Advanced Probability Theory
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Lecture 02

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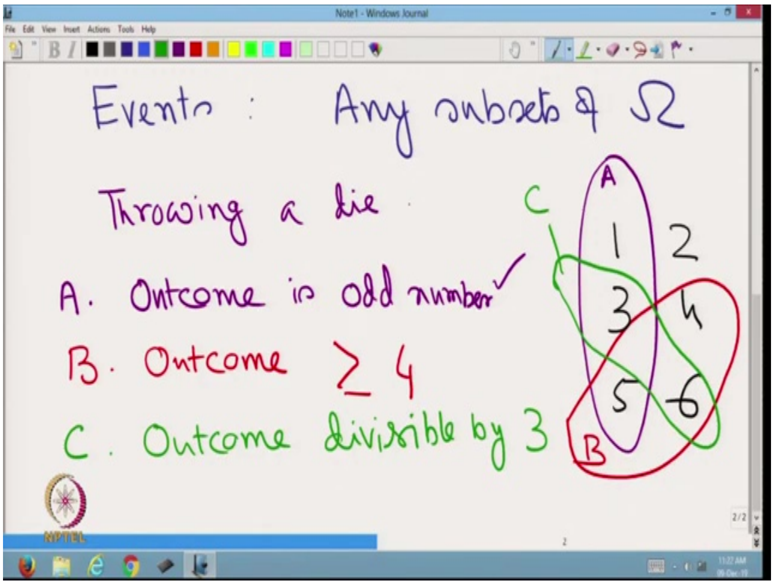

Welcome students to lecture number 2 of the MOOCs course on advanced probability theory. In the last lecture, we were talking about events any subsets of omega for illustration again consider throwing a die and suppose

- A is the event outcome is odd number,
- B is the event outcome ≥ 4
- C the event outcome divisible by 3.

So, if this is my Ω , then we can see that this subset of $\{1, 3, 5\}$ is giving us the event A this subset is giving us the event B and this subset is giving us the event C.

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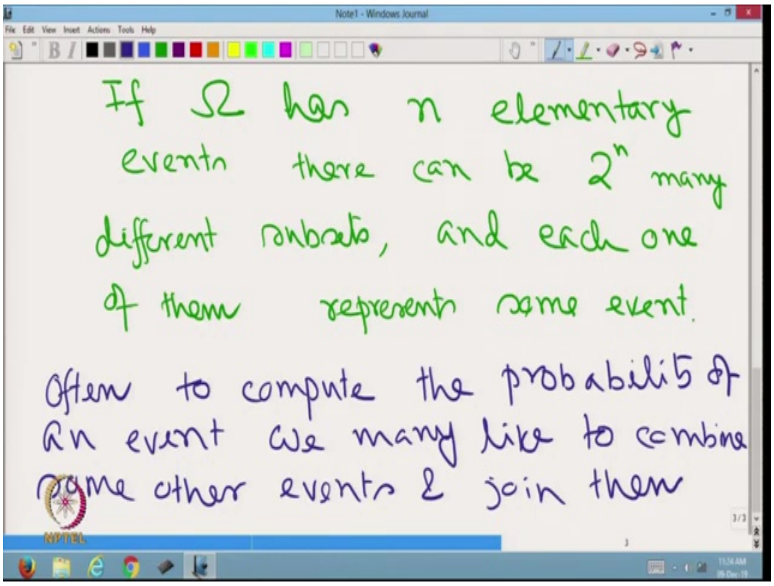

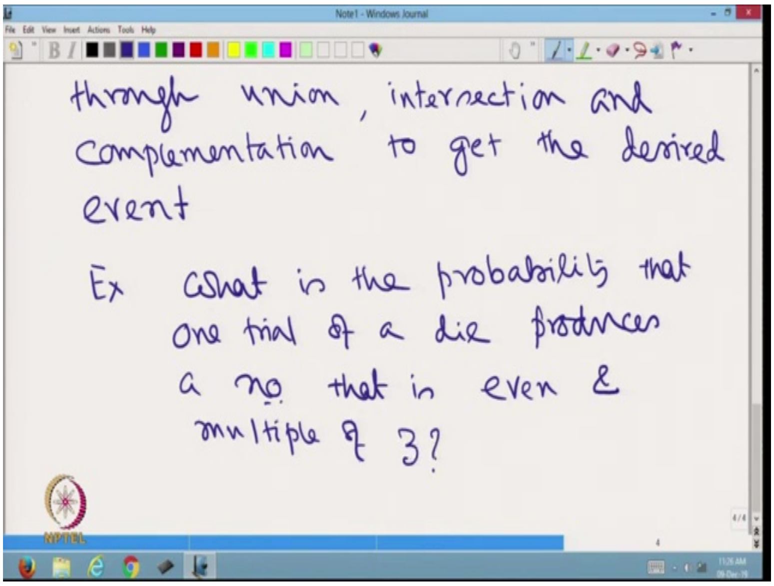

Therefore, if a set Ω has n elementary events there can be 2^n many different subsets and each one of them represents some event. Often to compute the probability of an event we may like to combine some other events and join them through union, intersection and complementation to get the desired event for example what is the probability that one trial of a die produces a number that is even and multiple of 3.

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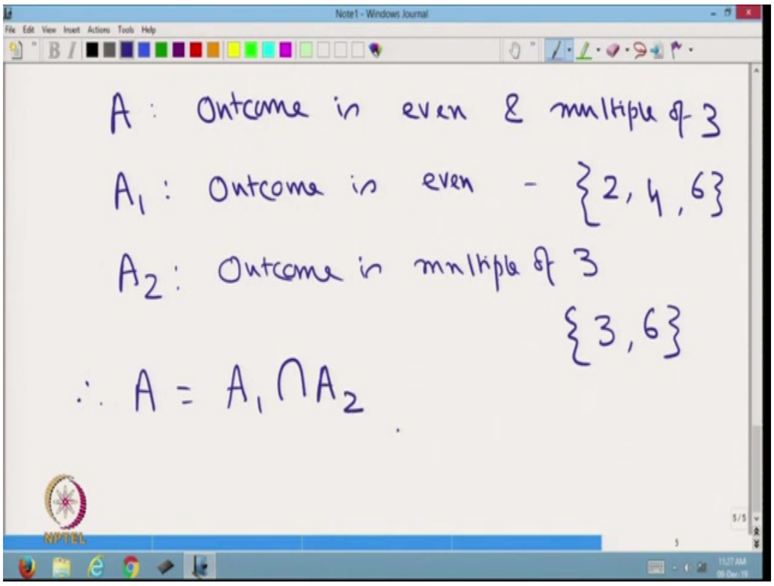

So,

- let A be the event outcome is even and multiple of 3
- let A_1 is equal to outcome is even which means essentially 2, 4 and 6 and
- A_2 be outcome is multiple of 3. Therefore, the outcome is 3 or 6.

Therefore, we can describe $A = A_1 \cap A_2$

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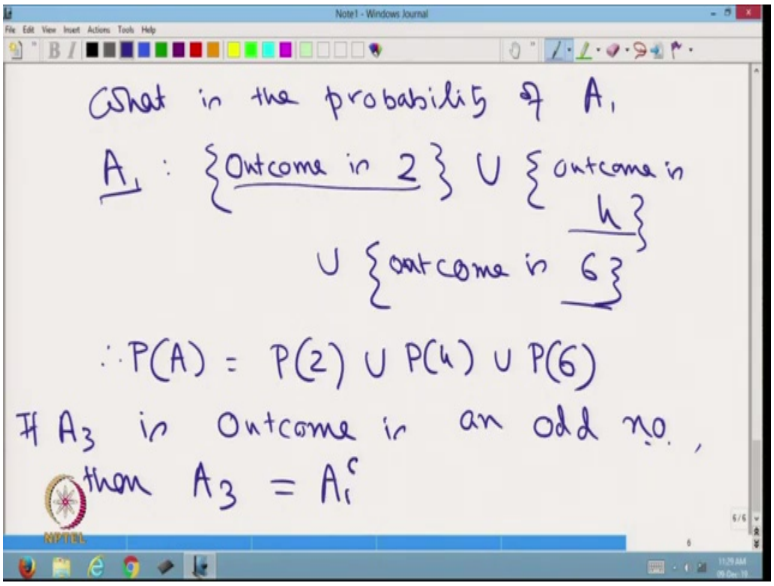

On the other end, what is the probability of A_1 , so

$$A_1 : \{outcome\ is\ 2\} \cup \{outcome\ is\ 4\} \cup \{outcpme\ is\ 6\}$$

Therefore, to obtain the probability of the event A, we may look at the probability of the 3 elementary events that 2, 4 and 6.

Therefore, $P(A) = P(2)\ \cup P(4)\ \cup P(6)$

Similarly, if A_3 is outcome of, outcome is an odd numbers then $A_3 = A_1^c$, thus the set operations union intersection and complementation are part and parcel of defining events on some omega.

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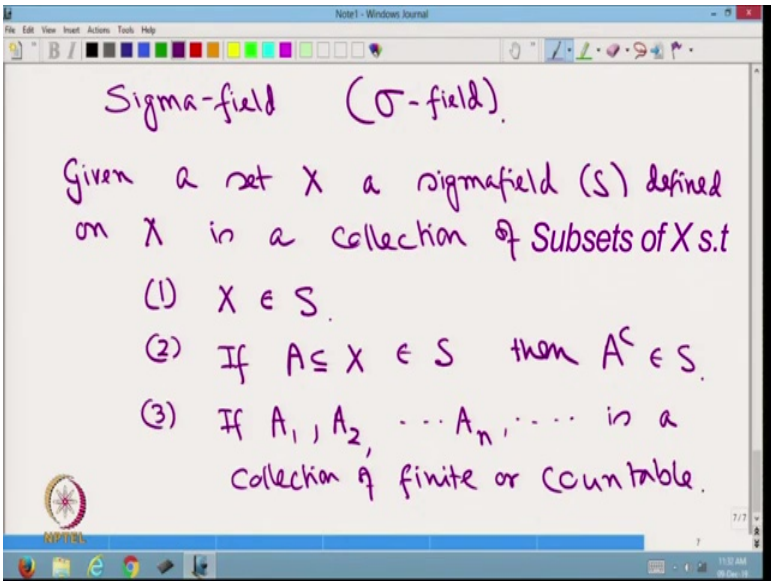

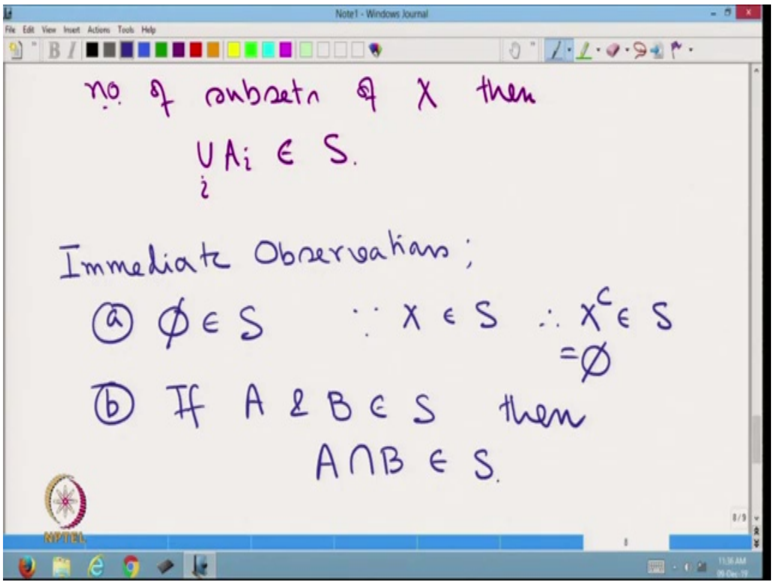

As I said in the last class, to handle them, we need the concept of $(\sigma - field)$ we will often write it as $\sigma - field$ given a set x a $\sigma - field$ let us called it as defined on x is a collection of subsets of x such that $X \in S$. If $A \subseteq X \in S$ $then$ $A^c \in S$ and 3 if –. $A_1, A_2, \dots A_n$

So, if – $A_1, A_2, \dots A_n$ is a collection of finite or countable number of subsets of X then $\cup_i A_i \in S$. So, once a collection of subsets of x satisfies these above properties, we called it a $\sigma - field$, some immediate results - $\emptyset \in S$, why? Since $\because X \in S$ $\therefore$ $X^c \in S$, X complement with respect to itself is -$\emptyset$,

(B) if $A \,\&\, B \in S$ then $A \cap B \in S$. This is not so obvious because we have talked about union but here we are saying that $A \cap B$ also a member of S.

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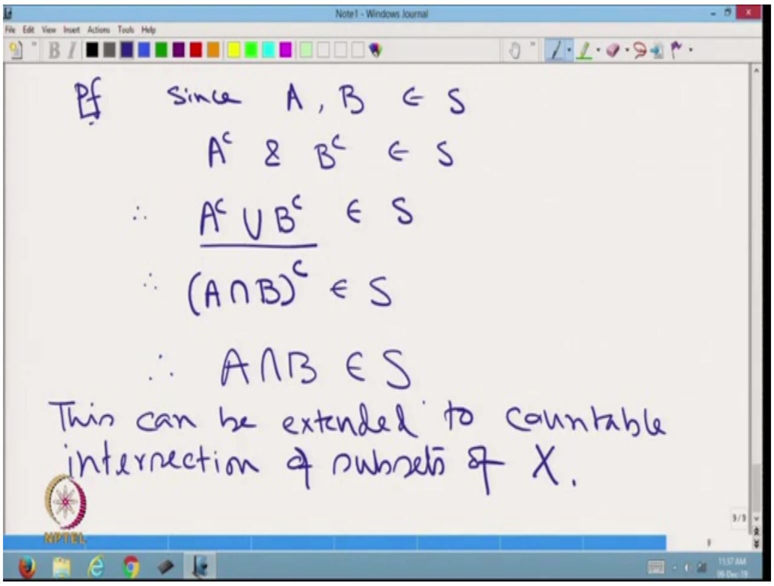

Proof,

since $A, B \in S$

$$A^c \,\&\, B^c \in S$$

$$A^c \cup B^c \in S$$

Therefore, we know that $A^c \cup B^c = (A \cap B)^c \in S$

Therefore, since $A, B \in S$

$$A^c \,\&\, B^c \in S$$

Therefore, $A^c \cup B^c \in S$

Therefore, $A \cap B \in S$. Therefore, its complement that is $A \cap B \in S$. This can be extended to countable intersections. That is, if A_1, A_2, A_3 $up\ to\ A_n$, $n \to \infty$ is a collection of subsets of $X \in S$ then the countable intersection also belongs to S.

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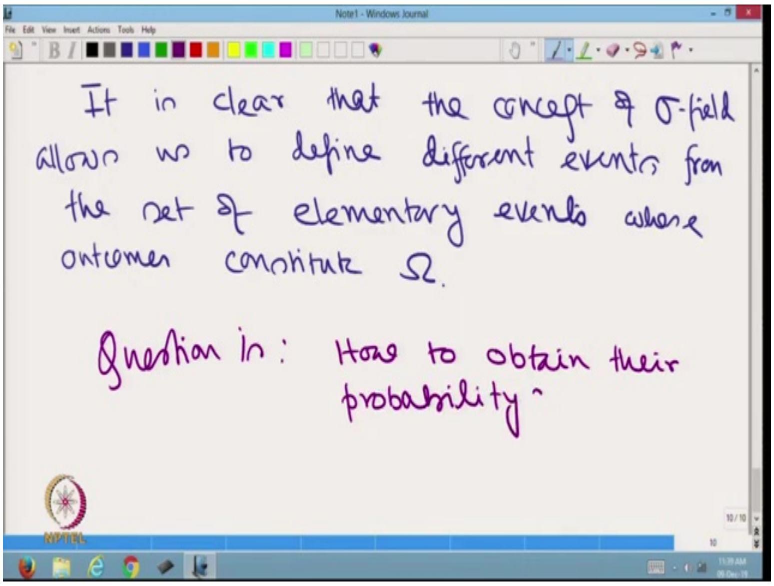

It is very clear the concept of sigma field allows us to define different events from the set of elementary events whose outcome constitute Ω .

Question is how to obtain their probability.

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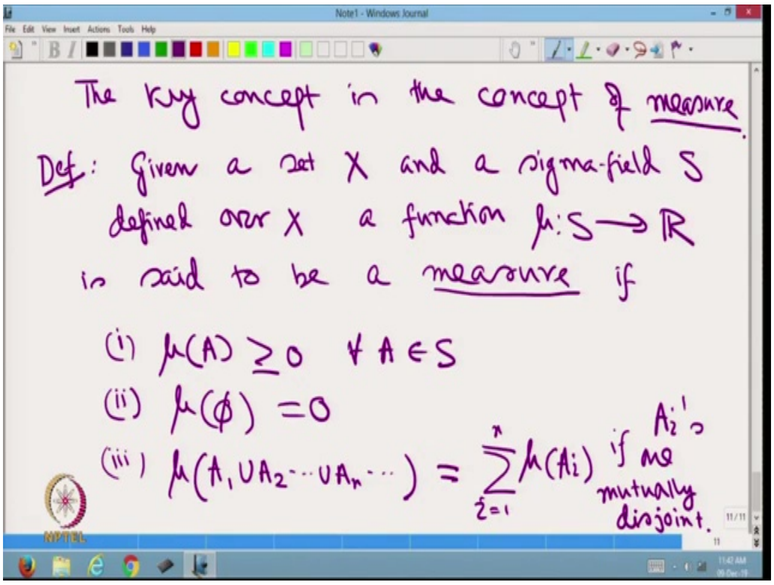

The key concept is the concept of measure. Definition, given a set X and a $\sigma - field$ S defined over X, a function new from S to the real line is set to be a measure if

(1) $\mu(A) \geq 0 \; \forall \; A \in S$.

(2) $\mu(\emptyset) = 0$, we all know that $\emptyset \in S$ and

(3) $\mu(A_1 \cup A_2 \ldots \cup A_n \ldots) = \sum_{i=1}^{n} \mu(A_i)$ $\; if \; A_i's$ are mutually disjoint. that is, any two A_i if we take then they have no intersection.

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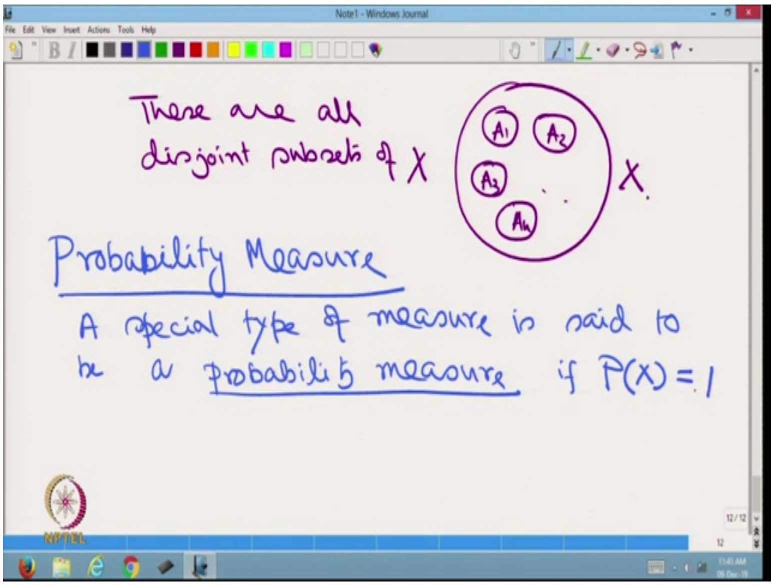

For example, if this is my A this is my - A_1 this is my - A_2 this is my A_3 this is my - A_4 like that we can see that. These are all disjoint subsets of X. Probability measure, a special type of measure is said to be a probability measure, if probability of the entire set is equal to one.

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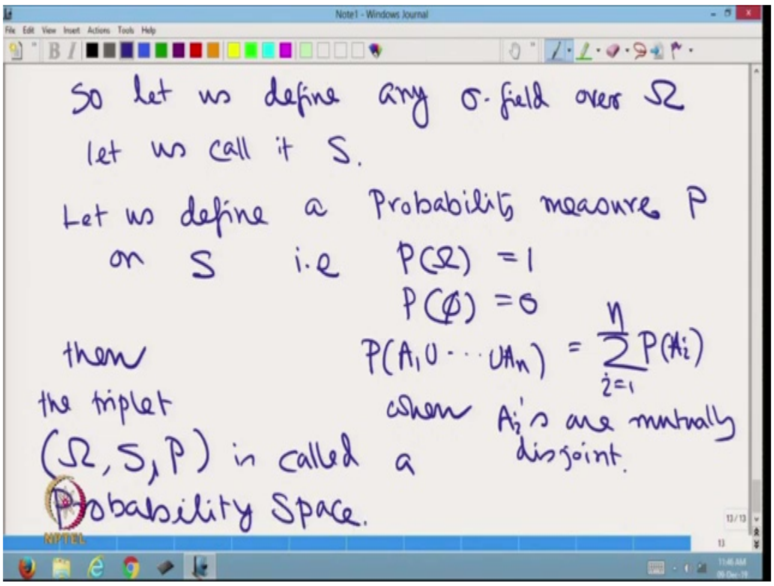

So, let us define any $\sigma - field$ over Ω . Let us call it as let us define a probability measure P on S

that is

- $P(\Omega) = 1$
- $P(\emptyset) = 0$
- $P(A_1 \cup \ldots A_n) = \sum_{i=1}^{n} P(A_i)$; $when\ A_i's$ are mutually disjoint

Then the triplet (Ω, S, P) is called a probability space.

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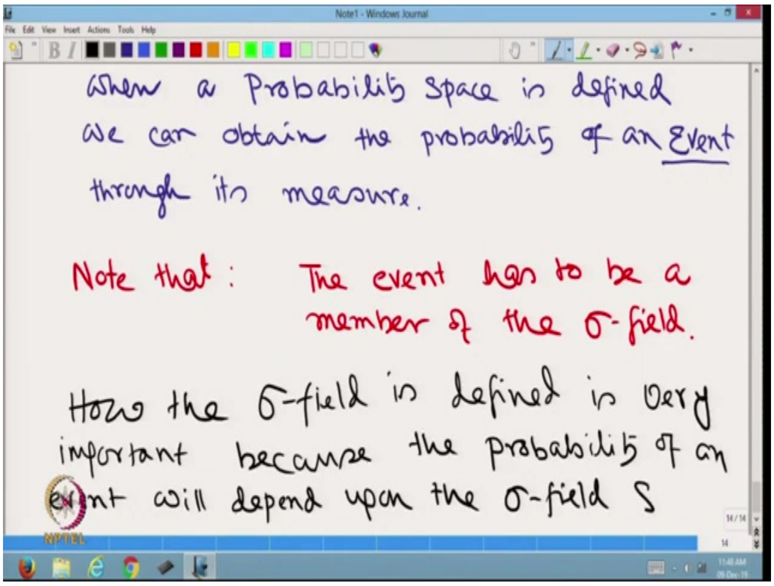

And when a probability space is defined, we can obtain the probability of an event through its measure. Note that: the event has to be a member of the $\sigma - field$. Because if you define an event in such a way that it is not a member of the $\sigma - field$ then we cannot assign the measured through the probability measure $\emptyset$ and there for you are not able to assign the probability to that particular event. Now, how the $\sigma - field$ is defined is very important because the probability of an event will depend upon the $\sigma - field$ S.

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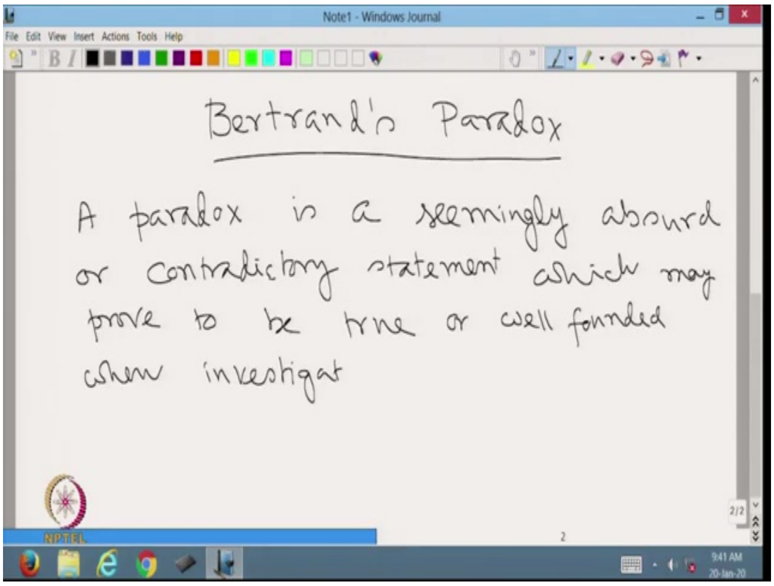

Let me give an example. The example is called Bertrand's Paradox. Now, what is a paradox? Now, paradox is a seemingly absurd or contradictory statement which may prove to be true or well founded, when investigated.

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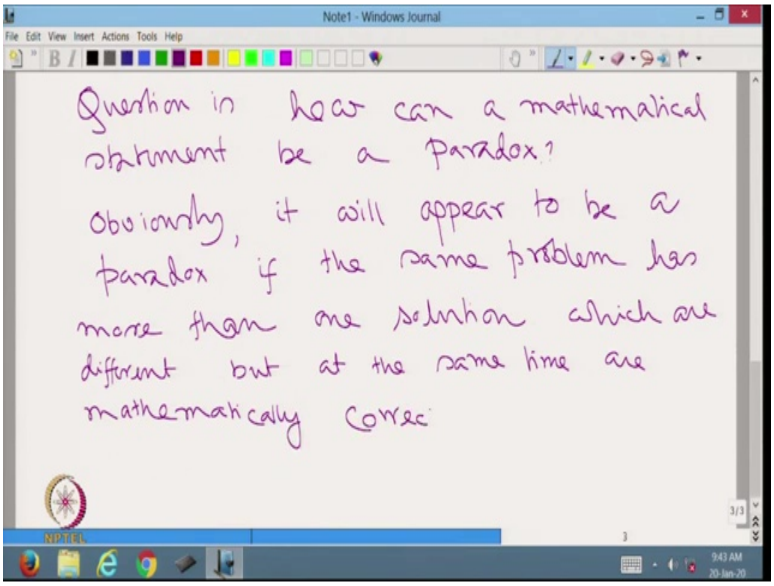

Question is how can a mathematical statement be a paradox? Obviously, it will appear to be a paradox if the same problem has more than one solution which are different, but at the same time are mathematically correct.

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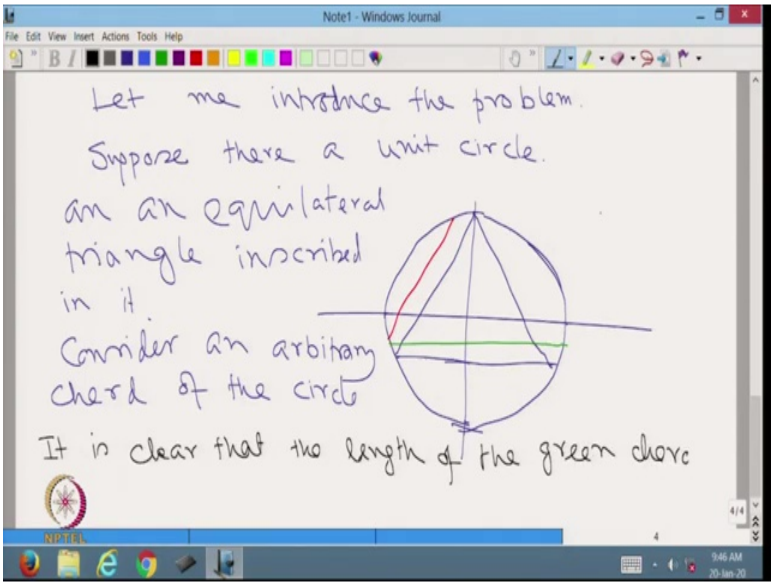

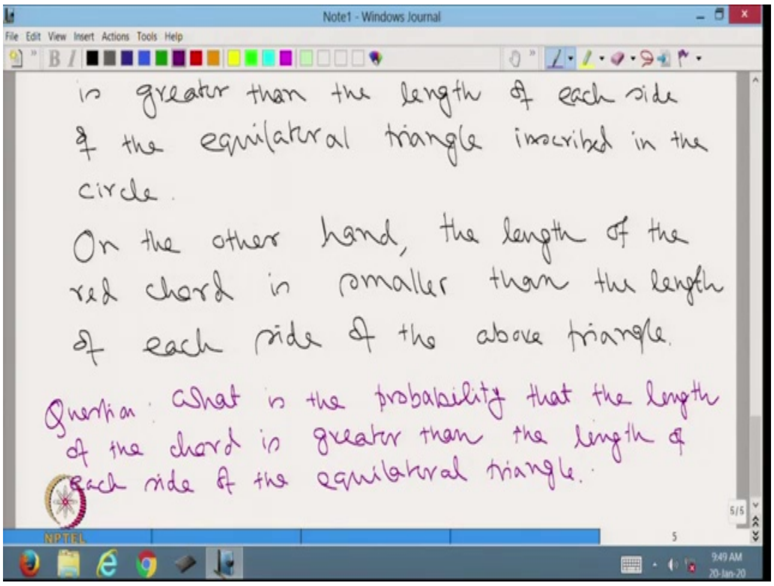

So, let me introduce the problem. Suppose, there is a unit circle, so let us assume that this is a unit circle and there is a equilateral triangle inscribed in it, consider an arbitrary chord of the circle say something like this or something like this. It is clear that, the length of the green chord is greater than the length of each side of the equilateral triangle inscribed in the circle.

On the other hand the length of the red chord is smaller than the length of each side of the above triangle. Question is what is the probability that the length of the chord is greater than the side length, the length of each side of the equilateral triangle that is the question.

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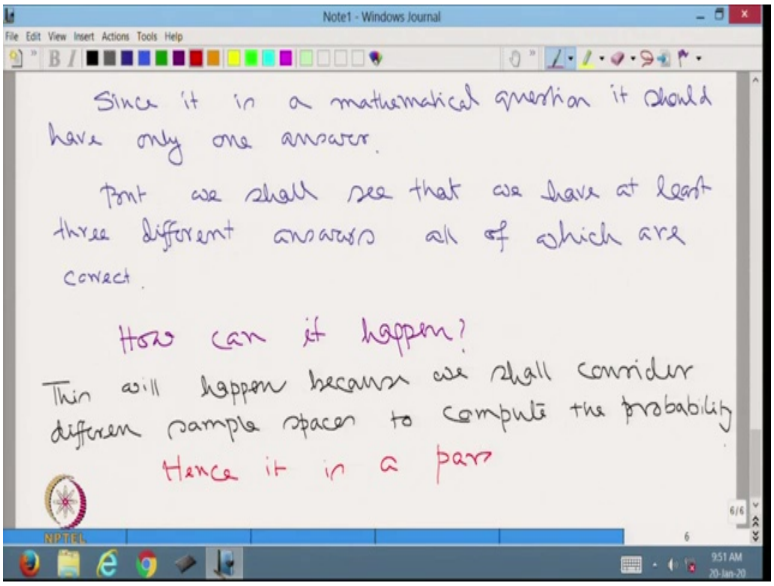

Since it is a mathematical question it should have only one answer, but we shall see that we have at least 3 different answers. All of which are correct. How can it happen? This will happen because we shall consider different sample spaces to compute the probability hence it is a paradox.

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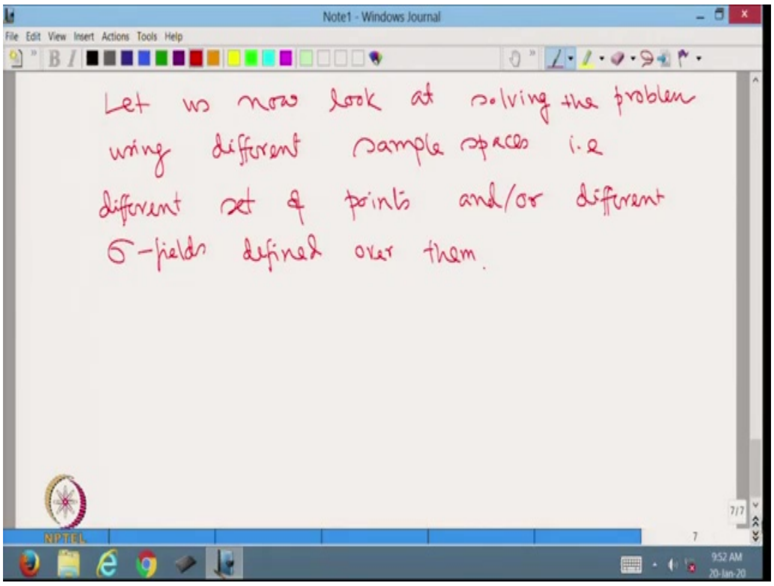

So, let us now look at solving the problem using different sample space that is different set of points and or different sigma fields defined over them.

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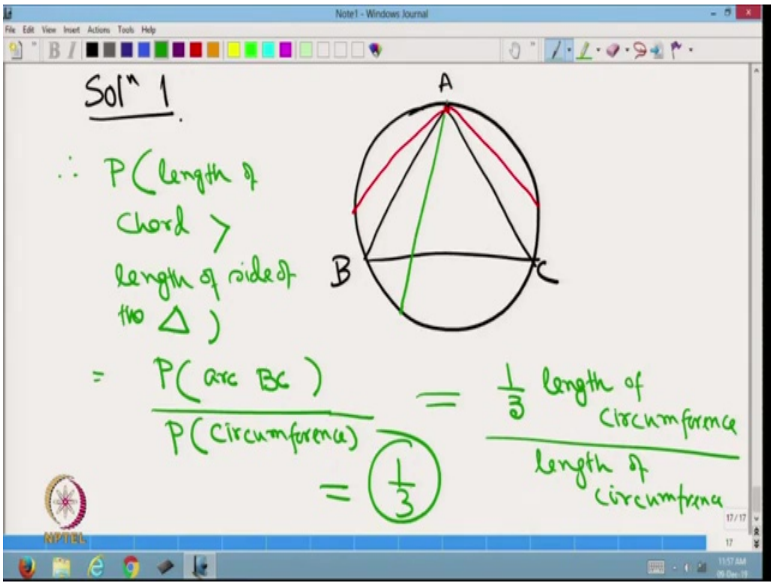

So, here is the solution one, so solution one. So, we have this circle we have this equilateral triangle, let this triangle be A, B, C and we are drawing a chord. It is obvious that suppose, we assume one end of the chord is at a point A, then if the end of the chord is in the arc A to B, then we can see that the length of this chord is less than the side of A. Similarly, if the chord is other end of the chord is any point on the arc A to C, then also its length is going to be less than this side of the triangle.

On the other hand, if the other end of the chord is on the arc B to C, then we can see that its length is going to be bigger than the side of the equilateral triangle.

Therefore,

$$P(length\ of\ chord\ > length\ of\ side\ of\ the\ \Delta)$$

is equal to a probability measure of the arc BC over the measures of the circumference of the circle. We are dividing by this P because if we considered the length of the circumference,

$$= \frac{P(arc\ BC)}{P(Circumference)}$$

this is going to be $2\pi \neq 1$. To make it 1 we need to divide everything by 2π.

This is $\frac{\frac{1}{3} length\ of\ circumference}{length\ of\ circumference} = \frac{1}{3}$

you have to understand that we have fixed one end of the chord at one of the vertices of the triangle. Therefore, any arbitrary triangle equilateral triangle that you take, we can always see that the same thing is going to happen, and therefore probability of the event is going to $\frac{1}{3}$.

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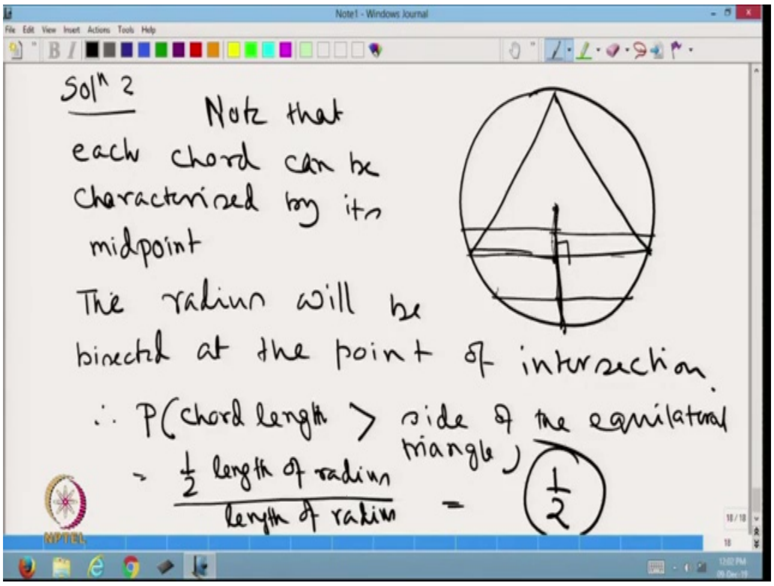

Solution 2,

so again as before I draw the circle, this is the center of it. This is a radius considered this equilateral triangle. Now, we want to draw a arbitrary chord note that each chord can be characterized by its midpoint because, if we take any chord and we draw a perpendicular from the center to that chord, it is going to intersect it at the midpoint. Now, we also know that this radius will be bisected at this point of intersection.

So, corresponding to this radius we can say that the cord will be of length bigger than the side of the equilateral triangle if its midpoint is somewhere above the point of intersection, and the length of the chord is going to be smaller than the side of the radius, if its intersection with the radius is somewhere below this point,

$\therefore P(chord\ length > side\ of\ the\ equilateral\ triangle)$

$$\frac{\frac{1}{2}\,length\ of\ radius}{length\ of\ radius} = \frac{1}{2}$$

this is because we have taken the $\sigma - field$ as points on the radius.

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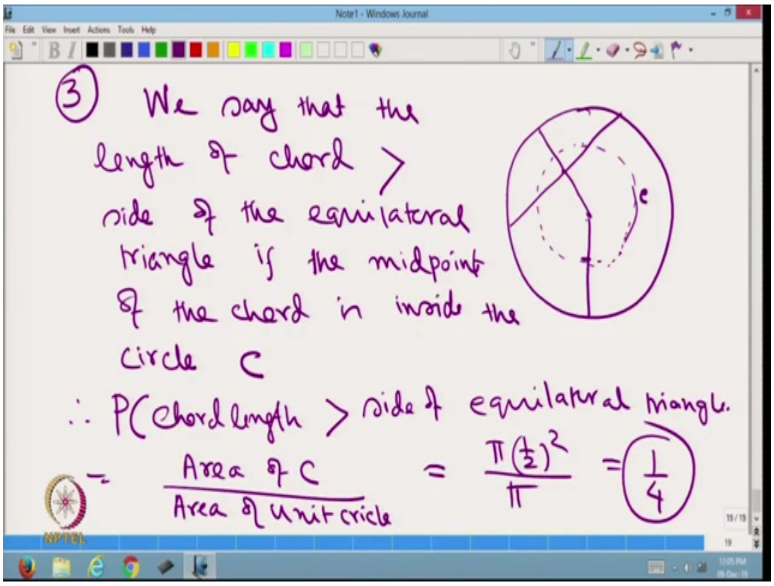

Let me give you a third solution very similar setup. So, here is the radius here is the midpoint of the radius, which is at a distance half from the origin, so consider this concentric circle, we say that the

$$length\ of\ chord > side\ of\ the\ equilateral\ triangle$$

If the midpoint of the chord falls somewhere inside this circle.

So if this is the midpoint of the chord then I will draw the radius like this and then I draw the perpendicular through that one. So, that is the chord we are looking at and obviously, its length is going to be bigger than the side of the equilateral triangle.

Midpoint of the chord is inside the circle C, so let us call it C.

Therefore, $P(chord\ length > side\ of\ the\ equilateral\ triangle)$

is equal to the measure of the circle, which in this case, we can write as

$$= \frac{Area\ of\ C}{Are\ of\ unit\ circle} = \frac{\pi\left(\frac{1}{2}\right)^2}{\pi} = \frac{1}{4}$$

Thus, we see that depending upon the interpretation of the $\sigma - field$, we are going to get different answers in different cases.

In the first case, we were looking at the $\sigma - field$ to be points on the circumference and the length of those arcs. In the second case, we are looking at the radius and the points on each radius and in the third point, we are looking at concentric circles and we are looking at the points inside it. Thus what I wanted to emphasize on is that depending upon how you define the sigma field, the probability of the event may change. Therefore, we need to be very careful about defining probability when things are complicated.

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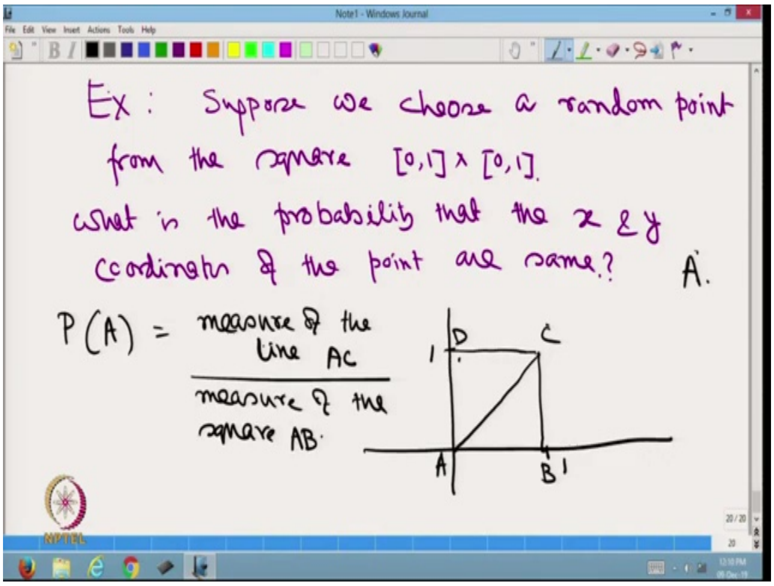

Problem, suppose we choose a random point from the square 0 to 1 cross 0 to 1 $[0,1]\times[0,1]$, what is the probability that the x and y coordinates of the point are same. So, that is the question. So, for our understanding consider the xy plane and suppose this is the square that we are talking about and we are looking at the probability that the x and y coordinate of the random point are same.

So, for example if I choose this point, obviously the x value is very small y value is very high and they are not same. It is very clear that the x and y coordinates will be same, if the point is lying on the line, this diagonal of the square. So, probability of the event so let us call the event A, the probability of A is equal to measure of the line AC over measure of the square ABCD.

$$P(A) = \frac{measure\ of\ the\ line\ AC}{measure\ of\ the\ square\ AB}$$

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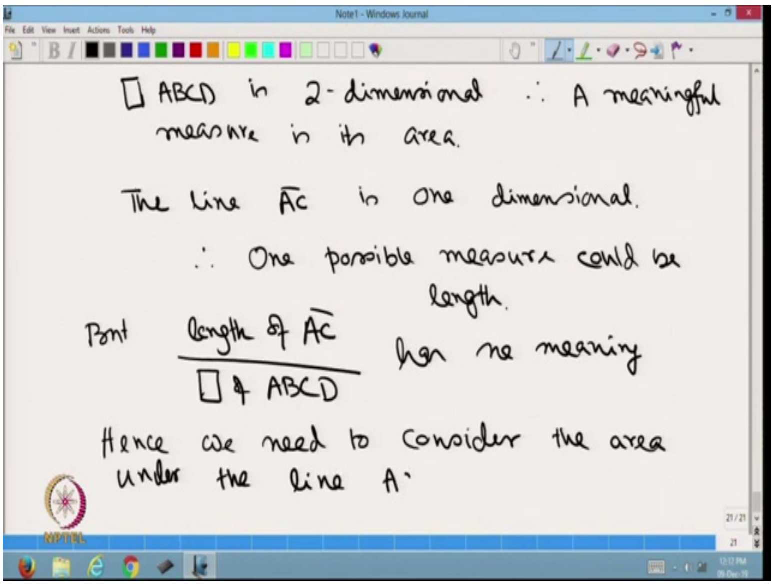

Now, squared ABCD is 2 dimensional, therefore a meaningful measure is its area. The line AC is one dimensional. Therefore, one possible measure could be length but then

$$\frac{length\ of\ AC}{Area\ of\ ABCD}$$

has no meaning because they had 2 different units. Hence we need to consider the area under the line AC.

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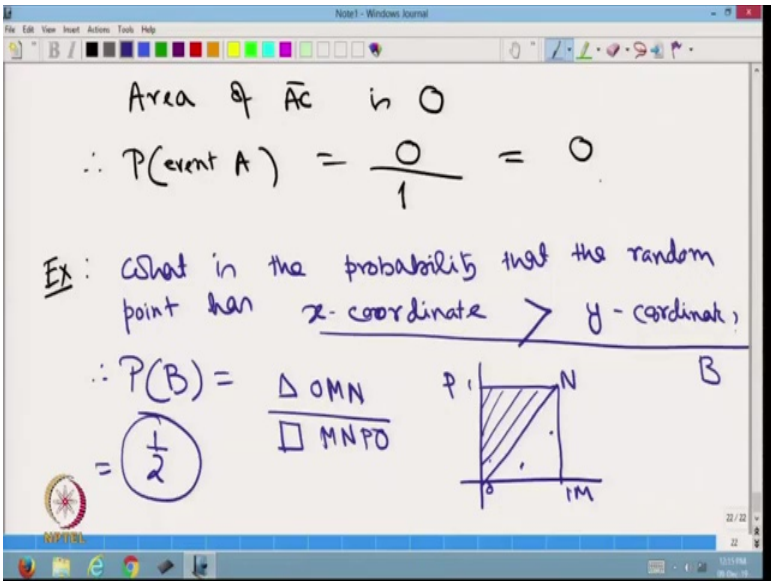

And we know that $Area\ of\ AC = 0$. Therefore, $P(event\ A) = \frac{0}{Area\ of\ ABCD} = \frac{0}{1} = 0$. A very similar question what is the probability that the random point has x coordinate greater than y coordinate. So, again if we look at the square and we can see that if this is this diagonal, then for each point in this triangle, the x value is greater than the y value on the other hand if we consider this triangle for each point inside the x value is smaller than the y value.

Therefore, probability of this event let us call it B,

$$P(B) = \frac{area\ of\ \Delta\ OMN}{area\ of\ square\ MNPO} = \frac{1}{2}$$

because the area of this triangle is going to be $\frac{1}{2}$. Area of the square is 1, so this probability is going to be $\frac{1}{2}$. So, I hope now you understand how we use measures for computing probability of an event.

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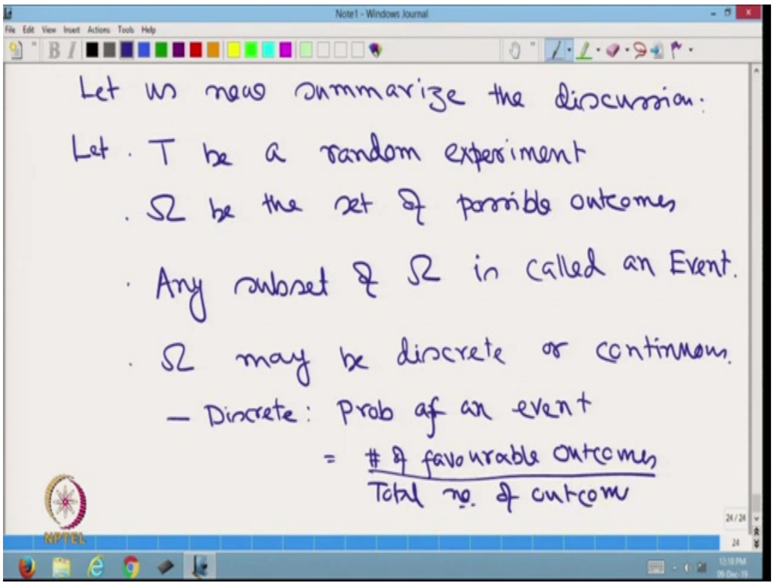

Let us now summarize the discussion. Let T be a random experiment, let Ω be the set of possible outcomes. Any subset of Ω is called an event, Ω may be discrete or continuous. When discrete, probability of an event is equal to number of favorable outcomes divided by total number of outcomes.

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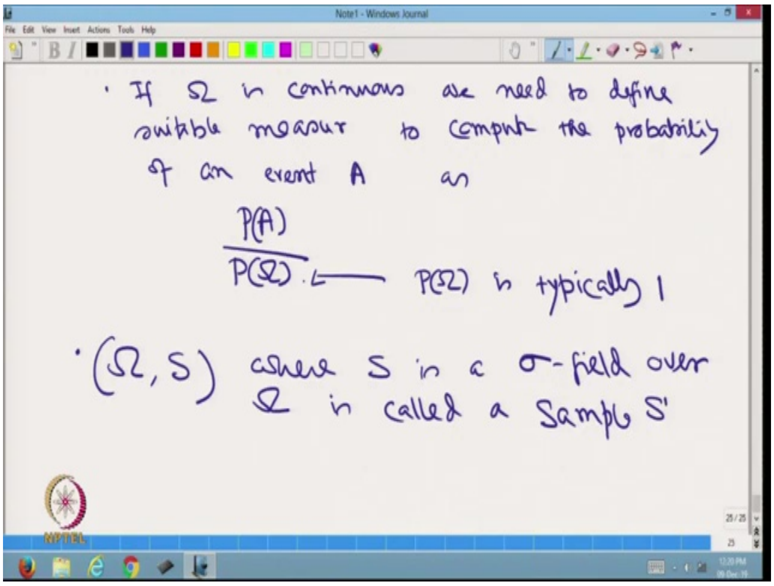

If Ω is continuous, we need to define suitable measure to compute the probability of an event A $P(A)$, if we call it by $P(\Omega)$. Typically we keep $P(\Omega)=1$, so if this is 1, then $P(A)$ itself will give us the probability of A. If it is not one, then we have to make the division and therefore, through that we will get the probability, Ω, S where S is a $\sigma - field$ over Ω is called a sample space.

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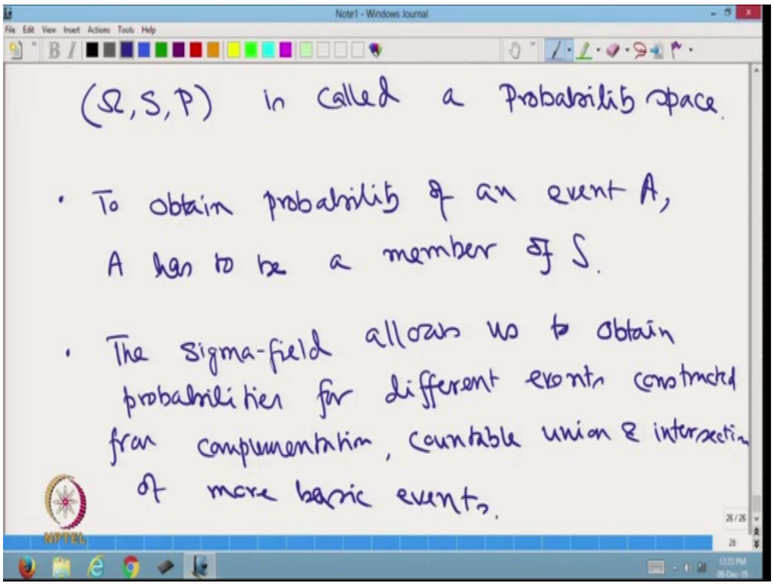

The triplet (Ω, S, P) is called a probability space to obtain probability of an event, A has to be a member of S. The $\sigma - field$ allows us to obtain probabilities for different events, constructed from complementation, countable, union and intersection of more basic events.

Okay friends, I stopped here today, in the next class. I shall start with Kolmogorov axioms on probability. And also we shall look at certain different mathematical functions, which will lead us to define certain probability of events, okay friends, thank you so much.