Examples, Counterexamples, Logic

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Outline

Examples

Counterexamples

Logic

Examples

- Sometimes, just one example is enough

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- If we want to prove that white horses exist...

Examples

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- If we want to prove that white horses exist...
- It is sufficient to show just one white horse

Examples

- Sometimes, just one example is enough
- If we want to prove that white horses exist...
- It is sufficient to show just one white horse
- However, such examples are not always easy to come up with

Problem

Is it possible that for three positive integer numbers a, b and c, $a^2 + b^2 = c^2$?

Solution

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$

To come up with this example, one can remember Pythagorean Theorem from school and a right triangle with sides 3, 4 and 5.

Problem

Is it possible that for three positive integer numbers a, b and c, $a^3 + b^3 = c^3$?

In fact, this is impossible, and so there is no example. Fermat's last theorem, one of the most famous mathematical conjectures, states that for any integer $n > 2$, there are no such integers a, b and c that $a^n + b^n = c^n$. Mathematicians were trying to prove unsuccessfully it for hundreds of years. These attempts led to a lot of useful mathematical theories. Finally, Andrew Wiles proved it in 1995.

Problem

Is it possible that for positive integers a, b, c and d, $a^4 + b^4 + c^4 = d^4$?

Solution

One could think that this is impossible due to Fermat's last theorem. However, it only says something about equations of the form $a^n + b^n = c^n$, so it is not applicable.

Solution

And in fact the answer is yes, this is possible. However, the smallest possible example is huge!

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

Computers are used to generate such examples, however, the number of possibilities can be so big that it is hard to find an example even with the help of a computer.

Problem

Is there a power of 2 that starts with 65?

Solution

$$2^{16} = 65536$$

This is the whole solution.

In fact, for any integer $n > 0$, there is a power of 2 that starts with this integer. But this is much harder to prove.

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- If we want to prove that all swans are white...

Counterexamples

- Just one counterexample is sufficient to disprove a statement
- If we want to prove that all swans are white...
- Just one black swan is sufficient to disprove it

Counterexamples

- Just one counterexample is sufficient to disprove a statement
- If we want to prove that all swans are white...
- Just one black swan is sufficient to disprove it
- However, it is often hard to find such counterexamples

Theorem

All rectangles are squares.

Counterexample

A rectangle with sides 1 and 2 is not a square — this is a counterexample for the theorem, so the theorem is wrong.

Theorem

All squares are rectangles.

In this case, there is no counterexample, and the theorem is true, as square is a rectangle with equal sides.

Euler's hypothesis

Euler came up with a generalization of Fermat's last theorem:

Theorem

For any $n > 2$, it is impossible for a n-th power of a positive integer to be represented as a sum of $n-1$ numbers which are n-th powers of positive integers.

For $n = 3$, it is the same as Fermat's last theorem: it is impossible that $a^3 + b^3 = c^3$.

Counterexample

However, in 1966 Lander found a counterexample for $n = 5$:

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

Counterexample

Also, in 1986 Elkies found another counterexample for $n = 4$:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

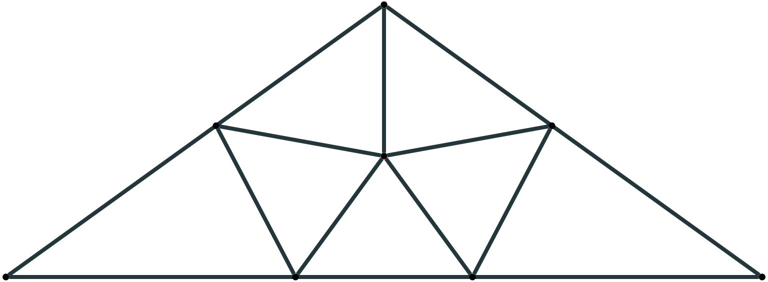
Counterexample

And in 1988 Frye found the smallest counterexample for $n = 4$:

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

We already know this as an example for the previous video about examples. The same example is an example for one statement and a counterexample for its opposite.

Reminder



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Logical Operations

- Negation
- Logical AND
- Logical OR
- If-then

Negation

Statement: all swans are white.

Negation: not all swans are white. Or, there are swans that are not white.

Negation

Statement: there exist three positive integers a, b and c, such that $a^3 + b^3 = c^3$.

Negation: there are no such positive integers a, b and c that $a^3 + b^3 = c^3$. Or, for any positive integers a, b and c, $a^3 + b^3 \neq c^3$.

Negation

Statement: $4 = 2 + 2$ Negation: $4 \neq 2 + 2$

Statement: $5 = 2 + 2$ Negation: $5 \neq 2 + 2$

Negation

Negation is true if and only if the initial statement is wrong, and vice versa.

Logical AND

$4 = 2 + 2$ AND $4 = 2 \cdot 2$

The logical AND of two statements is true if and only if both statements are true.

Logical AND

$4 = 2 + 2$ AND $4 = 2 \cdot 2$ — true

$4 = 2 + 2$ AND $5 = 2 \cdot 2$ — false

$5 = 2 + 2$ AND $4 = 2 \cdot 2$ — false

$5 = 2 + 2$ AND $5 = 2 \cdot 2$ — false

Logical OR

$4 = 2 + 2$ OR $5 = 2 + 2$

The logical OR of two statements is true if and only if at least one of them is true.

Logical OR

$4 = 2 + 2$ OR $4 = 2 \cdot 2$ — true

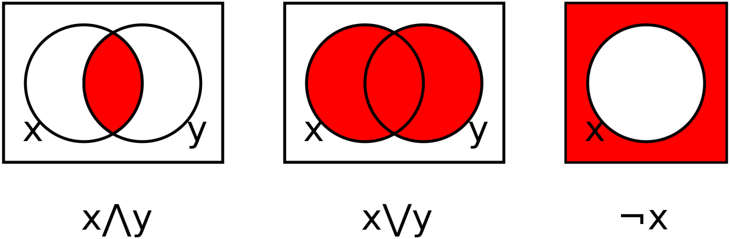
$4 = 2 + 2$ OR $5 = 2 \cdot 2$ — true

$5 = 2 + 2$ OR $4 = 2 \cdot 2$ — true

$5 = 2 + 2$ OR $5 = 2 \cdot 2$ — false

Venn Diagram

Logical NOT is denoted by ¬, logical AND is denoted by ∧, and logical OR is denoted by ∨.



wikipedia.org

Negation of AND

Negation of AND is OR of negations: negation of "A AND B" is "Not A OR Not B".

Negation of $4 = 2 + 2$ AND $4 = 2 \cdot 2$:

$4 \neq 2 + 2$ OR $4 \neq 2 \cdot 2$.

Negation of OR

Negation of OR is AND of negations: negation of “A OR B” is “Not A AND Not B”.
Negation of $4 = 2 + 2$ OR $5 = 2 + 2$:
$4 \neq 2 + 2$ AND $5 \neq 2 + 2$.

If-Then

Consider a promise: if there is an elephant in the refrigerator, then I'll give you 100 dollars.

If there is indeed an elephant in the refrigerator, and I indeed give you 100 dollars — the promise is kept, and the statement is true.

If-Then

Consider a promise: if there is an elephant in the refrigerator, then I’ll give you 100 dollars.

If there is no elephant, and I didn’t give you 100 dollars, I still kept the promise, and the statement is true.

If-Then

Consider a promise: if there is an elephant in the refrigerator, then I'll give you 100 dollars.

If there is an elephant in the refrigerator, but I don't give you 100 dollars, the promise is not kept, so the statement is false.

If-Then

Consider a promise: if there is an elephant in the refrigerator, then I'll give you 100 dollars.

The interesting case: there is no elephant in the refrigerator, but I gave you 100 dollars anyway, did I keep my promise?

If-Then

Consider a promise: if there is an elephant in the refrigerator, then I'll give you 100 dollars.

The interesting case: there is no elephant in the refrigerator, but I gave you 100 dollars anyway, did I keep my promise? Actually, in technical sciences, we consider it a kept promise, so the statement is true.

If-Then

In general, phrase “If P, Then Q” is true whenever either Q is true or P is false.

If-Then Examples

If $n = 6$, then n is even — true.

If $n = 5$, then n is even — false.

If $1 = 2$, then $2 = 3$ — true!

If $1 = 2$, then I am an elephant — true!

If-Then Generalization

“If n is divisible by 4, then n is divisible by 2” — actually means “for all n, if n is divisible by 4, then n is divisible by 2” — true:

For any number n, either it is divisible by 4, and so is divisible by 2, and both the “if” and the “then” parts are true, or it is not divisible by 4, so the “if” part is false, and if-then statement is true when the “if” part is false regardless of the “then” part.

If-Then Generalization

“If n is divisible by 2, then n is divisible by 4” — false, because $n = 2$ is divisible by 2, but not divisible by 4, so the “if” part is true, but the “then” part is false, so the if-then statement is false.

Direct and Converse

“If n is divisible by 4, then n is divisible by 2” — direct statement.
“If n is divisible by 2, then n is divisible by 4” — converse statement.

When both direct statement “If P, then Q” and converse statement “If Q, then P” are true, we write that they are equivalent like this: “P if and only if Q”.

Negation of If-Then

The negation of the phrase "If P, Then Q" is "P and not Q". Check that it is true whenever the corresponding If-Then is false, and vice versa.

Universal Quantification

Statements like

- All swans are white
- All integers ending with digit 2 are even
- For all n, $2 \cdot n = n + n$

are examples of universal quantification.

Fermat's last theorem states that for all $n > 2$ equation $a^n + b^n = c^n$ does not have solutions with positive integers a, b and c. Thus, it is also an example of universal quantification.

Existential Quantification

Statements like

- There are black swans
- There is a way to get a change of 12 cents with 4-cents and 5-cents coins
- There exist such integers a, b, c, d that $a^4 + b^4 + c^4 = d^4$
- There is a power of 2 starting with 65

are examples of existential quantification.

Combinations of Quantifiers

Mathematical statements are usually combinations of universal and existential quantifications. Here is a corollary from Fermat's last theorem:

Theorem

There exists such integer m that for any integer $n > m$ equation $a^n + b^n = c^n$ has no solutions with positive integers a, b and c.

Combinations of Quantifiers

Theorem

There exists such integer m that for any integer $n > m$ equation $a^n + b^n = c^n$ has no solutions with positive integers a, b and c.

If we take $m = 2$, then it follows from Fermat's last theorem. It is a combination of existential quantifier "there exists such integer m" and universal quantifier "for any integer $n > m$...".

Negation of Quantifications

Negation of universal quantification is a corresponding existential quantification, and vice versa.

Negation of “For all n, statement A is true” is “There exists such n that statement A is false”, and vice versa.

Negation of Quantifications

Euler's hypothesis is a combination of two universal quantifications: for any $n > 3$, for any positive integer a, it is impossible to represent a^n as a sum of $n - 1$ numbers which are n-th powers of positive integers.

Thus, negation of Euler's hypothesis is a statement that there exists such $n > 3$ and such positive integer a that a^n can be represented as a sum of $n - 1$ numbers which are n-th powers of positive integers.

Negation of Combinations

Negation of the statement “All positive integers are either even OR odd” is “There exists such positive integer n that it is not even AND not odd”.

To negate, we switch universal quantification to existential, and we switch OR to AND, and vice versa.

Examples and Counterexamples

Counterexample for Euler's hypothesis that we saw earlier is an example for the negation of Euler's hypothesis. By proving negation of Euler's hypothesis, we prove that Euler's hypothesis itself is false.

Reductio ad Absurdum

One popular way of proving mathematical statements is Reductio ad Absurdum — basically, proving that the negation of the initial statement is false, and so the statement itself is true.

You will learn this proof principle in the next lecture.

Conclusion

- Examples, counterexamples, and when one is enough
- Negation, logical AND, logical OR, If-Then
- Universal and existential quantifications
- Negation of different logical combinations
- Next lecture: reductio ad absurdum