Even and Odd Numbers

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Outline

Even and Odd Numbers

Piece on a Chessboard

Summing up Digits

Switching Signs

Advanced Signs Switching

Even and Odd Numbers

- Even numbers are integer numbers divisible by 2

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Even and Odd Numbers

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- What about 0? Also even

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Even and Odd Numbers

- The properties of a number to be even and odd are important invariants

geograph.org.uk

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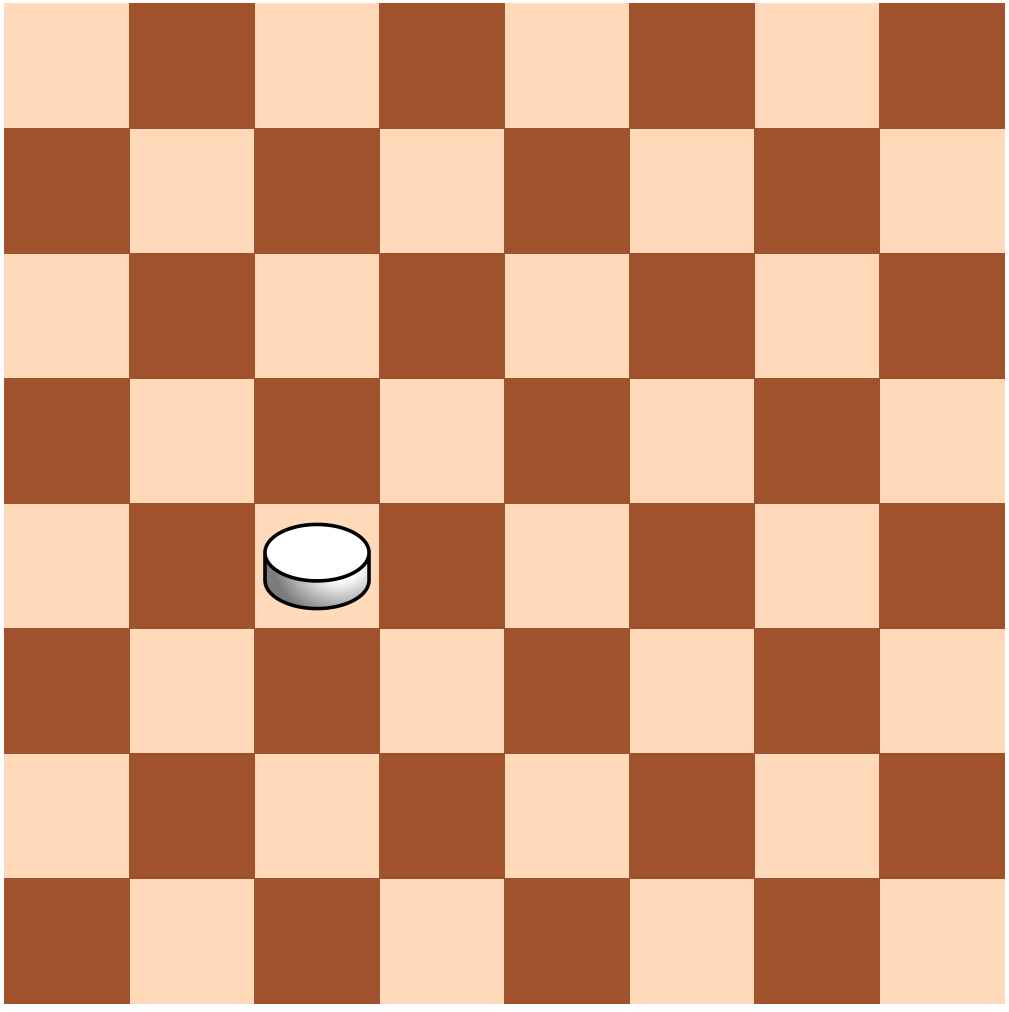
Puzzle

A piece on a chessboard can move to any cell adjacent by edge to the current one. Can it return to the original position after 17 moves? What about 18 moves?

Piece on a Chessboard

Let's start with a more simple case of 18 moves



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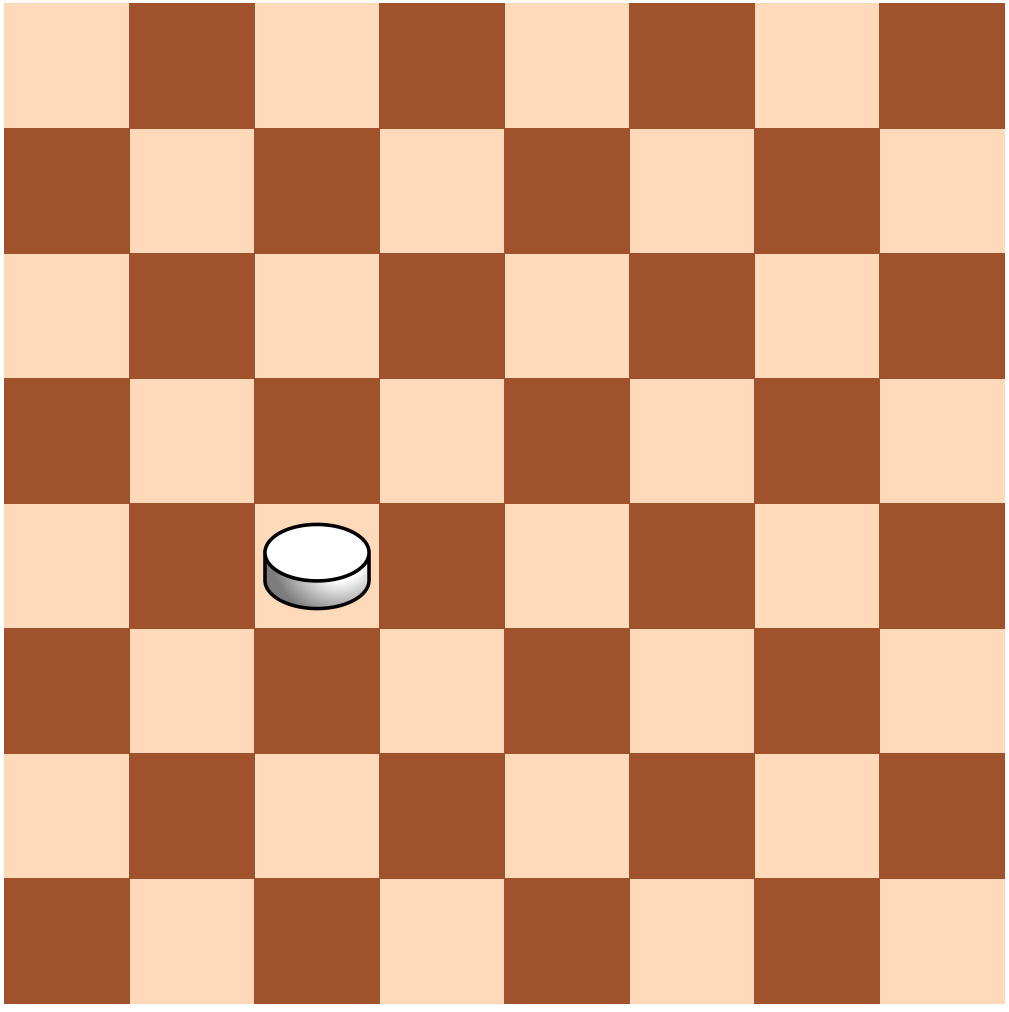

We can return after 2 steps

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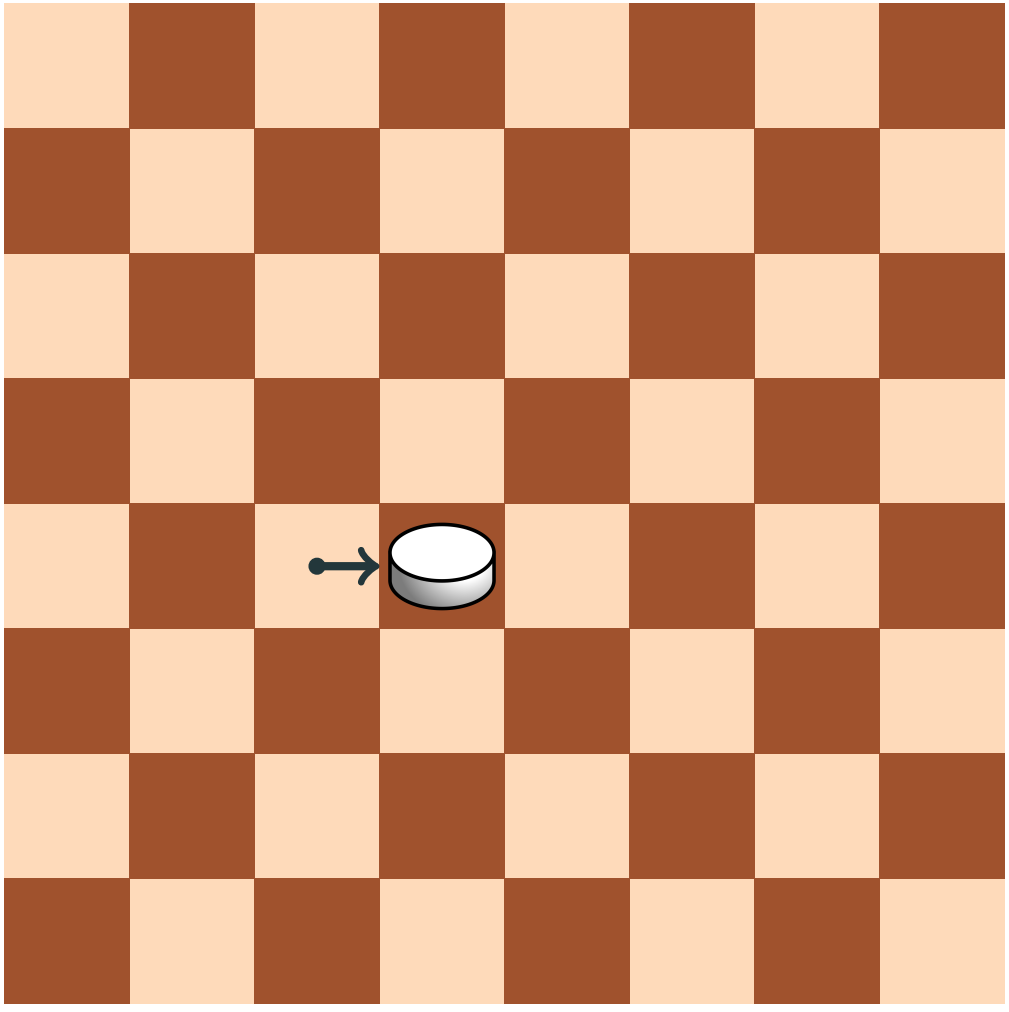

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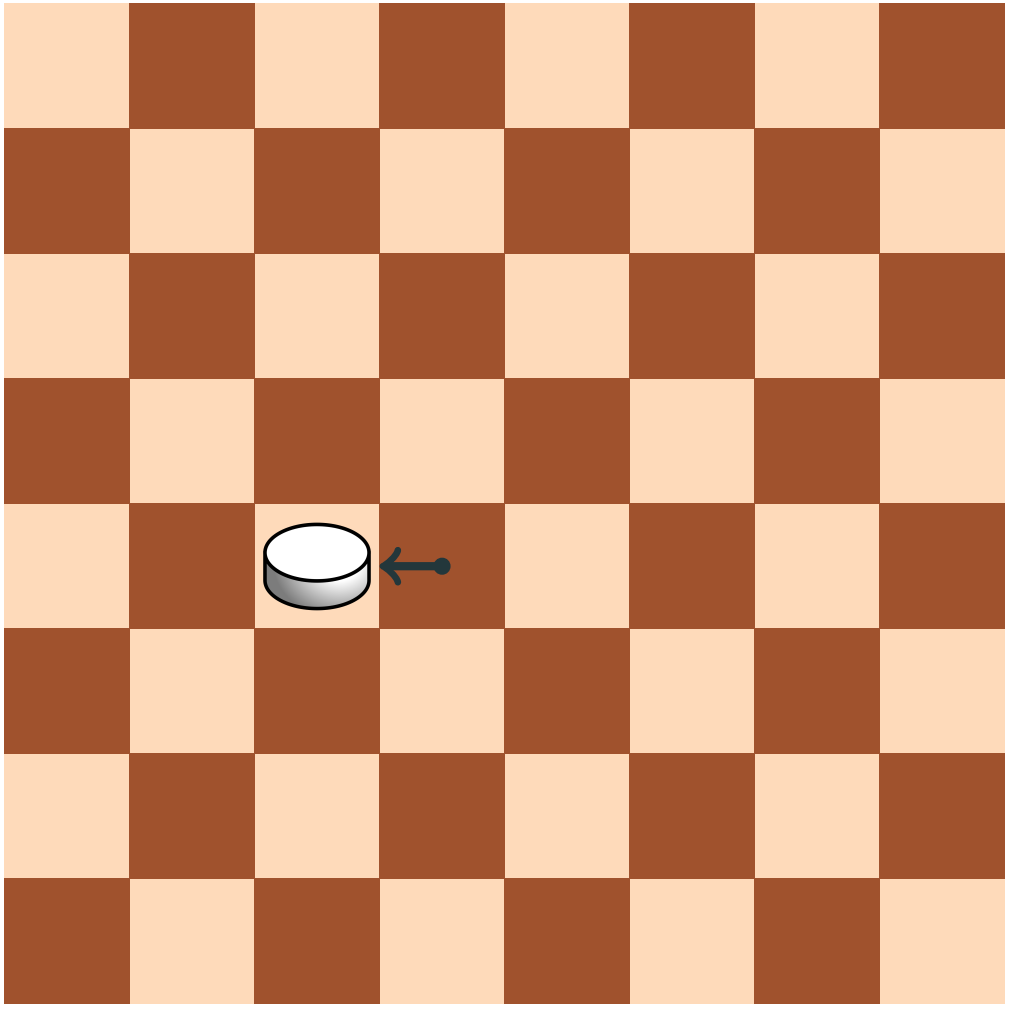

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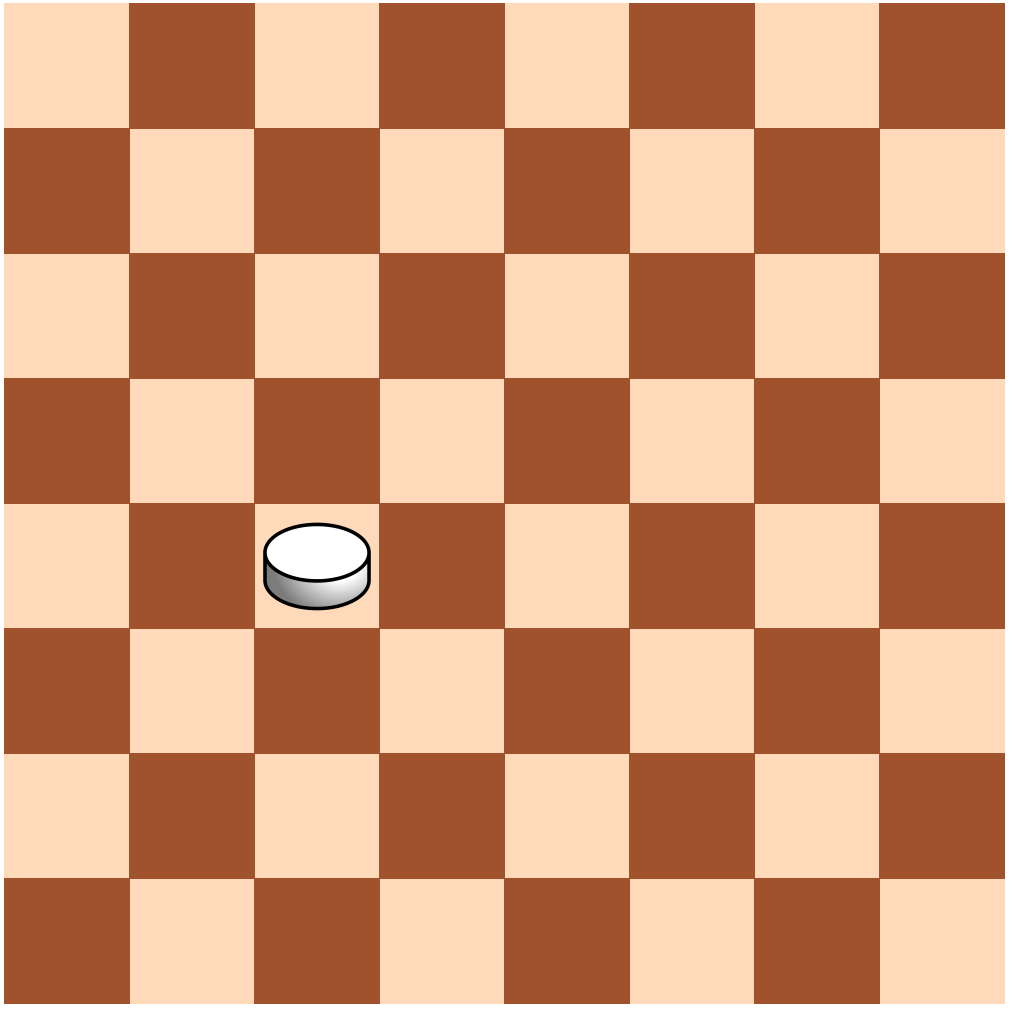
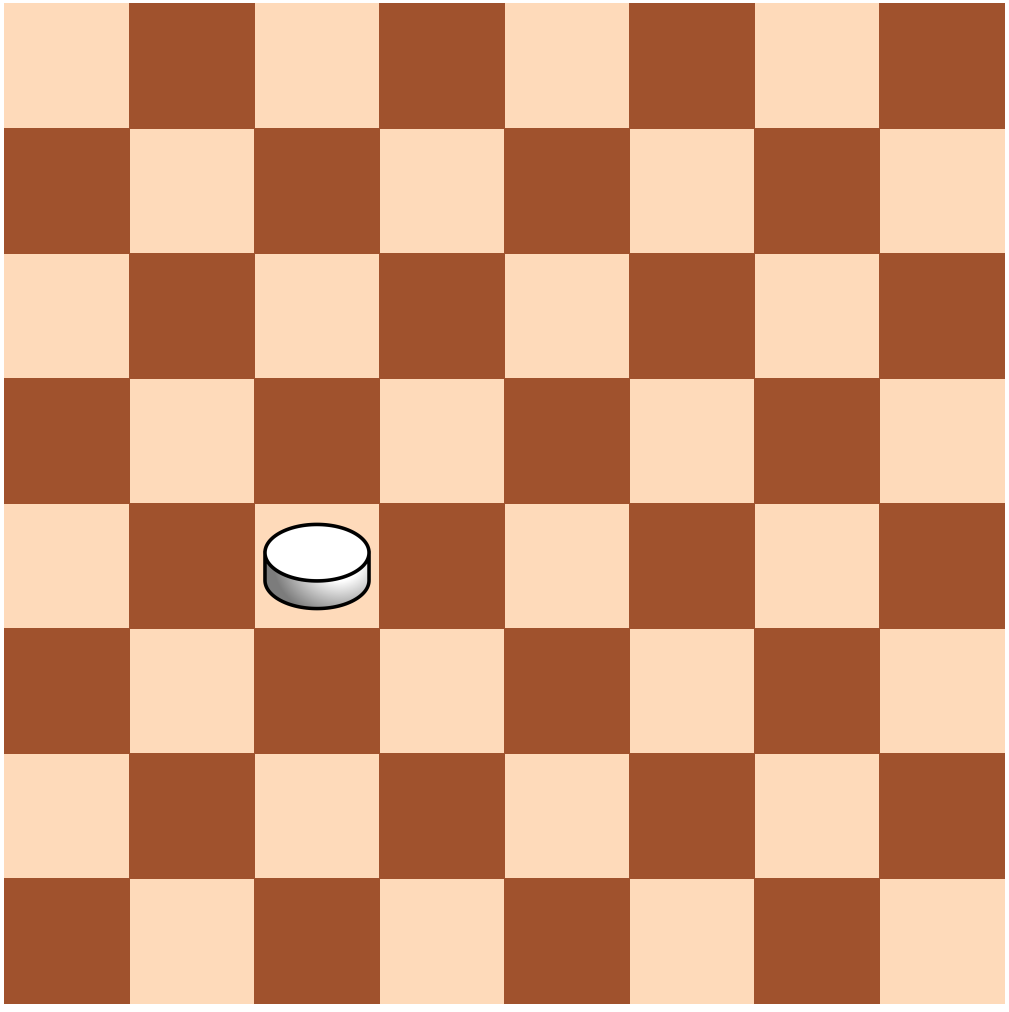


We can return after 2 steps

Repeat 9 times

Piece on a Chessboard

The same argument does not work for 17 steps



Indeed, 17 is odd

Piece on a Chessboard

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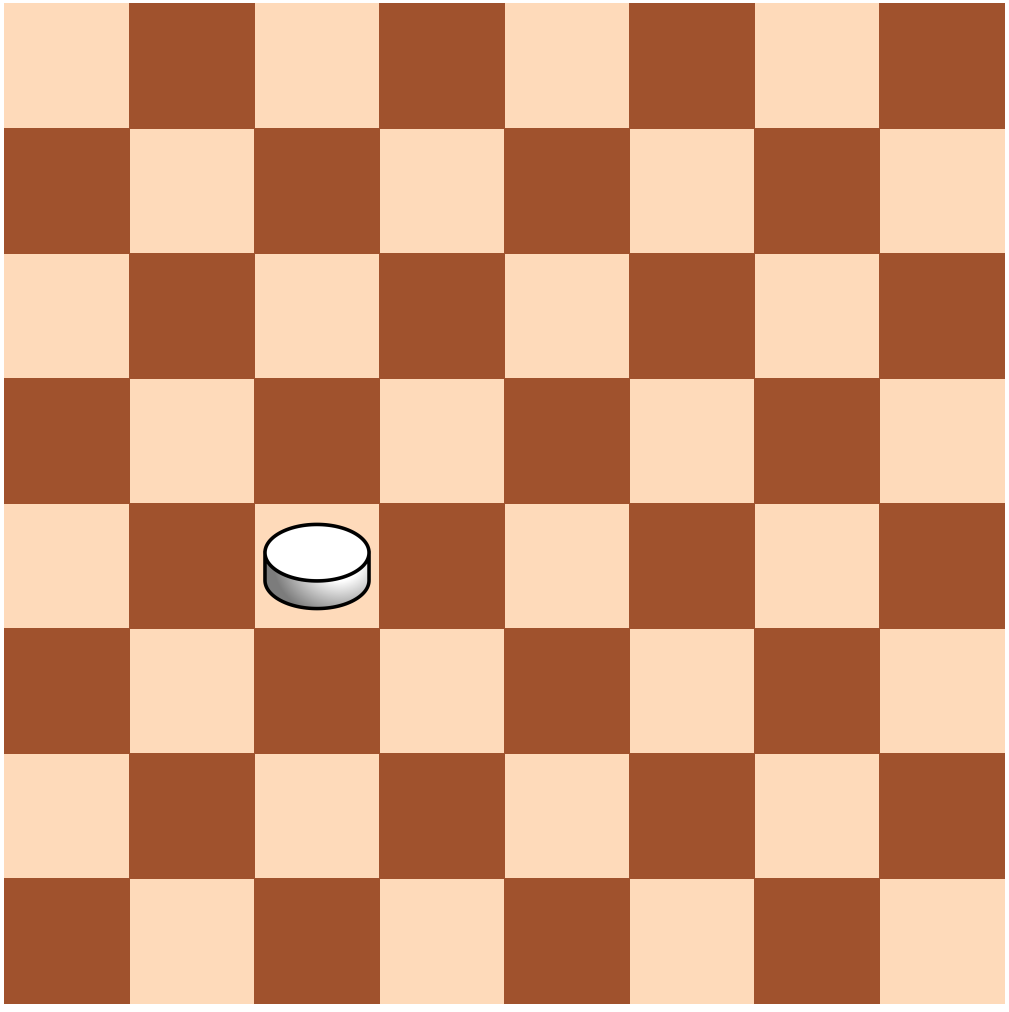

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Observation: after even number of steps a piece is on the light field

After odd number of steps it is on the dark field

Piece on a Chessboard

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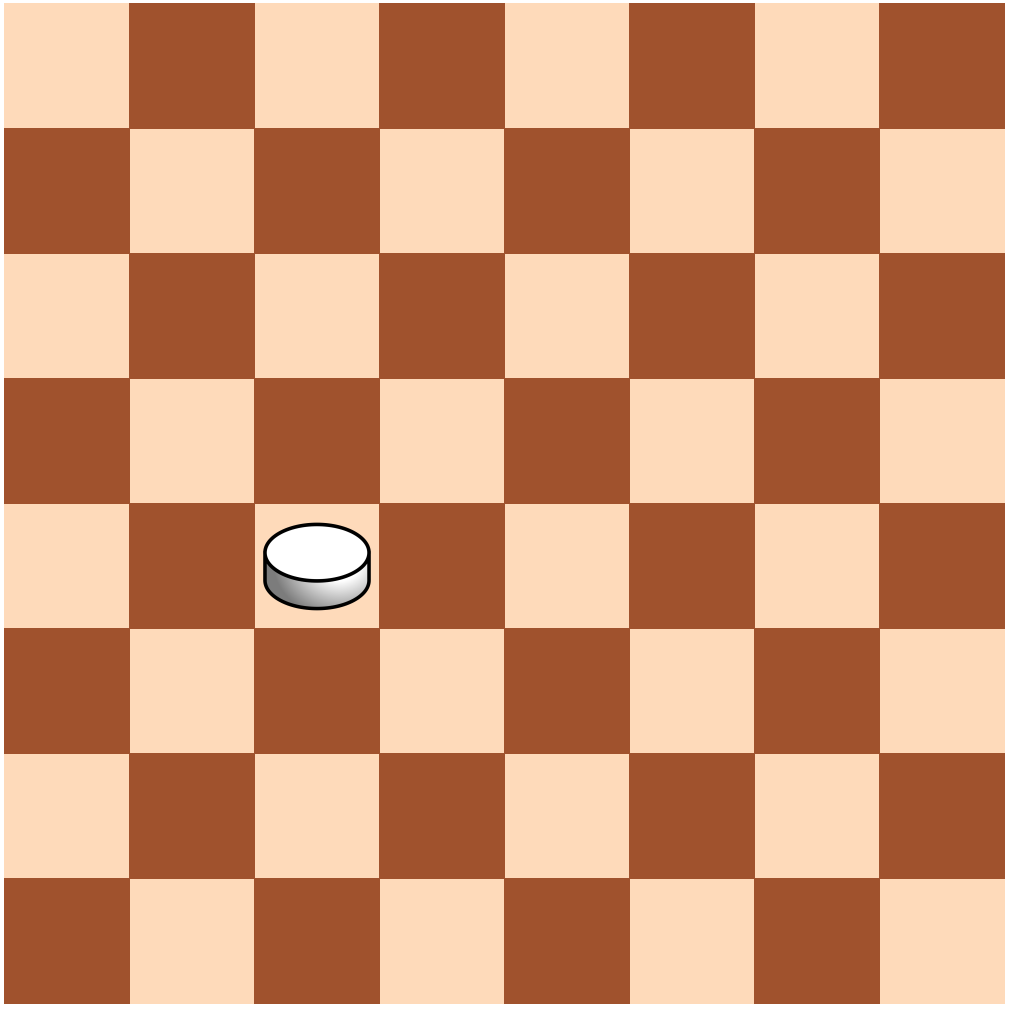

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Observation: after even number of steps a piece is on the light field

After odd number of steps it is on the dark field

No way to get back after odd number of steps

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Is it possible to place signs in the expression $\pm 1 \pm 2 \pm \ldots \pm 9$ to get as a result the sum 100? Can we get 2?

$$\pm\ 1\ \pm\ 2\ \pm\ 3\ \pm\ 4\ \pm\ 5\ \pm\ 6\ \pm\ 7\ \pm\ 8\ \pm\ 9$$

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- Let's start with 100

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- Let's start with 100
- We get the largest sum if place '$+$' everywhere

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- Let's start with 100
- We get the largest sum if place '$+$' everywhere
- So we cannot get anything greater than 45

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- What about the sum 2?
- Note that there are 4 even numbers and 5 odd numbers in the sequence

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- No matter how we put the signs, the sum will be odd

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- What about the sum 2?
- Note that there are 4 even numbers and 5 odd numbers in the sequence
- No matter how we put the signs, the sum will be odd
- So the sum 2 is impossible

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Which sums is it possible to get by place signs in the expression $\pm 1 \pm 2 \pm \ldots \pm 9$?

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- We have seen two obstacles: the sum is odd and the sum is at most 45 (and symmetrically, at least -45)

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- But what about other sums?
- It turns out, all sums satisfying restrictions are possible! We will not discuss it, but you can try to show it yourself

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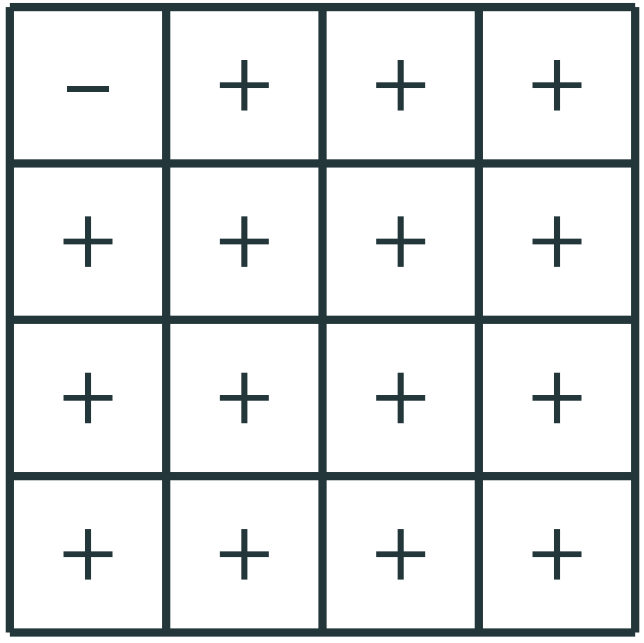
- But what about other sums?
- It turns out, all sums satisfying restrictions are possible! We will not discuss it, but you can try to show it yourself
- For this consider the following "greedy" algorithm. Start from right to left and place the signs greedily:
  if the current sum is less than the goal, increase the sum, otherwise decrease it

Outline

Switching Signs

Puzzle

We are given a 4×4 table. In left top corner there is '$-$' sign. In all other cells there are '$+$' signs. On one step we are allowed to switch all signs in some column or some row. Can we switch all signs to '$+$'?



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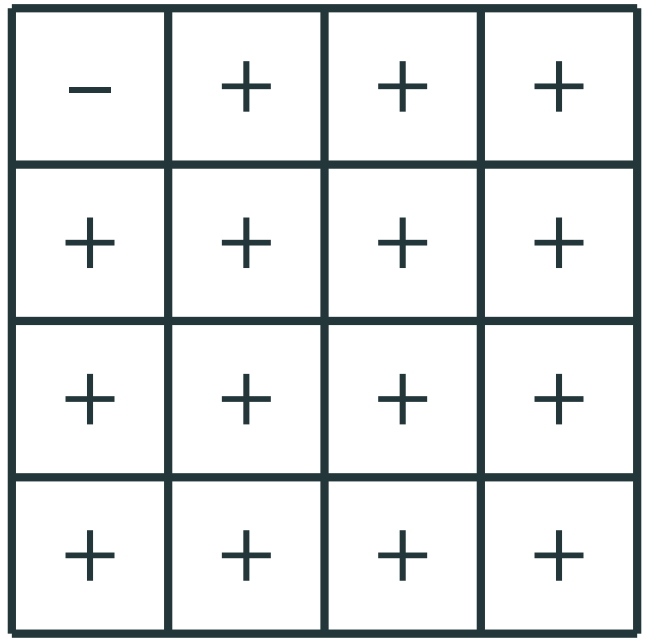

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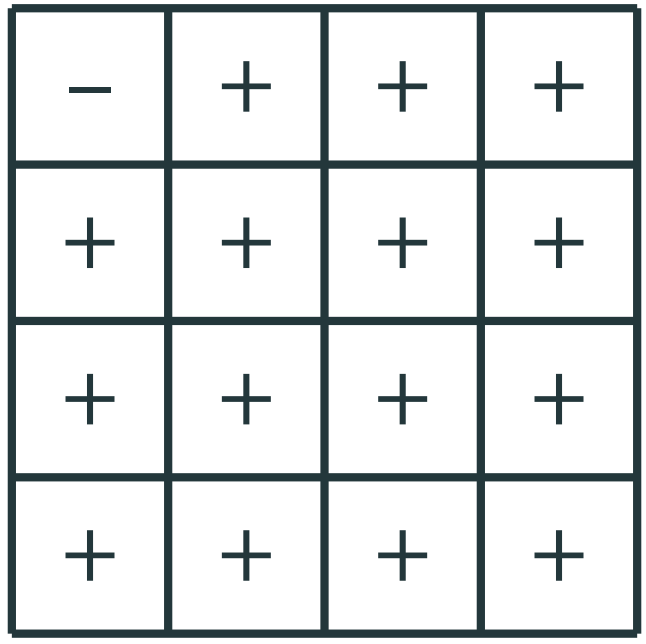

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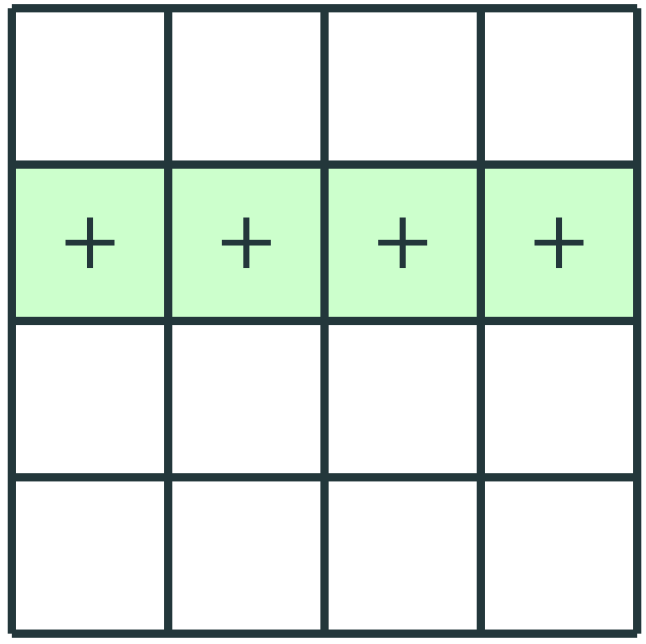

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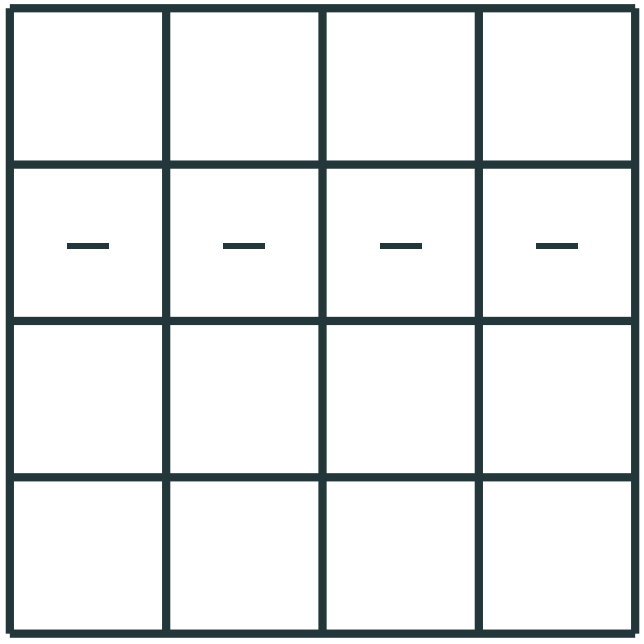

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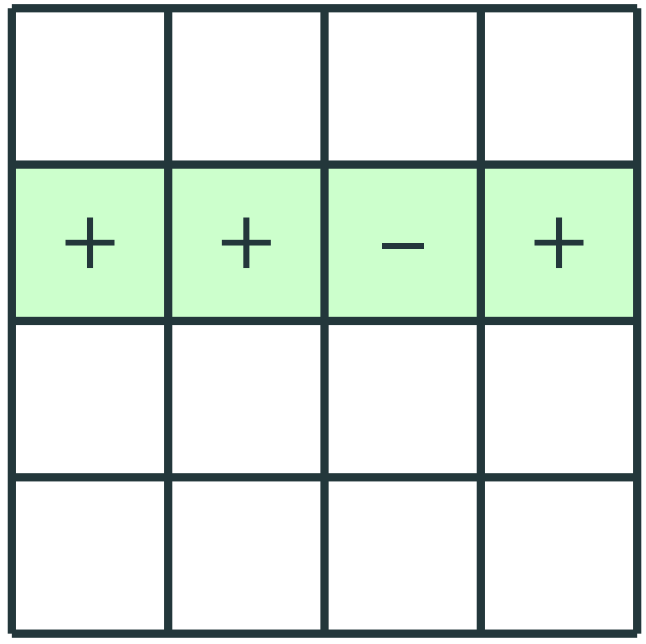

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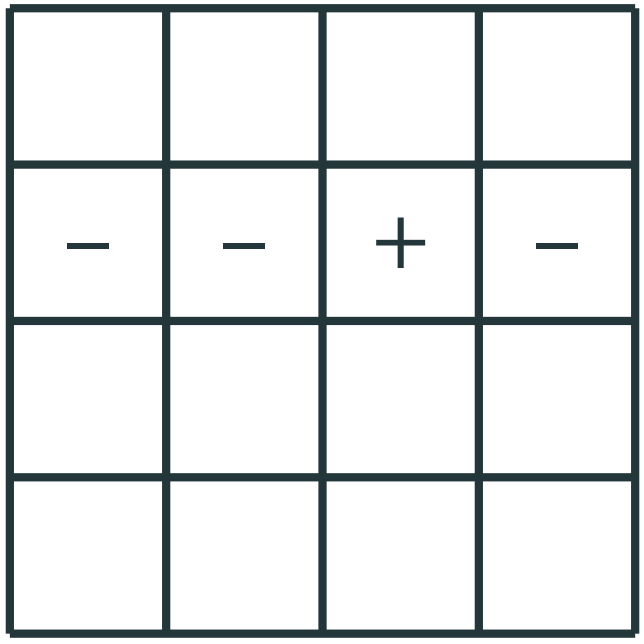

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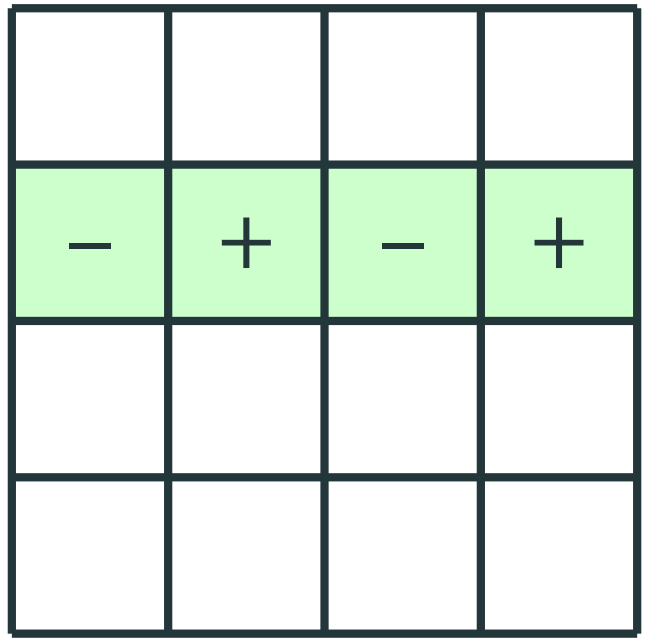

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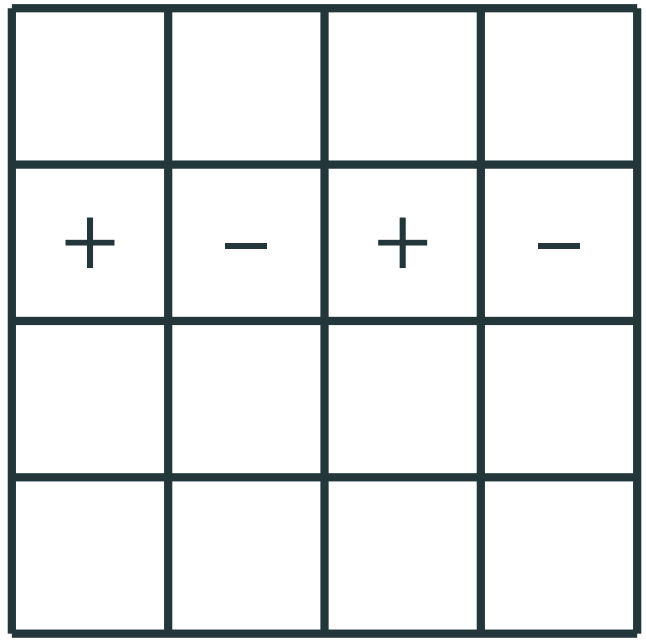

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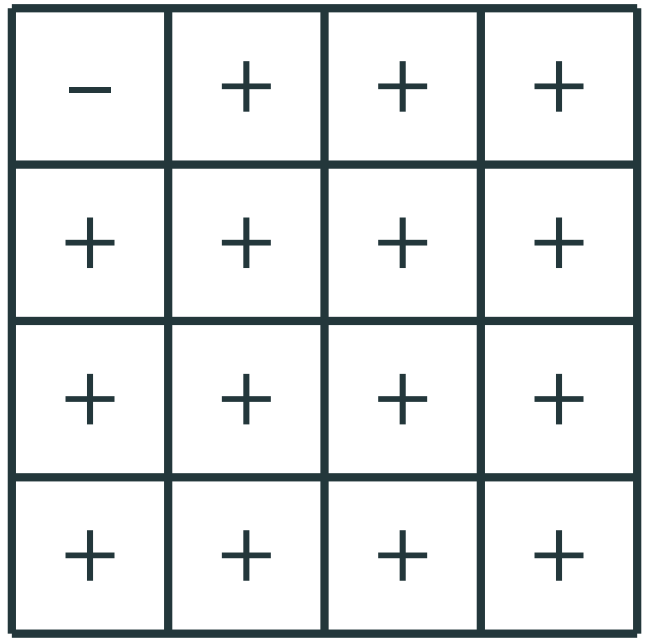

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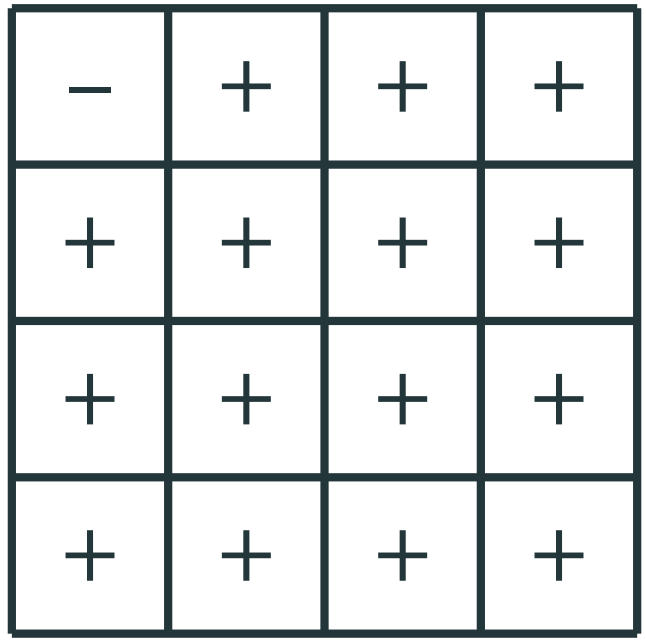

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- Cannot switch to all '+' since then the number of '−' would be even (zero)

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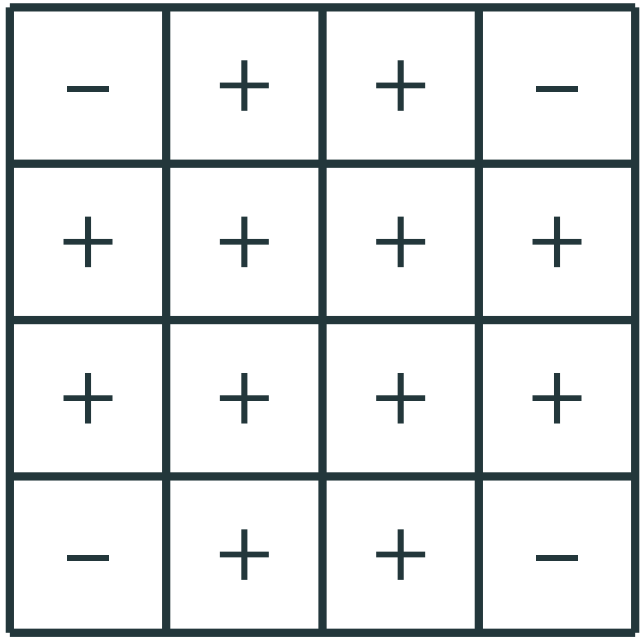

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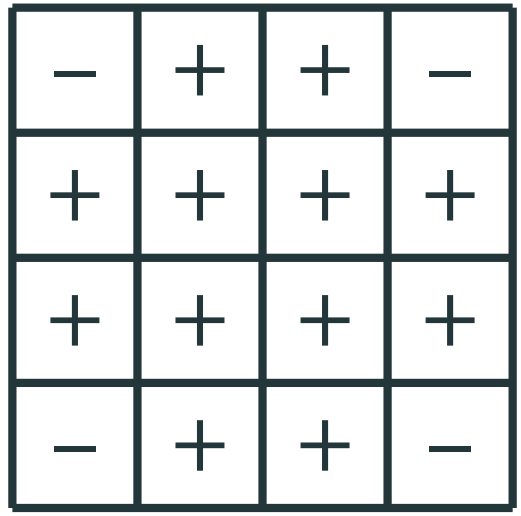

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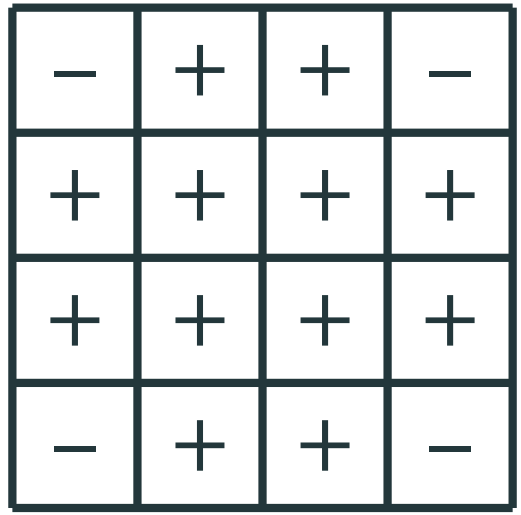

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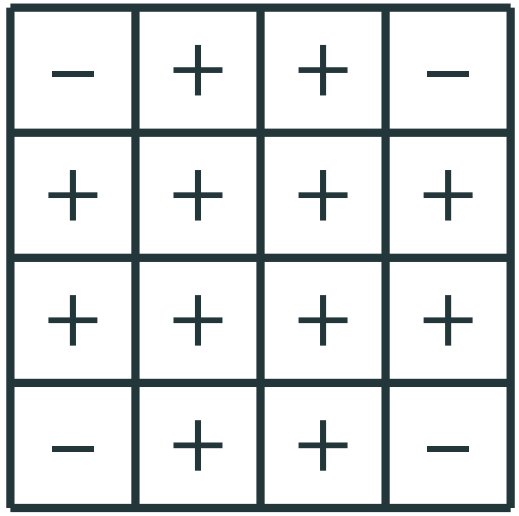

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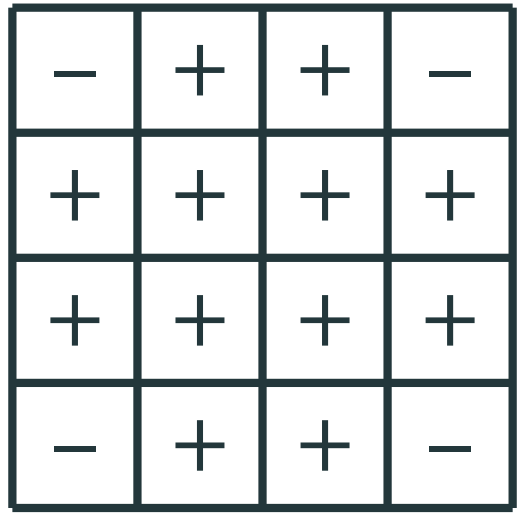

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- If you know about residues modulo 4 (we will learn later), you can try to use them.

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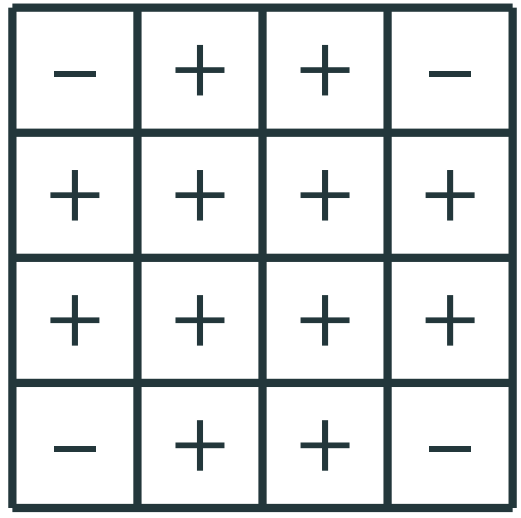

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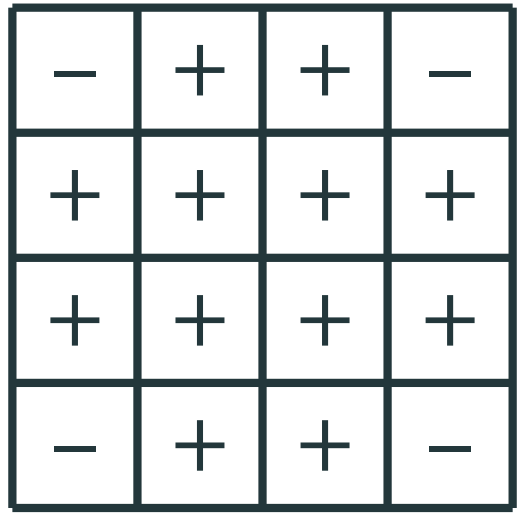

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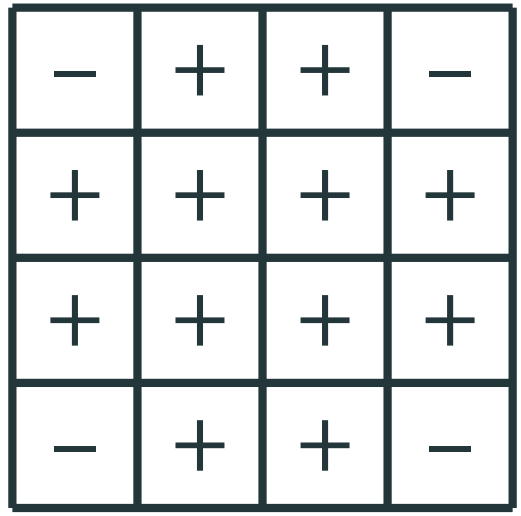

- Now, this looks way more tricky. The old solution does not work
- If you know about residues modulo 4 (we will learn later), you can try to use them. But it also does not help!
- Yet, there is a very simple solution
- If you do not know it, think a bit more before watching further for a better experience

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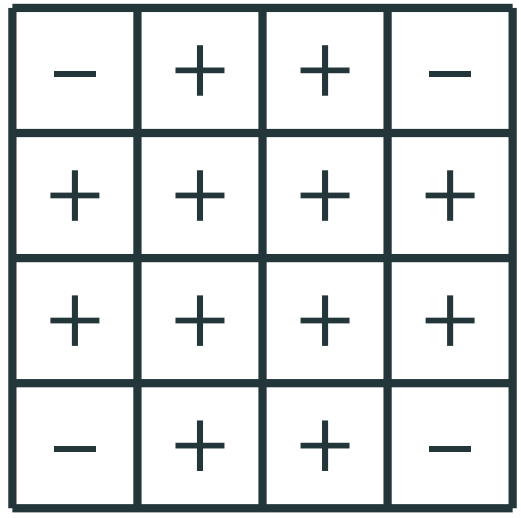

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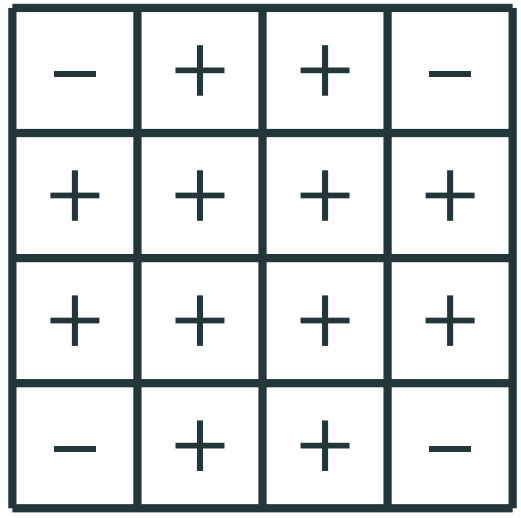

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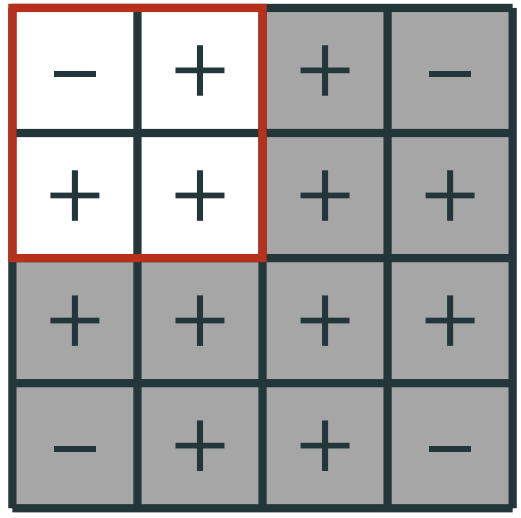

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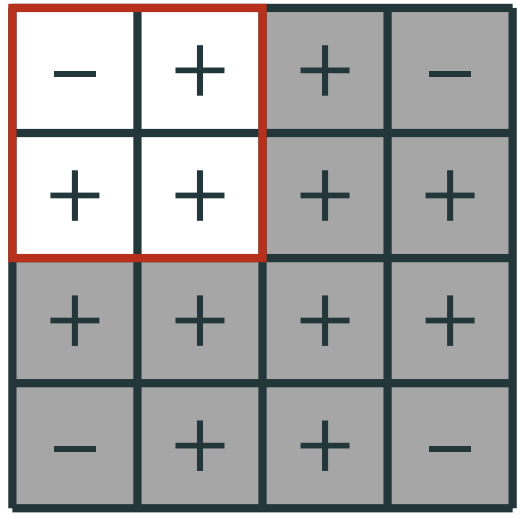

- The idea is simple. Let's look at the small part of the problem
- If we switch a column or a row in the large table, we switch a row or a column in the small too (or do nothing)

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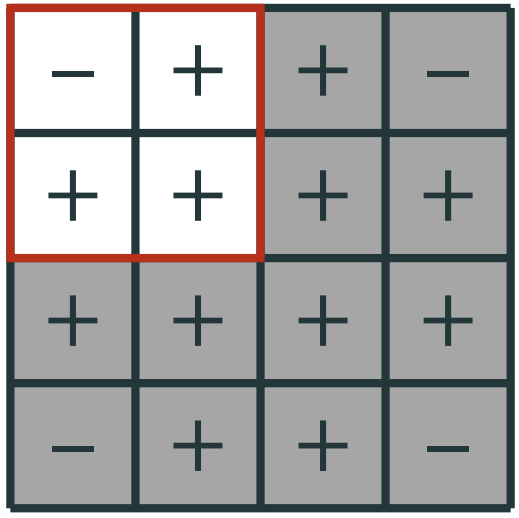

- To solve a big problem we have to solve this 2×2 problem

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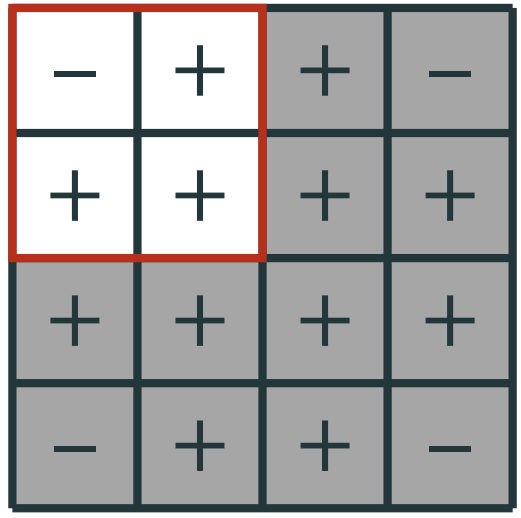

- To solve a big problem we have to solve this 2×2 problem
- And 2×2 problem is unsolvable by the old argument!

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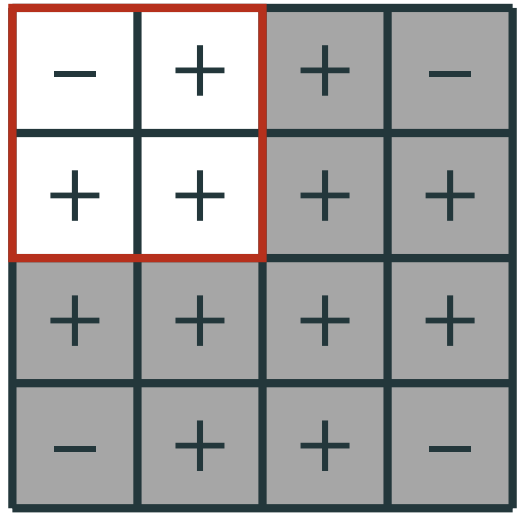
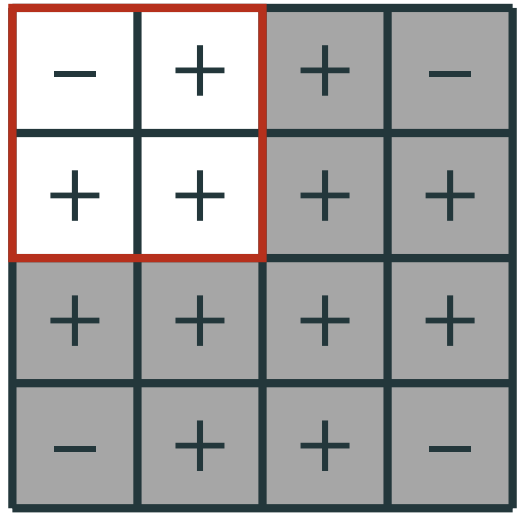


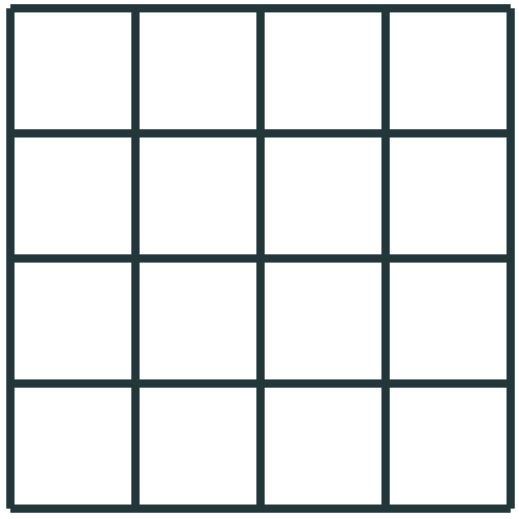
- We have found powerful obstacles: each 2×2 square should have even number of '$-$' for the puzzle to be solvable

Advanced Signs Switching



- We have found powerful obstacles: each 2×2 square should have even number of '$-$' for the puzzle to be solvable
- In fact these are the only obstacles! That is, if all 2×2 squares have even number of '$-$', it is possible to switch all signs to '$+$'.

Hint



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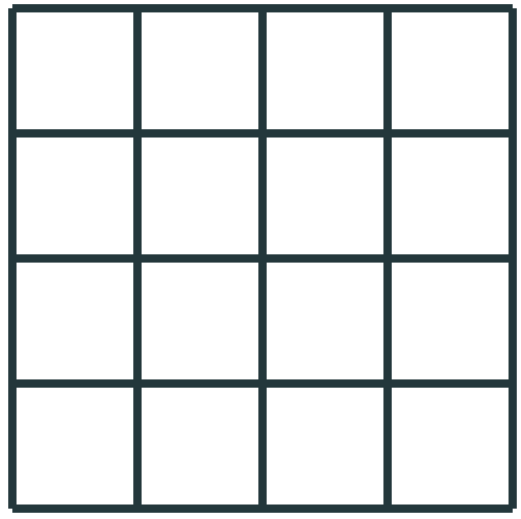

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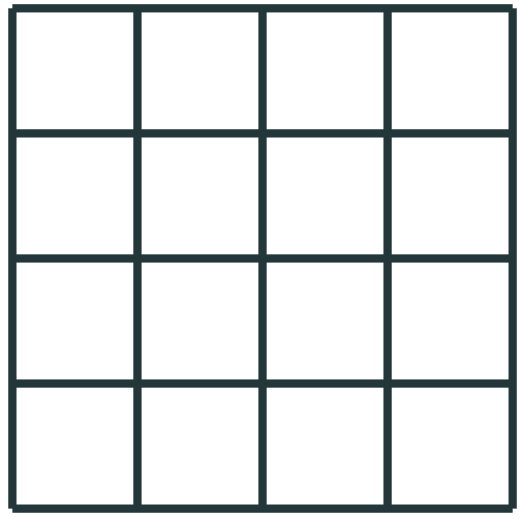

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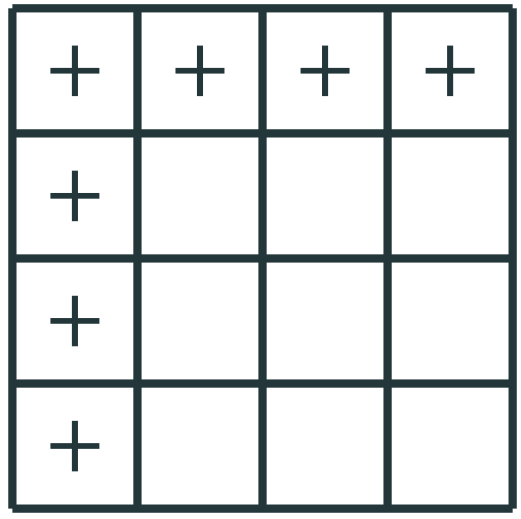

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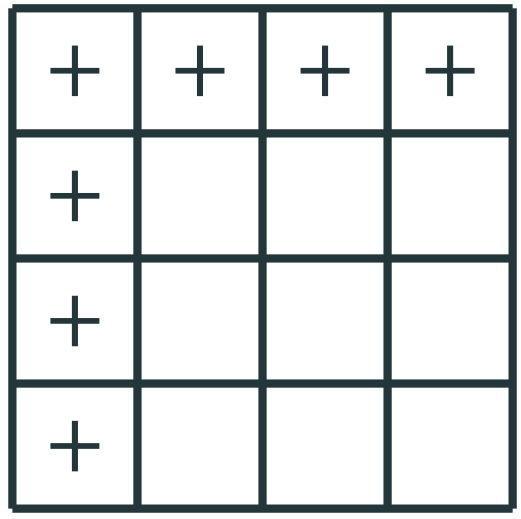

- Assume that our invariant holds: each 2×2 square has even number of '$-$'
- Switch the first row and the first column to all '$+$'
- Use invariant to show that the rest of the table is filled with '$+$'

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- And also to establish termination and running time bounds (important)
- Can take many forms: numbers, “being even or odd”, equations, inequalities, etc.
- Double counting: a special case
- This is not all: there are other forms of invariants (e.g. residues); we will see some more later on