Pascal’s Triangle

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Outline

Pascal’s Triangle

Symmetries

Row Sums

Binomial Theorem

Combinations

Question

There are n students. What is the number of ways of forming a team of k students out of them?

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Answer

$$\binom{n}{k}$$

Forming a Team

- Fix one of the students, call her Alice

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- There are two types of teams:
    1. Teams with Alice: $\binom{n-1}{k-1}$
    2. Teams without Alice: $\binom{n-1}{k}$
- Hence,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal’s Triangle

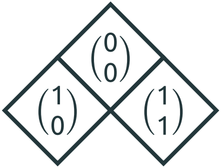
$n = 0$

$$\boxed{\binom{0}{0}}$$

Pascal's Triangle

$n = 0$

$n = 1$

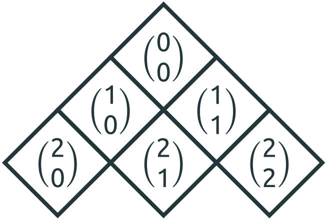


Pascal's Triangle

$n = 0$

$n = 1$

$n = 2$



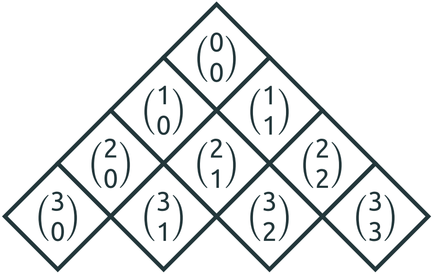
Pascal's Triangle

$n = 0$

$n = 1$

$n = 2$

$n = 3$



Pascal's Triangle

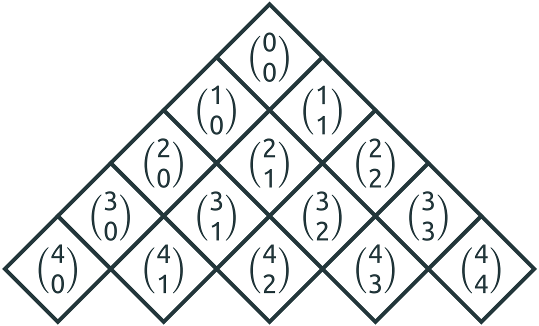
$n = 0$

$n = 1$

$n = 2$

$n = 3$

$n = 4$



Pascal’s Triangle

$n = 0$

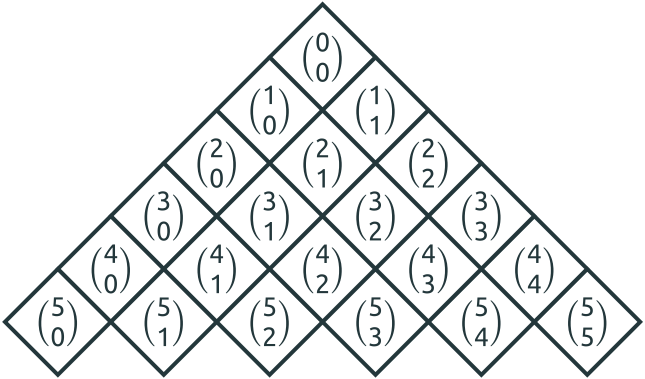
$n = 1$

$n = 2$

$n = 3$

$n = 4$

$n = 5$



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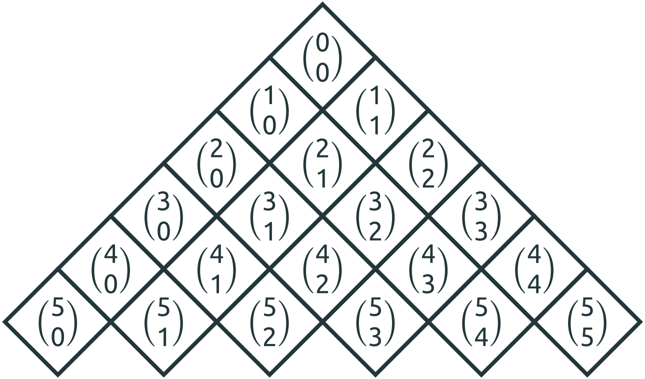

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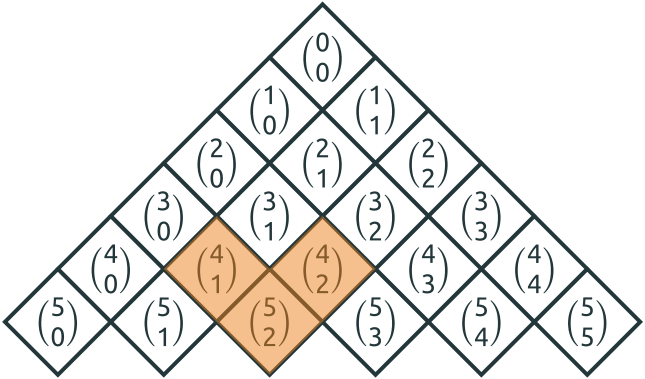

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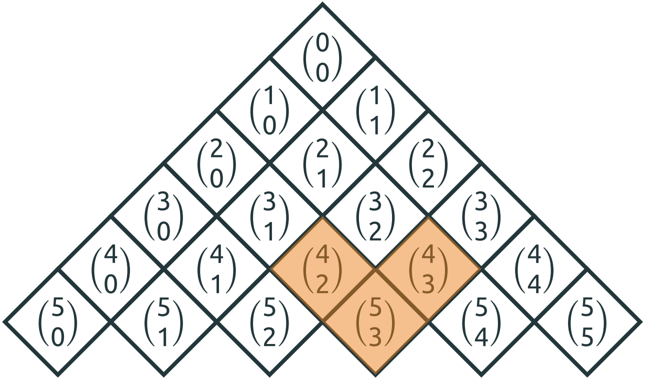

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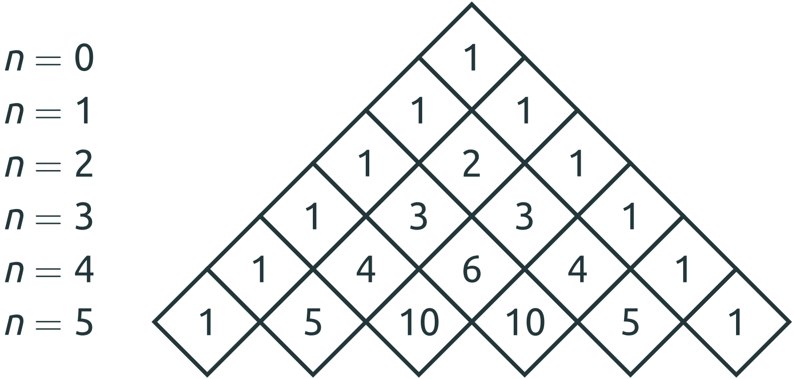

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Code

```python
C = dict() # C([n,k]) is equal to n choose k

for n in range(8):
    C[n, 0] = 1
    C[n, n] = 1

    for k in range(1, n):
        C[n, k] = C[n - 1, k - 1] + C[n - 1, k]

print(C[7, 4])
```

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35
```

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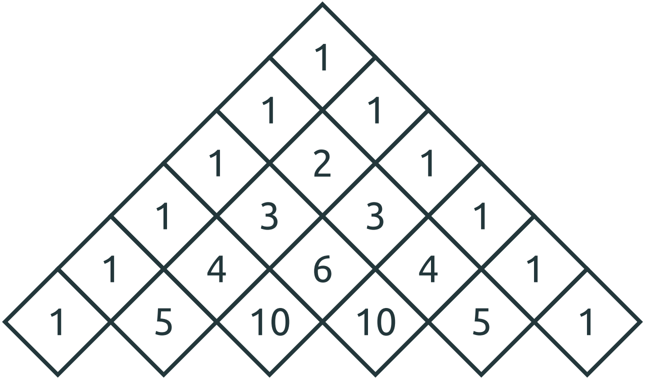
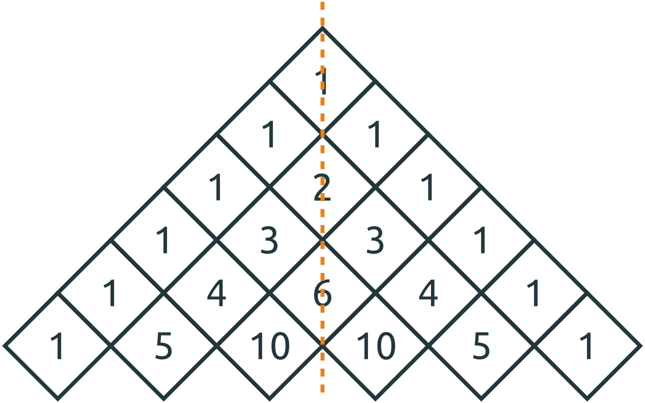
Binomial Theorem

Pascal’s Triangle is Symmetric



Pascal's Triangle is Symmetric



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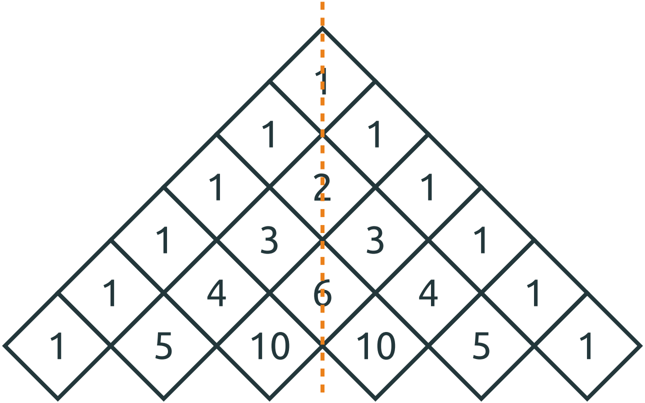

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Theorem

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Proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

Combinatorial Proof

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- this is just the number of ways of partitioning n students into two teams of size k and $n-k$

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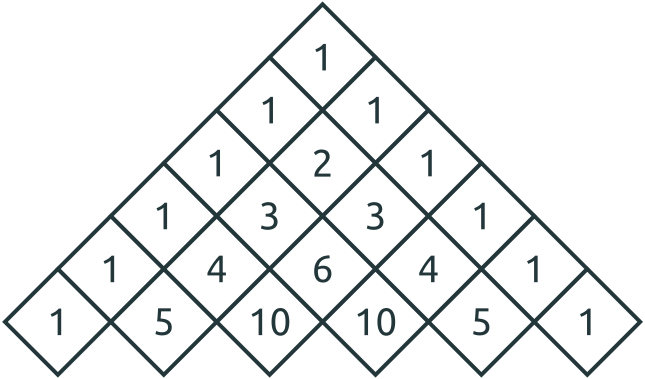

Row Sums

$$1$$
$$1 + 1$$
$$1 + 2 + 1$$
$$1 + 3 + 3 + 1$$
$$1 + 4 + 6 + 4 + 1$$
$$1 + 5 + 10 + 10 + 5 + 1$$

Row Sums

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & = 1 \\
 & & & & & 1 & + & 1 & & & & & = 2 \\
 & & & & 1 & + & 2 & + & 1 & & & & = 4 \\
 & & & 1 & + & 3 & + & 3 & + & 1 & & & = 8 \\
 & & 1 & + & 4 & + & 6 & + & 4 & + & 1 & & = 16 \\
 & 1 & + & 5 & + & 10 & + & 10 & + & 5 & + & 1 & = 32
\end{array}
$$

Theorem

The sum of all the numbers in the n-th row of Pascal's triangle is equal to 2^n:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof by Induction

- The base case (0-th row) holds

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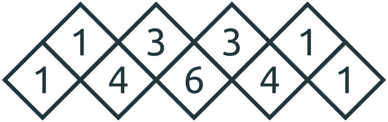
- We'll show that the sum of each row is twice the sum of the previous row:



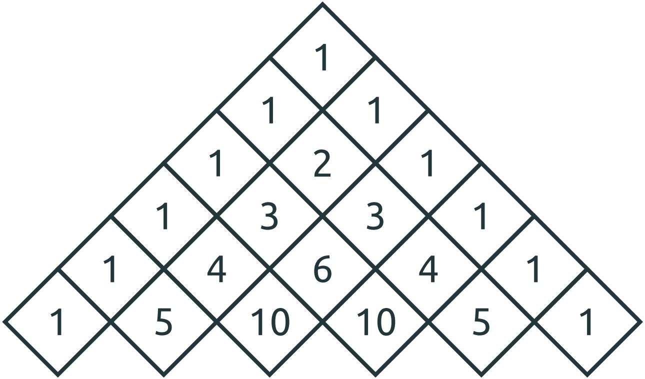
$$
\begin{array}{ccccc}
1 & 4 & 6 & 4 & 1 \\
1 & (1+3) & (3+3) & (3+1) & 1
\end{array}
$$

$$(1+1)\ (3+3)\ (3+3)\ (1+1)$$

Combinatorial Proof

- $\binom{n}{k}$ is the number of k-subsets of a set of size n
- the sum of $\binom{n}{k}$ for all k (from 0 to n) is the number of all subsets of an n element set
- this is 2^n by the product rule: each of the n elements is either included or not

Alternating Row Sums



Alternating Row Sums

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & - & 1 & & & & & \\
 & & & & 1 & - & 2 & + & 1 & & & & \\
 & & & 1 & - & 3 & + & 3 & - & 1 & & & \\
 & & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & \\
 & 1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 &
\end{array}
$$

Alternating Row Sums

$$
\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & \\
 & & & & 1 & - & 1 & & & & & & = 0 \\
 & & & 1 & - & 2 & + & 1 & & & & & = 0 \\
 & & 1 & - & 3 & + & 3 & - & 1 & & & & = 0 \\
 & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & & = 0 \\
1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
$$

Theorem

$$\text{For } n > 0, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\,.$$

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- For odd n, follows immediately from the symmetry property
- In general, can be shown by using the sum pattern of the triangle (each internal element is equal to the sum of the two elements above it)

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$$\binom{n}{1} + \binom{n}{3} + \cdots = \binom{n}{0} + \binom{n}{2} + \cdots$$

- Combinatorial meaning: the number of odd size subsets is the same as the number of even size subsets
- To prove this, we'll construct a one-to-one correspondence between odd size subsets and even size subsets

One-to-one Correspondence

- Fix any element x (can do this since $n > 0$)

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- Fix any element x (can do this since $n > 0$)
- Partition all subsets into pairs (A, B) where $A = B + x$ (more formally, $A = B \cup \{x\}$)
- One of A, B has odd size, the other one has even size

Example

$$S = \{a, b, c, d\}$$

Even size subsets	Odd size subsets
$\varnothing$	{a}
{a, b}	{b}
{a, c}	{c}
{a, d}	{d}
{b, c}	{a, b, c}
{b, d}	{a, b, d}
{c, d}	{a, c, d}
{a, b, c, d}	{b, c, d}

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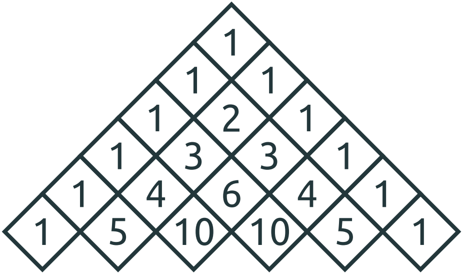

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$$

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Equivalently,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

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Can be shown by expanding the expression

$$(a+b)(a+b)\cdots(a+b)$$

Proof by Induction

$$
\begin{aligned}
&(a+b)^4 \\
&\qquad = (a+b)^3(a+b) \\
&\qquad = (a^3 + 3a^2b + 3ab^2 + b^3)(a+b) \\
&\qquad = a^4 + 3a^3b + 3a^2b^2 + ab^3 + \\
&\qquad \qquad + a^3b \quad + 3a^2b^2 + 3ab^3 + b^4 \\
&\qquad = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}
$$

Example

$$
\begin{aligned}
&(2a-b)^4 \\
&=((2a)+(-b))^4 \\
&=(2a)^4+4(2a)^3(-b)+6(2a)^2(-b)^2+4(2a)(-b)^3+(-b)^4 \\
&=16a^4-32a^3b+24a^2b^2-8ab^3+b^4
\end{aligned}
$$

Consequences

- Set $a = b = 1$. The number of subsets is 2^n:

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$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

- Set $a = 1$, $b = -1$. The number of odd size subsets is the same as the number of even size subsets:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

Another Consequence

- Set $a = 1$, $b = 2$:

$$3^n = \binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^n$$

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    - $\binom{n}{1}2$ is the number of words with exactly $n - 1$ letters x

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    - $\binom{n}{1}2$ is the number of words with exactly $n - 1$ letters x
    - $\binom{n}{2}2^2$ is the number of words with exactly $n - 2$ letters x