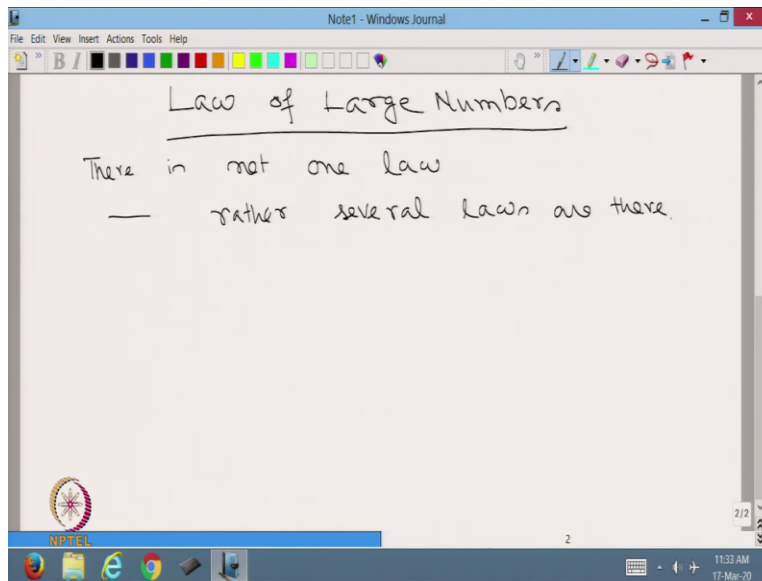


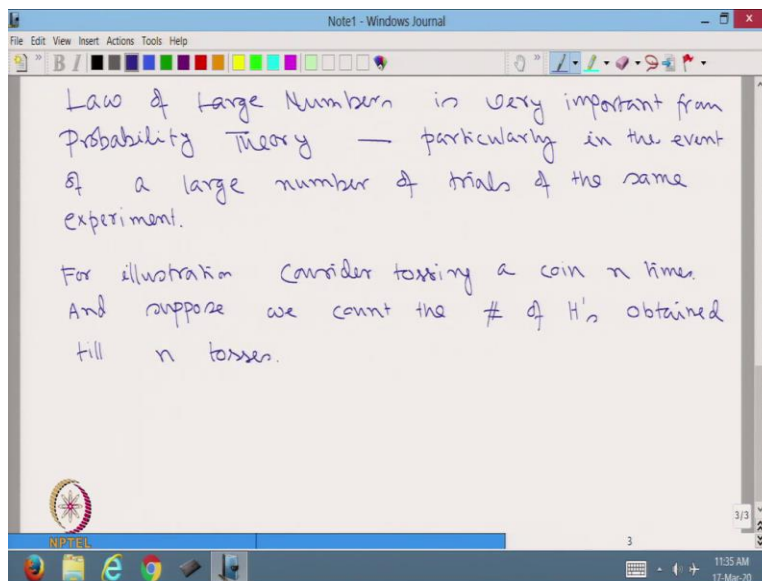
**Advanced Probability Theory**  
**Professor Niladri Chatterjee**  
**Department of Mathematics**  
**Indian Institute of Technology, Delhi**  
**Lecture 27**

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Welcome students to the mock lecture series on the Advance Probability Theory, this is lecture number 27, as I said at the end of the last class that today we will start Law of Large Numbers. In fact there is not one law rather several laws are there we shall look at some of them in detail.

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Now, Law of Large Numbers is very important from probability, probability theory particularly in the event of a large number of trials of the same experiment. For illustration consider tossing a coin  $n$  times and suppose we count the number of heads obtained till  $n$  tosses.

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The screenshot shows a Windows Journal window with a table of coin toss results. The table has 10 columns representing tosses and 4 rows representing different metrics. Below the table, there is a handwritten note explaining how to calculate the probability of success (heads) by dividing the total number of heads by the total number of tosses.

# of tosses	1	2	3	4	5	6	7	8	9	10
Outcome	H	H	T	T	H	T	T	T	H	T
# of H's till nth toss	1	2	2	2	3	3	3	3	4	4
Avg No. of tosses	1	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{4}{9}$	$\frac{4}{10}$

We obtain the probability 'p' of success (i.e. H) by dividing the total no. of H's by n.

Say for example we have number of tosses say 1, 2, 3, 4, 5, 6, 7, 8 and suppose the outcomes are like this, suppose the outcomes are head, head, tail, tail, head, tail, tail, let us go 2 more steps head, tail. Then the number of heads if we count till  $n$ th toss going to be 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, so if we consider average number tosses or then this is going to be 1, 1, 2 by 3, half, 3 by 5, half, 3 by 7, 3 by 8, 4 by 9 and 4 by 10, we know that we obtain the probability  $p$  of success that is head by dividing the total number of heads by  $n$ .

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The first screenshot shows handwritten text explaining the binomial distribution and the concept of probability as the ratio of favorable events to total events.

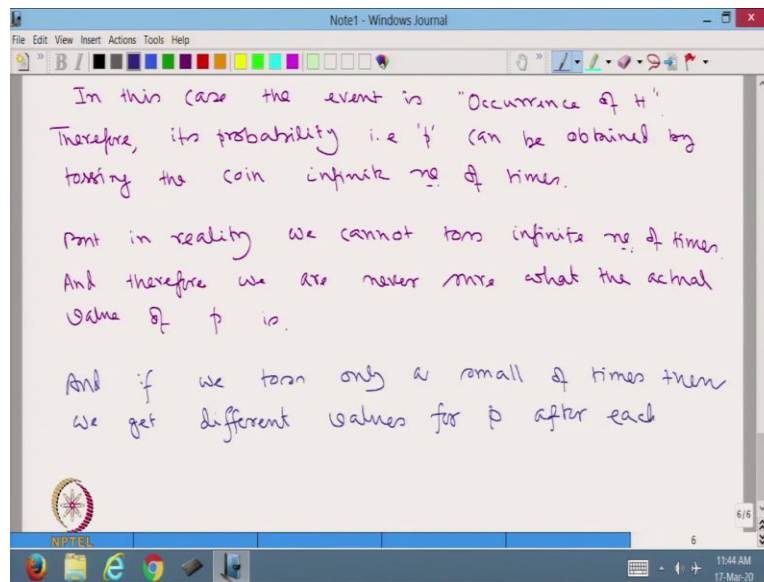
The second screenshot shows a table with 10 columns representing trials. The rows are labeled: '# of tosses', 'Outcome', '# of H's till nth toss', and 'Avg. No. of tosses'. The table data is as follows:

# of tosses	1	2	3	4	5	6	7	8	9	10
Outcome	H	H	T	T	H	T	T	T	H	T
# of H's till nth toss	1	2	2	2	3	3	3	3	4	4
Avg. No. of tosses	1	1	3/3	1/2	3/5	1/2	3/7	3/8	4/9	4/10

Below the table, it is noted that the probability 'p' of success (i.e. H) is obtained by dividing the total no. of H's by n.

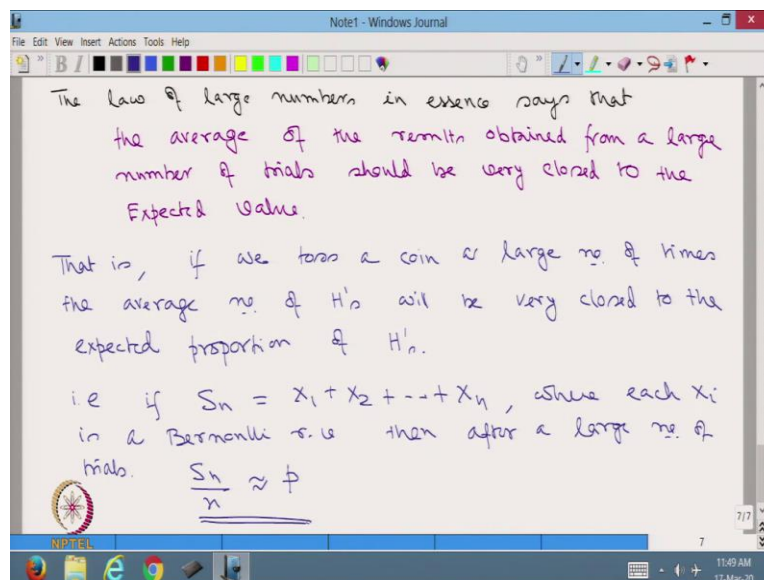
This is because the number of heads till nth toss follow binomially  $n, p$  where  $p$  is the probability of success. Therefore if we go back to the experiment just describe, therefore if we go back to the experiment just describe, we see that the number of or the proportion of heads is changing with the number of process. But we know that from the theory of probability that if a trail can be carried out infinitely number of times, then the number of favorable events divided by total number of events gives the probability of the event.

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In this case the event is occurrence of head, therefore its probability that is p can be obtain by tossing the coin infinite number of times. But in reality we cannot toss infinite number of times and therefore we are never sure what the actual value of p is and if we toss only a small number of times then we get different values for p after each toss.

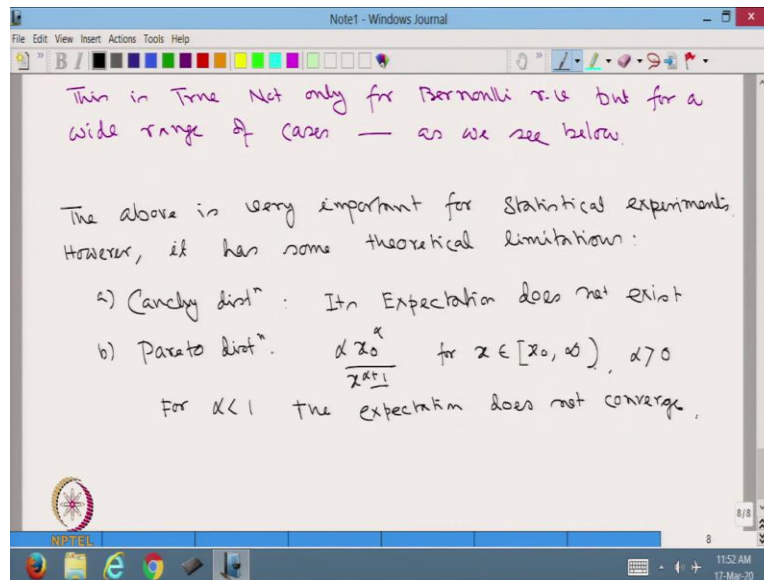
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The law of large number says that in essence says that the average of the results obtained from a large number of trials should be very closed to the expected value or in essence that is, if we toss a coin in large number of times, if we toss a coin in large number of times the average number of

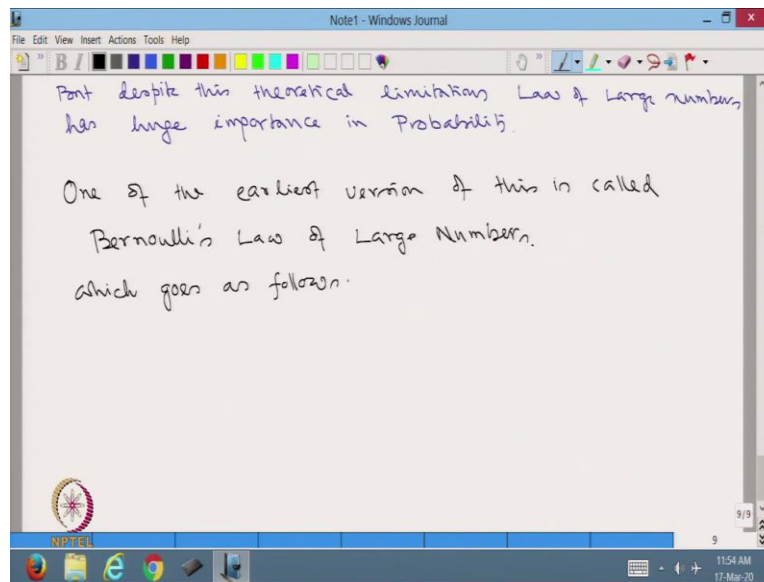
heads will be very close to the expected proportion of heads, that is if  $S_n$  is equal to  $X_1$  plus  $X_2$  plus  $X_n$ , where each  $X_i$  is a Bernoulli random variable then after a large number of trials  $S_n$  by  $n$  the average number of heads will be very close to the average proportion of heads will be very close to the actual proportion of heads that is  $p$ .

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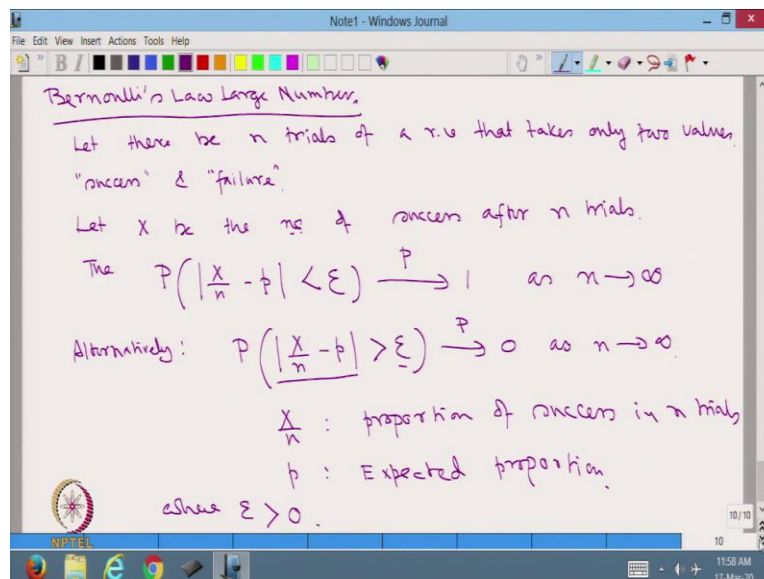
This is true not only for Bernoulli random variable but for a wide range of cases as we see below. Now, the above is very important for statistical experiments, however it has some theoretical limitations such as, Cauchy distribution, we know that its expectation does not exist, similarly Pareto distribution which is of the form  $\alpha x_0^\alpha$  to the  $\alpha$  upon  $x$  to the power  $\alpha$  plus 1 for  $x$  belonging to  $x_0$  comma infinity and  $\alpha$  is greater than 0, for  $\alpha$  less than 1 the expectation does not converge. Because this one will be canceled when we multiplied with  $x$ , therefore these series is not converge.

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But despite this theoretical limitations law of large numbers has huge importance in probability. One of the earliest version of this is called Bernoulli's law of large number, which goes as follows.

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So, Bernoulli's law of large number, which states that, let there be  $n$  trials of a random variables that takes only two values success and failure, let  $X$  be the number of success after  $n$  trials, then probability modulus of  $X/n$  minus  $p$  less than epsilon goes in probability to 1 as  $n$  goes to infinity, or alternately probability modulus of  $X/n$  minus  $p$  greater than epsilon goes to 0 as  $n$

goes to infinity. So, let us analysis the statement,  $x$  by  $n$  is the proportion of success in  $n$  trials and  $p$  is the expected proportion, so Bernoulli's law suggest that the difference between them being greater than some epsilon will probabilistically will go to 0, where epsilon ay positive quantity greater than 0.

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Proof: The proof comes from Chebyshev's Inequality, which states that:  
 if  $Z$  is r.v with  $E(Z) = \mu$   
 then  $P(|Z - \mu| < \epsilon) > 1 - \frac{\text{Var}(Z)}{\epsilon^2}$ .  
 Let us apply the above inequality for  $\frac{X}{n}$ .  
 We know  $X \sim \text{Bin}(n, p)$   $\therefore E\left(\frac{X}{n}\right) = \frac{np}{n} = p$   
 $V\left(\frac{X}{n}\right) = \frac{npq}{n^2} = \frac{pq}{n}$   
 $\therefore P\left(\left|\frac{X}{n} - p\right| < \epsilon\right) > 1 - \frac{pq}{n\epsilon^2}$   
 $> 1 - \frac{1}{4n\epsilon^2}$   $\therefore$  the maximum value for  $pq = \frac{1}{4}$ .

Proof, the proof comes from Chebyshev's Inequality which if you remember which I hope you remember, but let us still recollect that if which states that, if  $Z$  is a random variable with expected value of  $Z$  is equal to  $\mu$  then probability modulus of  $Z$  minus  $\mu$  less then epsilon is greater than 1 minus variance of  $Z$  upon epsilon square, this we have proved earlier.

Now, let us apply the above result the above inequality for  $X$  by  $n$ , we know  $X$  is distributed as binomial  $n$  comma  $p$ , therefore expected value of  $X$  by  $n$  is equal to  $np$  by  $n$  is equal to  $p$ . And variance of  $X$  by  $n$  is equal to  $npq$  upon  $n$  square is equal to  $pq$  upon  $n$ . Therefore probability modulus of  $X$  by  $n$  minus  $p$  less then epsilon is greater than 1 minus  $pq$  by  $n$  epsilon square, which is greater than 1 minus 1 by 4  $n$  epsilon square, since the maximum value for  $pq$  is equal to 1 by 4.



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$\therefore \text{for any fixed } \epsilon, P(|\frac{X}{n} - p| < \epsilon) > 1 - \frac{1}{4n\epsilon^2}$   
 $\xrightarrow{\text{as } n \rightarrow \infty} 1$   
 That is  $\frac{X}{n}$  converges to  $p$  in Probability

Therefore for any fixed epsilon probability modulus of  $X$  by  $n$  minus  $p$  less than epsilon is greater than  $1$  minus  $1$  upon  $4n$  epsilon square and this part going to  $0$  as  $n$  goes to infinity, that is  $X$  by  $n$  converges to  $p$  in probability. So, that is the proof that, that is the proof of Bernoulli's theorem.

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Weak Law of Large Numbers: (WLLN)  
 Let  $X_1, X_2, \dots, X_n$  be any sequence of r.v.s s.t.  
 $X_i$  has the Expectation  $\mu_i < \infty$  & has a Variance  
 $= \sigma_i^2$   
 Then  $P\left(\left|\frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^n \mu_i}{n}\right| < \epsilon\right) > 1 - \eta$   $\forall n \geq N_0$   
 where  $N_0$  depends upon the particular choice  
 of  $\epsilon$  &  $\eta$  however small positive  
 provided:  $\text{Var}\left(\sum_{i=1}^n X_i\right) < \infty$  and  $\frac{\text{Var}(X_i)}{n^2} \rightarrow 0$  as  $n \rightarrow \infty$

A more generalized version is weak law of large numbers which we shall write as W double L N and it is stated as follows, let  $X_1, X_2, \dots, X_n$  be any sequence of random variables such that  $X_i$  has the expectation  $\mu_i$  less than infinity and variance is equal to  $\sigma_i^2$ , then probability



that modulus of  $\sum_{i=1}^n X_i$  is equal to 1 to  $n$  upon  $n$  minus  $\sum_{i=1}^n \mu_i$ ,  $i$  is equal 1 to  $n$  upon  $n$  less than  $\epsilon$  is greater than  $1 - \eta$  for all  $n$  greater than equal to  $N$  naught, where  $N$  naught depends upon the particular choice of  $\epsilon$  and  $\eta$  however small positive quantity they are.

So, you understand that this is a much stronger statement, we are looking at a sequence of random variables we are not talking about whether they are independent, we are not talking about whether they are identically distributed only thing that if each one of them has a finite mean and a finite variance then the average of values of the random variable minus the average of their expectations will be very close that means less than  $\epsilon$  with a very high probability.

Provided variance of  $\sum_{i=1}^n X_i$  is equal to 1 to  $n$  is finite and variance of  $\sum_{i=1}^n X_i$  upon  $n^2$  goes to 0 as  $n$  goes to infinity. So, this is a very important assumption for weak law of large numbers as we will see some examples later that this is very very important.

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proof: Using Chebyshev's inequality:

$$P\left(\left|\frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^n E(X_i)}{n}\right| < \epsilon\right) \geq 1 - \frac{\text{Var}\left(\frac{\sum_{i=1}^n X_i}{n}\right)}{\epsilon^2}$$

i.e.  $P(|\bar{X} - \bar{\mu}| < \epsilon) \geq 1 - \frac{\text{Var}(\sum_{i=1}^n X_i)}{n^2 \epsilon^2}$

where  $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$  &  $\bar{\mu} = \frac{\sum_{i=1}^n \mu_i}{n}$

$$P(|\bar{X} - \bar{\mu}| < \epsilon) \geq 1 - \frac{\sum_{i=1}^n \sigma_i^2}{n^2 \epsilon^2}$$

Given condition is that  $\frac{\text{Var}(\sum_{i=1}^n X_i)}{n^2} \rightarrow 0$  : Given  $\epsilon$ ,  $\frac{\sum_{i=1}^n \sigma_i^2}{n^2 \epsilon^2} \rightarrow 0$

Proof again using Chebyshev's inequality probability modulus of  $\sum_{i=1}^n X_i$  is equal to 1 to  $n$  divided by  $n$  minus  $\sum_{i=1}^n$  expected value of  $X_i$  is equal to 1 to  $n$  divided by  $n$  this less than  $\epsilon$  is going to be greater than  $1 - \text{variance of } \sum_{i=1}^n X_i \text{ divided by } n^2 \epsilon^2$ .

That is probability modulus of  $\bar{X}$  minus  $\bar{\mu}$  less than  $\epsilon$  is greater than equal to  $1$  minus variance of  $\sum X_i$  upon  $n^2 \epsilon^2$ , where  $\bar{X}$  is equal to  $\sum X_i$   $i$  is equal to  $1$  to  $n$  divided by  $n$  and  $\bar{\mu}$  is equal to  $\sum \mu_i$   $i$  is equal to  $1$  to  $n$  divided by  $n$ . Therefore we get probability modulus of  $\bar{X}$  minus  $\bar{\mu}$  less than  $\epsilon$  is greater than equal to  $1$  minus summation  $\sigma_i^2$   $i$  is equal to  $1$  to  $n$  upon  $n^2 \epsilon^2$ . Now, the given condition is that variance of  $\sum x_i$  upon  $n^2$  goes to  $0$ , therefore for a given  $\epsilon$  this quantity goes to  $0$ .

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Handwritten notes in a Windows Journal window:

ie  $\exists N_0 \Rightarrow \forall n \geq N_0 \quad \frac{\sum \sigma_i^2}{n^2 \epsilon^2} < \eta$

however small  $\eta$  is.

$\therefore$  We see that  $P\left(\left|\frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^n \mu_i}{n}\right| < \epsilon\right) \geq 1 - \eta$

$\forall n \geq N_0$

proved

Note that :

If  $X_i$ 's are independent then

$$\text{Var}(\sum X_i) = \sum_{i=1}^n \sigma_i^2$$

If  $X_i$ 's are Not independent

$$\text{then } \text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \sigma_i^2 + \sum_{\substack{i,j \\ i \neq j}} \text{cov}(X_i, X_j)$$

But because of the assumption

$$\frac{\text{Var}(\sum X_i)}{n^2} \rightarrow 0$$



WLLN holds.

That is there exist  $N$  naught such that for all  $n$  greater then equal to  $N$  naught, summation of  $\sigma_i^2$  upon a square epsilon square is less then eta, however small eta is therefore we see that probability modulus of  $\sigma(\bar{X}_n)$  is equal to 1 to  $n$  upon  $n$  minus sigma mu i i is equal to 1 to  $n$  upon, that probability less then epsilon is greater than equal to 1 minus eta for all  $n$  greater then equal to  $N$  naught, hence proved.

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What happens if  $X_i$ 's are iid's.

In that case each  $X_i$  has Expectation  $\mu_i = \mu$  Variance  $\sigma_i^2 = \sigma^2$

$\therefore \text{Var}\left(\frac{\sum X_i}{n}\right) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \rightarrow 0$  as  $n \rightarrow \infty$

$\therefore$  WLLN holds trivially.

In the following examples  
 $X_i$ 's are considered independent.

What happens if  $X_i$ 's are iid's. In that case each  $X_i$  has expectation  $\mu_i$  is equal to  $\mu$  and variance  $\sigma_i^2$  is equal to  $\sigma^2$ . Therefore variance of  $\sigma(\bar{X}_n)$  is equal

to  $n$  sigma square upon  $n$  square is equal to sigma square by  $n$  goes to 0 as  $n$  goes to infinity. Therefore weak law of large number holds trivially. So, let me give you some examples.

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Ex ①  $X_i = \begin{cases} \sqrt{i} & \frac{1}{2} \\ -\sqrt{i} & \frac{1}{2} \end{cases} \therefore E(X_i) = 0 \quad \forall i$   
 $V(X_i) = E(X_i^2) = i$   
 $\therefore \frac{V(\sum X_i)}{n^2} = \frac{\sum_{i=1}^n i}{n^2} = \frac{n(n+1)}{2n^2} \rightarrow \frac{1}{2}$   
 Since the condition  $\frac{V(\sum X_i)}{n^2} \rightarrow 0$  does not hold  
 $\therefore$  WLLN does not hold.

②  $X_i = \begin{cases} i^\alpha & \frac{1}{2} \\ -i^\alpha & \frac{1}{2} \end{cases} \therefore$  As before  $E(X_i) = 0$   
 $V(X_i) = i^{2\alpha}$   
 $\therefore V(\sum X_i) = \sum_{i=1}^n i^{2\alpha} \approx \int_0^n x^{2\alpha} dx = \frac{x^{2\alpha+1}}{2\alpha+1} \Big|_0^n = \frac{n^{2\alpha+1}}{2\alpha+1}$

Suppose  $X_i$  is distributed as root over  $i$  with probability half and minus of root over  $i$  with probability half, therefore expected value of  $X_i$  is equal to 0 for all  $i$  and variance of  $X_i$  is equal to expectation of  $X_i$  square is equal to  $i$ , therefore variance of sigma  $X_i$  upon  $n$  square is equal to sigma over  $i$   $i$  is equal to 1 to  $n$  upon  $n$  square is equal to  $n$  into  $n$  plus 1 by 2  $n$  square which converges to half, since the condition variance of sigma  $X_i$  upon  $n$  square goes to 0 does not hold, therefore weak law of larger number does not hold.

Example 2,  $X_i$  is equal to  $i$  to the power  $\alpha$  with probability half and minus  $i$  to the power  $\alpha$  with probability half. Therefore as before expected value of  $X_i$  is equal to 0 and variance of  $X_i$  is equal to  $i$  the power 2  $\alpha$ , therefore sigma variance of  $X_i$  is equal to sigma  $i$  to the power 2  $\alpha$   $i$  is equal to 1 to  $n$  is equal to integration 0 to  $n$   $x$  to the power 2  $\alpha$   $dx$  which is equal to  $x$  to the power 2  $\alpha$  plus 1 upon 2  $\alpha$  plus 1 from 0 to  $n$  is equal to  $n$  to the power 2  $\alpha$  plus 1 upon 2  $\alpha$  plus 1.

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$$\therefore \frac{1}{n^2} \sum V(X_i)$$

$$\because V(\sum X_i) = \sum V(X_i)$$
 if  $X_i$ 's are independent  
 So in the above set up  $X_i$ 's are independent  
 let us assume.  

$$\therefore \frac{1}{n^2} \sum V(X_i) = \frac{n^{2\alpha+1}}{n^2} \rightarrow 0 \text{ if } 2\alpha+1 < 2$$

$$\text{i.e. } \alpha < \frac{1}{2}$$

Therefore  $1$  by  $n$  square into sigma variance of  $X_i$  I am writing is sigma variance of  $X_i$  since variance of sigma  $X_i$  is equal to sigma variance of  $X_i$  if  $X_i$ 's are independent and so in the above set up  $X_i$ 's are independent let us assume. Therefore  $1$  by  $n$  square into sigma variance of  $X_i$  is equal to  $n$  to the power  $2\alpha + 1$  upon  $n$  square into  $2$  to the power  $\alpha + 1$ ,  $n$  to the power  $2\alpha + 1$  upon  $n$  square  $2\alpha + 1$  which will go to  $0$  if  $2\alpha + 1$  is less than  $2$  that means this quantity has to be dominated by this quantity that is  $\alpha$  is less than half.

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Ex ①  $X_i = \begin{cases} \sqrt{i} & \frac{1}{2} \\ -\sqrt{i} & \frac{1}{2} \end{cases} \therefore E(X_i) = 0 \quad \forall i$   
 $V(X_i) = E(X_i^2) = i$   

$$\therefore \frac{V(\sum X_i)}{n^2} = \frac{\sum_{i=1}^n i}{n^2} = \frac{n(n+1)}{2n^2} \rightarrow \frac{1}{2}$$
  
 Since the condition  $\frac{V(\sum X_i)}{n^2} \rightarrow 0$  does not hold  
 $\therefore$  WLLN does not hold.

②  $X_i = \begin{cases} i^{2\alpha} \cdot \frac{1}{2} & \therefore \text{As before } E(X_i) = 0 \\ -i^{2\alpha} \cdot \frac{1}{2} & V(X_i) = i^{2\alpha} \end{cases}$   

$$\therefore \sum_{i=1}^n V(X_i) = \sum_{i=1}^n i^{2\alpha} \approx \int_0^n x^{2\alpha} dx = \frac{x^{2\alpha+1}}{2\alpha+1} \Big|_0^n = \frac{n^{2\alpha+1}}{2\alpha+1}$$

Hence no wonder that  $\alpha$  is equal to half weak law of large number does not hold.

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Ex Consider  $X_i \sim N(0, 1)$   
 Since these are iid's with finite Mean & Variance  
 WLLN holds.

What about  $\{X_i^2\}$ ?

We know that  $X_i^2 \sim \chi^2_1$   
 $\therefore E(X_i^2) = 1$  &  $Var(X_i^2) = \frac{1}{(\frac{1}{2})^2} = 2$ .

$\therefore \sum_{i=1}^n V(X_i) = 2n$ .

$\therefore \frac{1}{n^2} \left( \sum_{i=1}^n V(X_i) \right) = \frac{2}{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

$\therefore$  WLLN holds.

Another example, consider  $X_i$  following normal 0, 1. Since these are iid's with finite mean and variance weak law of large number holds, what about the sequence  $\sigma X_i$ ? We know that  $X_i$  square is distributed as Chi square with 1 degree of freedom, therefore expected value of  $X_i$  square is equal to 1 and variance of  $X_i$  square is equal to lambda upon alpha square is equal to 2.

Therefore sigma variance of  $X_i$  i is equal to 1 to n is equal to 2n. Therefore 1 by n square sigma variance of  $X_i$  i is equal to 1 to n is equal to 2 by n which goes to 0 as n goes to infinity, therefore weak law of large number holds.

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Ex  $X_i \sim N(\mu, \sigma^2)$   
 Let  $Y_i = X_1 + \dots + X_i$   
 Does WLLN hold for  $\{Y_i\}$ ?

$Y_i \sim N(i\mu, i\sigma^2)$   
 $\therefore V(Y_1 + Y_2 + \dots + Y_n) = \sigma^2(1 + 2 + \dots + n) = \frac{n(n+1)}{2} \sigma^2$

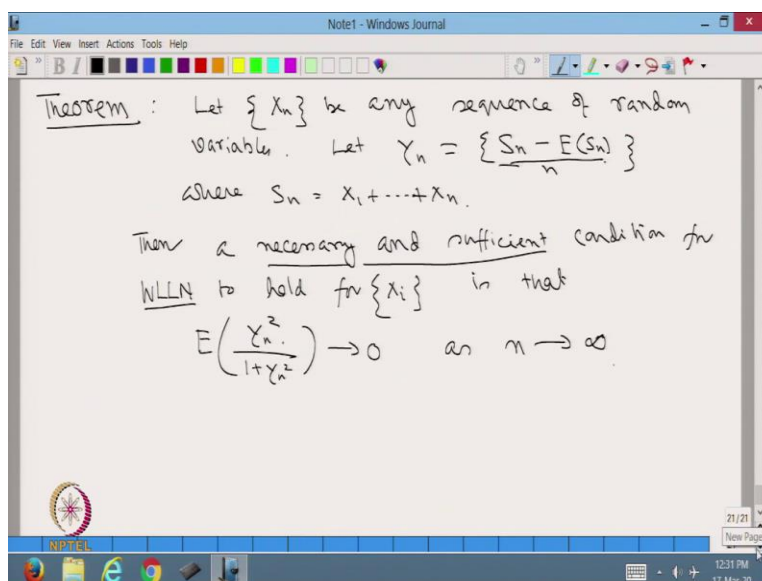
$\therefore \frac{1}{n^2} V(Y_1 + Y_2 + \dots + Y_n) = \frac{n(n+1)}{2n^2} \sigma^2 \not\rightarrow 0$  as  $n \rightarrow \infty$

$\therefore$  WLLN does not hold

Another example, suppose  $X_i$  is equal to normal with mean  $\mu$  variance  $\sigma^2$  let  $Y_i$  is equal to  $X_1$  plus  $X_2$  up to  $X_i$ , does weak law of large number hold for the sequence  $Y_i$ ? That is the question. Now, each  $Y_i$ ,  $Y_i$  is distributed as normal with  $i\mu$  and variance is equal to  $i\sigma^2$ , therefore variance of  $Y_1$  plus  $Y_2$  plus  $Y_n$  is equal to  $\sigma^2$  into  $1$  plus  $2$  up to  $n$  is equal to  $n$  into  $n+1$  by  $2$   $\sigma^2$ .

Therefore  $1$  by  $n$  square into variance of  $Y_1$  plus  $Y_2$  plus up to  $Y_n$  is equal to  $n$  into  $n+1$  by  $2$   $\sigma^2$  into  $n$  square into  $\sigma^2$ , which does not go to  $0$  as  $n$  goes to infinity. So, this is the notation to imply that it is not converging to  $0$ . Therefore weak law of large number does not hold.

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Let me now state a very important theorem, which states as follows, let  $X_n$  be a sequence, let  $X_n$  be any sequence of random variables, let  $Y_n$  be  $S_n$  minus expected value of  $S_n$  divided by  $n$ , where  $S_n$  is equal to  $X_1$  plus  $X_2$  up to  $X_n$ , then a necessary and sufficient condition for weak law of large numbers to hold for  $X_i$  is that expected value of  $Y_n$  square upon  $1$  plus  $Y_n$  square goes to  $0$  as  $n$  goes to infinity.

So, this is a very important result and it is a necessary and sufficient condition for weak law of large numbers to hold. Now, let us analysis the result,  $Y_n$  is equal to the partial sum  $X_1$  up to  $X_n$  minus its expectation divided by  $n$ , therefore yields expectation is going to  $0$  means expected value of  $Y_n$  square is becoming very very small, so that is the implication in such a case  $X_n$  is equal to obey weak law large numbers.



(Refer Slide Time: 44:20)

The image shows a handwritten mathematical proof in a Windows Journal window titled "Note1 - Windows Journal". The text is written in black ink on a light gray background. The proof is as follows:

Proof: Suppose  $E\left(\frac{X_n^2}{1+X_n^2}\right) \rightarrow 0$  as  $n \rightarrow \infty$

We want to show that  $\{X_i\}$  satisfies WLLN.

i.e. given  $\epsilon > 0$   $\exists N \ni \forall n > N$   $P\left(\left|\frac{\sum X_i}{n} - \frac{\sum \mu_i}{n}\right| > \epsilon\right) \rightarrow 0$  as  $n \rightarrow \infty$ .

Consider the event  $A = |X_n| > \epsilon$

$\therefore P(A) = P(|X_n| > \epsilon)$

Note that on  $A$   $X_n^2 > \epsilon^2$

We know that given  $a > b$ ,  $a + ab > b + ab$   
i.e.  $a(1+b) > b(1+a)$   
i.e.  $\frac{a}{1+a} > \frac{b}{1+b}$ .

The Windows Journal window includes a menu bar (File, Edit, View, Insert, Actions, Tools, Help), a toolbar with various drawing tools, and a taskbar at the bottom showing the date (22/2/22) and time (12:35 PM).

Proof, suppose expected value of  $Y_n$  square upon  $1 + Y_n$  square goes to 0 as  $n$  goes to infinity we want to show that  $X_i$  satisfies weak law of large numbers that is given epsilon greater than 0 they are exist in such that for all  $n$  greater than  $N$  probability modulus of  $\sigma \sum X_i$  upon  $n$  minus  $\sigma \mu_i$  upon  $n$  greater than epsilon that probability will go to 0 as  $n$  goes to infinity.

Consider the event  $A$  is equal to modulus of  $Y_n$  is greater than epsilon, therefore probability of  $A$  is equal to probability modulus of  $Y_n$  greater than epsilon. Note that on  $A$   $Y_n$  square is greater than epsilon square. Now, we know that given  $a$  greater than  $b$ ,  $a$  plus  $ab$  is greater than  $b$  plus  $ab$ , that is  $a$  into  $1 + b$  is greater than  $b$  into  $1 + a$  that is  $a$  upon  $1 + a$  is greater than  $b$  upon  $1 + b$ , we want to apply the same result here.

(Refer Slide Time: 46:54)

The image shows a handwritten derivation in a Windows Journal window. The text is as follows:

We have  $X_n^2 > \epsilon^2$  on A ✓

$\therefore$  on A  $\frac{X_n^2}{1+X_n^2} > \frac{\epsilon^2}{1+\epsilon^2}$

Let B be the event  $\frac{X_n^2}{1+X_n^2} > \frac{\epsilon^2}{1+\epsilon^2}$

$\therefore P(A) \leq P(B)$

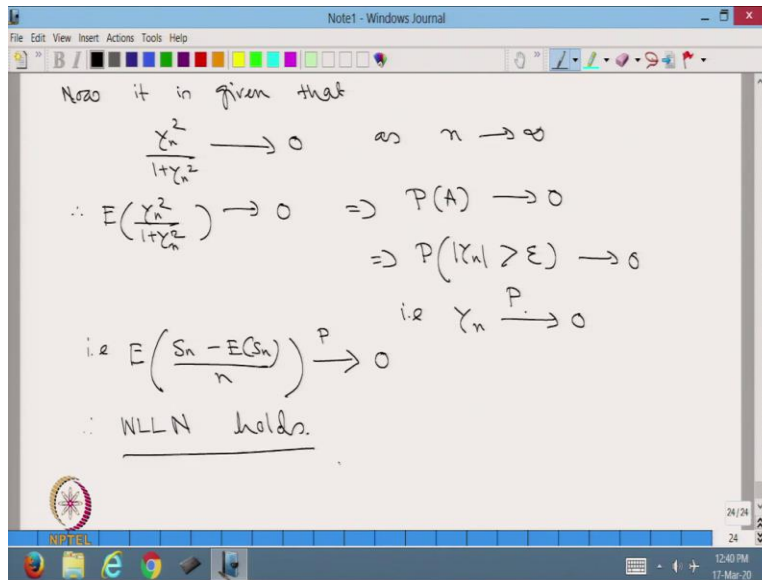
$\therefore P(A) \leq P\left(\frac{X_n^2}{1+X_n^2} > \frac{\epsilon^2}{1+\epsilon^2}\right)$  i.e.  $P\left(\frac{X_n^2(1+\epsilon^2)}{(1+X_n^2)\epsilon^2} > 1\right)$

$\therefore$  by using Markov inequality:  $\leq E\left(\frac{X_n^2(1+\epsilon^2)}{(1+X_n^2)\epsilon^2}\right)$

So, we have  $Y_n$  square is greater than epsilon square on A therefore on A  $Y_n$  square upon 1 plus  $Y_n$  square is greater than epsilon square upon 1 plus epsilon square, let B be the event  $Y_n$  square upon 1 plus  $Y_n$  square is greater than epsilon square upon 1 plus epsilon square here probability of A is less than equal to probability of B, because this is contained in this event.

Therefore probability of A is less than or equal to probability  $Y_n$  square upon 1 plus  $Y_n$  square greater than epsilon square upon 1 plus epsilon square that is probability  $Y_n$  square into 1 plus epsilon square upon 1 plus  $Y_n$  square into epsilon square is greater than 1. Therefore by using Markov inequality which we have done several classes back we can write it as this probability is less than equal to expected value of  $y_n$  square into 1 plus epsilon square upon 1 plus  $Y_n$  square into epsilon square because the threshold there is 1.

(Refer Slide Time: 49:13)



The image shows a handwritten mathematical proof in a Windows Journal window titled "Note1 - Windows Journal". The text is written in black ink on a light gray background. The proof starts with "Now it is given that" followed by the equation  $\frac{Y_n^2}{1+Y_n^2} \rightarrow 0$  as  $n \rightarrow \infty$ . This is followed by  $\therefore E\left(\frac{Y_n^2}{1+Y_n^2}\right) \rightarrow 0 \Rightarrow P(A) \rightarrow 0$ , then  $\Rightarrow P(|Y_n| > \epsilon) \rightarrow 0$ , and finally  $\therefore Y_n \xrightarrow{P} 0$ . Below this, it says "i.e.  $E\left(\frac{S_n - ES_n}{n}\right) \xrightarrow{P} 0$ ". The final conclusion is " $\therefore$  WLLN holds.". The window has a standard Windows interface with a menu bar (File, Edit, View, Insert, Actions, Tools, Help), a toolbar with various drawing tools, and a taskbar at the bottom showing the date and time (12:40 PM, 17-Mar-20).

Now it is given that

$$\frac{Y_n^2}{1+Y_n^2} \rightarrow 0 \text{ as } n \rightarrow \infty$$
$$\therefore E\left(\frac{Y_n^2}{1+Y_n^2}\right) \rightarrow 0 \Rightarrow P(A) \rightarrow 0$$
$$\Rightarrow P(|Y_n| > \epsilon) \rightarrow 0$$

i.e.  $Y_n \xrightarrow{P} 0$

$$\therefore E\left(\frac{S_n - ES_n}{n}\right) \xrightarrow{P} 0$$

$\therefore$  WLLN holds.

Now, it is given that  $Y_n^2$  upon  $1 + Y_n^2$  going to 0 as  $n$  goes to infinity, therefore expected value of  $Y_n^2$  upon  $1 + Y_n^2$  goes to 0 imply probability of  $A$  is going to 0 implies probability modulus of  $Y_n$  greater than  $\epsilon$  is going to 0 that is  $Y_n$  converges to 0 in probability that is expected value of  $S_n$  minus expected value of  $S_n$  divided by  $n$  converges in probability is 0.

Therefore weak law of large number holds. okay friends I stop here today we have just proved the theorem in one direction we shall look at the other side that given that  $X_n$  holds weak law of large numbers we shall show that expected value of  $Y_n^2$  upon  $1 + Y^2$  goes to 0, also we shall do some more examples and some theorem in the next class. Thank you.