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Algebraic Extension: A field extension K of L is called algebraic if every element of K is algebraic over L, i.e, if every element of K is a root of some non-zero polynomial with coefficients in L.

→ Field extensions that are not algebraic are called transcendental.

→ 1. $R(Q)$ is transcendental

2. $C(\mathbb{R})$ is a algebraic

3. $C(Q)$ is not algebraic

Integral element definition (146)

Integral Extension: Let R be a subring of the ring S. $\alpha \in S$ is integral over R if α is a root of a monic polynomial with coefficients in R, i.e, $\exists\ f(x) \in R[x]$ s.t $f(\alpha) = 0$.

Def: If every element $\alpha \in S$ is integral over R, then S is said to be integral over R or S is integral extension of R.

→ Integral Extension is also called Integral closure.

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$K = Q[\sqrt{d}]$

$f(x) = x^2 - d$

Basis $= \{1, \sqrt{d}\}$

Let $\alpha = a + b\sqrt{d} \in K, \quad a, b \in Q$

$Tr(\alpha) = 2a = \alpha + \bar{\alpha}$

$N(\alpha) = a^2 - db^2 = \alpha\bar{\alpha}$

Also $\quad \sigma_i : K \to C \quad , i = 1, 2$

$[m_\alpha] = ?$

$\sigma_1(\alpha) = 1 \cdot \alpha = a + b\sqrt{d}$

$\sigma_2(\alpha) = \sqrt{d} \cdot \alpha = db + a\sqrt{d}$

$$[m_\alpha] = \begin{pmatrix} a & db \\ b & a \end{pmatrix}$$

$Tr(\alpha) = Tr([m_\alpha]) = 2a$

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$$N(\alpha) = \det([m_\alpha])$$

$$= a^2 - db^2$$

<u>Disoriminant:</u> Δ

$$B = \{1, \sqrt{d}\} = \{\alpha_1, \alpha_2\}$$

$$\alpha_1 = 1$$

$$\alpha_2 = \sqrt{d}$$

$$\Delta = \det(Tr(\alpha_i \alpha_j))$$

$$= \begin{pmatrix} Tr(\alpha_1\alpha_1) & Tr(\alpha_1\alpha_2) \\ Tr(\alpha_2\alpha_1) & Tr(\alpha_2\alpha_2) \end{pmatrix}$$

$$= \begin{pmatrix} Tr(1) & Tr(\sqrt{d}) \\ Tr(\sqrt{d}) & Tr(d) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix} = 4d$$

In general if

$$B = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$$

$$\Delta = \begin{pmatrix} Tr(\alpha_1\alpha_1) & Tr(\alpha_1\alpha_2) & \cdots & Tr(\alpha_1\alpha_n) \\ Tr(\alpha_2\alpha_1) & Tr(\alpha_2\alpha_2) & \cdots & Tr(\alpha_2\alpha_n) \\ \vdots & & & \\ Tr(\alpha_n\alpha_1) & Tr(\alpha_n\alpha_2) & \cdots & Tr(\alpha_n\alpha_n) \end{pmatrix}$$

<u>Exc:</u> $K = \mathbb{Q}[\sqrt[3]{d}]$

$$B = \{ 1, \sqrt[3]{d}, \sqrt[3]{d^2} \}$$

<u>HW</u>