

# Combinations with Repetitions

---

Vladimir Podolskii

Computer Science Department, Higher School of Economics

# Outline

Review

Salad

Combinations with Repetitions

# Review

We considered selections of  $k$  items out of  $n$  possible options

# Review

We considered selections of  $k$  items out of  $n$  possible options

	With repetitions	Without repetitions
Ordered		
Unordered		

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Consider  $k = 2$  and  $n = 3$  options: a, b, c

	With repetitions	Without repetitions
Ordered		
Unordered		

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Consider  $k = 2$  and  $n = 3$  options: a, b, c

	With repetitions	Without repetitions
Ordered	(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)	
Unordered		

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Consider  $k = 2$  and  $n = 3$  options: a, b, c

	With repetitions	Without repetitions
Ordered	(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)	(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)
Unordered		

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Consider  $k = 2$  and  $n = 3$  options: a, b, c

	With repetitions	Without repetitions
Ordered	(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)	(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)
Unordered		{a, b}, {a, c}, {b, c}



# Review

We considered selections of  $k$  items out of  $n$  possible options  
Consider  $k = 2$  and  $n = 3$  options: a, b, c

	With repetitions	Without repetitions
Ordered	(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)	(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)
Unordered	{a, b}, {a, c}, {b, c} {a, a}, {b, b}, {c, c}	{a, b}, {a, c}, {b, c}

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Current state of things:

	With repetitions	Without repetitions
Ordered		
Unordered		

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Current state of things:

	With repetitions	Without repetitions
Ordered	Tuples $n^k$	
Unordered		

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Current state of things:

	With repetitions	Without repetitions
Ordered	Tuples $n^k$	$k$ -permutations $\frac{n!}{(n-k)!}$
Unordered		

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Current state of things:

	With repetitions	Without repetitions
Ordered	Tuples $n^k$	$k$ -permutations $\frac{n!}{(n-k)!}$
Unordered		Combinations $\binom{n}{k}$

# Review

We considered selections of  $k$  items out of  $n$  possible options  
Current state of things:

	With repetitions	Without repetitions
Ordered	Tuples $n^k$	$k$ -permutations $\frac{n!}{(n-k)!}$
Unordered	?	Combinations $\binom{n}{k}$

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

Ballot	
<input type="checkbox"/>	Candidate 1
<input type="checkbox"/>	Candidate 2
...	...
<input type="checkbox"/>	Candidate n



# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

Ballot	
<input checked="" type="checkbox"/>	Candidate 1
<input type="checkbox"/>	Candidate 2
...	...
<input type="checkbox"/>	Candidate n

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

Ballot

<input type="checkbox"/>	Candidate 1
<input checked="" type="checkbox"/>	Candidate 2
...	...
<input type="checkbox"/>	Candidate n

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

Ballot

☐ Candidate 1

☐ Candidate 2

... ..

☒ Candidate n

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

- All votes equally matter

Ballot	
<input type="checkbox"/>	Candidate 1
<input type="checkbox"/>	Candidate 2
...	...
<input type="checkbox"/>	Candidate n

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

- All votes equally matter
- So votes are unordered

Ballot	
<input type="checkbox"/>	Candidate 1
<input type="checkbox"/>	Candidate 2
...	...
<input type="checkbox"/>	Candidate n

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

- All votes equally matter
- So votes are unordered
- Candidates can be voted for several times

Ballot	
<input type="checkbox"/>	Candidate 1
<input type="checkbox"/>	Candidate 2
...	...
<input type="checkbox"/>	Candidate n

# Example: Voting

There are  $k$  voters that vote for one of  $n$  candidates

- All votes equally matter
- So votes are unordered
- Candidates can be voted for several times
- So voters as a group pick  $k$  people out of  $n$  with repetitions

Ballot	
<input type="checkbox"/>	Candidate 1
<input type="checkbox"/>	Candidate 2
...	...
<input type="checkbox"/>	Candidate $n$

# Outline

Review

Salad

Combinations with Repetitions



# Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers and lettuce. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

# Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers and lettuce. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

- We pick 4 items out of 3 options **with repetitions**

# Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers and lettuce. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

- We pick 4 items out of 3 options **with repetitions**
- **Order does not matter**

# Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers and lettuce. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

- We pick 4 items out of 3 options **with repetitions**
- **Order does not matter**
- So this is our setting

# Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers and lettuce. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

- We pick 4 items out of 3 options **with repetitions**
- **Order does not matter**
- So this is our setting
- Still do not know how to count

# Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers and lettuce. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

- We pick 4 items out of 3 options **with repetitions**
- **Order does not matter**
- So this is our setting
- Still do not know how to count
- We will **list** all possible salads, then count them

# Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers and lettuce. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

- We pick 4 items out of 3 options **with repetitions**
- **Order does not matter**
- So this is our setting
- Still do not know how to count
- We will **list** all possible salads, then count them
- But we want to do it wisely

# Salad



— tomato



— bell pepper



— lettuce





# Salad



— tomato



— bell pepper



— lettuce



# Salad



— tomato



— bell pepper



— lettuce



The same salad

# Salad



— tomato



— bell pepper



— lettuce



The same salad

- The order does not matter

# Salad



— tomato



— bell pepper



— lettuce



The same salad



- The order does not matter
- So let's draw tomatoes first, then bell peppers, then lettuce

# Salad



— tomato



— bell pepper



— lettuce



- The order does not matter
- So let's draw tomatoes first, then bell peppers, then lettuce

# Salad



- The order does not matter
- So let's draw tomatoes first, then bell peppers, then lettuce
- Let's consider **all possible numbers of tomatoes** in the salad and count in each case separately

Salad



# Salad



Case 1: 4 tomatoes



# Salad



Case 1: 4 tomatoes

# Salad



Case 1: 4 tomatoes

- 4 tomatoes: 1 salad

# Salad



Case 2: 3 tomatoes

- 4 tomatoes: 1 salad

# Salad



Case 2: 3 tomatoes

- 4 tomatoes: 1 salad

# Salad



Case 2: 3 tomatoes

- 4 tomatoes: 1 salad

# Salad



Case 2: 3 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

# Salad



Case 3: 2 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

# Salad



Case 3: 2 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads



# Salad



Case 3: 2 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

# Salad



Case 3: 2 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

# Salad



Case 3: 2 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads

# Salad



Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads

# Salad



Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads

# Salad



Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads

# Salad



Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads

# Salad



Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads



# Salad



Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads

# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads

# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads

# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads

# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads

# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads

# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads

# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads
- 0 tomatoes: 5 salads



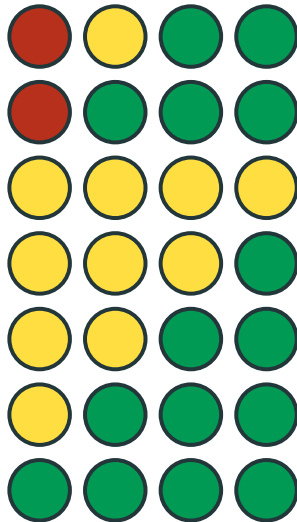
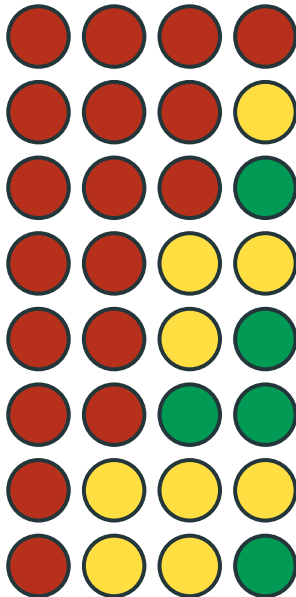
# Salad



Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads
- 0 tomatoes: 5 salads
- In total: 15 salads

# List of all Salads



# Summary

- The solution looks very structured

# Summary

- The solution looks very structured
- Same structure for larger salads

# Summary

- The solution looks very structured
- Same structure for larger salads
- But more complicated for more ingredients

# Summary

- The solution looks very structured
- Same structure for larger salads
- But more complicated for more ingredients
- Yet, the same strategy works for recursive counting for any salad size and any number of ingredients

# Outline

Review

Salad

Combinations with Repetitions

# Large Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?



# Large Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

- We can use recursive counting here as well

# Large Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

- We can use recursive counting here as well
- But now we will obtain a formula

# Large Salad

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

- We can use recursive counting here as well
- But now we will obtain a formula
- This will be a general solution

# Large Salad



— tomato



— bell pepper



— lettuce



— eggplant



# Large Salad



— tomato



— bell pepper



— lettuce



— eggplant



- The order does not matter

# Large Salad



— tomato



— bell pepper



— lettuce



— eggplant



- The order does not matter
- Let's list first tomatoes, then bell pepper, then lettuce, then eggplant

## Large Salad



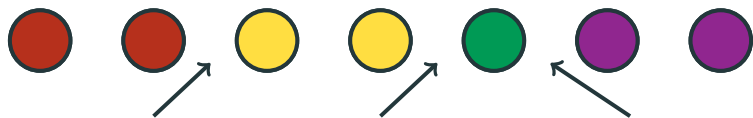
# Large Salad



- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch



# Large Salad



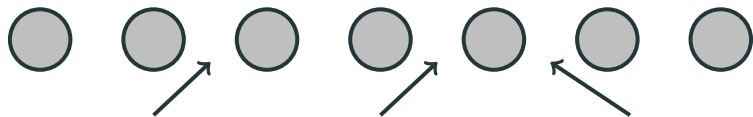
Tomatoes are  
on the left

Bell peppers are  
on the left

Eggplants are  
on the left

- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch

# Large Salad



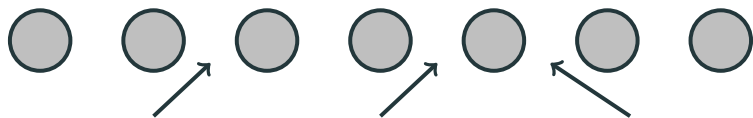
Tomatoes are  
on the left

Bell peppers are  
on the left

Eggplants are  
on the left

- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch

# Large Salad



Tomatoes are  
on the left

Bell peppers are  
on the left

Eggplants are  
on the left

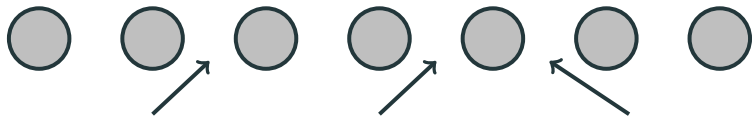
- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch
- **Idea 2:** Do not even need the text descriptions

# Large Salad



- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch
- **Idea 2:** Do not even need the text descriptions

# Large Salad



- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch
- **Idea 2:** Do not even need the text descriptions
- **Idea 3:** Can represent places of switch as delimiter signs

# Large Salad



- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch
- **Idea 2:** Do not even need the text descriptions
- **Idea 3:** Can represent places of switch as delimiter signs

# Large Salad



- **Idea 1:** to specify the list it is enough to indicate where the ingredients switch
- **Idea 2:** Do not even need the text descriptions
- **Idea 3:** Can represent places of switch as delimiter signs
- The salad can still be restored: tomatoes are on the left from the left delimiter, bell peppers are next, and so on

## Large Salad



- What if there are no, say, bell peppers in the original salad?



# Large Salad



- What if there are no, say, bell peppers in the original salad?
- This is fine

# Large Salad



- What if there are no, say, bell peppers in the original salad?
- This is fine
- Now, to specify the salad we need to pick three positions among 10 to place delimiters

# Large Salad



- What if there are no, say, bell peppers in the original salad?
- This is fine
- Now, to specify the salad we need to pick three positions among 10 to place delimiters
- These are **combinations**! The answer to the problem is  $\binom{10}{3} = 120$ !

# How We Got There

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

Main ideas:

# How We Got There

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

Main ideas:

- Order salad in a convenient way

# How We Got There

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

Main ideas:

- Order salad in a convenient way
- Salad is determined by delimiters between types of ingredients

# How We Got There

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

Main ideas:

- Order salad in a convenient way
- Salad is determined by delimiters between types of ingredients
- Place delimiters in the line with ingredients

# How We Got There

## Problem

We have an unlimited supply of tomatoes, bell peppers, lettuce and eggplant. We want to make a salad out of 7 units among these four ingredients (we do not have to use all ingredients). How many different salads we can make?

Main ideas:

- Order salad in a convenient way
- Salad is determined by delimiters between types of ingredients
- Place delimiters in the line with ingredients
- It is left to choose delimiters in the line — old problem



# General case

## Combinations with Repetitions

The number of combinations of size  $k$  of  $n$  objects with repetitions is equal to  $\binom{k+n-1}{n-1}$

# General case

## Combinations with Repetitions

The number of combinations of size  $k$  of  $n$  objects with repetitions is equal to  $\binom{k+n-1}{n-1}$

- Size of the combination = size of salad

# General case

## Combinations with Repetitions

The number of combinations of size  $k$  of  $n$  objects with repetitions is equal to  $\binom{k+n-1}{n-1}$

- Size of the combination = size of salad
- Number of objects = number of ingredients

# General case

## Combinations with Repetitions

The number of combinations of size  $k$  of  $n$  objects with repetitions is equal to  $\binom{k+n-1}{n-1}$

- Size of the combination = size of salad
- Number of objects = number of ingredients
- The same argument works

# General case

## Combinations with Repetitions

The number of combinations of size  $k$  of  $n$  objects with repetitions is equal to  $\binom{k+n-1}{n-1}$

- Size of the combination = size of salad
- Number of objects = number of ingredients
- The same argument works
- Why  $k+n-1$  and  $n-1$ ?

# General case

## Combinations with Repetitions

The number of combinations of size  $k$  of  $n$  objects with repetitions is equal to  $\binom{k+n-1}{n-1}$

- Size of the combination = size of salad
- Number of objects = number of ingredients
- The same argument works
- Why  $k+n-1$  and  $n-1$ ?
- $n$  ingredients mean  $n-1$  delimiters; choosing  $(n-1)$  element in the line of  $k+(n-1)$  elements

# Standard Settings

We considered selections of  $k$  items out of  $n$  possible options

# Standard Settings

We considered selections of  $k$  items out of  $n$  possible options

	With repetitions	Without repetitions
Ordered	Tuples $n^k$	$k$ -permutations $\frac{n!}{(n-k)!}$
Unordered	Combinations with repetitions $\binom{k+n-1}{n-1}$	Combinations $\binom{n}{k}$