Random Variables and Expectations

Vladimir Podolskii

Computer Science Department, Higher School of Economics

Outline

Random Variables

Average

Expectation

Random Variables

- We have studied probability distributions

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- It is important to study numerical characteristics of random outcomes
- So we introduce random variables

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- To define f we assign a number a_i to each outcome
- Then f has value a_i with probability p_i

Random Variables

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- Outcomes of the dice throw are labeled by numbers

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Random Variables

Other examples:

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- Tossing a coin: heads=0, tail=1
- An age of a random person in the class
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- Sum of outcomes of two dice throws

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- It is called arithmetic mean in mathematics

Example

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- We got lucky and the answer is integer; this is not guaranteed

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- We will fire everyone except one best employee

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- No, it is a random variable
- But we can give an approximation that is good with high probability

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- Thus the average is approximately 3.5
- This is an expected value or expectation of a dice throw

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- Let's repeat the random experiment many times

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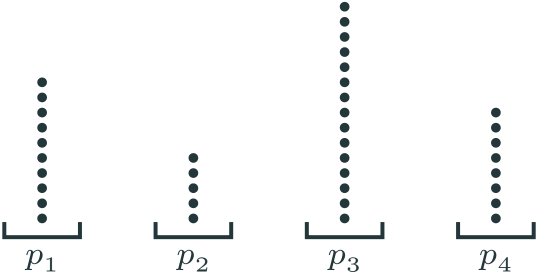

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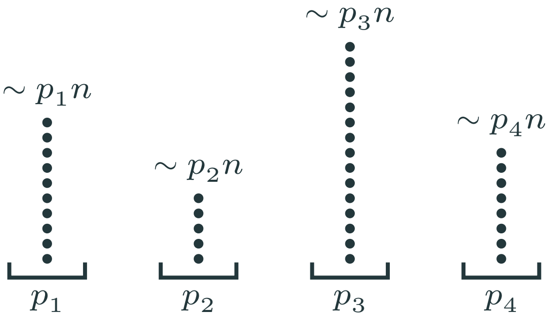

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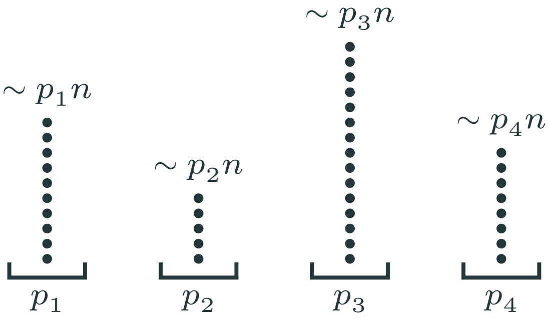

- Repeat n times, where n is large
- What is the average value of f on these outcomes?

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- Does not depend on n
- An approximation to what we would expect as an average outcome of an experiment repeated many times

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- Expectation is a number!
- An important characteristic of a random variable

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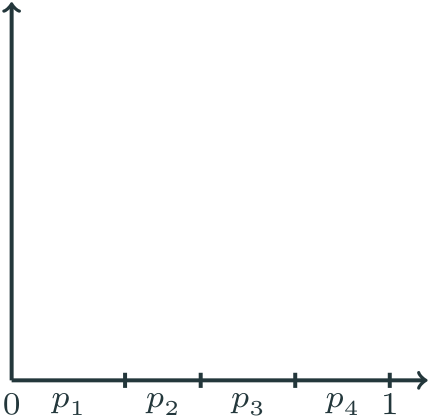

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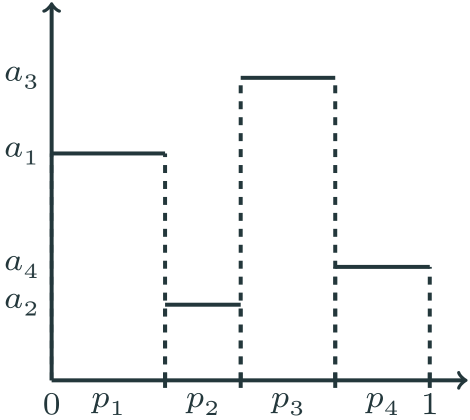

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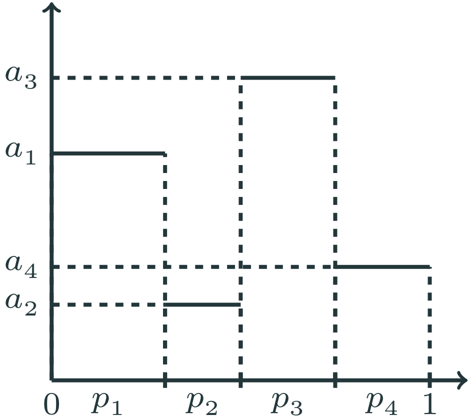

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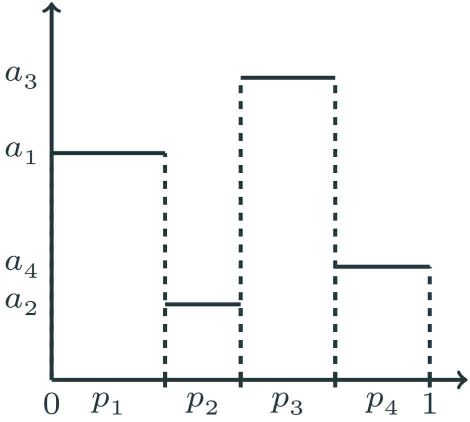

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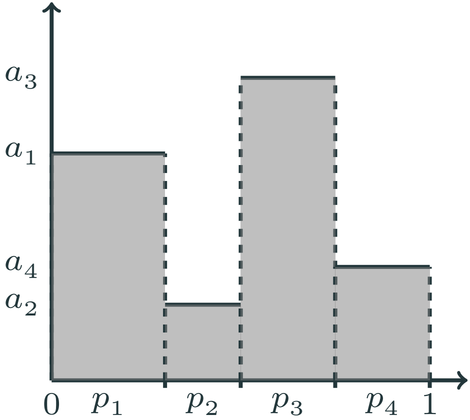

- $\mathsf{E}f$ is the area of the gray region

Real Life Examples

- Expectations occur everywhere in statistics and sociology

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- Average age

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- Average grades and evaluations