Advanced Probability Theory
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Lecture 7

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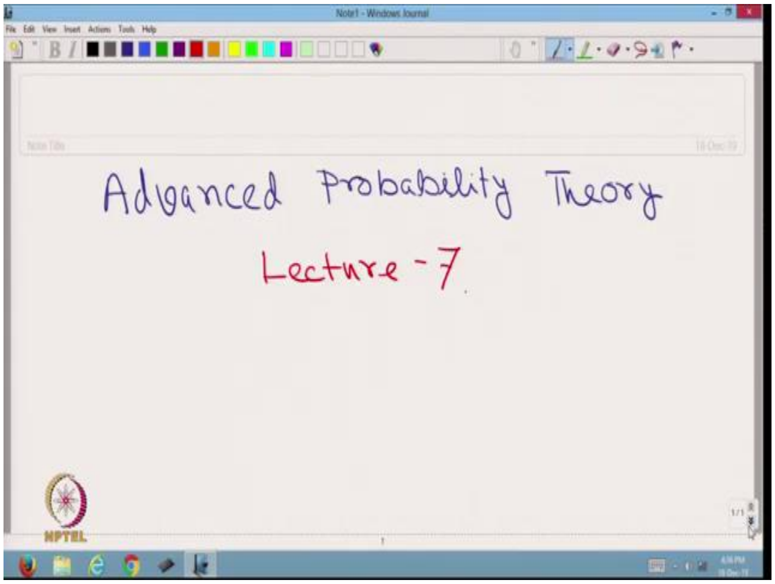

Welcome students. The MOOCs course on Advanced Probability Theory. This is lecture number seven.

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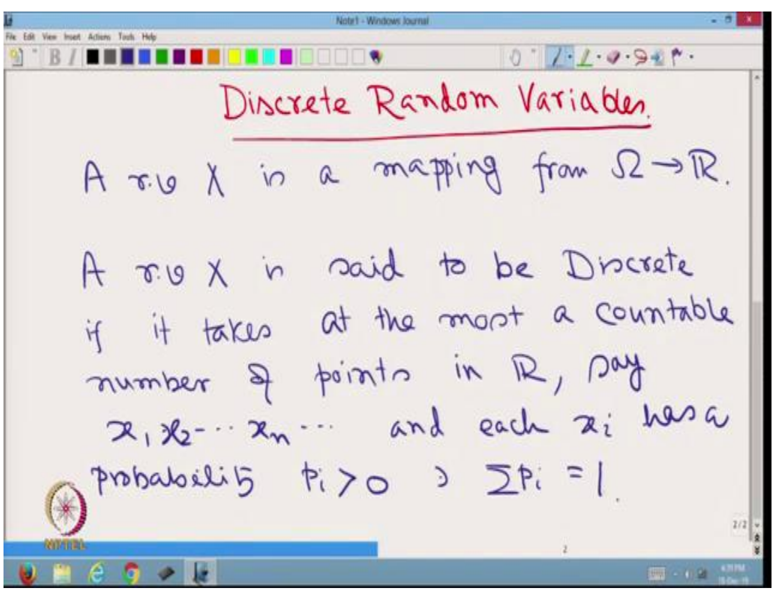

As I discussed in my last class, that in this lecture, we shall start Discrete Random Variables. We know that a random variable X is a mapping from omega to real line, of course, there are certain other properties. I am not going into that. Now, let us look at a discrete random variable. A random variable X is set to be discrete if it takes at the most a countable number of points in R, say x1, x2… xn and each xi has a probability Pi greater than 0 such that sigma over Pi is equal to 1.

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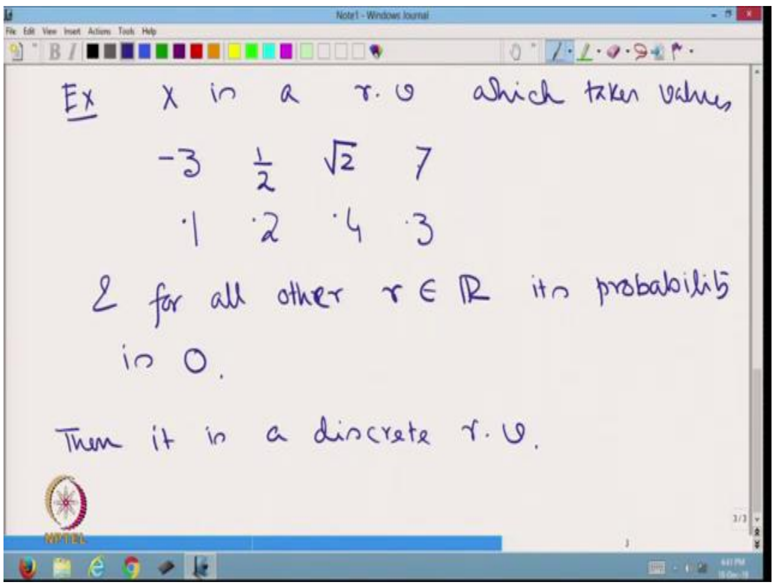

Example, suppose X is a random variable which takes values say minus 3, half, root 2 and 7 with probabilities 0.1, 0.2, 0.4 and 0.3 and for all other r belonging to R its probability is 0, then it is a Discrete Random Variable because it can take only four values and their corresponding probabilities are 0.1, 0.2, 0.4 and 0.3, note that the sum of this is equal to 1.

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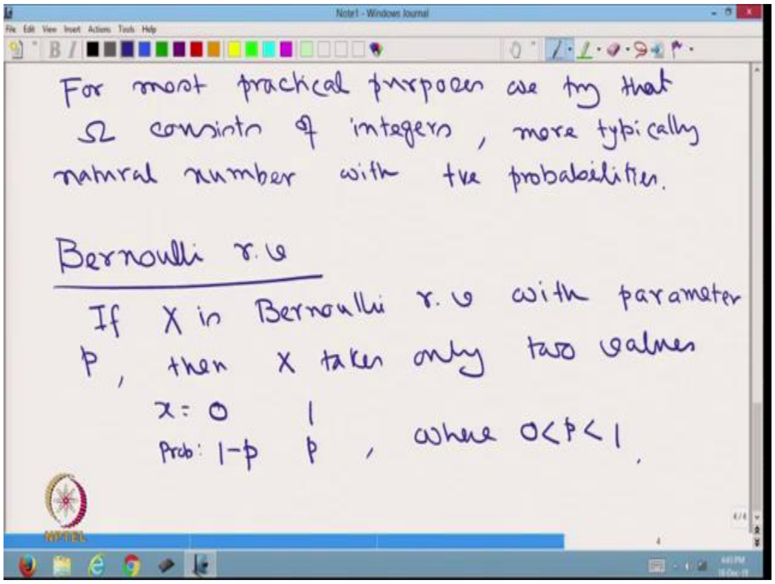

For most practical purposes we try that omega consists of integers, more typically natural numbers with positive probabilities. So, let me now give you several different types of discrete random variables, perhaps, one of the most simplest one is Bernoulli random variable. If X is a Bernoulli random variable with parameter p, then X takes only 2 values 0 and 1 and their probabilities are 1 minus p and p where 0 less than p less than 1.

If you remember then we have seen such a random variable, when we were talking about tossing a coin and the corresponding observations were head and tail, when head is mapped into 1 and tail is mapping to 0.

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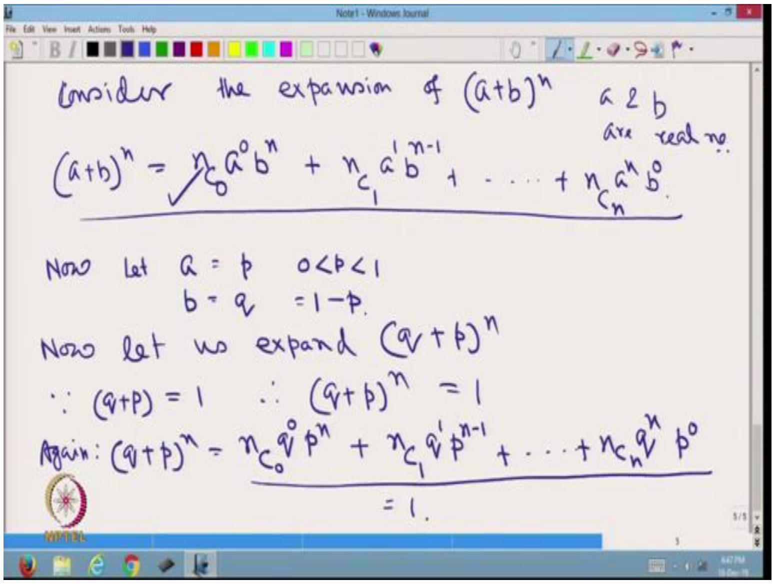

Now, let us consider the expansion of the a plus b whole to the power n, when a and b are real numbers. We know that a plus b whole to the power n is equal to nc0 a to the power 0 b to the power n plus nc1 a to the power 1, b to the power n minus 1 plus up to ncn a to the power n b to the power 0, all of us have seen such a binomial expansion. Now, let a equal to p, 0 less than p less than 1, and b is equal to q is equal to 1 minus p.

Now, let us expand q plus p whole to the power n therefore, since, q plus p is equal to 1. Therefore, q plus p whole to the power n is equal to 1. Again by using this formula q plus p whole to the power n is equal to nc0 q to the power 0 p to the power n plus nc1 q to the power 1 p to the power n minus 1 plus up to ncn q to the power n p to the power 0. Therefore, this sum is going to be equal to 1.

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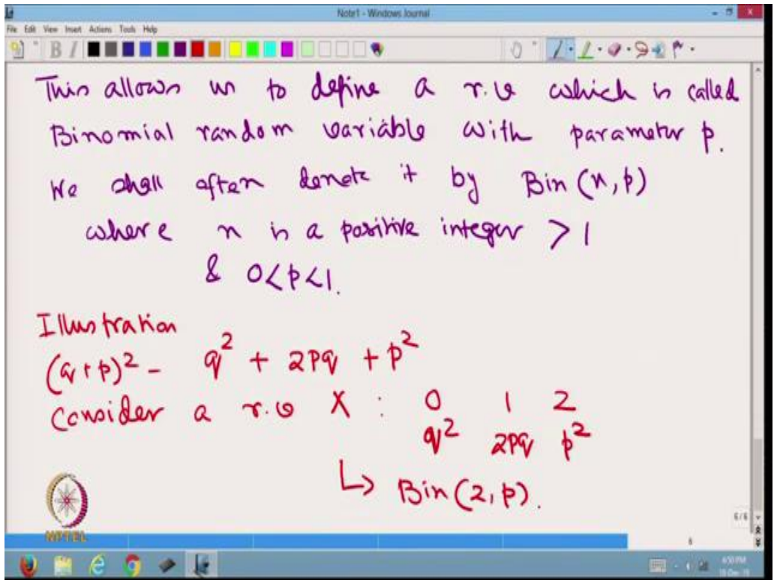

This allows us to define a random variable which is called Binomial random variable with parameter p. We shall often denote it by Binomial n comma P where n is a positive integer greater than 1, and 0 less than p less than 1. Illustration, q plus P whole square is equal to q square plus 2Pq plus P square. Consider a random variable X such that it takes values 0, 1 and 2, 0 with probability q square, 1 with probability 2Pq and 2 with probability P square. Then this X is a binomial random variable with 2 comma p.

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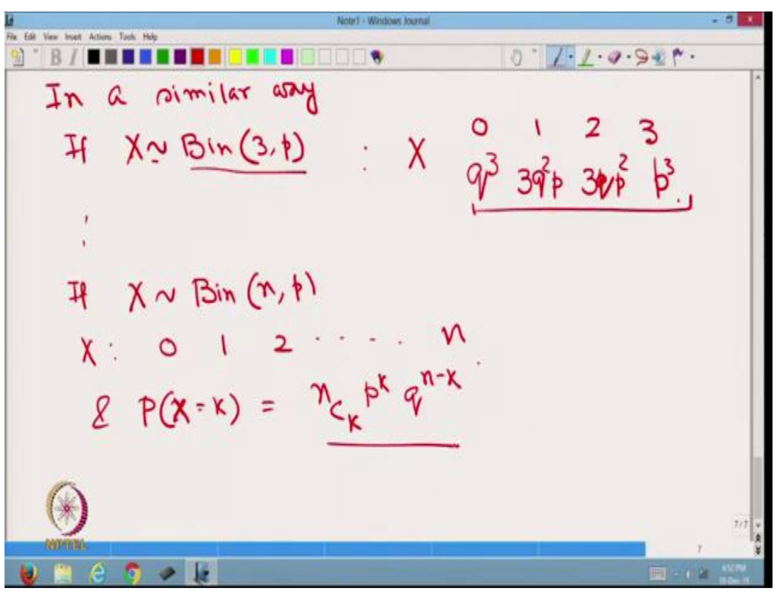

In a similar way, if X is following binomial 3, p this notation I will use to denote that X is a random variable, which is following this distribution, then X has the following values 0, 1, 2 and 3 and the corresponding probabilities are q to the power 3, 3q square p, 3q p square and pq. It is very clear that sum of these is going to be 1. In general if X is a binomial random variable with parameter n comma p then X takes values 0, 1, 2 up to n and probability X is equal to k is equal to nck p to the power k q to the power n minus k.

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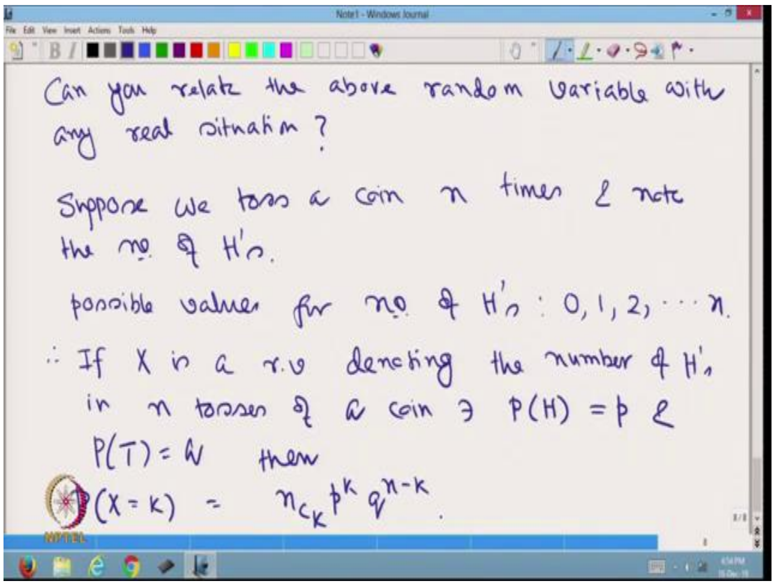

Now, if I ask you can you relate the above random variable with any real situation? Perhaps, you can. But still let me explain. Suppose, we toss a coin n times and note the number of heads, then possible values for number of heads can be 0, 1, 2 up to n. Therefore, if X is a random variable denoting the number of heads in n tosses of a coin such that probability of head is equal to p and probability of tail is equal to q. Then probability X is equal to k is nck p to the power k q to the power n minus k.

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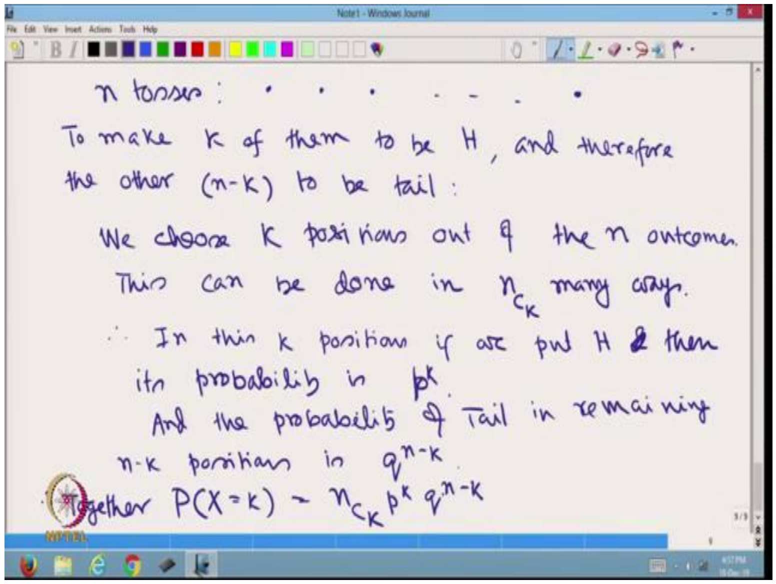

Why? Because there are n tosses. So, there may be n outcomes. To make k of them to be H, and therefore, the other n minus k to be tail. What we will do? We choose k positions out of the n outcomes. This can be done in nck many ways. Therefore, in this k positions if we put H and its probability is p to the power k because the tosses are independent.

Therefore, to get k heads we have to have the probability p into p into p k times and therefore, it is p to the power k. And the probability of tail in remaining n minus k positions is q to the power n minus k. Therefore, together probability X is equal to k is equal to nck p to the power k q to the power n minus k.

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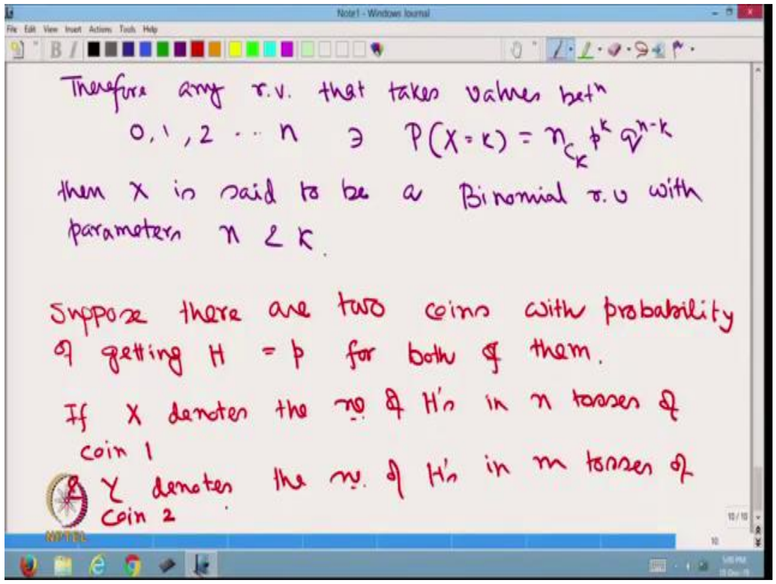

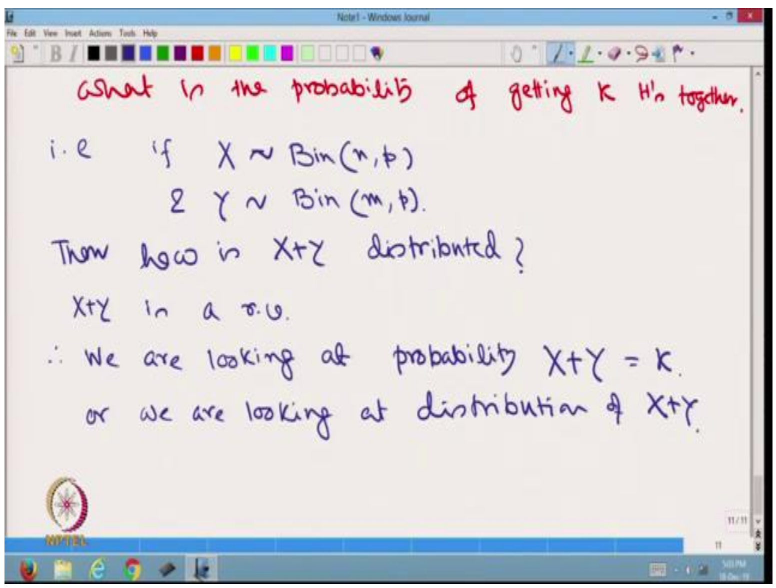

Therefore, any random variable that takes values between 0, 1, 2… n such that probability X is equal to k is equal to nck p to the power k q to the power n minus k. Then X is said to be a Binomial random variable with parameters n and k. Now, suppose there are two coins with probability of getting a head is equal to p for both of them. If X denotes the number of heads in n tosses of coin 1 and Y denotes the number of heads in m tosses of coin 2, what is the probability of getting k heads together?

If X is equal to Binomial n, p and Y is binomial m, p then how is X plus Y distributed? That is the question. And if you remember, in our last class, we have said that X plus Y is a random variable therefore, we are looking at probability of X plus Y is equal to k or we are looking at distribution of X plus Y.

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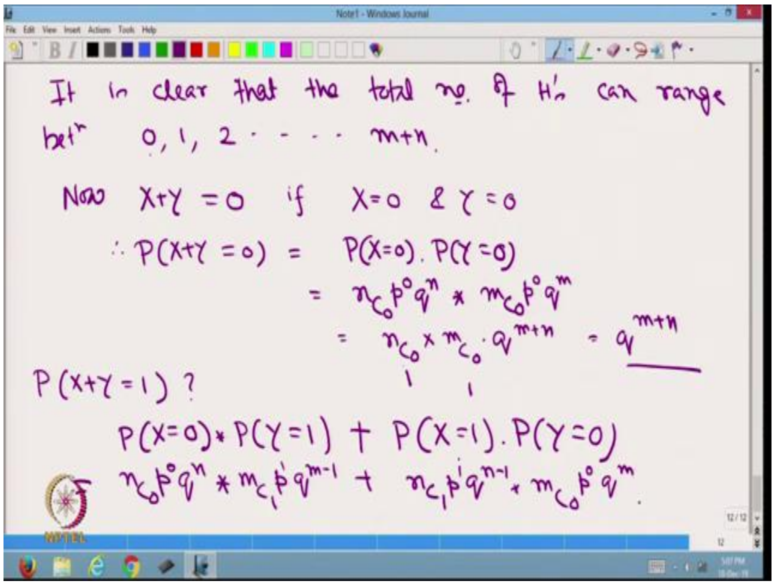

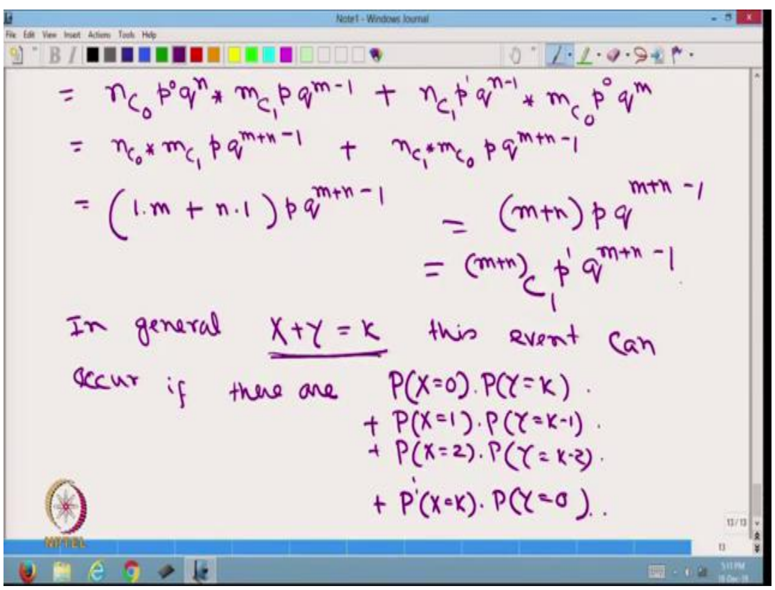

It is clear that the total number of heads can range between 0, 1, 2 up to m plus n, it will take 0 when there is no head in n tosses of X and m tosses of Y it is going to be m plus n if there are all heads in n tosses of X and the m tosses of Y or m tosses of coin 2. Now, X plus Y is equal to 0 if X is equal to 0, and Y is equal to 0, therefore, probability X plus Y is equal to 0 is equal to probability X is equal to 0 multiplied by probability Y is equal to 0, because these are independent events.

Therefore, we can multiply their probabilities and this is going to be nc0 p to the power 0, q to the power n multiplied by mc0 p to the power 0 q to the power m is equal to nc0 into mc0 into q to the power m plus n since, p to the power 0 is equal to 1 is equal to q into m plus n. since, nc0 is equal to 1 and mc0 is equal to 1. Let us go one more step. What is the probability X plus Y is equal to 1?

We understand that X plus Y can be 1 in two possible ways. Probability X is equal to 0 multiplied by probability Y is equal to 1 plus probability X is equal to 1 multiplied by probability Y is equal to 0. Because, if we get one head in the total number of tosses, then it will be either there is 0 head from coin 1, but one head from coin 2, or there is 1 head from coin 1 and no heads for coin 2.

So, this probability is going to be nc0 p to the power 0 q to the power n into mc1 p to the power 1 q to the power m minus 1 plus mc1 p to the power 1 q to the power n minus 1 multiplied by mc0 p to the power 0 q to the power m, which is equal to nc0 p to the power 0 q to the power n multiplied by mc1, p to the power 1 q to the power m minus 1 plus nc1 p to the power 1 q to the power n minus 1 multiplied by mc0 p to the power 0, q to the power m.

Is equal to nc0 into mc1 multiplied by p into q to the power m plus n minus 1 plus nc1 into mc0 p into q to the power m plus n minus 1 which is equal to nc0 is 1 multiplied by m plus nc1 is then multiplied by 1 p q to the power m plus n minus 1 is equal to m plus n pq to the power m plus n minus 1, which we can write it as n plus n c1 p to the power 1 q to the power m plus n minus 1.

In general X plus Y is equal to k, this event can occur if there are 0 heads from coin 1, and k heads from coin 2, so we can write it at probability X is equal to 0 into probability Y is equal to k plus probability X is equal to 1 into probability Y is equal to k minus 1 plus probability X is equal to 2 multiplied by probability Y is equal to k minus 2, up to probability X is equal to k into probability Y is equal to 0. Thus, there are k plus 1 disjoint events which gives rise to the event X plus Y is equal to k.

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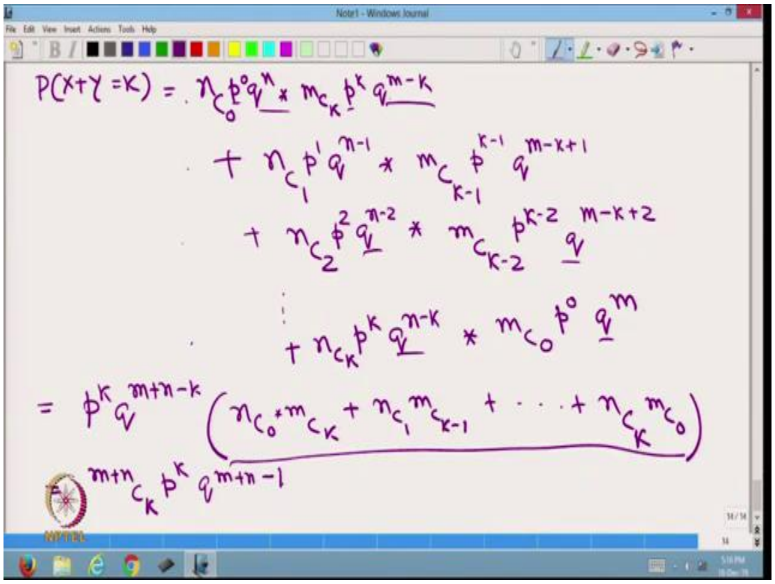

And therefore, this probability is going to be nc0 p to the power 0 q to the power n multiplied by mck p to the power k q to the power m minus k plus nc1 p to the power 1 q to the power n minus 1 multiplied by mck minus 1 p to the power k minus 1 q to the power m minus k plus 1 plus nc2 p square q to the power n minus 2 multiplied by mck minus 2 p to the power k minus 2 q to the power m minus k plus 2 up to nck p to the power k q to the power n minus k multiplied by mc0 p to the power 0 q to the power m.

Thus, there are k plus 1 terms and if you look at the total power of p is k and total power of q is m plus n minus k, that you can verify for all the k plus 1 terms therefore, we can write it as p to the power k q to the power m plus n minus k multiplied by nc0 into mck plus nc1 into mck minus 1 plus up to nck into mc0. Now, from our high school mathematics we know that this term is m plus nck. Therefore, this probability is equal to m plus nck p to the power k q to the power m plus n minus k.

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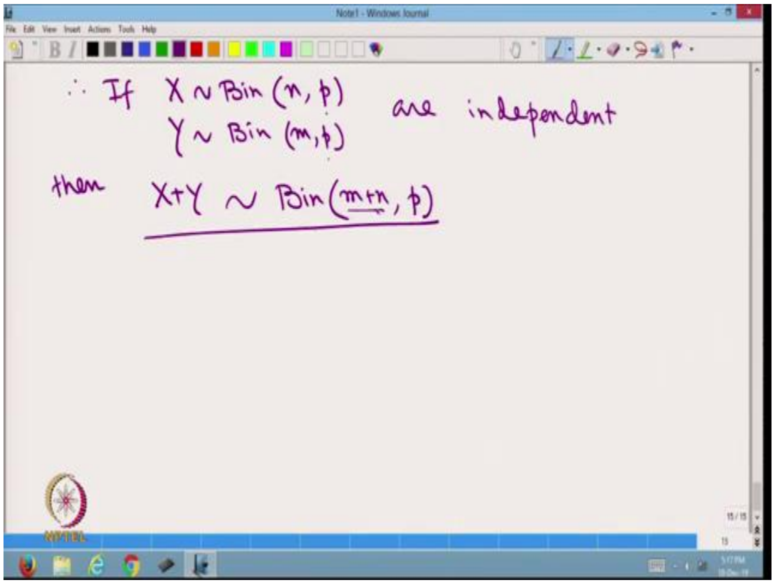

Therefore, what we get? We get that if the X is Binomial n comma p and Y is Binomial m comma p are independent. Then X plus Y follows Binomial m plus n comma p. That is an important result. However, you have to remember that the p has to be the same. The probability of success has to be the same. In that case, the sum of two binomial random variables, if they are independent is going to be a binomial with the first parameter being the sum of their individual values.

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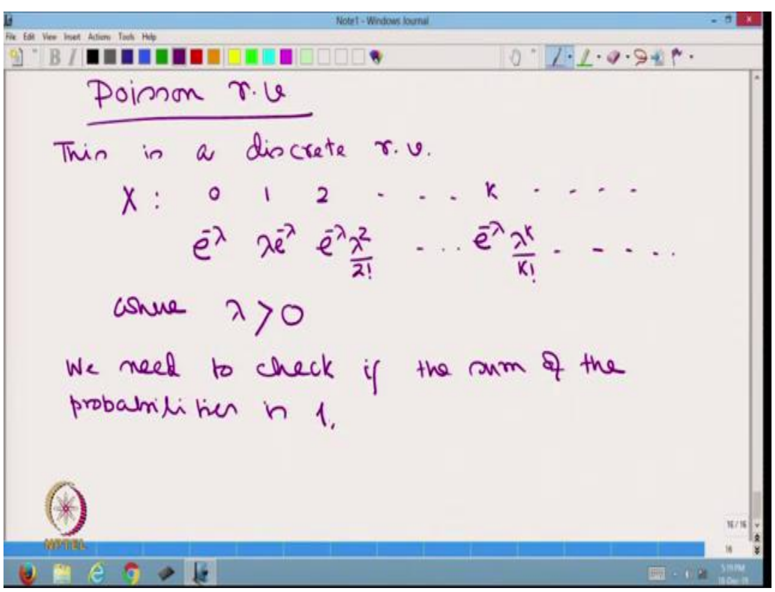

Let us look at another discrete random variable, which is called Poisson random variable. This is a discrete random variable such that X takes values 0, 1, 2… k up to infinity. That means X takes all positive integers or all non-negative integers from 0 to infinity such that probability X is equal to 0 is e to the power minus lambda, probability X is equal to 1 is lambda e to the power minus lambda, probability X is equal to 2 is e to the power minus lambda lambda square upon factorial 2.

Probability X is equal to k is e to the power minus lambda lambda to the power k upon factorial k. Like that, where lambda is greater than 0. Since, lambda is greater than 0, all these individual terms are greater than 0. Only thing we need to check if the sum of the probability is 1.

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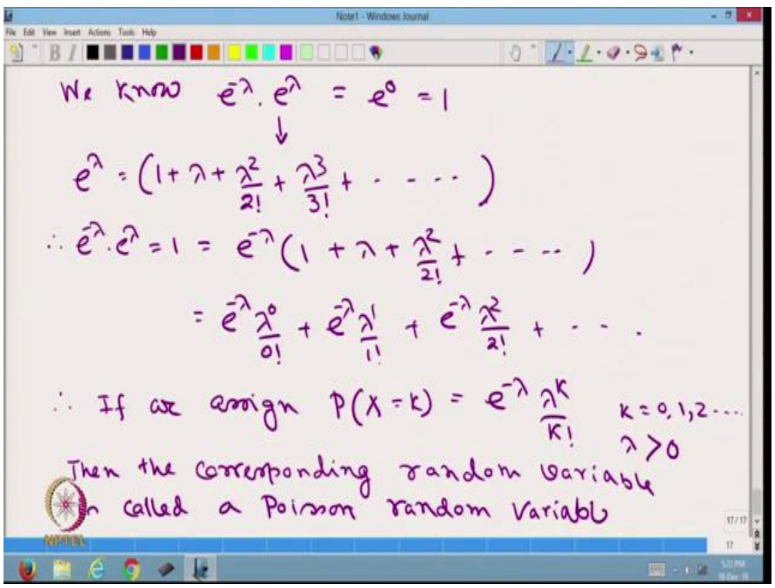

We know e to the power minus lambda into e to the power lambda is equal to e to the power 0 is equal to 1. Now, we know that we can expand it to the power lambda as follows e to the power lambda is equal to 1 plus lambda plus lambda square upon factorial 2 plus lambda cube upon factorial 3 up to infinity. Therefore, e to the power minus lambda into e to the power lambda is equal to 1 is equal to e to the power minus lambda into 1 plus lambda plus lambda square upon factorial 2 like that.

Therefore, we are writing it as e to the power minus lambda lambda power 0 upon factorial 0, e to the power minus lambda lambda power 1 upon factorial 1 plus e to the power minus lambda lambda square upon factorial 2 like that. Therefore, if we assign probability X is equal to k is equal to e to the power minus lambda lambda power k upon factorials k, where k is equal to 0, 1, 2 up to infinity and lambda is greater than 0 any real number, then the corresponding random variable is called a Poisson random variable.

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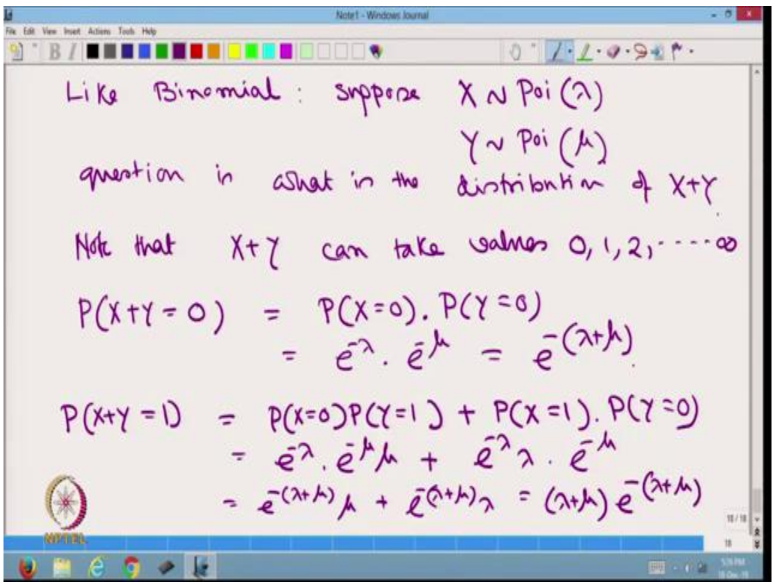

Natural question is what is the sum of two Poisson random variables? So, like binomial suppose, X is a Poisson random variable with parameter lambda and Y is Poisson random variable with say parameter mu, question is what is the distribution of X plus Y? Note that X plus Y can take values 0, 1, 2 up to infinite. So, probability X plus Y is equal to 0 is equal to probability X is equal to 0 multiplied by probability Y is equal to 0 is equal to e to the power minus lambda multiplied by e to the power minus mu, is equal to e to the minus lambda plus mu.

What is probability? X plus Y is equal to 1. We know that the event X plus Y is equal to 1 can happen if X is equal to 0 and Y is equal to 1 or X is equal to 1 and Y is equal to 0. Probability X is equal to 0 into probability Y is equal to 1, because these are independent, we can write as a product plus probability X is equal to 1 multiplied by probability Y is equal to 0.

Is equal to e to the power minus lambda multiplied by e to the power minus mu into mu plus probability X is equal to 1, which is e to the power minus lambda lambda power 1 upon factorial 1 into it to e to the power minus mu, is equal to e to the power minus lambda plus mu multiplied by mu plus e it to the power minus lambda plus mu multiplied by lambda is equal to lambda plus mu into e to the power minus lambda plus mu. I hope now, you are understanding in which direction it is moving. In fact, we are going to move towards that X plus Y is also Poisson random variable with parameters lambda plus mu.

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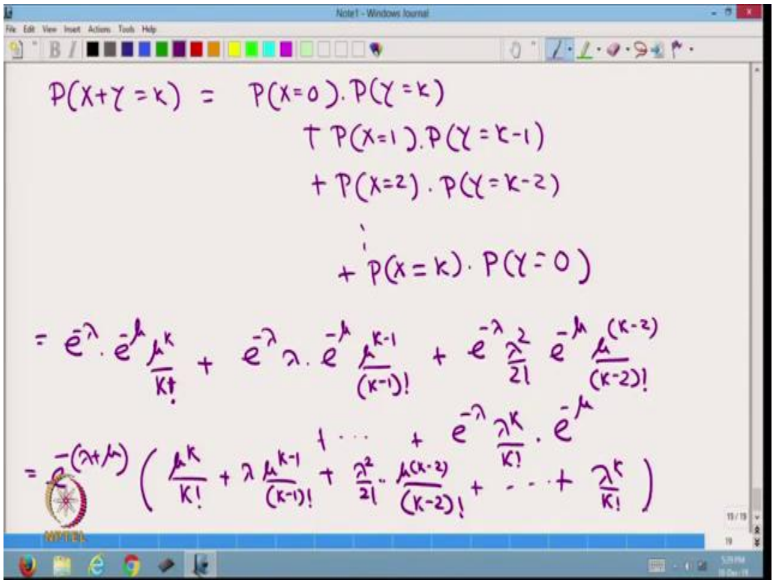

So, to verify that let us go one step further then probability X plus Y is equal to k is like the previous example, we are writing probability X is equal to 0 multiplied by probability Y is equal to k plus probability X is equal to 1 multiplied by probability Y is equal to k minus 1 plus probability X is equal to 2 multiplied by probability Y is equal to k minus 2 up to probability X is equal to k multiplied by probability Y is equal to 0.

This is equal to e to the power minus lambda multiplied by e to the power minus mu mu to the power k upon factorial k plus e to the power minus lambda into lambda multiplied by e to the power minus mu mu to the power k minus 1 upon factorial k minus 1 plus e to the power minus lambda lambda square upon factorial 2 multiplied by e to the power minus mu mu to the power k minus 2 upon factorial k minus 2 plus up to e to the power minus lambda lambda to the power k upon factorial k multiplied by e to the minus mu.

Is equal to e to the power minus lambda plus mu multiplied by mu to the power k upon factorial k plus lambda mu to the power k minus 1 upon k minus 1 factorial plus lambda square upon factorial 2 multiplied by mu to the power k minus 2 upon factorial k minus 2 up to lambda to the power k upon factorial k.

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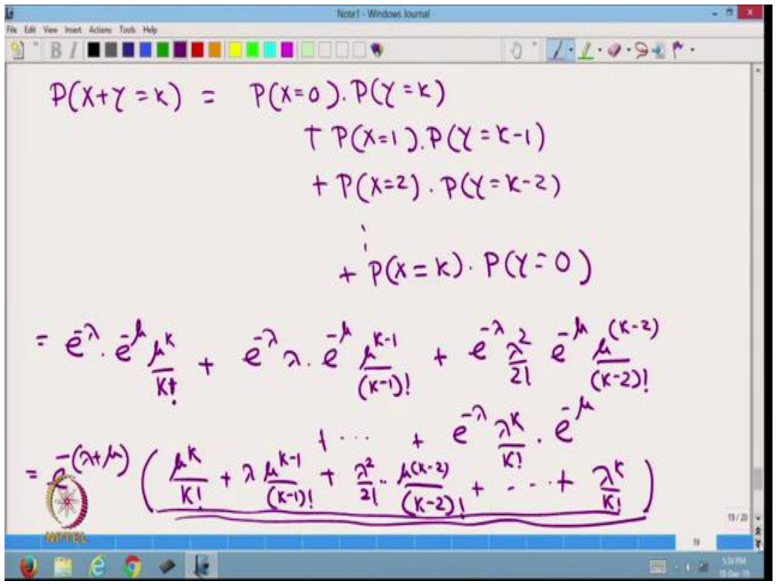

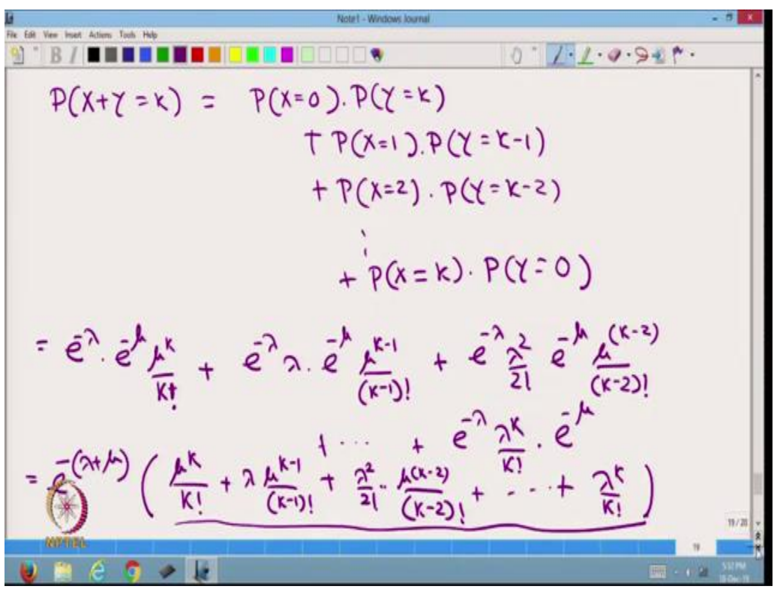

Now, lambda plus mu whole to the power k is equal to kc0 lambda power 0 mu to the power k plus 1 lambda power 1 mu to the power k minus 1 plus kc2 lambda power 2 mu to the power k minus 2 plus up to kck lambda power k mu to the power k minus k, is equal to k factorial into 0 factorial into k factorial lambda power 0 mu to the power k plus factorial k factorial upon factorial 1 factorial k minus 1 lambda power 1 into mu to the power k minus 1 plus factorial k upon factorial 2 factorial k minus 2 lambda square mu to the power k minus 2 up to factorial k upon factorial k into 0 factorial lambda power k mu to the power 0.

Is equal to k factorials into lambda power 0 mu to the power k upon k factorial plus lambda mu to the power k minus 1 upon 1 factorial into k minus 1 factorial plus lambda square mu to the power k minus 2 upon 2 factorial into k minus 2 factorial up to 1 upon k factorial lambda power k.

Therefore, if we compared, we see that this term is nothing but the term within the bracket therefore, mu to the power k upon factorial k plus lambda mu to the power k minus 1 upon k minus 1 factorial plus lambda square mu to the power k minus 2 upon 2 factorial into k minus 2 factorial plus up to lambda to the power k upon k factorial this term is nothing but lambda plus mu whole to the power k upon factorial k.

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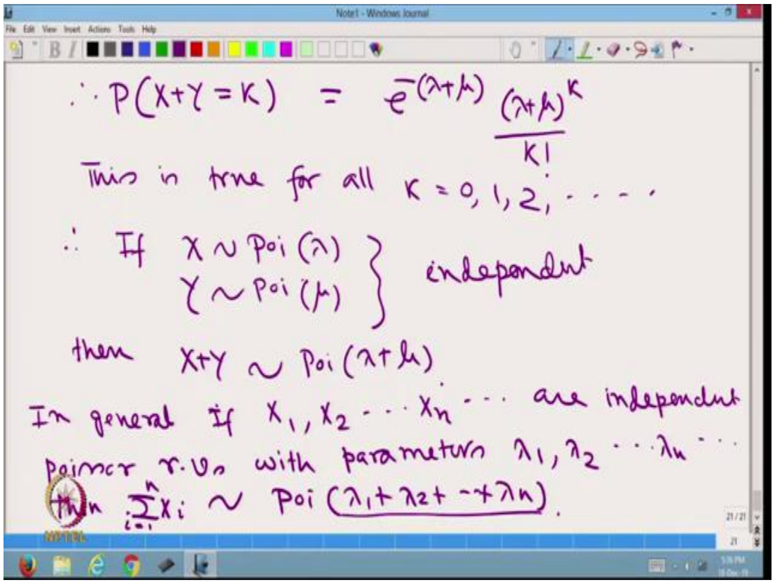

Therefore, if we replace it with lambda plus mu to the power k upon k factorial, then we get that probability X plus Y is equal to k is nothing but e to the power minus lambda plus mu lambda plus mu to the power k upon factorial k. This is true for all k is equal to

0, 1, 2 up to infinity. Therefore, what do you find that if X is Poisson with lambda and Y is Poisson with mu are independent then X plus Y is Poisson with lambda plus mu.

In general, if X1, X2… Xn are independent Poisson random variables with parameters lambda 1, lambda 2… lambda n, then sigma Xi, i is equal to 1 to n is distributed as Poisson with lambda 1 plus lambda 2 plus lambda n. This is a very interesting result which we shall see later. Because we use Poisson random variables for counting the number of arrivals and we will see that that naturally comes with the summation of the parameters.

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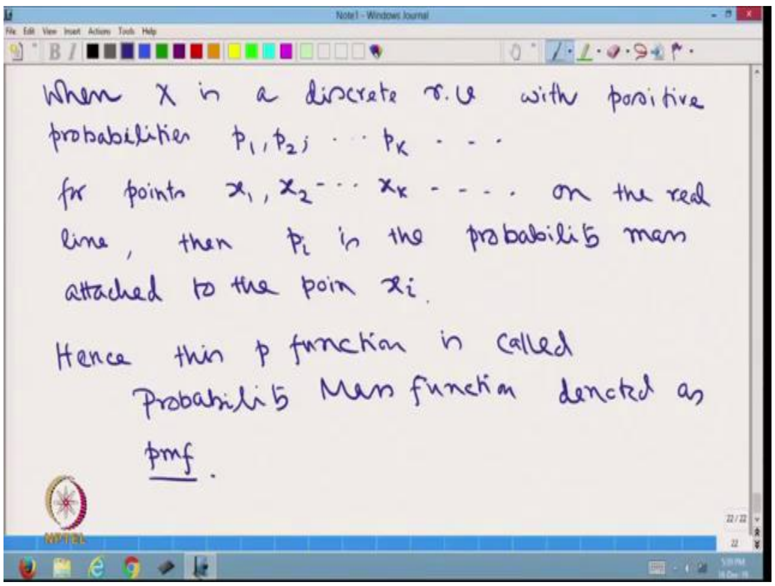

When X is a discrete random variables with positive probabilities that means non-zero probabilities p1, p2… pk for points x1, x2… xk on the real line, then pi is the probability mass attached to the point xi hence, this function is called Probability Mass Function which we often denote as pmf, for each point the corresponding pmf gives the quantum of probability associated with it.

Okay friends, I stopped here today. So, in this class, we have seen different discreet random variables like Bernoulli, Binomial, and Poisson. In the next class, I shall talk about some more discrete random variables namely, Geometric, Hyper Geometric and Negative Binomial random variables. These are all discrete random variables, which are very useful for modeling different practical phenomena. Okay friends, thank you so much.