Number field (Recall):

Number field K is a finite degree field extension of $\mathbb{Q}$.

$[K;\mathbb{Q}]$ = dimension of K as a vector space over $\mathbb{Q}$.

Algebraic Integer:

An algebraic integer in ~~an~~ a number field K is an element $\alpha \in K$ which is a root of monic polynomial with coefficients in $\mathbb{Z}$.

e.g $\sqrt{2}$ is an algebraic integer in $\mathbb{Q}[\sqrt{2}]$ as $\sqrt{2}$ is a solution of $x^2-2=0$.

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Algebraic Number: An algebraic number is an element $\alpha \in K$ which is a root of monic polynomial $f(x) \in Q[x]$.

e.g $\sqrt{2}$, $K = Q[\sqrt{2}]$

$$f(x) = x^2 - 2 \in Q[x]$$

$f(\sqrt{2}) = 0 \Rightarrow \sqrt{2}$ is an algebraic number.

Transcendental Number: The number $\alpha \in K$ which is not algebraic is called transcendental number.

e.g e, π

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Liouville Theorem:

Def: (Liouville Number) A real number x is a Liouville number if $\forall\ n \in \mathbb{N}$, $\exists\ p, q \in \mathbb{Z}$ with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}$$

(Liouville Theorem)

Statement of Theorem: Let $x \in \mathbb{R}$ be an algebraic number with degree $n \geq 2$ (i.e. x is irrational), $\exists$ a constant $c = c(x) > 0$ depending only on x such that

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^n} \quad \forall\ \frac{p}{q} \in \mathbb{Q},\ q \neq 0$$

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Proof: Let $r_1, r_2, \ldots r_k$ be the rational roots of a polynomial P of degree n that has x as a root. Since x is irrational, $\therefore$ $x \neq r_i$ for any i.

Let $c_1 = \min\{|x-r_1|, |x-r_2|, \ldots |x-r_k|\}$

If there is no r_i, $c_1 = 1$

Let $\alpha = \frac{p}{q}$, $\alpha \neq r_1, r_2, \ldots, r_k$

Then $P(\alpha) \neq 0$, $P(x) = 0$

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

$$P(\alpha) = \sum_{k=0}^{n} a_k \alpha^k$$

$$|P(\alpha)| \geq \frac{1}{q^n}$$

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$$\Rightarrow \quad |P(x) - P(\alpha)| \geq \frac{1}{q^n} \quad \text{as } P(x) = 0$$

Since $\quad x^k - \alpha^k = (x-\alpha) \sum_{i=0}^{k-1} x^{k-1-i} \alpha^i$

$$P(x) - P(\alpha) = \sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{n} a_k \alpha^k$$

$$= \sum_{k=0}^{n} a_k (x^k - \alpha^k)$$

$$= (x-\alpha) \sum_{k=1}^{n} a_k \sum_{i=0}^{k-1} x^{k-1-i} \alpha^i$$

Suppose $\quad |x - \alpha| \leq 1$

$$|\alpha| \leq |x| + 1$$

$$|P(x) - P(\alpha)| \leq |x-\alpha| \sum_{k=1}^{n} |a_k| \sum_{i=0}^{k-1} |x^{k-1-i}|.$$

$$(|x|+1)^i$$

$$\leq |x-\alpha| \sum_{k=1}^{n} |a_k| \, k \, (|x|+1)^{k-1}$$

$$= |x-\alpha| \, c_x$$

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$$C_x = \sum_{k=1}^{n} |a_k| \, k \left(|x|+1\right)^{k-1}$$

for such $\alpha \neq r_i$

$$|x-\alpha| \geq \frac{|P(x) - P(\alpha)|}{C_x} \geq \frac{1}{C_x q^n}$$

Case 2:

If $\alpha = r_i$, for some i

$$|x-\alpha| \geq c_1 \geq \frac{c_1}{q^n}$$

Case 3:

If $|x-\alpha| \geq 1$ then

$$|x-\alpha| \geq \frac{1}{q^n}$$

choose $c = \min\left\{1, \frac{1}{C_x}, c_1\right\}$

then $|x-\alpha| \geq \dfrac{c}{q^n}$

ie $\left|x - \dfrac{p}{q}\right| \geq \dfrac{c}{q^n}$; $\alpha = \dfrac{p}{q}$

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<u>Corollary</u>: Liouville numbers are transcendental. Converse is not true.

<u>Proof</u>: Suppose $\exists$ a Liouville number α that is algebraic for some degree $n > 1$ and α is irrational.

By Liouville theorem $\exists$ $c \geq 1$ s.t

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n} \quad \forall \text{ integers } p, q > 0$$

Choose an integer $k \geq n$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^k} < \frac{c}{q^n}$$

This is a contradiction to the assumption that α is algebraic.

$\Rightarrow$ α is transcendental.