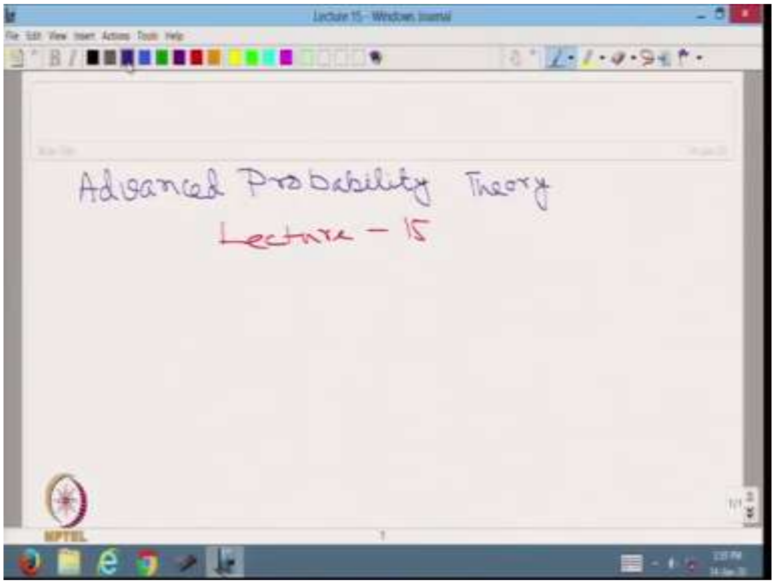
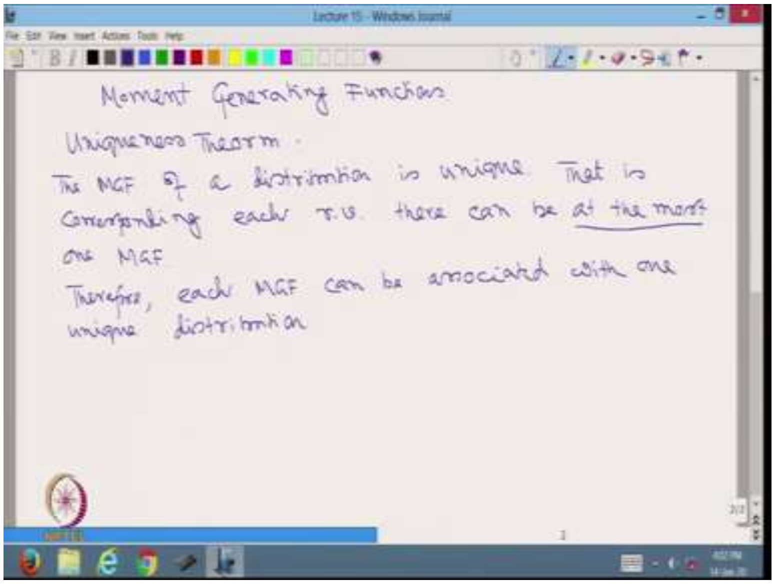
Advanced Probability Theory
Professor Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture 15

(Refer Slide Time: 00:30)



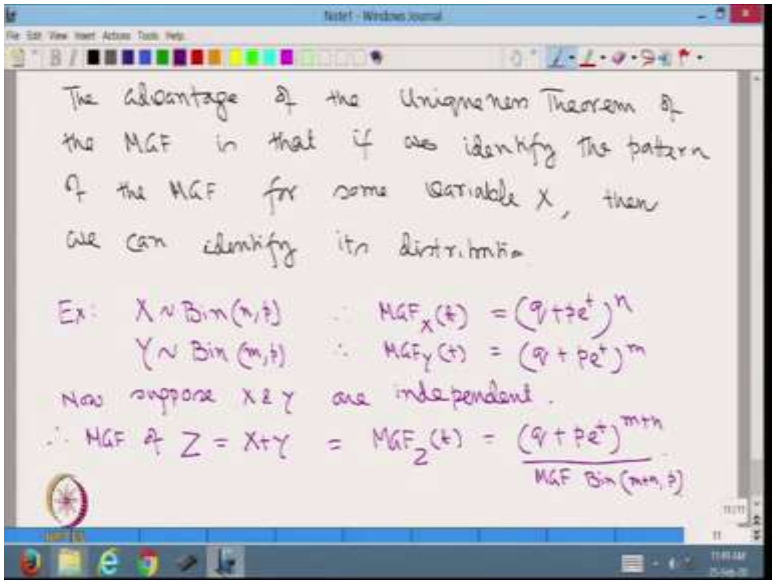
Welcome students to the MOOC lecture series on Advanced Probability Theory, this is lecture number 15.

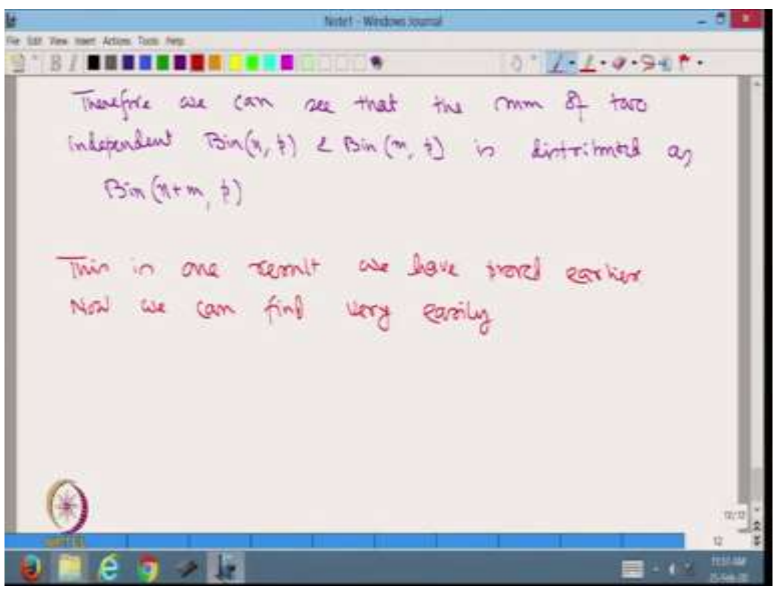
(Refer Slide Time: 00:34)



In the last class, we have seen Moment Generating Functions and some properties of it. These are important because of the uniqueness theorem which says that the moment generating function of a distribution is unique. That is corresponding to each random variable there can be at the most one MGF. I am saying at the most, because there may be distributions, which do not have the moment generating function. Therefore each moment generating function can be associated with one unique distribution.

(Refer Slide Time: 02:28)

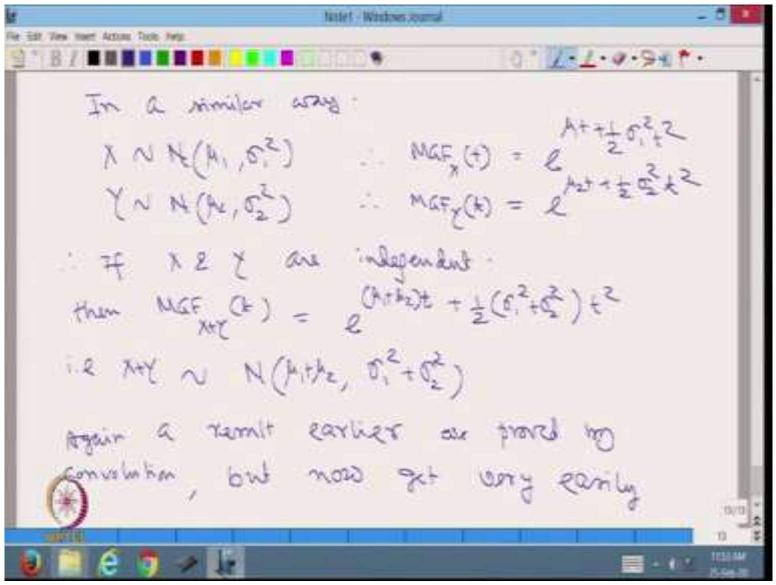




The advantage of the uniqueness theorem of the moment generating function is that if we identify the pattern of the moment generating function for some variable x, then we can identify its distribution. For example, suppose x is binomial n comma p. Therefore MGF of x at t is equal to q plus p, e to the power t, whole to the power n. Suppose y is distributed as binomial m p, therefore MGF, MGF of y t is equal to q plus p e to the power t whole to the power m.

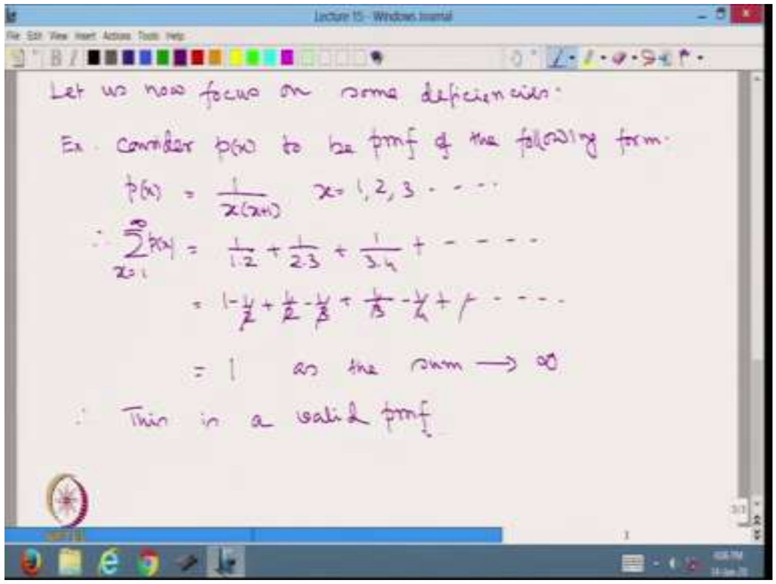
Now, suppose x and y are independent therefore MGF of z is equal to x plus y is equal to, we know that it is the product of their individual moment generating functions that is q plus p e to the power t whole to the power m plus n which is the MGF of binomial m plus n comma p. Therefore, we can see that this sum of 2 independent binomial n comma p and binomial m comma p is distributed as binomial n plus m comma p, this is one result we have proved earlier. But now, we can find very easily.

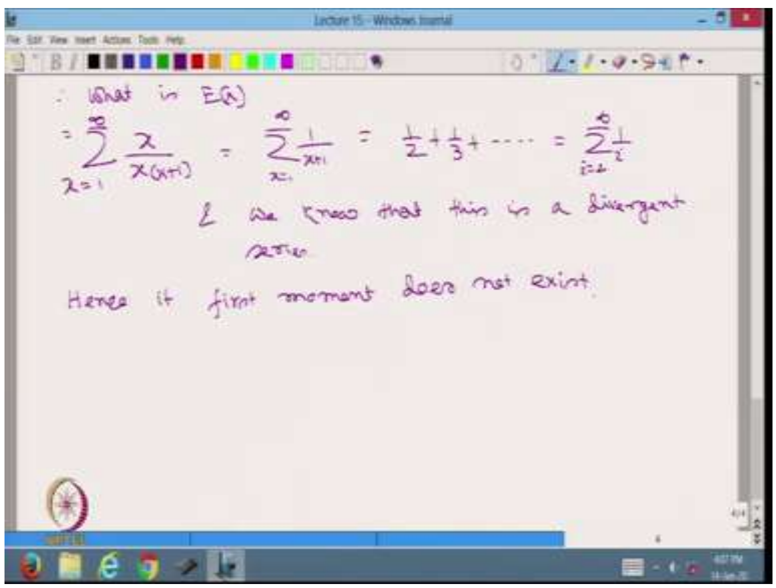
(Refer Slide Time: 06:28)



In a similar way, suppose x is normal with mu 1 sigma 1 square therefore its MGF at t is equal to e to the power mu 1 t plus half sigma 1 square t square suppose y is normal mu 2 comma sigma 2 square. Therefore MGF of y is equal to e to the power mu 2 t plus half sigma 2 square t square. Therefore, if x and y are independent then MGF of x plus y at t is equal to their product and it is equal to e to the power mu 1 plus mu 2 t plus half sigma 1 square plus sigma 2 square into t square that is x plus y is distributed as normal with mu 1 plus mu 2 and variance is equal to sigma 1 square plus sigma 2 square. Again a result, earlier we proved by convolution, but now get very easily.

(Refer Slide Time: 08:43)

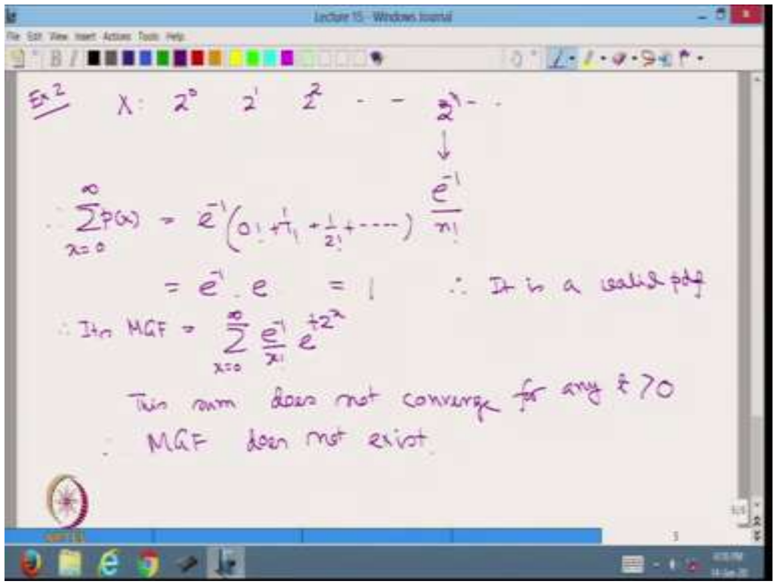




Let us now focus on some deficiencies for example, consider px to be a PMF of the following form p x is equal to 1 upon x into x plus 1, x is equal to 1, 2, 3. Therefore, sigma p x, x is equal to 1 to infinity is equal to, which is equal to 1 minus half plus half minus one third, plus one third minus one fourth and if we notice that alternating terms keep on canceling. Therefore, this is going to be 1 as the sum goes to infinity, therefore this is a valid PMF.

Therefore, what is expected value of x? This is equal to sigma x upon x into x plus 1, x is equal to 1 to infinity is equal to sigma 1 upon x plus 1, x is equal to 1 to infinity is equal to half plus one third up to infinity is equal to sigma 1 over i,i is equal to 2 to infinity and we know that this is a divergent series. Hence, its first moment does not exist.

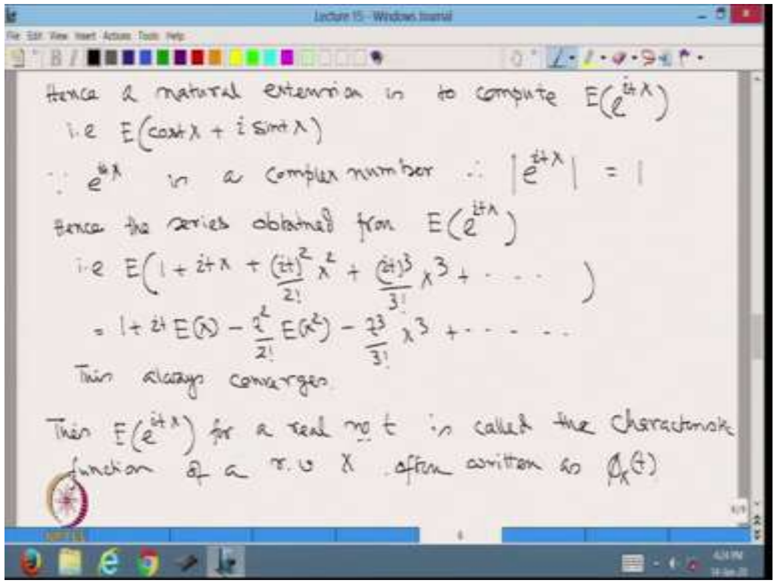
(Refer Slide Time: 11:37)



Another example, x is a random variable, which takes values 2 to the power 0,2 to the power 1, 2 to the power 2, etc with probability of 2 to the power n is equal to e to the power minus 1 into n factorial. Therefore, sigma p x, x is equal to 0 to infinity is equal to e to the power minus 1 into 0 factorial plus 1 upon 1 factorial plus 1 upon 2 factorial plus etc is equal to e to the power minus 1 into e is equal to 1. Therefore, it is a valid PDF.

Therefore, its MGF is equal to sigma e to the power minus 1 x factorial into e to the power t into 2 to the power x, x is equal to 0 to infinity. This sum does not converge for any t greater than 0 therefore, moment generating function does not exist. Like that, we can show many examples, where there are problems with respect to MGF. However MGF is important because of its uniqueness.

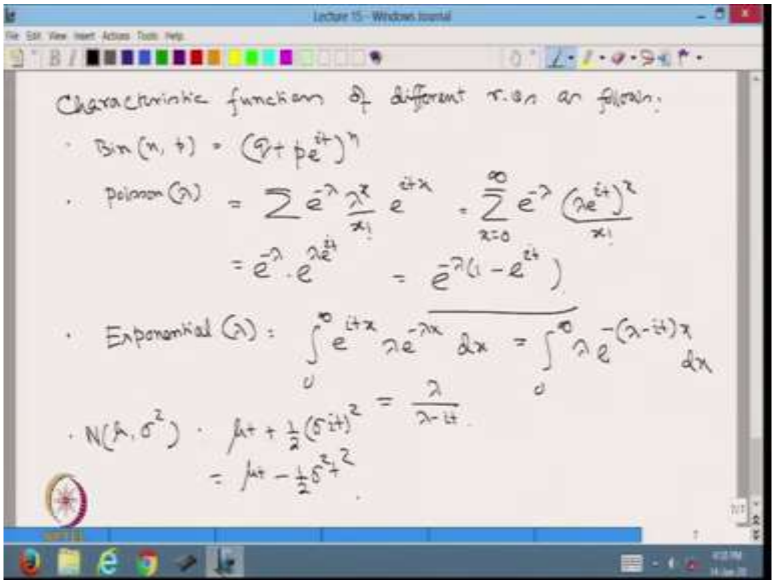
(Refer Slide Time: 13:34)



Hence a natural extension is to compute expected value of e to the power itx that is expected value of cos t x plus i sine t x. Since e to the power itx is a complex number. Therefore, modulus of e to the power itx is equal to 1 hence this series obtained from expected value of e to the power itx, that is expected value of 1 plus itx plus i t whole square upon factorial 2 x square plus i t whole cube upon factorial 3 x cube which is equal to 1 plus i t times expected value of x minus t square upon factorial 2 expected value of x square minus t cube upon factorial 3 x cube.

Like that, we get an infinite series, this always converges, this expectation of e to the power itx for a real number t is called the characteristic function of a random variable x which we often write as, as phi x.

(Refer Slide Time: 16:38)



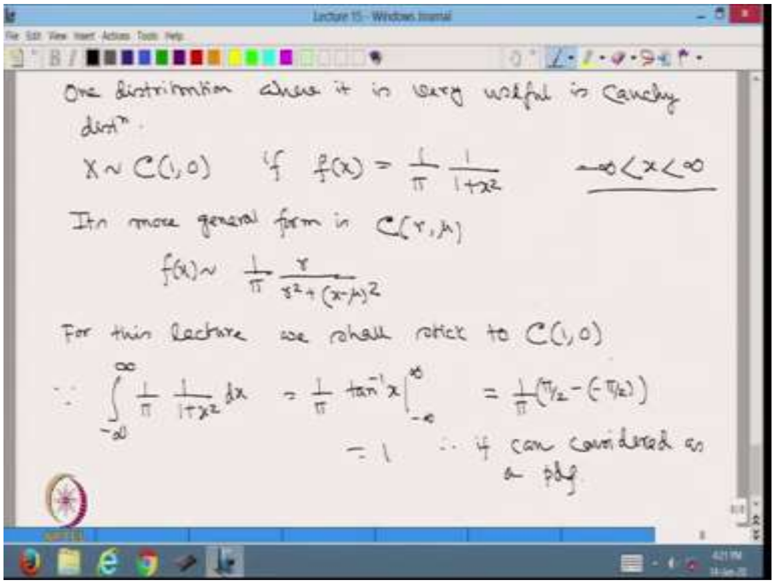
Since we have already computed moment generating function for different random variables we can easily find that characteristic functions of different random variables as follows binomial n comma p it is going to be q plus p e to the power i t whole to the power n Poisson lambda is equal to sigma e to the power minus lambda.

Lambda power x upon factorial x, e to the power itx is equal to sigma say x is equal to 0 to infinity e to the power minus lambda, lambda e to the power i t whole to the power x upon factorial x is equal to e to the power minus lambda into e to the power lambda, e to the power i t is equal to e to the power minus lambda into 1 minus e to the power i t that is the characteristic function of Poisson random variable.

Similarly, for exponential random variable this is going to be 0 to infinity e to the power i t x lambda e to the power minus lambda x dx is equal to 0 to infinity lambda into e to the power minus lambda minus i t x dx is equal to lambda upon lambda minus i t. Finally, for normal mu comma sigma square is moment generating function is mu t plus half sigma square t square.

Therefore, here we should get half sigma i t whole square is equal to mu t minus half sigma square t square, okay friends in a similar way, we can calculate the characteristic function of different random variables. Again like the moment generating function, the characteristic function is also unique and given any distribution, it will have a unique characteristic function and given any characteristic function, it should correspond with one particular distribution.

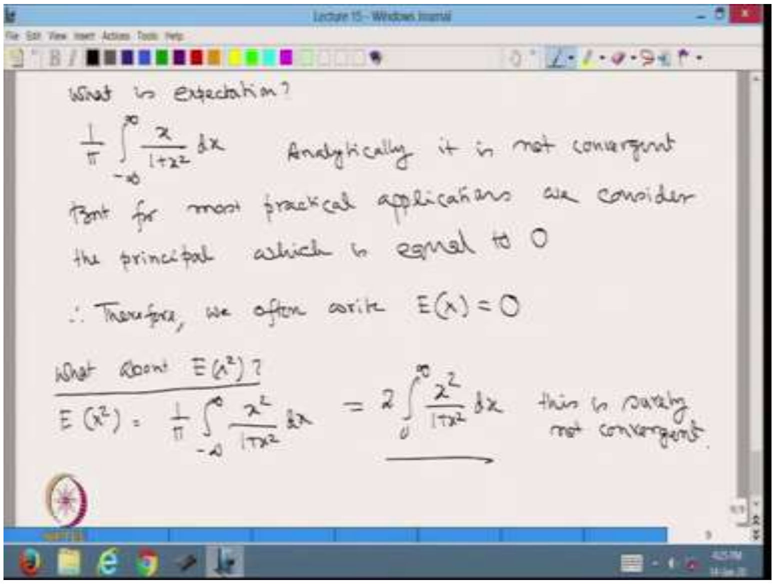
(Refer Slide Time: 19:49)



One particular distribution, where it is very useful is Cauchy distribution, x is distributed as Cauchy with 1 comma 0 if fx is equal to 1 over Pi into 1 upon 1 plus x square. Its more general form is C r comma mu and its f x is equal to 1 over Pi r, r square plus x minus mu whole square. For this lecture we shall stick to C 1 comma 0 since, so, I missed a point let us write that minus infinity less than x less than infinity.

That is it is defined over the entire r. Since integration minus infinity to infinity, 1 over Pi, 1 over 1 plus x square dx is equal to 1 over Pi tan inverse x from minus infinity to infinity is equal to 1 upon Pi into Pi by 2 minus, minus pi by 2 is equal to 1 therefore, it can be considered as a PDF.

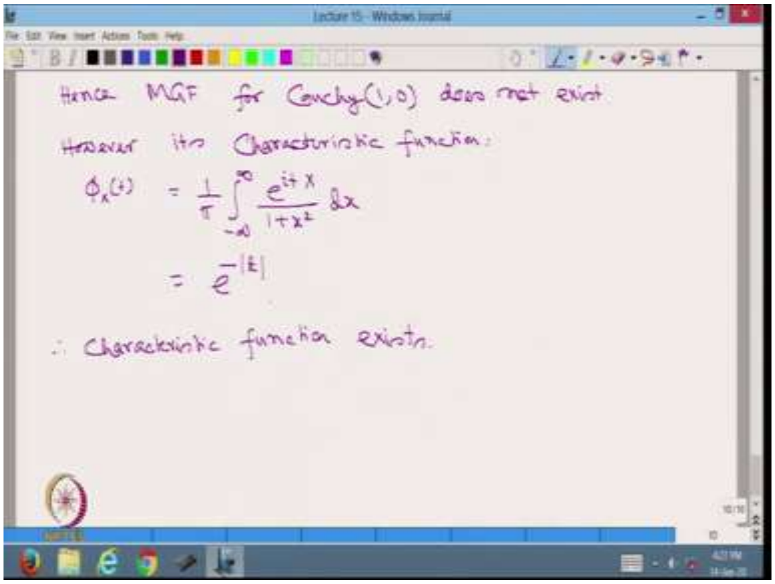
(Refer Slide Time: 22:12)



So, what is its expectation? 1 over Pi integration minus infinity to infinity x upon 1 plus x square dx analytically it is not convergent. But for most practical applications we consider with the principle value, which is equal to 0, and therefore, you often find that expected value of x is equal to 0. What about expected value of x square?

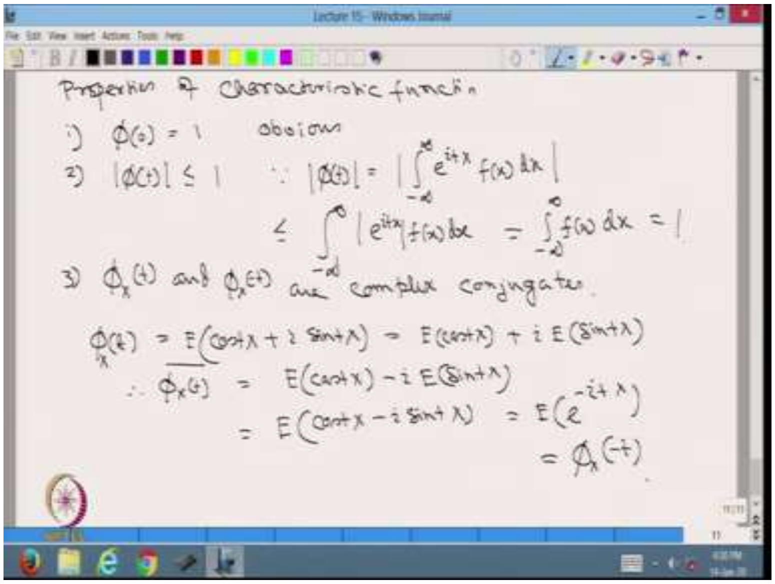
Expected value of x square is equal to 1 over Pi minus infinity to infinity x square upon 1 plus x square dx is equal to 2 times 0 to infinity x square upon 1 plus x square dx and this is surely not convergent. Thus we can show that higher order moments for a Cauchy distribution do not exist.

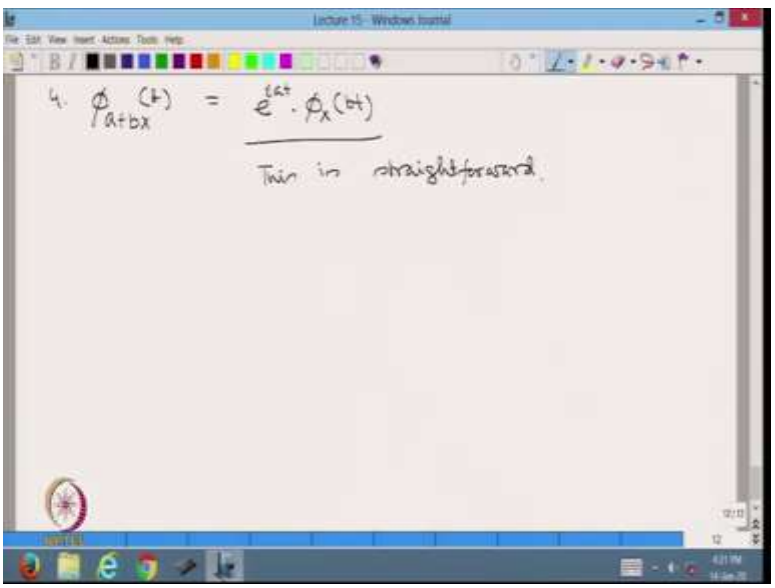
(Refer Slide Time: 24:30)



Hence, moment generating function for Cauchy 1 comma 0 does not exist. However, its characteristic function which is defined as phi x of t is equal to 1 over Pi into minus infinity to infinity e to the power i t x upon 1 plus x square dx and I am not computing it, but this converges to e to the power minus modulus of t. Therefore, characteristic function exists. Although its moment generating function does not exist.

(Refer Slide Time: 25:48)



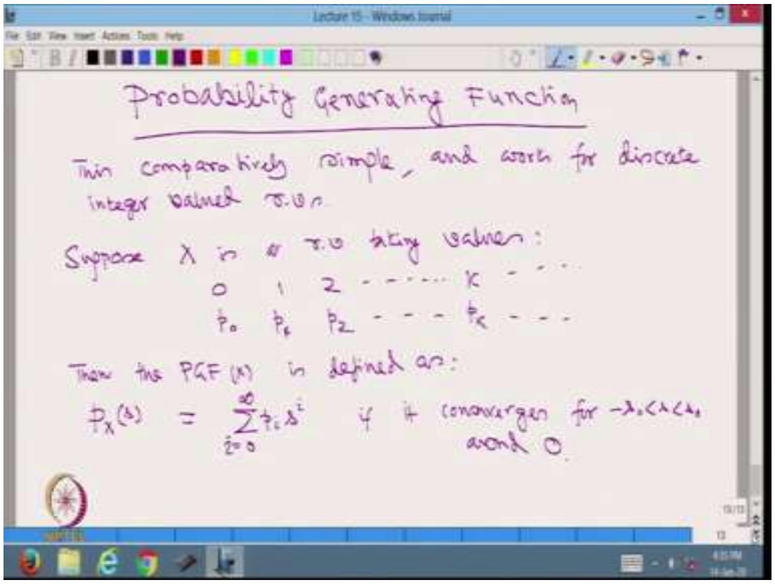


Properties of characteristic function, 1, phi 0 is equal to 1 at the point 0 it is 1 which is obvious that is why I am not doing it. 2, Modulus of phi t is less than equal to 1, since modulus of phi t is equal to modulus of minus infinity to infinity e to the power i t x fx dx which is less than equal to integration minus infinity to infinity modulus of e to the power i t x fx dx which is equal to minus infinity to infinity fx dx is equal to 1.

3, phi x at t and phi x at minus t are complex conjugates, phi x of t is equal to expected value of cos tx plus i sine tx is equal to expected value of cos tx plus i times expected value of sine tx.

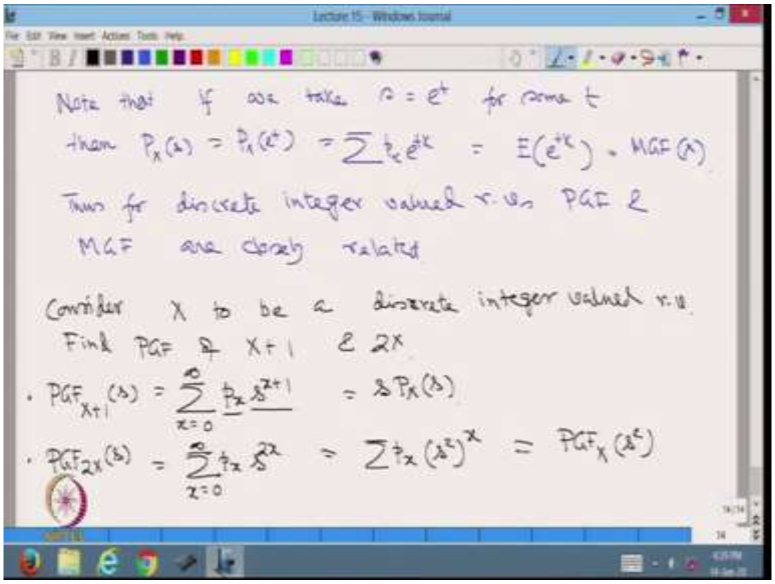
Therefore, phi xt is conjugate is equal to expected value of cos tx minus i expected value of sine tx is equal to expected value of cos tx minus i sine tx is equal to expected value of e to the power minus i t x which is equal to phi x at minus t. Thus, we get the result. 4, phi of a plus bx at t is equal to e to the power i a t into phi x at bt this is straight forward and I leave it as an exercise.

(Refer Slide Time: 29:18)



The next generating function that we study is Probability Generating Function. This is comparatively simple and works for discrete integer value random variables. So, suppose x is a random variable taking values 0, 1, 2, k like that, and probability of x taking the value i is equal to say pi, then the probability generating function of x is defined as px at s is equal to sigma pi s to the power i, i is equal to 0 to infinity if it converges for some interval minus s knot less than s less than s knot around 0.

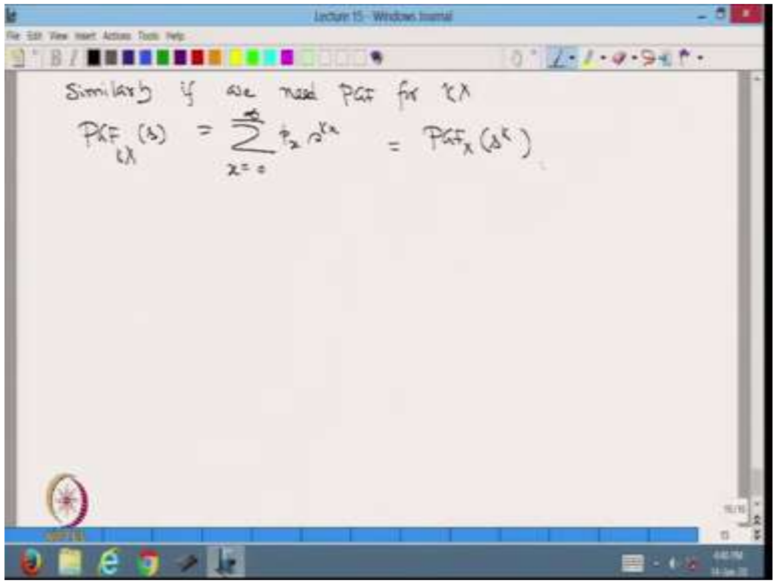
(Refer Slide Time: 31:32)



Note that if we take s is equal to e to the power t for some t, then Px of s is equal to px at e to the power t is equal to sigma pk e to the power tk is equal to the expected value of e to the power tk is equal to moment generating function of x thus for discrete integer valued random variables, probability generating function and moment generating function are closely related. Now consider x to be a discrete integer valued random variable.

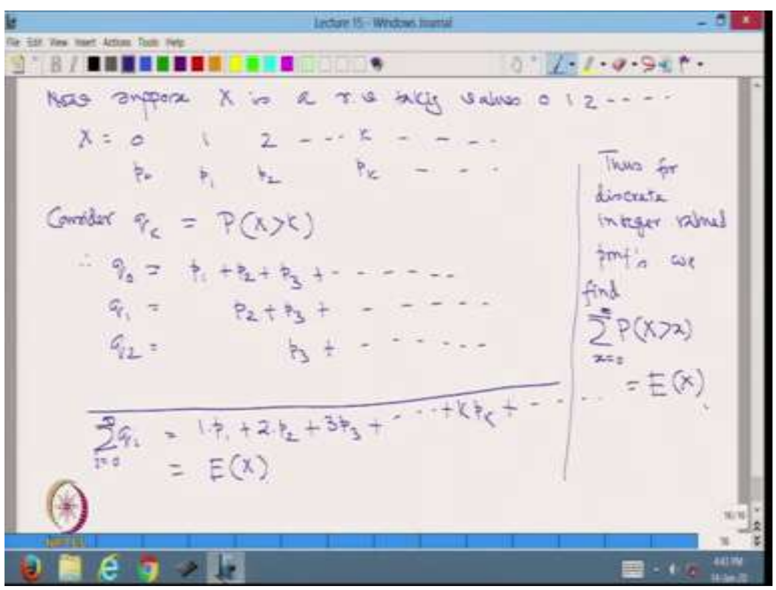
Find PGF of x plus 1 and 2 x, PGF of x plus 1 at s is equal to sigma x is equal to 0 to infinity px into s to the power x plus 1 because x plus 1 takes the value say 5 when x is equal to 4, therefore s to the power 5 will be multiplied by the probability of 4, same is for all x is equal to s times probability generating function of x around the point s PGF of 2 x at a point s is equal to sigma x is equal to 0 to infinity px into s to the power 2 x is equal to sigma px into a square whole to the power x is equal to PGF of x at the point a square.

(Refer Slide Time: 34:30)



Similarly, if we need PGF for Kx that is PGF of K times x at a point s is equal to sigma x is equal to 0 to infinity px into s to the power kx is equal to PGF of x at s the power k.

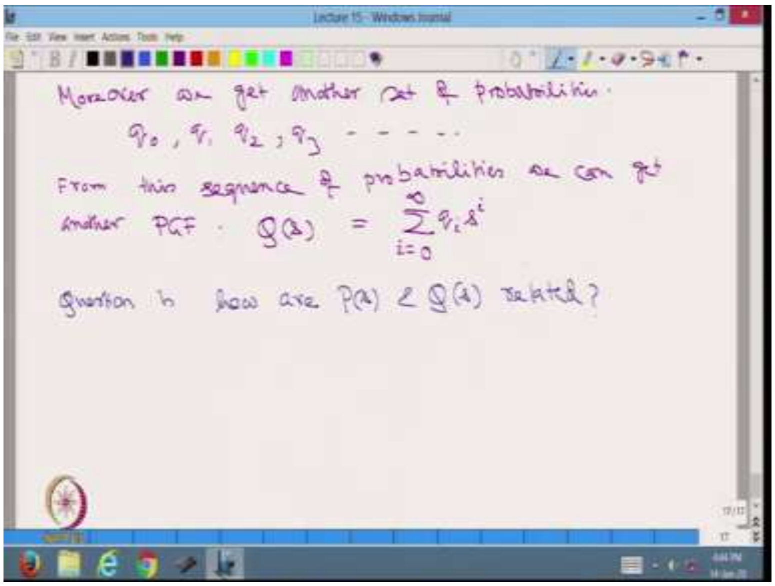
(Refer Slide Time: 35:12)



Now, suppose x is a random variable taking values 0, 1, 2 up to infinity like this x takes the values 0, 1, 2, K with p0, p1, p2, pk. Consider qk is equal to probability x greater than k. Therefore, q 0 is equal to p1 plus p2 plus p3 up to infinity q1 is equal to p2 plus p3 up to infinity, q2 is equal to p3 plus up to infinity.

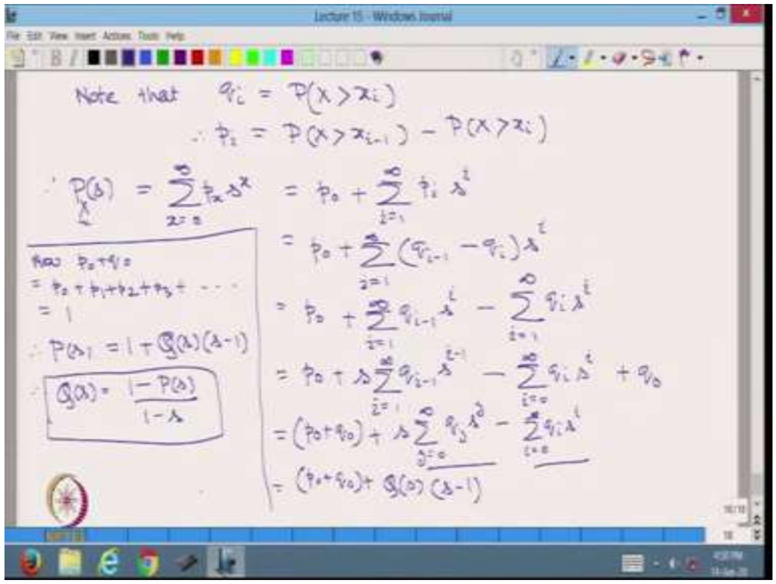
Like that if we sum then we get sigma qi i is equal to 0 to infinity is equal to 1 times p1 plus 2 times p2 plus 3 times p3 plus k times pk plus up to infinity. This is nothing but expected value of x thus for discrete integer valued PMFs we find sigma over x probability x greater than x, x is equal to 0 to infinity is equal to expected value of x. So this is an interesting property of such variables.

(Refer Slide Time: 37:40)



But for me, for the time being, we find that another set of probabilities namely, q0, q1, q2, q3 like that, from this sequence of probabilities we can get another probability generating function namely Q s is equal to sigma qi s to the power i, i is equal to 0 to infinity, question is how are P s and Q s related that is a very interesting question and we solve it as follows.

(Refer Slide Time: 39:08)

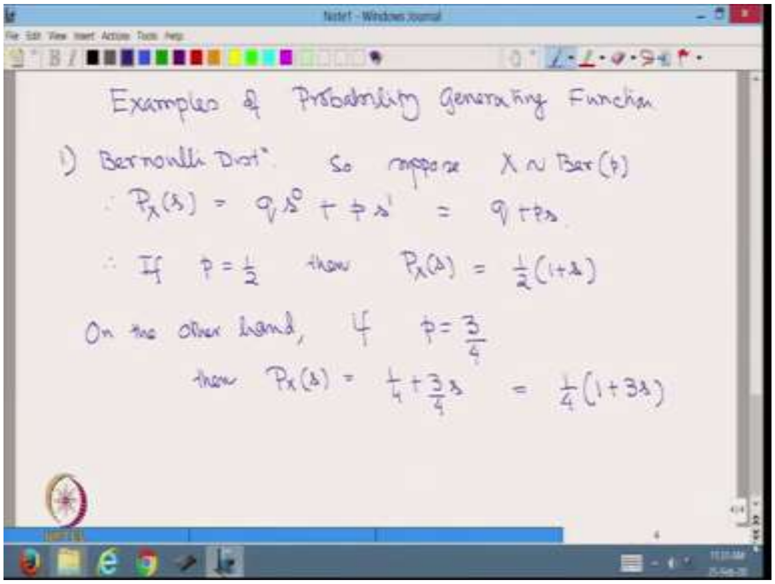


Qi is equal to probability x greater than xi. Therefore, pi is equal to probability x greater than xi minus 1 minus probability x greater than xi. Therefore, Ps of x is equal to sigma px s to the power x, x is equal to 0 to infinity is equal to p0 plus sigma i equal to 1 to infinity pi s to the power i is equal to p0 plus sigma i is equal to 1 to infinity qi minus 1 minus qi s to the power i is equal to p0 plus sigma i is equal to 1 to infinity qi minus 1 s to the power i minus sigma i is equal to 1 to infinity qi minus 1 s to the power i is equal to p0 plus s sigma i is equal to 1 to infinity, qi minus 1 s to the power i minus 1 minus sigma i is equal to 0 to infinity qi, qi s to the power i.

Since we have added the terms q0 s to the power 0, which is q0 with a minus sign to compensate, we had one q0 is equal to p0 plus q0 plus s times now, we understand that since summing from i is equal to 1 to infinity, qi minus 1 into s to the power i, i minus 1, we can write it as j is equal to 0 to infinity, qj s to the power j minus sigma qi s to the power i, i is equal to 0 to infinity this is equal to p 0 plus q 0 plus this is qs and this is also qs. So, we can write it as qs into s minus 1.

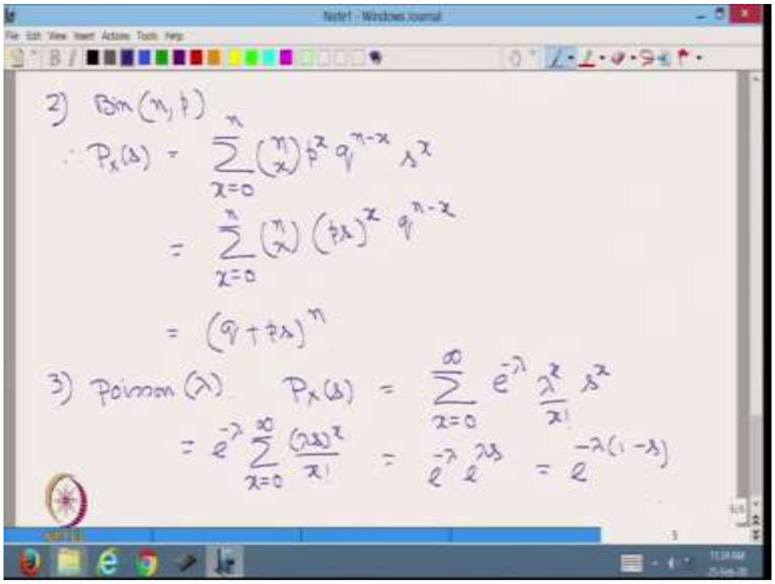
Now, p0 plus q0 is equal to p0 plus probability is greater than 0, which is equal to p1 plus p2 plus p3 up to infinity is equal to 1. Therefore, Ps let me write it as Ps is equal to 1 plus Qs into s minus 1. Therefore, Qs is equal to 1 minus Ps upon 1 minus s. So, these two different probability generating functions are related by this relationship. So, this is an interesting generating function which is comparatively simpler than moment generating function or characteristic function.

(Refer Slide Time: 43:30)



Let us now see examples of probability generating function, 1, Bernoulli distribution so, suppose x is distributed as Bernoulli P therefore, Px s at s is equal to q times s to the power 0 plus p times s to the power 1 is equal to q plus ps. Therefore, if p is equal to half then Px s is equal to half times 1 plus s. On the other hand if P is equal to say 3 by 4, then Px s is equal to 1 by 4 plus 3 by 4 times s is equal to 1 by 4 into 1 plus 3 s.

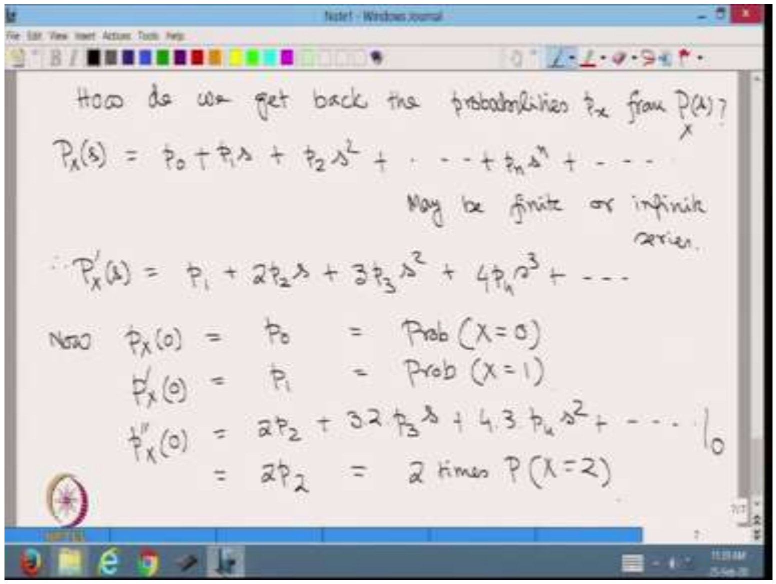
(Refer Slide Time: 45:29)

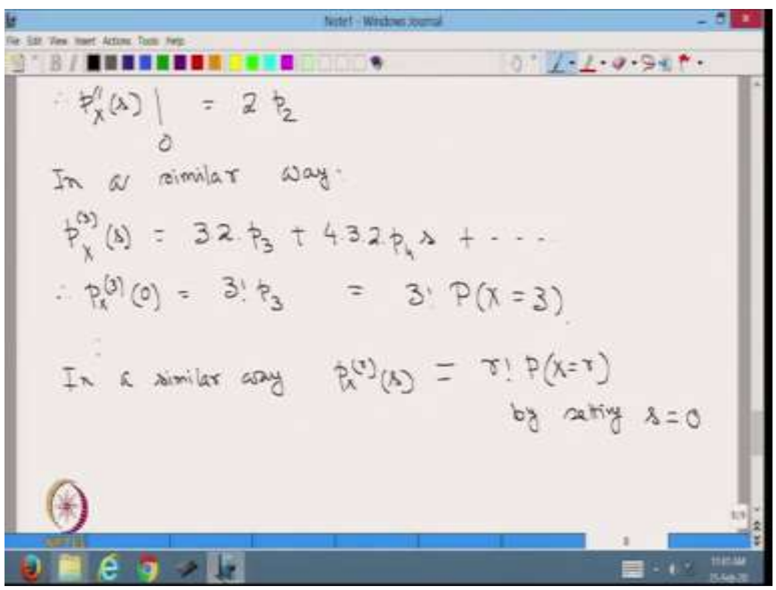


Binomial distribution with parameter n comma p therefore, Px s is equal to sigma x is equal to 0 to n, ncx p to the power x, q to the power n minus x into s to the power x is equal to sigma x is equal to 0 to n, ncx ps to the power x q to the power n minus x which is equal to q plus ps whole to the power n.

Poisson distribution with lambda therefore, Px of s is equal to sigma x is equal to 0 to infinity e to the power minus lambda, lamda power x upon factorial x into s to the power x which is equal to e to the power minus lambda sigma x is equal to 0 to infinity lambda s to the power x upon factorial x which is equal to e to the power minus lambda into e to the power lambda s which is equal to e to the power minus lambda into 1 minus s.

(Refer Slide Time: 47:45)



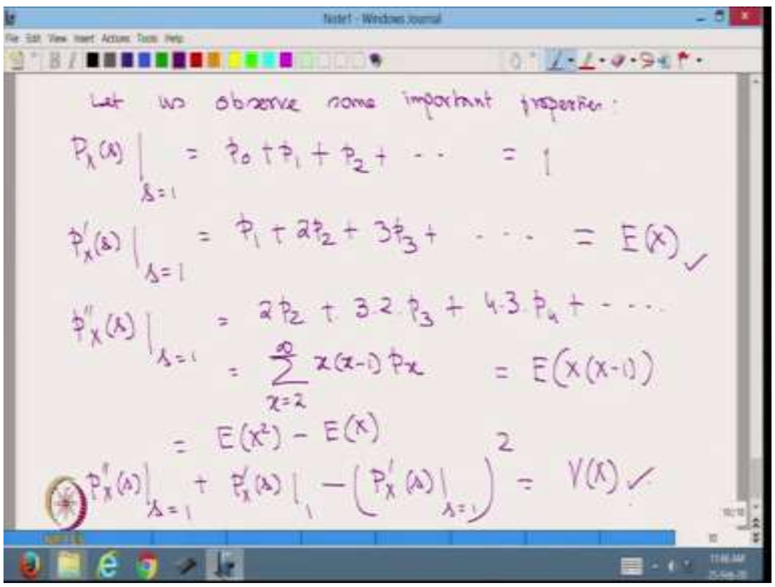


How do we get back our probabilities? Px from the probability generating function ps of x that is very simple px s is equal to p0 plus p1 s plus p 2 s square up to say pn sn pn s to the power n. It may be finite or infinite series P prime x at s is equal to p1 plus 2 p2 s plus 3 p3 s square plus 4 p 4 s cube, like that.

Now, Px at as 0 is equal to p0 is equal to probability x is equal to 0, p prime x at 0 is equal to p1 is equal to probability x is equal to 1. P double prime x at 0 is equal to 2 p2 plus 3 into 2 into p3 s plus 4 into 3 into p4 s square plus, etc at 0 is equal to 2 p2 is equal to 2 times probability x is equal to 2.

Therefore, p double prime x s at 0 is equal to 2 times p2. In a similar way the third derivative of probability generating function is equal to 3 into 2 into p3 plus 4 into 3 into 2 into p4 s, etc or px 3 at 0 is equal to factorial 3 into p3 is equal to factorial 3 into probability x is equal to 3. In a similar way, we can find the r^{th} derivative of the probability generating function will give us r factorial into probability x is equal to r by setting s equal to 0.

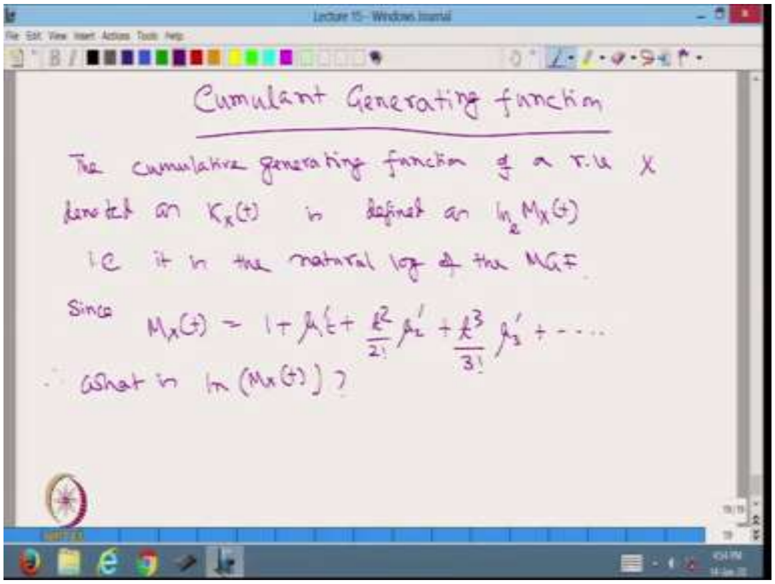
(Refer Slide Time: 52:17)



Let us now observe some important properties px at s at s is equal to 1 is equal to p0 plus p1 plus p2 is equal to 1, p prime x at s is equal to 1 is equal to p1 plus 2 p2 plus 3 p3 etc is equal to expectation of x. Similarly, p double prime x at s at s is equal to 1 is equal to 2 p2 plus 3 into 2 into p3 plus 4 into 3 into p4 like that is equal to sigma x is equal to 2 to infinity x into x minus 1 into px which is equal to expected value of x into x minus 1 which is equal to expected value of x square minus expected value of x.

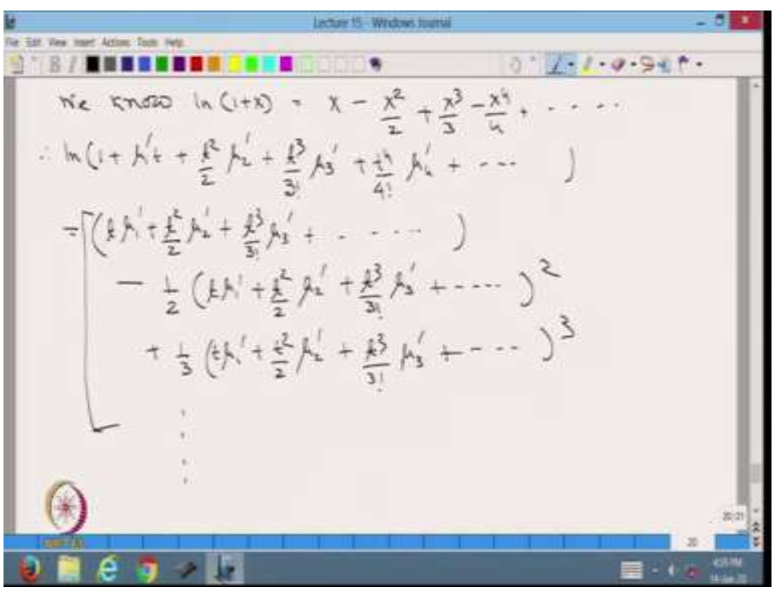
Therefore, p double prime x of s at s is equal to 1 plus p first derivative of x at s is equal to 1 minus p prime x s at s is equal to 1 square is equal to variance of x. I like you to verify this results for different discrete distributions.

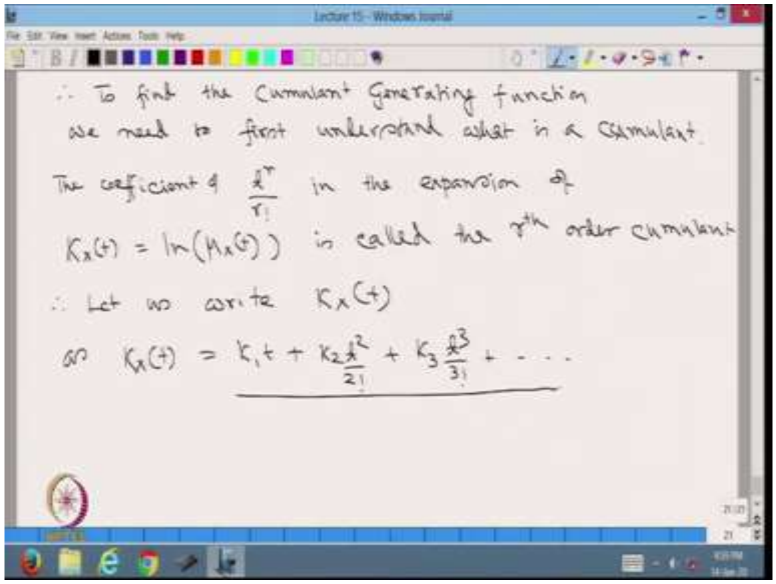
(Refer Slide Time: 54:39)

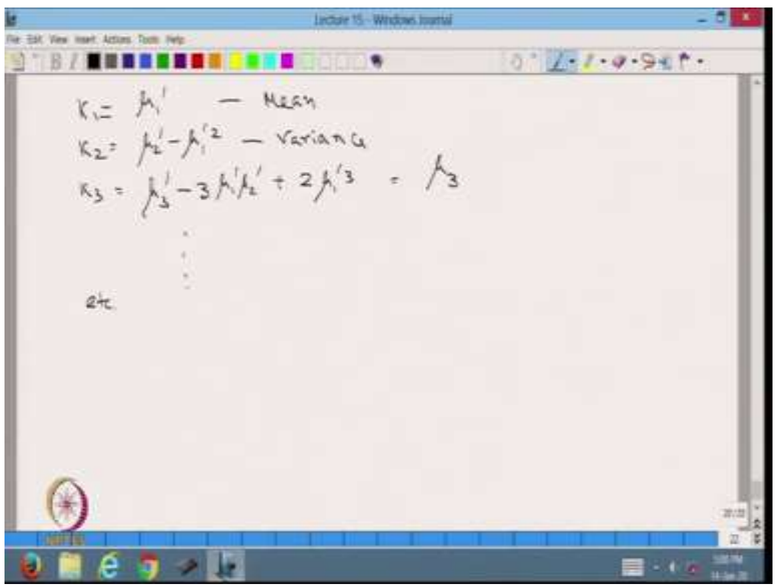


The last topic on this is called Cumulant Generating Function, the cumulative generating function of a random variable x denoted as kx of t is defined as log of Mxt to the base e. That is, it is the natural log of the moment generating function since Mxt is equals 1 plus mu 1 prime t plus t square upon factorial 2 mu 2 prime plus t cube upon factorial 3 mu 3 prime like that. Therefore, what is log of Mxt? That is the question.

(Refer Slide Time: 56:34)





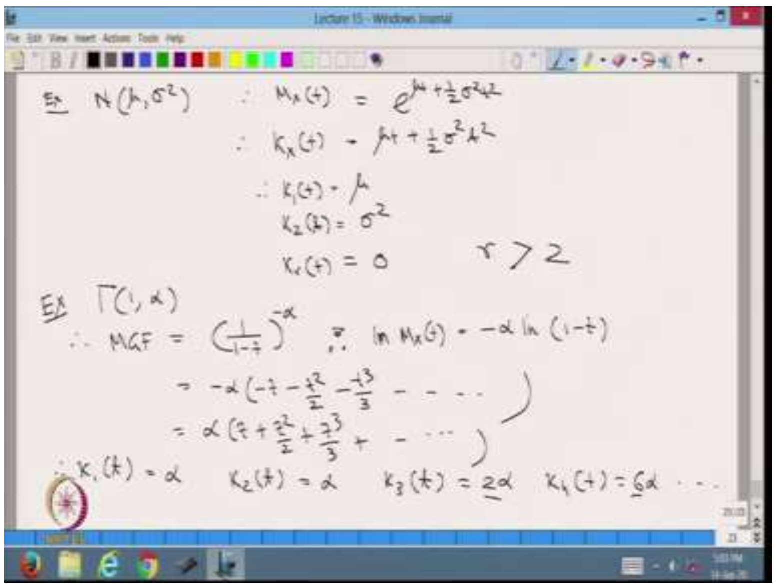


We know natural log of 1 plus x is equal to x minus x square upon 2 plus x cube upon 3 minus x 4 upon 4 etc therefore, log of 1 plus mu 1 prime t plus t square upon 2 mu 2 prime plus t cube upon factorial 3 mu 3 prime plus t 4 upon factorial 4 mu 4 prime etc is equal to t mu 1 prime plus t square by 2, mu 2 prime plus t cube by factorial 3 mu prime minus half of t mu1 prime plus t square by 2 mu 2 prime plus t cube by 3 factorial 3 mu 3 prime whole square plus 1 by 3 into t mu 1 prime plus t square by 2 Mu 2 prime plus t cube by factorial 3 mu 3 prime whole cube like that, we can have the infinite sum.

Therefore, to find the Cumulant Generating Function we need to first understand what is a cumulant. The coefficient of t to the power r upon r factorial in the expansion of Kxt, which is equal to log of Mxt is called the r^{th} order cumulant. So, let us write Kxt as Kxt is equal to k 1 times t plus k 2 times t square factorial 2 plus k 3 times t cube factorial 3 like that.

And if we equate this with this expression that we got here, what we are getting is k1 is equal to mu 1 prime, which is the mean k2 is equal to mu 2 prime minus mu 1 prime square, which is equal to variance k3 is equal to mu 3 prime minus 3 mu 1 prime, mu 2 prime plus 2 mu 1 prime q, which is equal to the third order central moment etc. Thus from the Cumulant generating function also we can get moments of different orders.

(Refer Slide Time: 61:35)

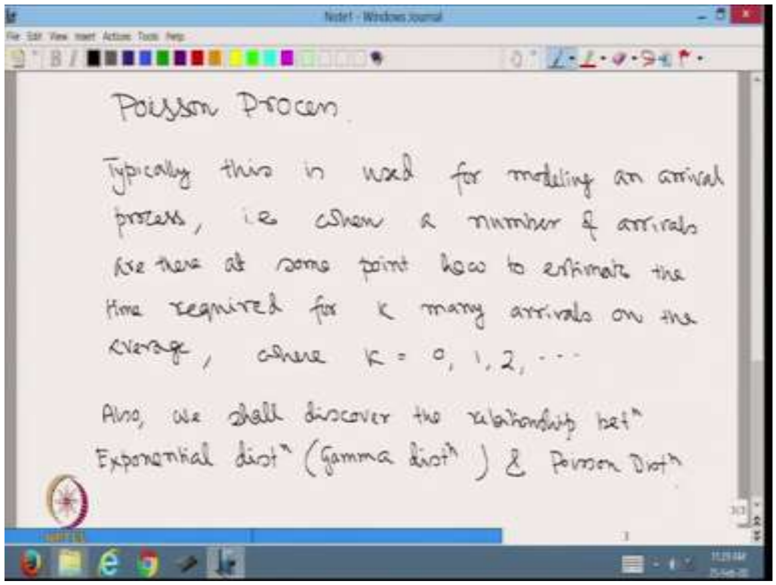


So, before I stop, I give you small examples normal mu sigma square. Therefore, Mxt is equal to e to the power mu t plus half sigma square t square. Therefore, Kxt is equal to mu t plus half sigma square t square. Therefore, k 1 t is equal to mu k2 t is equal to sigma square Krt is equal to 0 or greater than 2.

Similarly, another example say gamma with 1 comma alpha therefore MGF is equal to 1 upon 1 minus t whole to the power minus alpha therefore, log of Mxt is equal to minus alpha into log of 1 minus t is equal to minus alpha into minus t minus t square by 2 minus t cube by 3 is equal to alpha times t plus t square by 2 plus t cube by 3 etc.

Therefore, K1 t is equal to alpha k2 t is equal to alpha k3 t is equal to 2 alpha, k4 t is equal to 6 alpha etc. This is very simple, because we have to consider the coefficient of t to the power r upon factory r and to compensate for that factorial we need this extra coefficients. Okay friends, I stop here today.

(Refer Slide Time: 64:12)



In the next class we shall discuss an important concept of probability which is called Poisson process. Typically this is used for modeling an arrival process that is when a number of arrivals are there at some point how to estimate the time required for K many arrivals on the average, where K can be 0, 1, 2 etc. Also we shall discover the relationship between exponential distribution or gamma distribution in a more generalized way and Poisson distribution. Thank you.