Recursion

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Outline

Recursion

Coin Problem

Hanoi Towers

Line to Louvre

By Edal Anton Lefterov, `https://commons.wikimedia.org/w/index.php?curid=11806863`

Computing Queue Length

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Computing Queue Length

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- She wants to know the number of people before her in the line
- She asks a person (call him Bob) standing before her in the line: Could you please tell me the number of people before you?
- Now, Bob faces exactly the same problem
- Bob somehow computes that the number of people before him is 239
- Now, Alice knows that the number of people before her is 240

Recursive Program

```
numberOfPeopleBefore(A):
if there is nobody before A:
  return 0
B ← person right before A
return numberOfPeopleBefore(B) + 1
```

Factorial Function

Definition

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Recursive definition

$$n! = \begin{cases} 1 & \text{if } n = 1 \\ n \times (n-1)! & \text{if } n > 1 \end{cases}$$

Iterative Program

```python
def factorial(n):
    assert(n > 0)
    result = 1
    for i in range(1, n + 1):
        result *= i
    return result
```

Recursive Program

```
def factorial(n):
    assert(n > 0)
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)
```

Termination

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- One must ensure that a recursive program (or definition) terminates after a finite number of steps
- This is usually achieved by decreasing some parameter until it reaches a base case
    - Line length: the length of the line decreases by 1 with each recursive call, until it becomes 1
    - Factorial: n decreases to 1

Example of Infinite Recursion

```python
def infinite(n):
    if n == 1:
        return 0
    return n * infinite(n + 1)
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- In theory, will never stop: the parameter increases to infinity
- In practice, will cause an error message like `stack overflow` or `maximum recursion depth exceeded`

More Examples of Infinite Recursion

- No base case:

```
def factorial(n):
    return n * factorial(n - 1)
```

More Examples of Infinite Recursion

- No base case:

```
def factorial(n):
    return n * factorial(n - 1)
```

- Parameter does not change:

```
def infinite(n):
    if n == 1:
        return 0
    return 1 + infinite(n)
```

More (Jokey) Examples

- To understand recursion, you must first understand recursion

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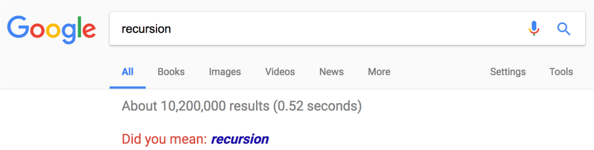
- By Google:



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- Looks plausibly: $8 = 3 + 5$, $9 = 3 + 3 + 3$, $10 = 5 + 5$, $11 = 3 + 3 + 5$
- But how can we be sure that it is always possible?

Speculating

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- Similarly, 9 gives 12, 15, 18, 21, etc

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- OK, we can definitely pay 8
- But then we can also pay $11 = 8 + 3$ (adding one 3-coin)
- Then, 14, 17, 20, etc
- Similarly, 9 gives 12, 15, 18, 21, etc
- While 10 gives 13, 16, 19, 22, etc

Recursive Program

```
def change(amount):
    assert(amount >= 8)
    if amount == 8:
        return [3, 5]
    if amount == 9:
        return [3, 3, 3]
    if amount == 10:
        return [5, 5]

    coins = change(amount - 3)
    coins.append(3)
    return coins
```

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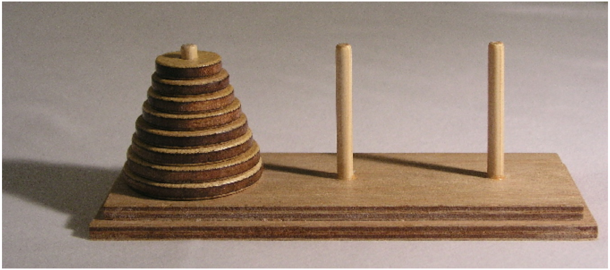
Hanoi Towers

Puzzle

There are three sticks with n discs sorted by size on one of the sticks. The goal is to move all n discs to another stick subject to two constraints: move one disc at a time and don't place a larger disc on a smaller one.



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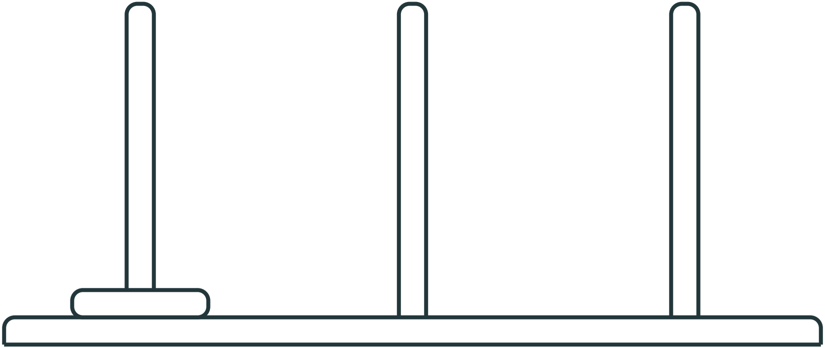
Is It Even Possible?

- For what n this is possible?
- For all!
- But how can we be sure?
- We will design a recursive program that will solve the puzzle for every n

Simplest Possible Scenario



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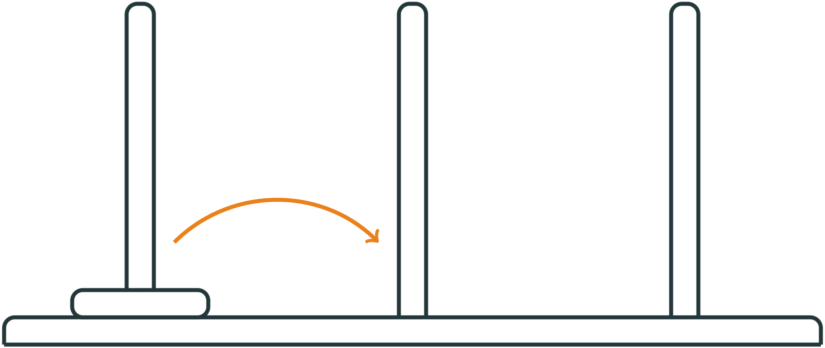

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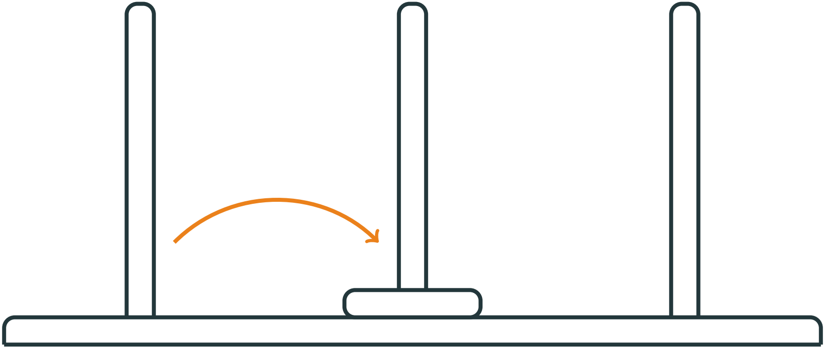
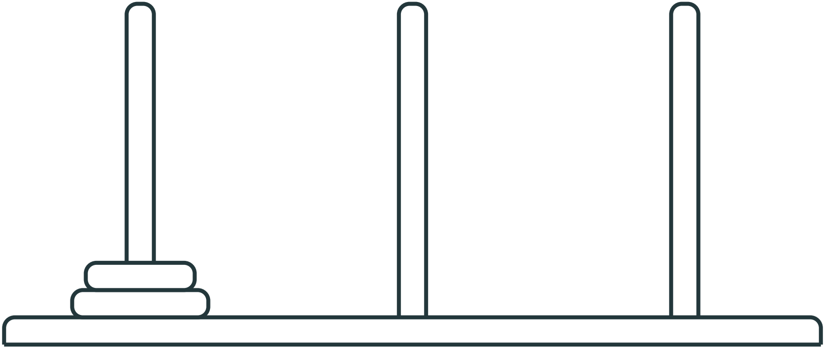


Two Discs



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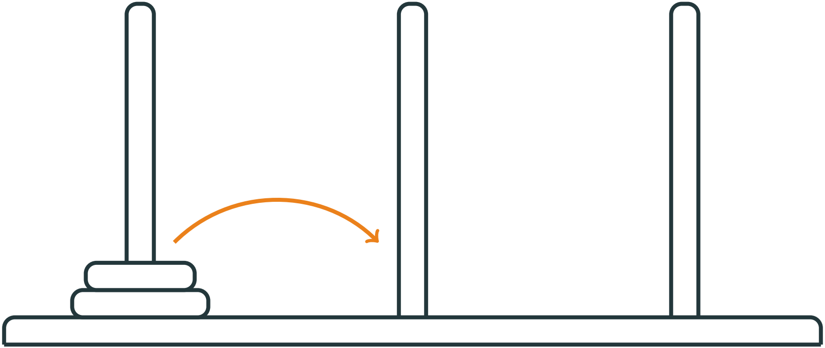

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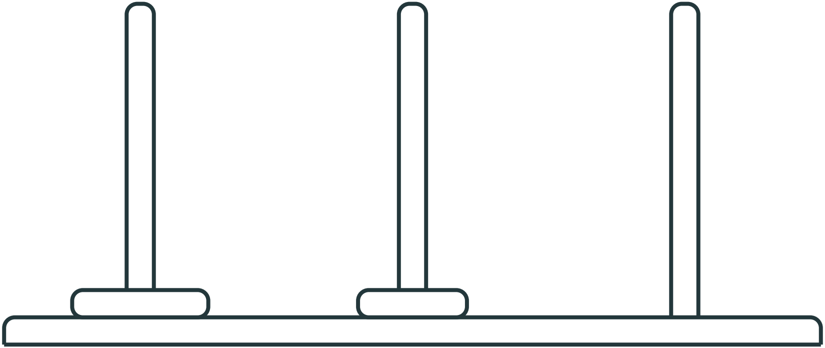

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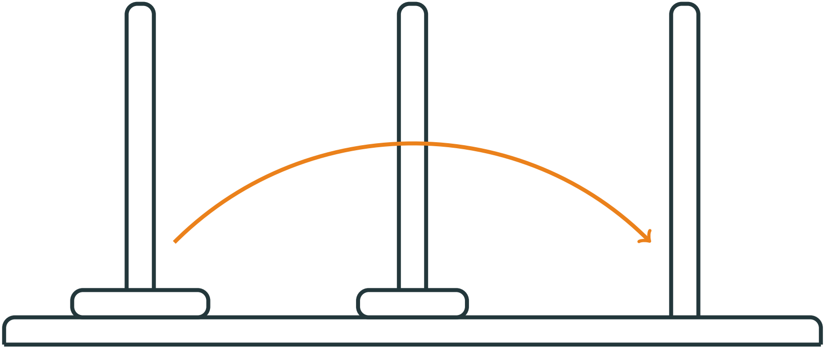

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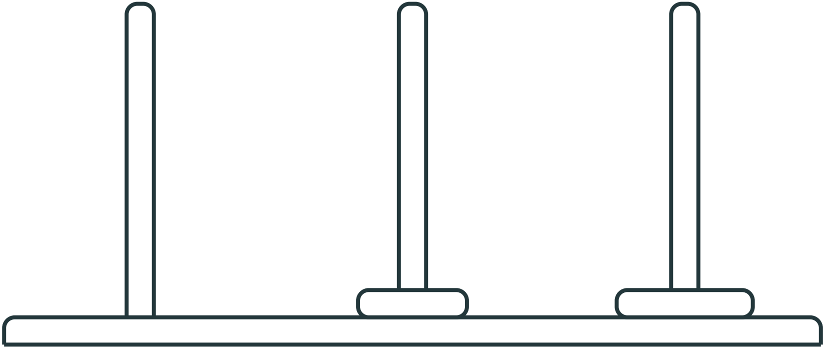

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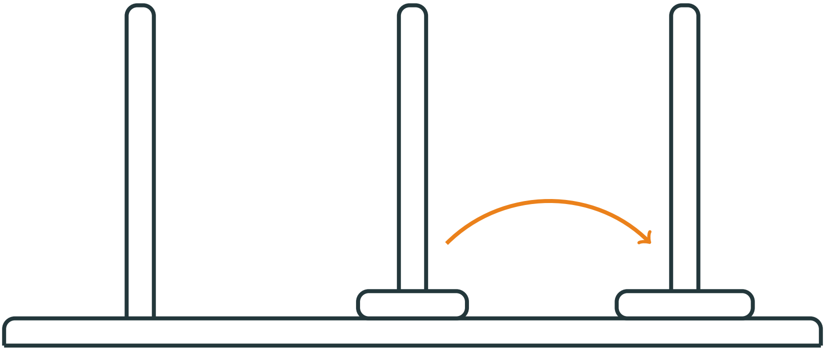

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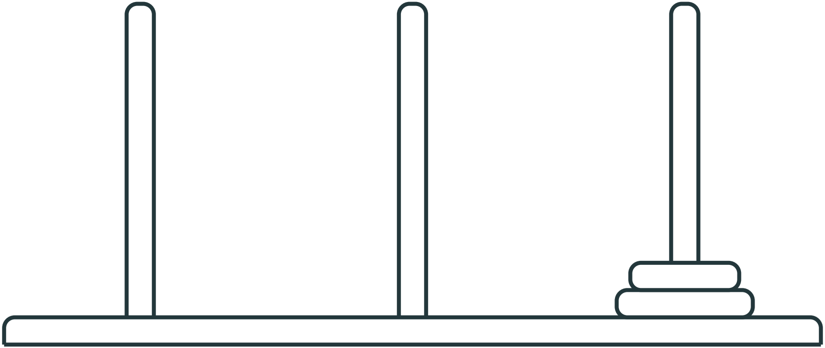
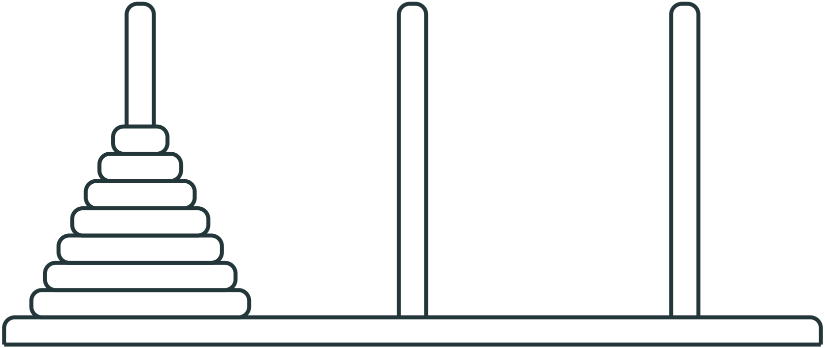


Speculating



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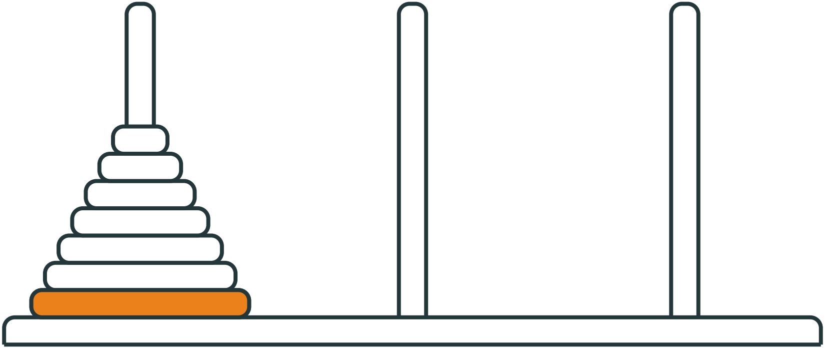
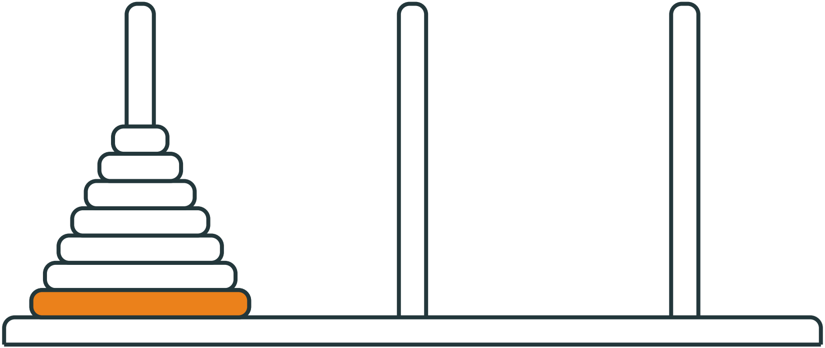


consider the largest disc

Speculating



when can we move it?

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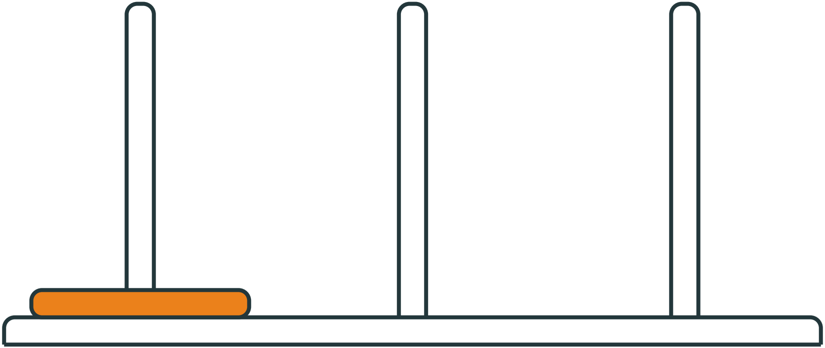
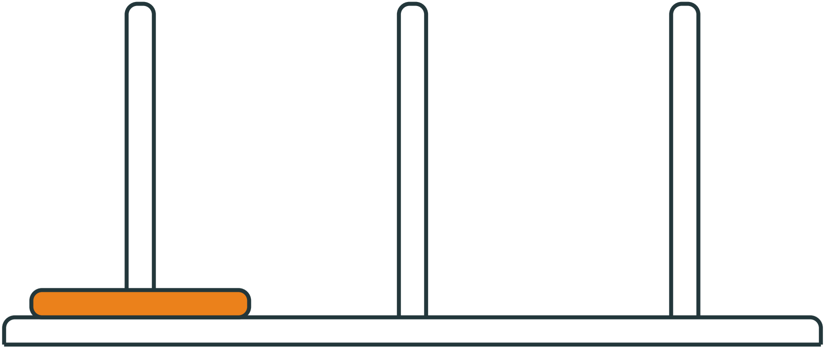


when there are no other discs on this stick!

Speculating



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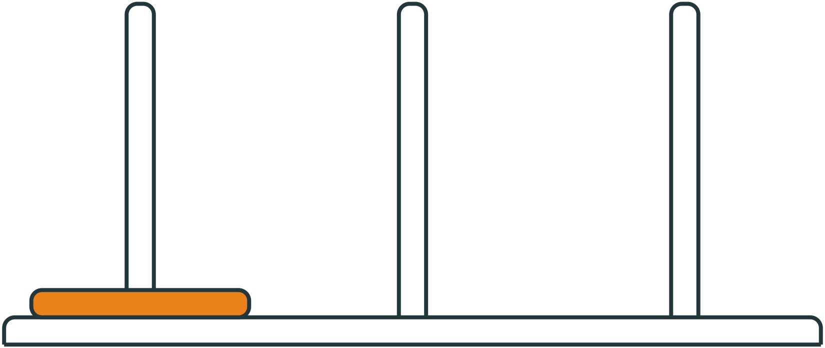

only to an empty stick!

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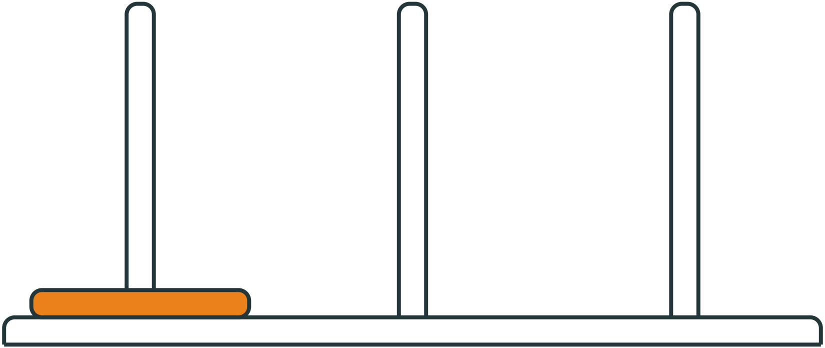

all other discs should be at another stick

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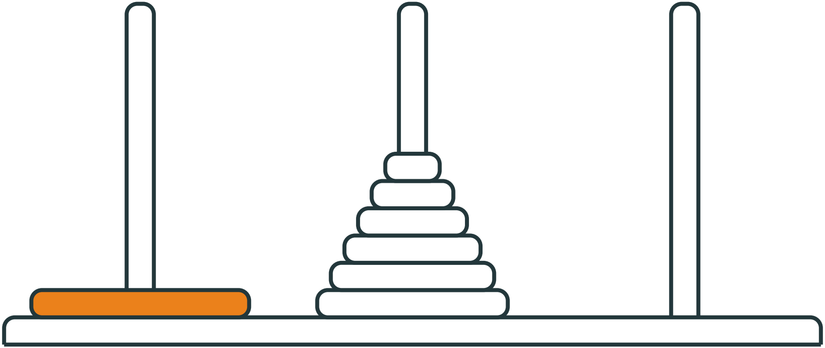

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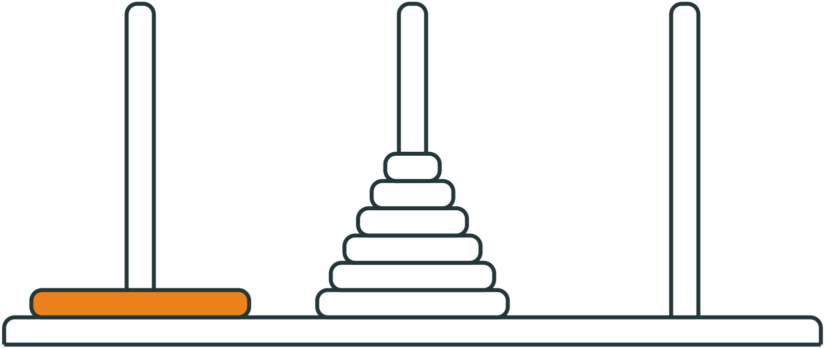
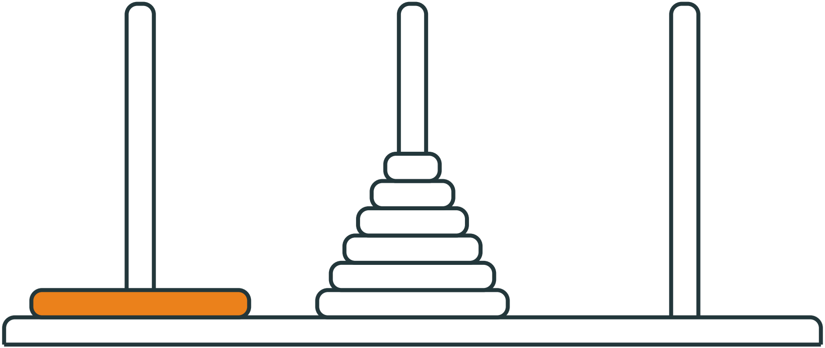


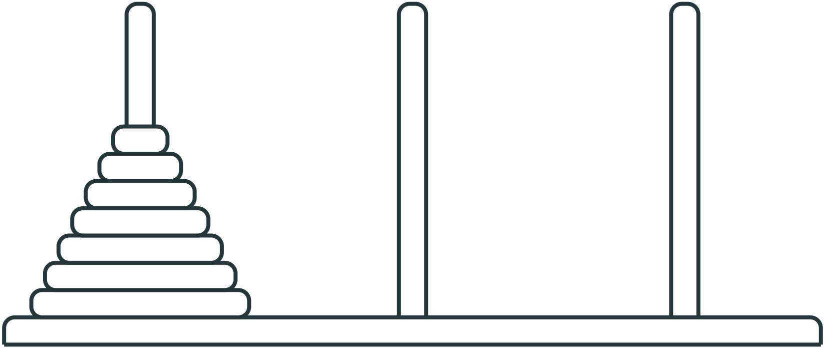
need to move $n-1$ discs

Speculating



this is the same problem!

Recursive Solution



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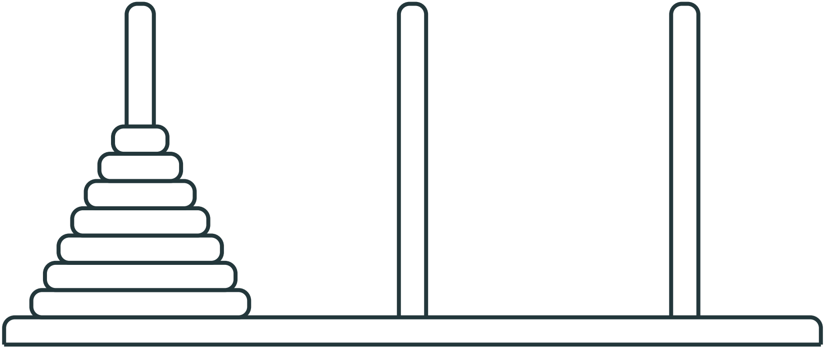

move $n - 1$ discs recursively

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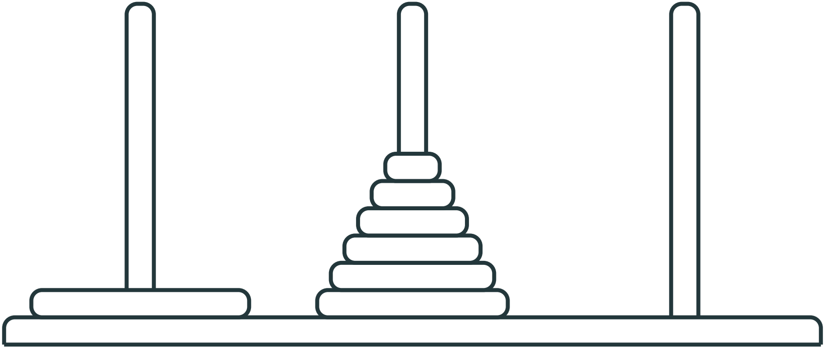

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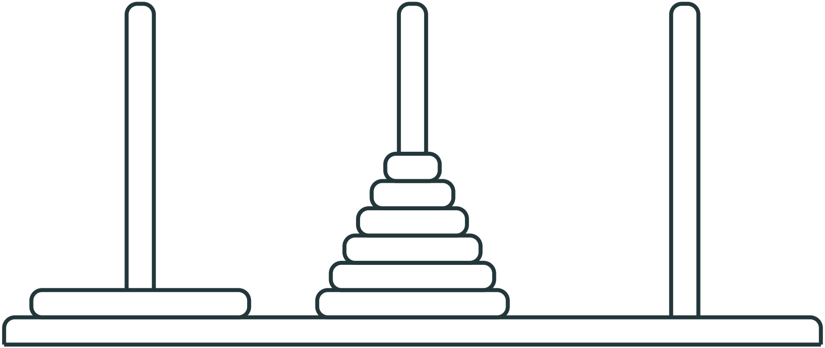
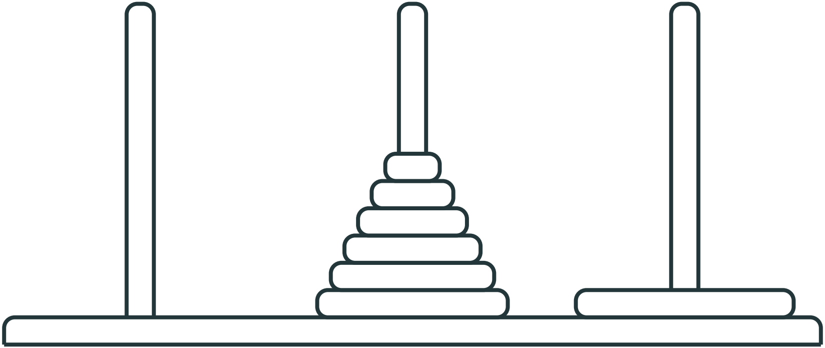


move the largest disk

Recursive Solution



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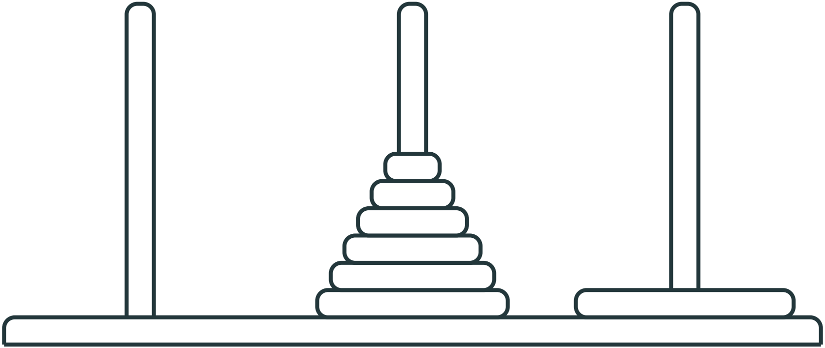

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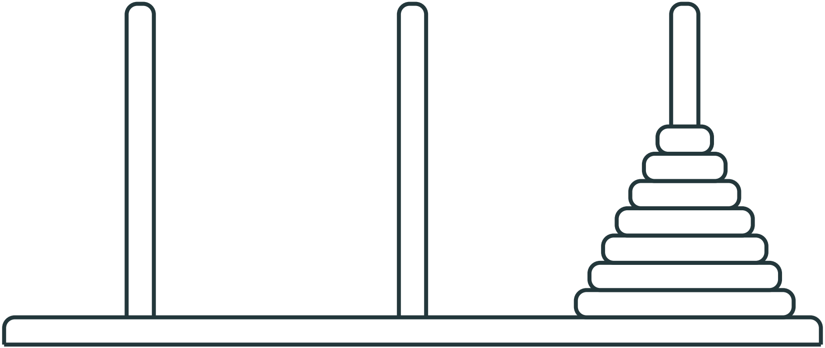

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Summary

- We've constructed a solution for all n:
  - Base case: it is possible for $n = 1$
  - Hence, it is possible for $n = 2$
  - Hence, it is possible for $n = 3$
  - $\ldots$
- We'll later learn a related notion of induction