Dice Game Problem

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Outline

Dice Game

Playing the game

Dice Game

- Consider the following situation

Bosch, “The Conjurer”

Dice Game

- Consider the following situation
- You are in a bad neighborhood

Bosch, "The Conjurer"

Dice Game

- Consider the following situation
- You are in a bad neighborhood
- There is a shady person on the corner of the street who offers bypassers to play a game with him

Bosch, “The Conjurer”

Dice Game

The rules of the game

- There are several dices with various numbers on their sides

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Dice Game

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- Both of you throw your dices

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- You and the shady person pick one dice each
- Both of you throw your dices
- Whoever has the larger number wins

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Dice Game

The rules of the game

- There are several dices with various numbers on their sides
- You and the shady person pick one dice each
- Both of you throw your dices
- Whoever has the larger number wins
- The winner gets 1 dollar from the loser

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Dice Game

- To give you advantage the shady person lets you to pick your dice first

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- So you can pick the one you find the best

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- So you can pick the one you find the best
- And he will have to pick from the remaining options
- So why not to win all shady person's money?
- Where is the catch?

Dice Game

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- No one will hit you on the head during the game
- No one will pick your pocket
- Disclamer: beware, all of these are not guaranteed to you in real life games with scammers!
- In our problem the shady person is not cheating: the game will be played exactly as described

Dice Game

- The game seems favorable to us

Dice Game

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- Yet, the shade person is eager to play

Dice Game

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- So, what’s wrong with this situation?

Dice Game

- The game seems favorable to us
- Yet, the shade person is eager to play
- So, what's wrong with this situation?
- It turns out that there is purely mathematical answer to this puzzle

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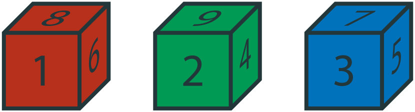
- Once we come closer we see that the shady person has just three dices

Playing the game

- Once we come closer we see that the shady person has just three dices
- And here they are:



- Dice 1 has numbers: 1, 1, 6, 6, 8, 8
  Dice 2 has numbers: 2, 2, 4, 4, 9, 9
  Dice 3 has numbers: 3, 3, 5, 5, 7, 7

Playing the game

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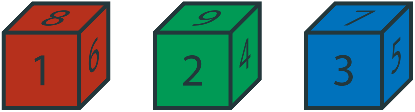
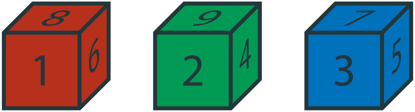


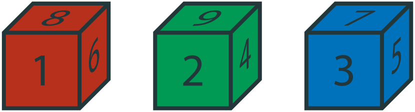
- Dice 1 has numbers: 1, 1, 6, 6, 8, 8
  Dice 2 has numbers: 2, 2, 4, 4, 9, 9
  Dice 3 has numbers: 3, 3, 5, 5, 7, 7
- Which one should we pick?

Playing the game



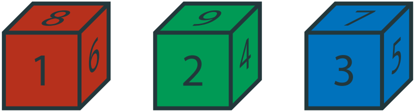
- We are educated

Playing the game



- We are educated
- Should compare dices and compute the probabilities

Playing the game



- We are educated
- Should compare dices and compute the probabilities
- Let’s start with Dice 1 and Dice 2

Dice 1 vs. Dice 2

- We have to consider all outcomes and count winning outcomes for each of the dices

Dice 1 vs. Dice 2

- We have to consider all outcomes and count winning outcomes for each of the dices

1,2	1,2	1,4	1,4	1,9	1,9
1,2	1,2	1,4	1,4	1,9	1,9
6,2	6,2	6,4	6,4	6,9	6,9
6,2	6,2	6,4	6,4	6,9	6,9
8,2	8,2	8,4	8,4	8,9	8,9
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- Dice 1 wins in 16 outcomes

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8,2	8,2	8,4	8,4	8,9	8,9

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- Dice 2 wins in 20 outcomes

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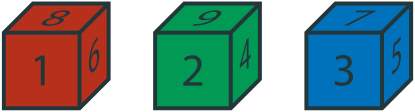
- Dice 2 wins in 20 outcomes
- Dice 2 wins with probability $\frac{20}{36} = \frac{5}{9} > \frac{1}{2}$

Playing the game



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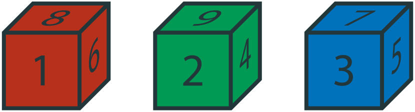
Playing the game



- Dice 2 is better than Dice 1
- Let’s compare Dice 2 with Dice 3 and find the best dice!

Dice 2 vs. Dice 3

- We have to consider all outcomes and count winning outcomes for each of the dices

Dice 2 vs. Dice 3

- We have to consider all outcomes and count winning outcomes for each of the dices

2,3	2,3	2,5	2,5	2,7	2,7
2,3	2,3	2,5	2,5	2,7	2,7
4,3	4,3	4,5	4,5	4,7	4,7
4,3	4,3	4,5	4,5	4,7	4,7
9,3	9,3	9,5	9,5	9,7	9,7
9,3	9,3	9,5	9,5	9,7	9,7

Dice 2 vs. Dice 3

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4,3	4,3	4,5	4,5	4,7	4,7
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4,3	4,3	4,5	4,5	4,7	4,7
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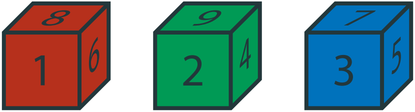

- Dice 2 is better than Dice 1

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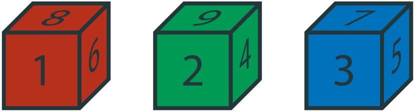

- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2

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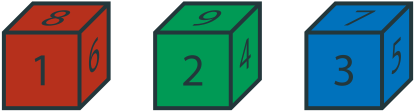
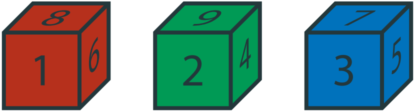


- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Clearly, Dice 3 is better than Dice 1 and we are done

Playing the game



- Dice 2 is better than Dice 1
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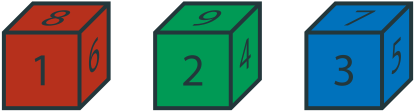
- Or are we?

Playing the game



- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Clearly, Dice 3 is better than Dice 1 and we are done
- Or are we?
- Let's check

Dice 3 vs. Dice 1

- We have to consider all outcomes and count winning outcomes for each of the dices

Dice 3 vs. Dice 1

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3,1	3,1	3,6	3,6	3,8	3,8
3,1	3,1	3,6	3,6	3,8	3,8
5,1	5,1	5,6	5,6	5,8	5,8
5,1	5,1	5,6	5,6	5,8	5,8
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- Dice 1 wins in 20 outcomes
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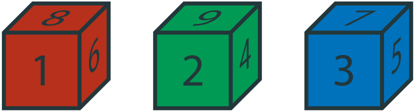
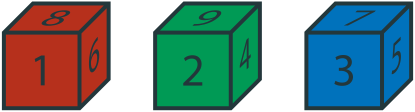
Playing the game



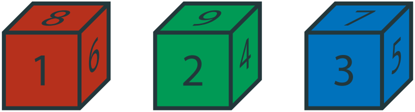
- Dice 2 is better than Dice 1

Playing the game



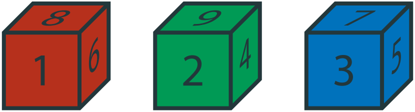
- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2 ...

Playing the game



- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2 ...
- But Dice 1 is better than Dice 3!

Playing the game



- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2 ...
- But Dice 1 is better than Dice 3!
- How is this even possible?

Numbers vs. Random Variables

- We are used to compare numbers

Numbers vs. Random Variables

- We are used to compare numbers
- And we are used that certain properties hold

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Numbers vs. Random Variables

- We are used to compare numbers
- And we are used that certain properties hold
- One of them: if $a > b$ and $b > c$, then $a > c$
- This is called transitivity
- This translates to real life experience: faster, higher, stronger

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- But random variables are not numbers!

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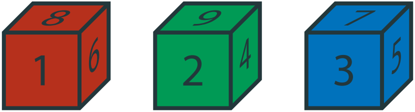
Numbers vs. Random Variables

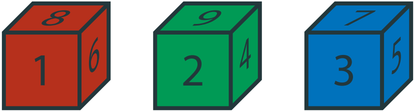
- But random variables are not numbers!
- It is way harder to compare them
- If we find some way of comparison, still usual properties are not guaranteed!
- For instance: no transitivity in our game!

Who Wins the Game?



- But what our discovery means for the game?

Who Wins the Game?



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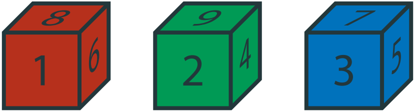

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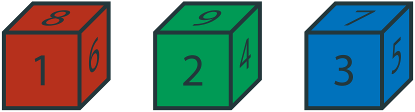

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- Dice 1 is better than Dice 3

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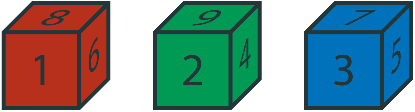

- But what our discovery means for the game?
- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Dice 1 is better than Dice 3
- The shady person, who allowed us to choose a dice first, is actually having an advantage because of that!

Who Wins the Game?

Dice 2 is better than Dice 1

Dice 3 is better than Dice 2

Dice 1 is better than Dice 3

How the shady person should play the game:

Who Wins the Game?

Dice 2 is better than Dice 1

Dice 3 is better than Dice 2

Dice 1 is better than Dice 3

How the shady person should play the game:

- If we pick Dice 1 , the shady person picks Dice 2

Who Wins the Game?

Dice 2 is better than Dice 1

Dice 3 is better than Dice 2

Dice 1 is better than Dice 3

How the shady person should play the game:

- If we pick Dice 1 , the shady person picks Dice 2
- If we pick Dice 2 , the shady person picks Dice 3

Who Wins the Game?

Dice 2 is better than Dice 1
Dice 3 is better than Dice 2
Dice 1 is better than Dice 3

How the shady person should play the game:

- If we pick Dice 1 , the shady person picks Dice 2
- If we pick Dice 2 , the shady person picks Dice 3
- If we pick Dice 3 , the shady person picks Dice 1

Main Lessons

- The probability is tricky!

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- One should be very careful when applying usual intuition to probability

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- The probability is tricky!
- One should be very careful when applying usual intuition to probability
- One should avoid scam games