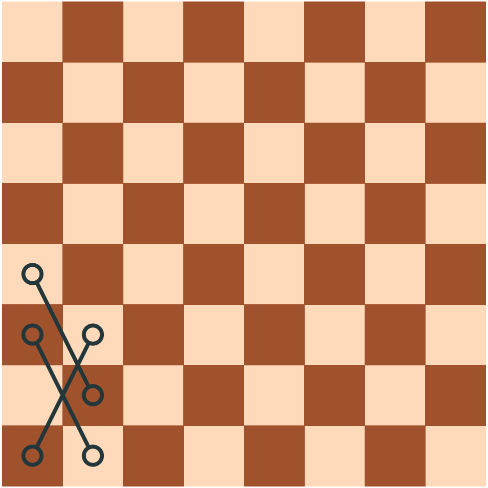

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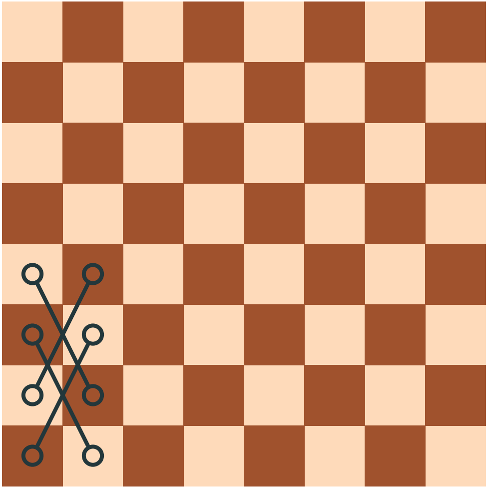

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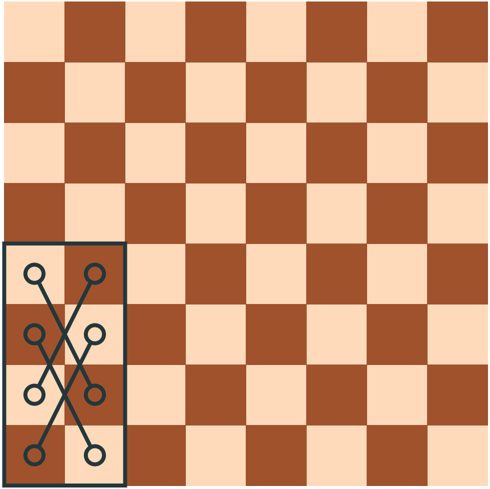

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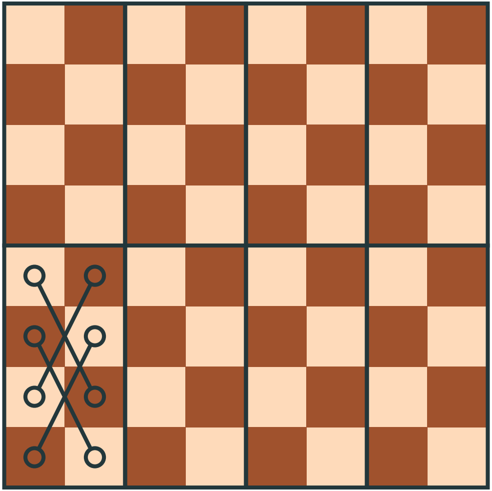

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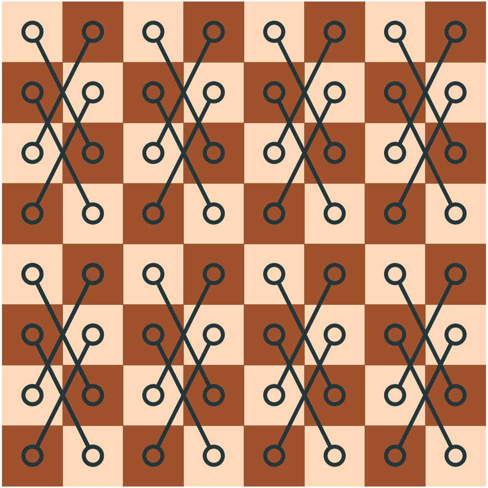

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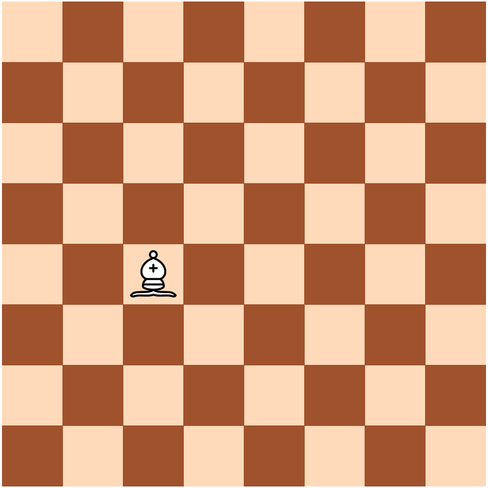
Chess Bishop



A chess bishop moves diagonally

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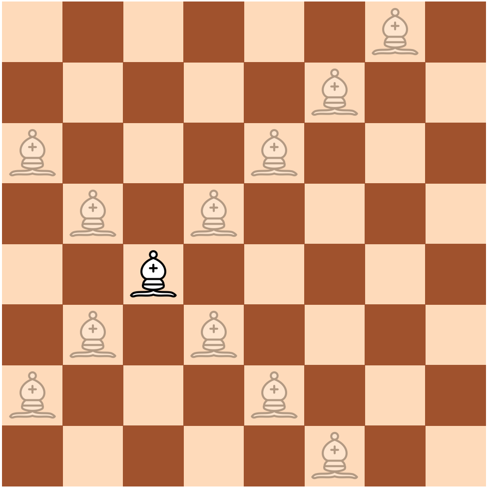

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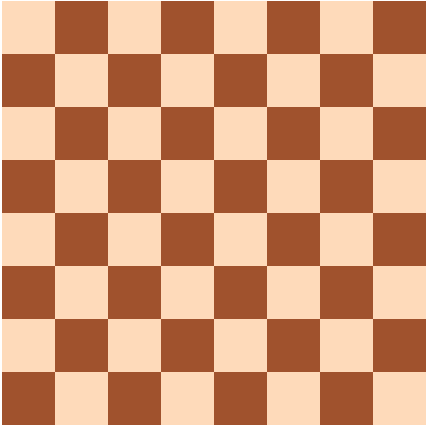
Maximum Number of Bishops

Problem

What is the maximum number of bishops on a chessboard s.t. no two attack each other?

Solving the Problem



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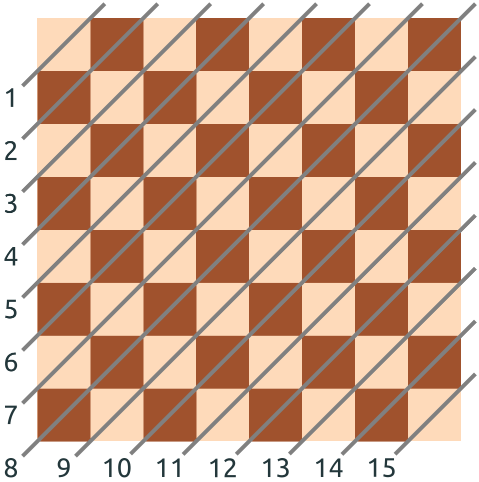

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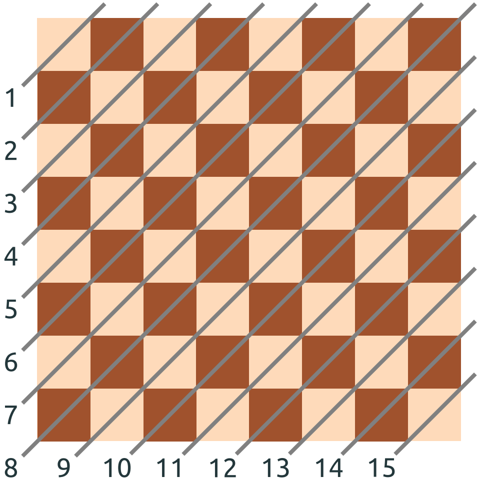

- partition the board into 15 diagonals
- each diagonal contains at most one bishop, so at most 15 bishops

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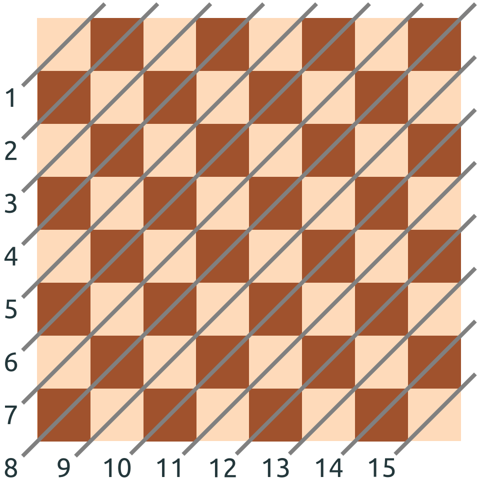
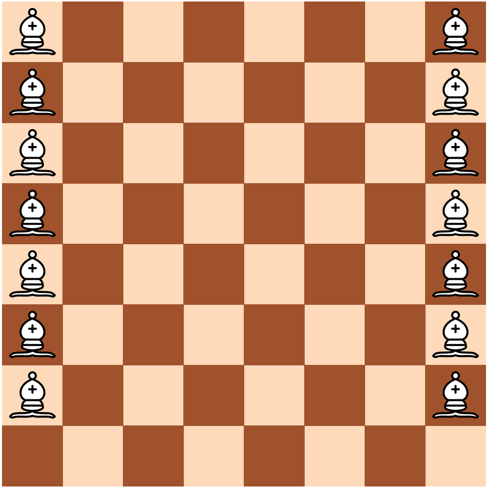


- partition the board into 15 diagonals
- each diagonal contains at most one bishop, so at most 15 bishops
- but! the diagonals 1 and 15 cannot both contain a bishop, so at most 14

14 Bishops



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Subset without x and $2x$

Problem

What is the maximum number of integers among $1, 2, \ldots, 50$ that one can select, if it is not allowed to select simultaneously x and $2x$?

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- 1 and 2 cannot be taken simultaneously

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- 2 and 4 cannot be taken simultaneously

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- Another chain: 3 – 6 – 12 – 24 – 48

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- 2 and 4 cannot be taken simultaneously
- More generally, no two neighbors from the following chain can be taken simultaneously: $1 - 2 - 4 - 8 - 16 - 32$
- Another chain: $3 - 6 - 12 - 24 - 48$
- The integers $1, 2, \ldots, 50$ can be partitioned into such chains, each chain starting with an odd number

Partitioning into Chains

1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	$\cdots$	49
2	6	10	14	18	22	26	30	34	38	42	46	50				
4	12	20	28	36	44											
8	24	40														
16	48															
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- Optimum size: 33