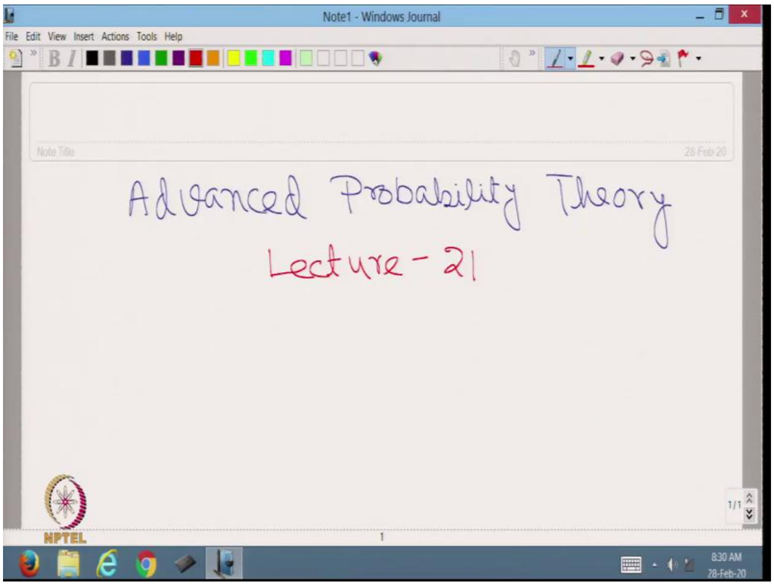
Advanced Probability Theory
Professor Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture 21

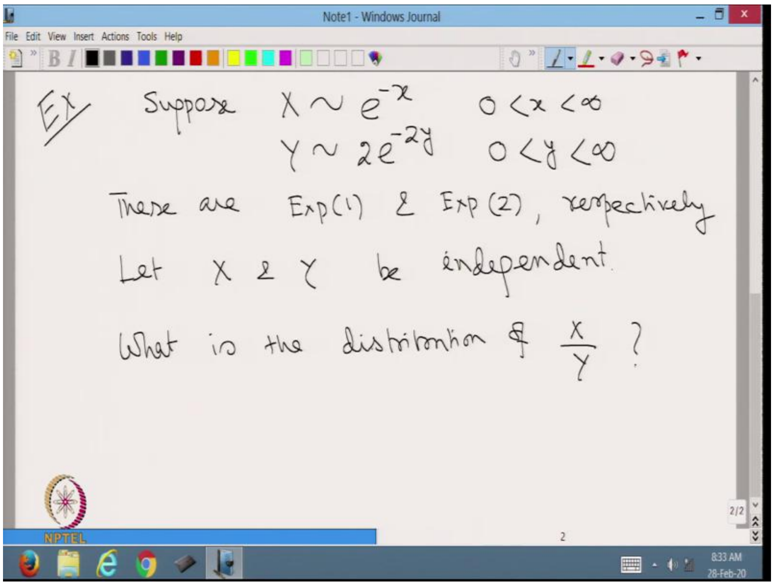
(Refer Slide Time: 0:27)



Welcome students, the MOOCS series of lectures on Advance Probability Theory. This is lecture number 21. Hope you remember that in the last class we started with functions of two random variables. In this class, I shall continue with that and in particular I will give you several examples and also I will try to derive the distribution of two very important statistical distributions namely T and F.

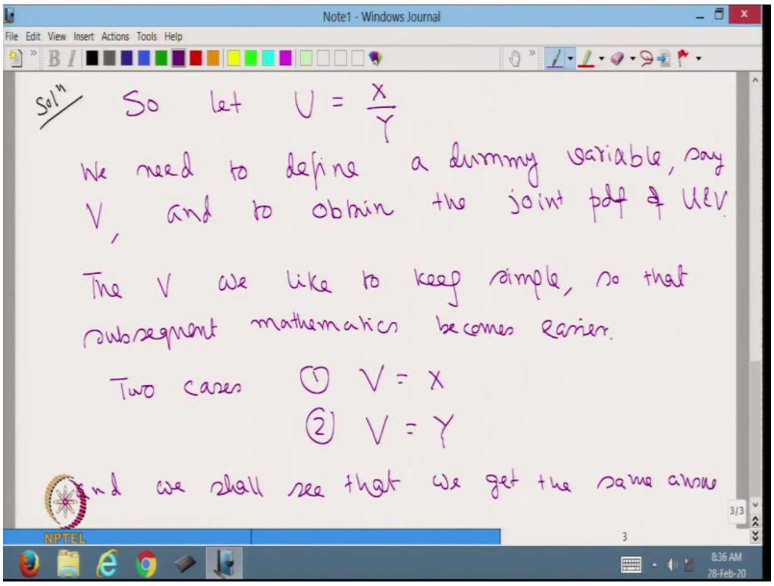
Those who were working in statistics, they will know that these two are very important for different tests, but here in this course, we shall look at it mathematically and we shall try to see how to derive their pdf from their basic definitions. So, let me start with a simple example.

(Refer Slide Time: 1:17)



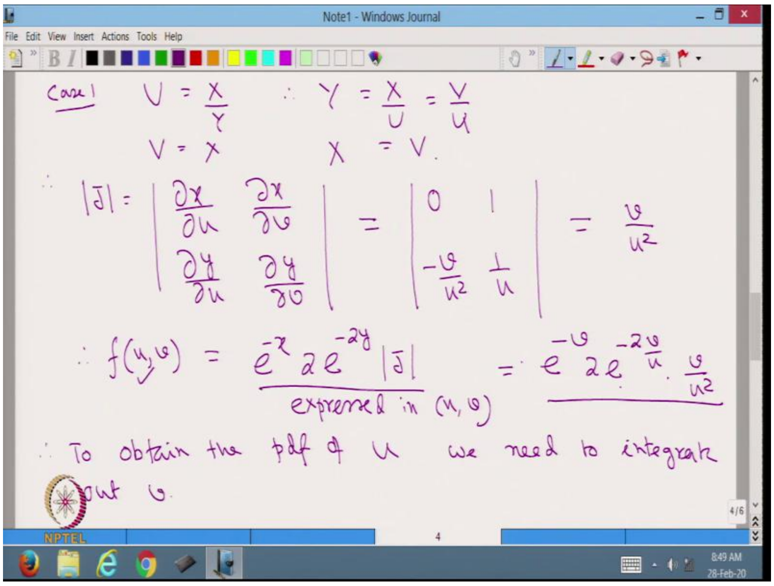
Suppose, X is a random variable which is distributed as e to the power minus x, 0 less than x less than infinity and Y is another random variable which is distributed with this pdf. Again 0 less than y less than infinity. So, you understand that these are exponential with parameter 1 and exponential with parameter 2 respectively. Let X and Y be independent. So, what is the distribution of X by Y? So, that is the question that we want to solve.

(Refer Slide Time: 2:36)



So, solution. So, let U is equal to X by Y. We need to define a dummy variable, say V, and to obtain the joint pdf of U and V. As I said, the V we like to keep simple so that subsequent mathematics becomes easier. So, I will take two cases. One is V is equal to X and second is I will take V is equal to Y and we will see that we get the same answer.

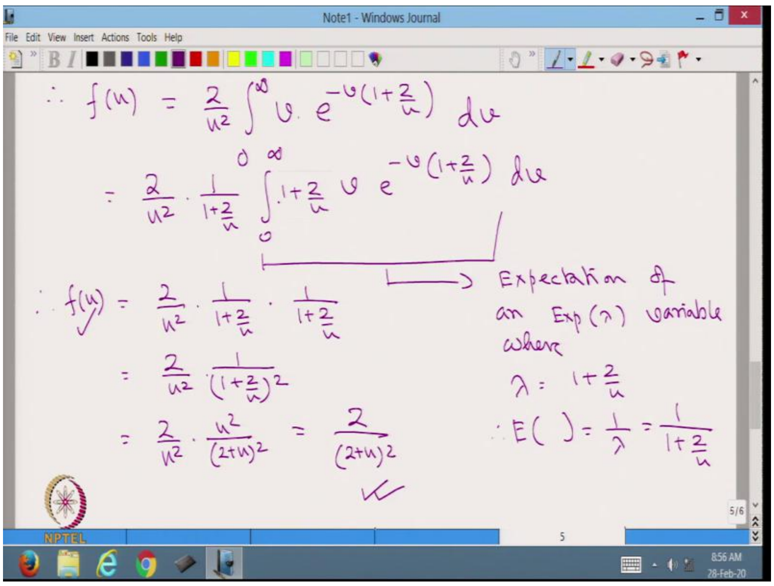
(Refer Slide Time: 4:21)

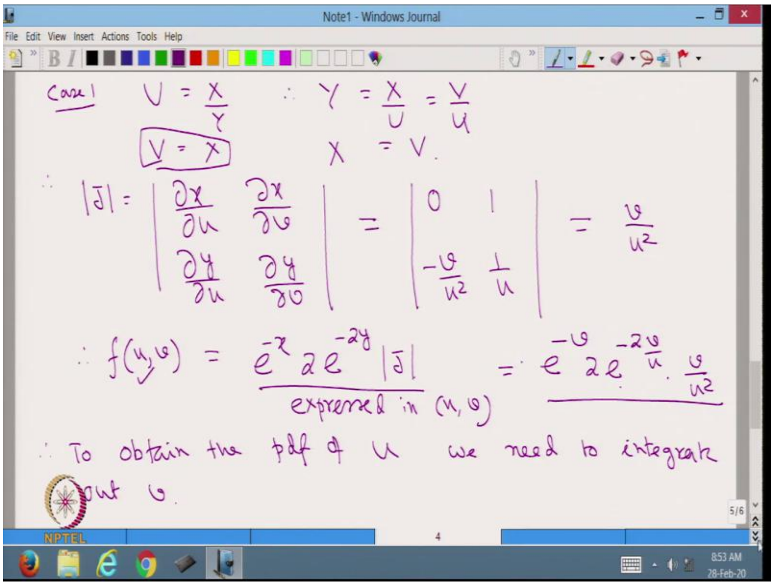


So, case 1. U is equal to X by Y, V is equal to X. Therefore, Y is equal to X by U is equal to V by U. And X is equal to V. Therefore, Jacobian is equal to del x del u del x del v del y del u and del y del v is equal to 0 del x del u is equal to del v del u is equal to 0 1, del y del u is equal to minus v up on u square and del y del v is equal to 1 up on u. Therefore, the Jacobian is coming out to be v up on u square, which is the determinant to this matrix.

Therefore, joint pdf f u, v is equal to e to the power minus x 2 e to the power minus 2 y into determinant of J which is expressed in u, comma v is equal to e to the power minus v to e to the power minus 2 v by u into v upon u square. So, that is the pdf of f u, v. Therefore, to obtain the pdf of u, we need to integrate out v.

(Refer Slide Time: 6:52)

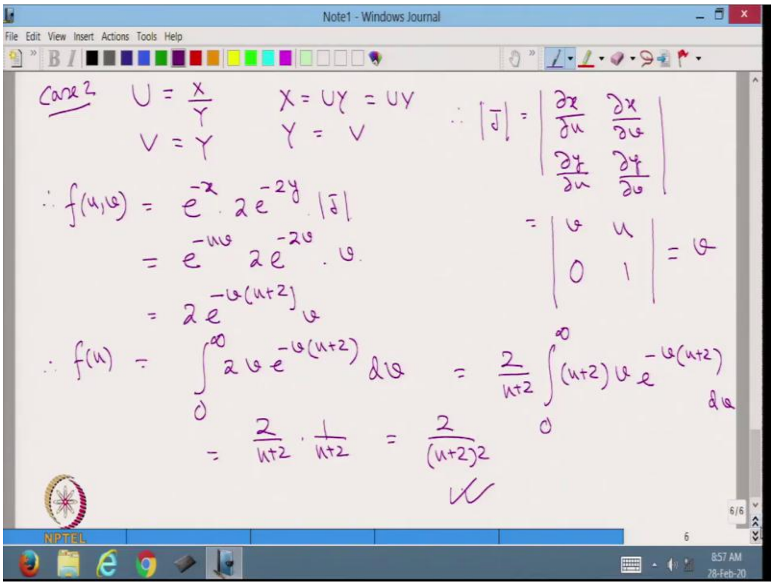




Therefore, f of u is equal to 2 by u square integration 0 to infinity v e to the power minus v into 1 plus 2 by u dv. Is equal to 2 by u square into 1 upon 1 plus 2 by u multiplied by 0 to infinity 1 plus 2 by u into v e to the power minus v into 1 plus 2 by u dv.

Now, this is the expectation of an exponential lambda variable where lambda is equal to 1 plus 2 by u. Therefore, expected value will come out to be 1 upon lambda is equal to 1 upon 1 plus 2 by u. Therefore, f u is equal to 2 upon u square into 1 plus 2 by u into 1 upon 1 plus 2 by u is equal to 2 by u square into 1 upon 1 plus 2 by u whole square which is is equal to 2 upon u square multiplied by u square upon 2 plus u whole square is equal to 2 upon 2 plus u whole square. So, that is the distribution of u. By considering if you remember, V is equal to X. Now, I go to the second case.

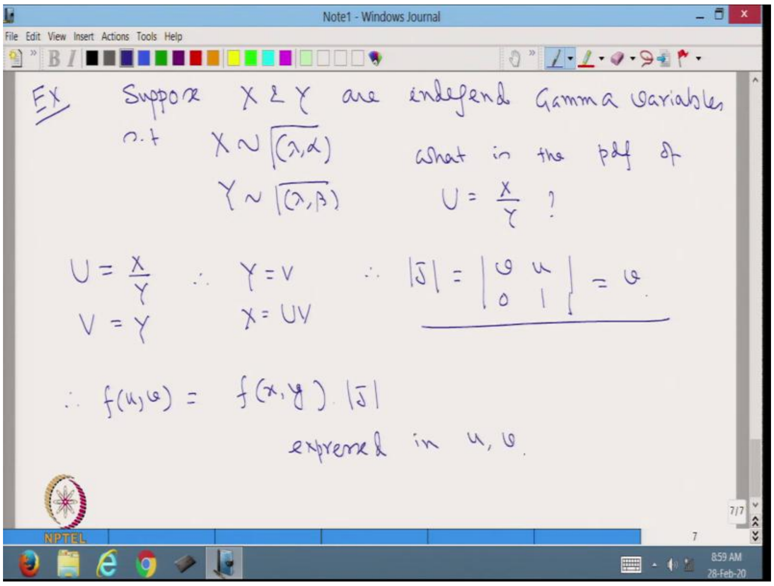
(Refer Slide Time: 9:23)



Therefore, as before, U is equal to X by Y, but let us take V is equal to Y. In that case what is going to happen? We can see that X is equal to UY is equal to UV. And Y is another V. Therefore, Jacobian is equal to determinant of del x del u del x del v del y del u del y del v is equal to del x del u is equal to v, del x del v is equal to u, del y del u is equal to 0 and this is 1 therefore, the Jacobian is equal to v.

Therefore, f of u, v is equal to e to the power minus x 2 e to the power minus 2 y into the Jacobian which we can write now as e to the power minus u v 2 e to the power minus 2 v into v is equal to 2 e to the power minus v into u plus 2 into v. Therefore, f of u is equal to integration 0 to infinity 2 v e to the power minus v u plus 2 dv is equal to 2 upon u plus 2 integration 0 to infinity u plus 2 v e to the power minus v u plus 2 dv is equal to as before, we can see is equal to 2 upon u plus 2 into 1 upon u plus 2 is equal to 2 upon u plus 2 whole square. Thus, we see that we get the same result by taking two different choices of V.

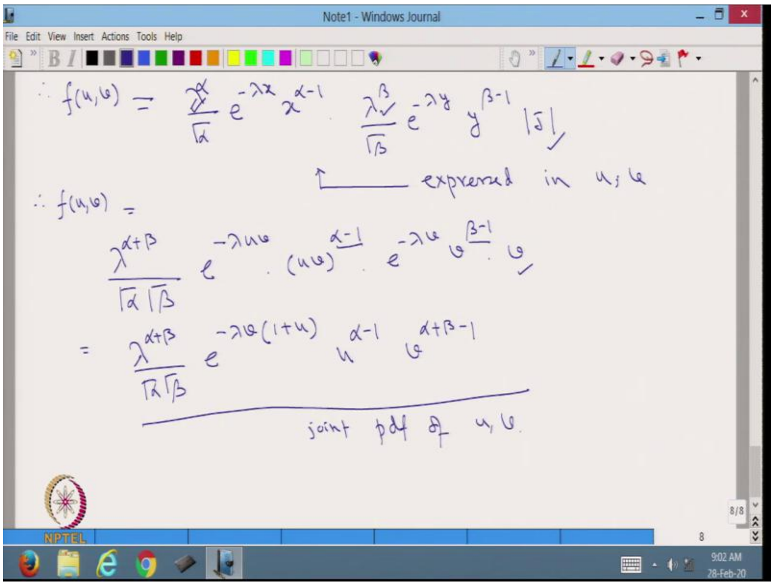
(Refer Slide Time: 12:09)



Let us now consider another example. Suppose, X and Y are independent gamma variables such that X is distributed as gamma lambda alpha and Y is distributed as gamma lambda beta. What is the pdf of U is equal to X by Y? So, that is the question. Therefore, as before, we go U is equal to X by Y, and let us consider V is equal to Y. Therefore, as before, Y is equal to V and X is equal to UV.

Therefore, the Jacobian we have just seen will come out to be v u 0 1 determinant of this matrix is equal to v. Since, we have just worked it out in the previous example, I am not going into detail. Therefore, f of u, v is equal to f x, y into the Jacobian expressed in u, v.

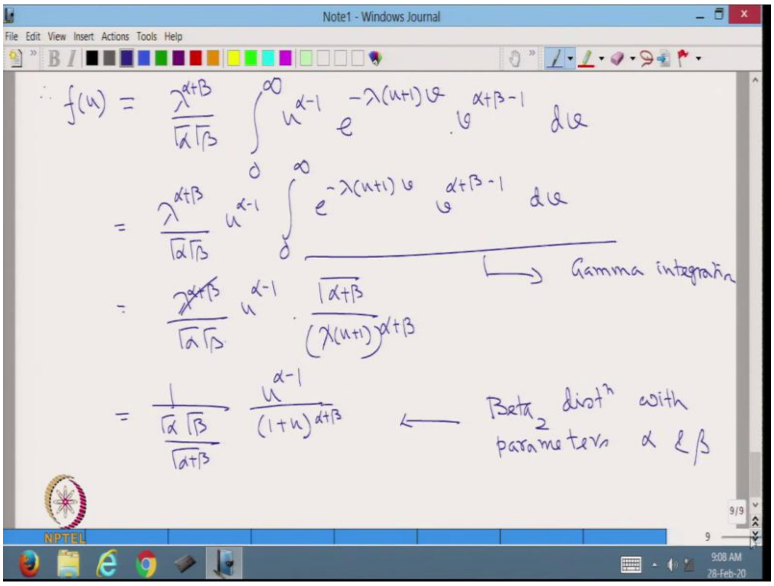
(Refer Slide Time: 14:05)



Therefore, joint pdf f u, v is equal to lambda power alpha upon gamma alpha e to the power minus lambda x, x to the power alpha minus 1 multiplied by lambda power beta upon gamma beta e to the power minus lambda y, y to the power beta minus 1 J expressed in u, v.

Therefore, f u, v is equal to lambda power alpha plus beta, I take out the constants upon gamma alpha gamma beta e to the power minus lambda u v into u v to the power alpha minus 1, e to the power minus lambda v, v to the power beta minus 1 into v. This is from the Jacobian. Is equal to lambda power alpha plus beta upon gamma alpha gamma beta e to the power minus lambda v into 1 plus u u to the power alpha minus 1, v to the power alpha minus 1 plus beta minus 1 plus 1. Therefore, alpha plus beta minus 1 and therefore that is the joint pdf of u, v.

(Refer Slide Time: 15:58)

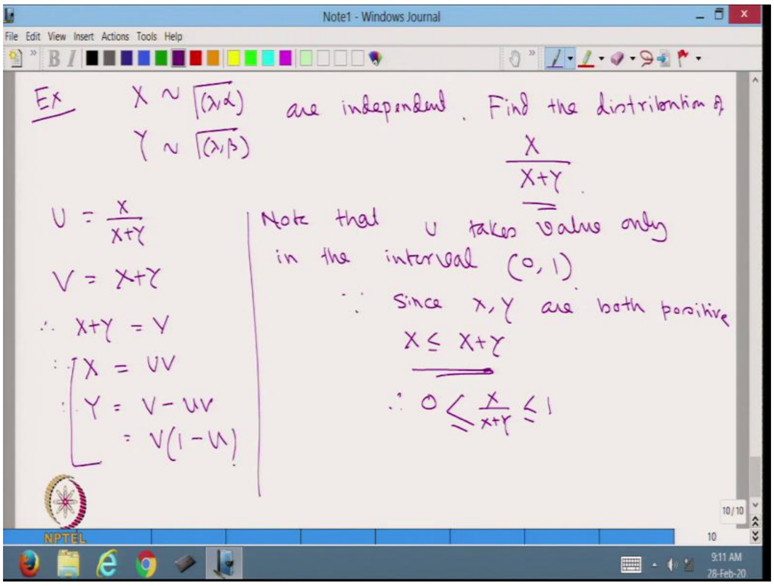


Therefore, f of u is equal to lambda power alpha plus beta upon gamma alpha gamma beta integration of 0 to infinity u to the power alpha minus 1, e to the power minus lambda u plus 1 v, v to the power alpha plus beta minus 1 dv. Is equal to lambda power alpha plus beta upon gamma alpha gamma beta u to the power alpha minus 1 integration 0 to infinity e to the power minus lambda u plus 1 v, v to the power alpha plus beta minus 1 dv.

Is equal to lambda power alpha plus beta upon gamma alpha gamma beta u to the power alpha minus 1 into gamma alpha plus beta divided by lambda u plus 1 whole to the power alpha plus beta. This is because this is our gamma integration and this result we have seen many times.

This is is equal to 1 upon gamma alpha gamma beta upon gamma alpha plus beta, lambda power alpha plus beta cancels therefore what is remaining is u to the power alpha minus 1 upon 1 plus u whole to the power alpha plus beta. So, we get this is our beta 2 distribution with parameters alpha and beta.

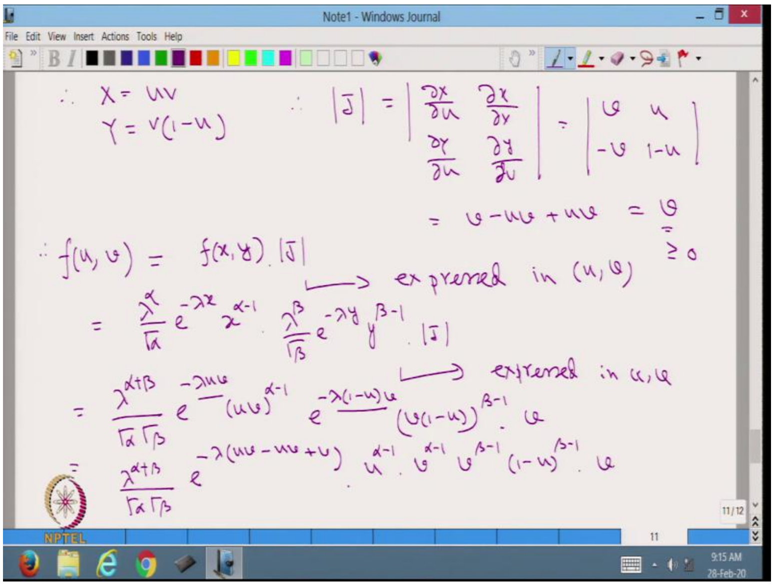
(Refer Slide Time: 18:23)



Let us consider a very similar example under the same of that is, X is gamma lambda, comma alpha, Y is gamma lambda, comma beta are independent. Find the distribution of X upon X plus Y. Slightly tricky. So, let us call U is equal to X upon X plus Y. Note that U takes value only in the interval 0 to 1 because since X and Y are both positive, X is always less than equal to X plus Y. Hence, this ratio X upon X plus Y is between 0 to 1.

Let us consider V is equal to X plus Y. Therefore, what we get. X plus Y is equal to V. Therefore, X is equal to U V. Therefore, Y is equal to V minus uv is equal to V into 1 minus u. So, this is important because this is the inverse transformation from u v to x 1.

(Refer Slide Time: 20:40)

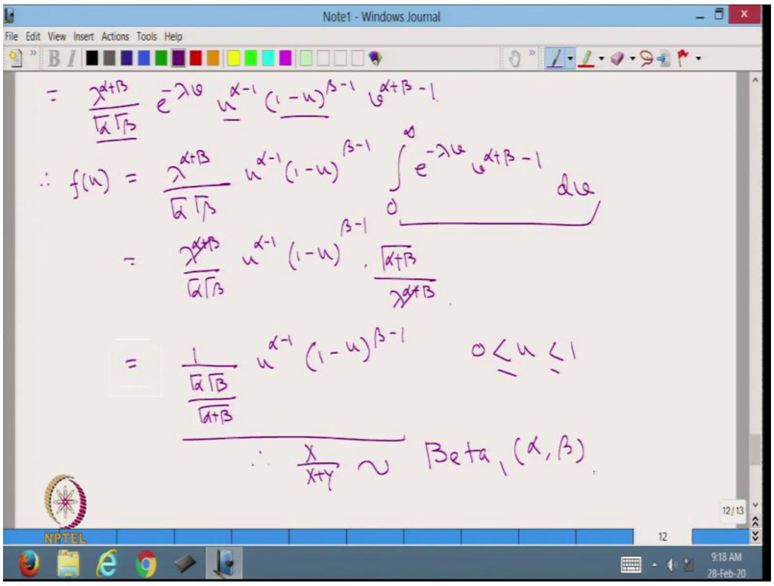


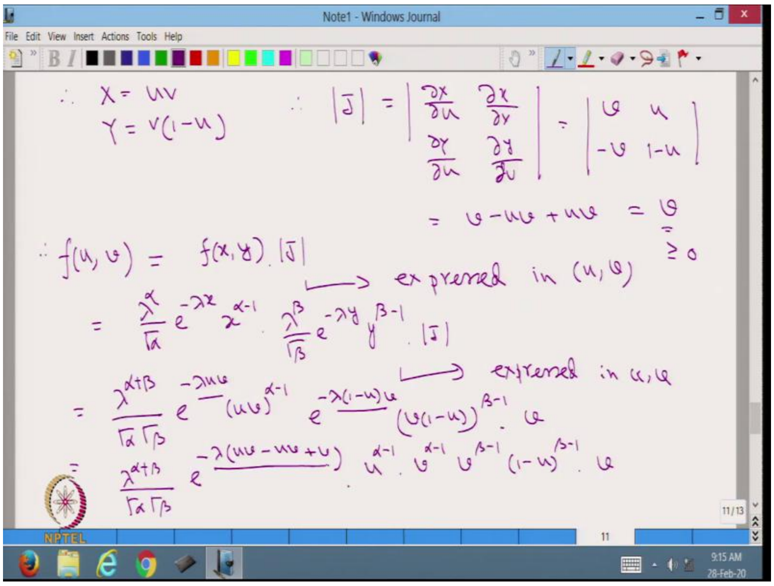
Therefore, X is equal to uv, Y is equal to v into 1 minus u. Therefore, Jacobian is equal to del x del u del x del v del y del u del y del v is equal to v u minus v 1 minus u is equal to v minus uv plus uv is equal to v, which is greater than equal to 0. Therefore, we need not take any modulus here. Therefore, joint distribution of u and v is equal to f x, y into Jacobian expressed in u, v.

Is equal to lambda power alpha upon gamma alpha e to the power minus lambda x, x to the power alpha minus 1 multiplied by lambda power beta upon gamma beta e to the power minus lambda y, y to the power beta minus 1 multiplied by the Jacobian expressed in u, v, is equal to lambda power alpha plus beta upon gamma alpha gamma beta e to the power minus lambda u v, u v to the power alpha minus 1, e to the power minus lambda 1 minus u into v into v into 1 minus u whole to the power beta minus 1 into v.

Is equal to lambda power alpha plus beta upon gamma alpha gamma beta e to the power minus lambda u v minus u v plus v. By combining these two into u to the power alpha minus 1, v to the power alpha minus 1, v to the power beta minus 1 1 minus u to the power beta minus 1 into v.

(Refer Slide Time: 24:07)





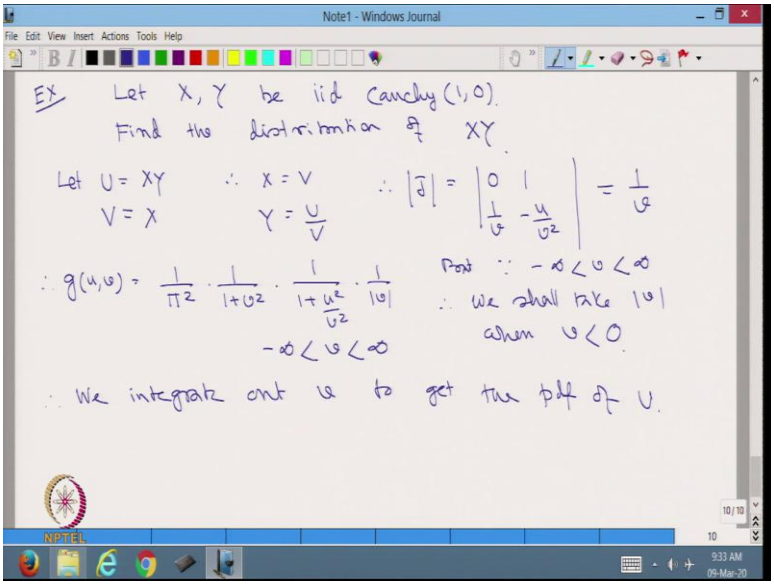
Is equal to lambda to the power alpha plus beta upon gamma alpha gamma beta e to the power minus lambda v. This I am getting from here. u to the power alpha minus 1, 1 minus u to the power beta minus 1 and v to the power alpha plus beta minus 1. U to the power alpha minus 1, 1 minus u to the power beta minus 1, v to the power alpha plus beta minus 1.

Therefore, f of u is equal to, let me take out the terms independent of v to be out of the integration. Therefore, lambda to the power alpha plus beta gamma alpha gamma beta u to the power alpha minus 1, 1 minus u to the power beta minus 1, integration 0 to infinity e to the power minus lambda v, v to the power alpha plus beta minus 1 dv.

Is equal to lambda to the power alpha plus beta gamma alpha gamma beta u to the power alpha minus 1, 1 minus u to the power beta minus 1, and this again using gamma integration, we are getting gamma alpha plus beta upon lambda to the power alpha plus beta. Is equal to, because this gets cancelled, we have 1 upon gamma alpha gamma beta upon gamma alpha plus beta, u to the power alpha minus 1, 1 minus u to the power beta minus 1, when 0 less than u less than 1.

Therefore, we get X upon X plus Y is distributed as beta 1 with parameter alpha, beta. So, we have seen these distributions before, but now we can see from how, from known distributions particularly gamma distribution with same parameters lambda, we can get both beta 1 and beta 2 distributions.

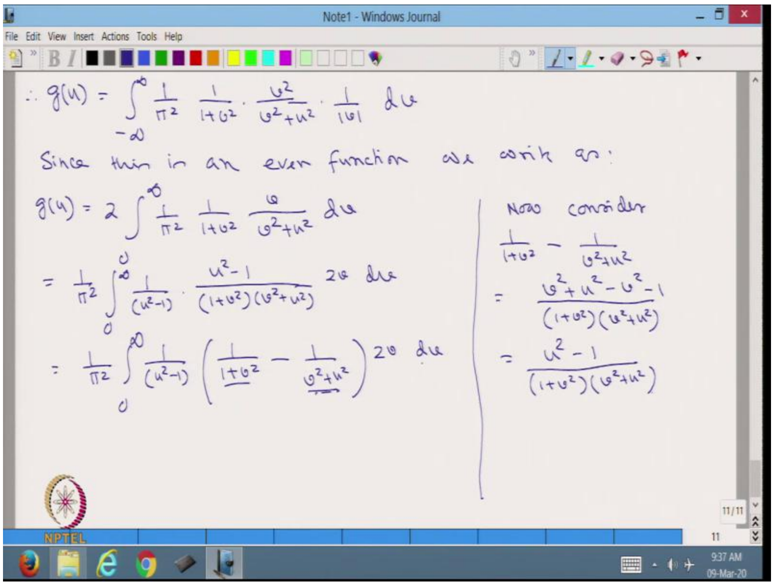
(Refer Slide Time: 26:54)



So, let us take another example which is slightly difficult but still let us do it. Let X, Y be iid Cauchy with 1, 0 that is standard Cauchy distribution. Find the distribution of XY. So, let us take U is equal to XY and V is equal to X. Therefore, X is equal to v, Y is equal to u by v. Therefore, Jacobian is going to be del x del u del x del v del y del u del y del v is equal to 1 upon v. But since minus infinity less than v less than infinity, therefore we shall take modulus of v when v is less than 0.

Therefore, g u, v is equal to 1 upon pi square into 1 upon 1 plus v square into 1 upon 1 plus u square by v square into 1 upon modulus of v, when minus infinity less than v less than infinity. Therefore, we integrate out v to get the pdf of u.

(Refer Slide Time: 29:25)



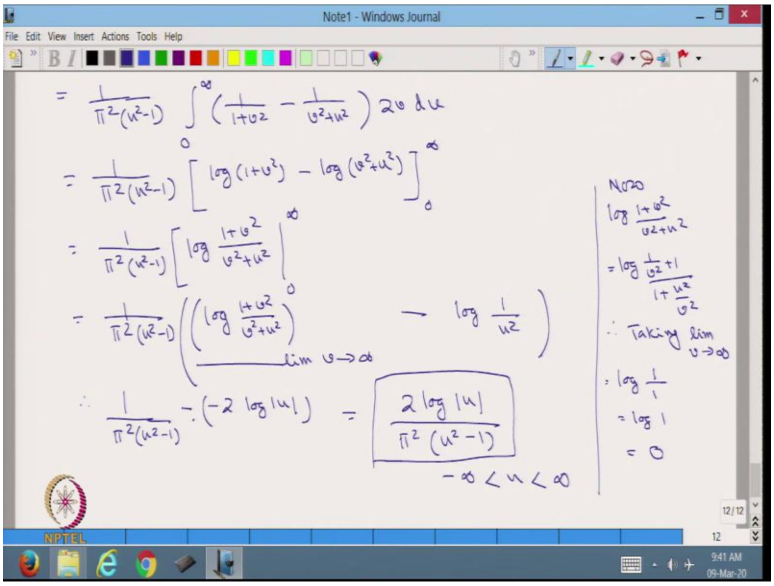
Therefore, g of u is equal to integration minus infinity to infinity 1 upon pi square into 1 upon 1 plus v square into v square upon v square plus u square into 1 upon modulus of v dv. And v is going from minus infinity to infinity. Since, this is an even function, we write as g of u is equal to 2 times 0 to infinity 1 upon pi square 1 upon 1 plus v square, v upon v square plus u square dv.

Now, consider 1 upon 1 plus v square minus 1 upon v square plus u square. So, this is is equal to 1 plus v square into v square plus u square in the denominator and in the numerator we have v square plus u square minus v square minus 1 is equal to u square minus 1 upon 1 plus v square into v square plus u square.

Therefore, this we write as 1 upon pi square, we take out of the integration, 0 to infinity 1 upon u square minus 1 into u square minus 1 upon 1 plus v square into v square plus u square 2v dv. Is equal to 1 upon pi square 0 to infinity 1 upon u square minus 1 into 1 upon 1 plus v square minus 1 upon v square plus u square 2v dv.

Note that in both of them, v square is there in the denominator so by replacing 1 plus v square say as z, we shall get 2v dv is equal to dz and in a similar way for v square plus u square we shall get 2v dv to be say, d of w.

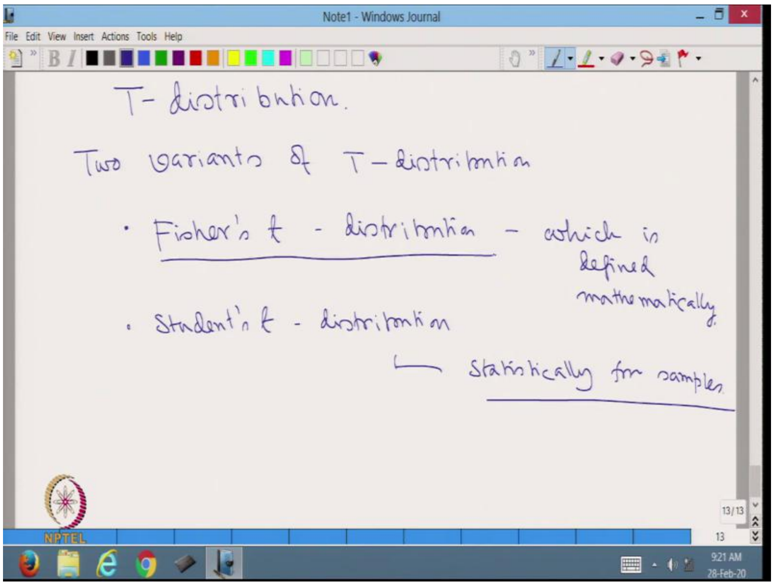
(Refer Slide Time: 32:40)



So, this we write as, 1 upon pi square into u square minus 1 integration 0 to infinity 1 upon 1 plus v square minus 1 upon v square plus u square into 2v dv. Is equal to 1 upon pi square into u square minus log of 1 plus v square minus log of v square plus u square 0 to infinity, is equal to 1 upon pi square into u square minus 1 into log of 1 plus v square upon v square plus u square 0 to infinity, is equal to 1 upon pi square into u square minus 1 log of 1 plus v square to v square plus u square limit v going to infinity minus by putting the value v is equal to 0 we have log of 1 upon u square.

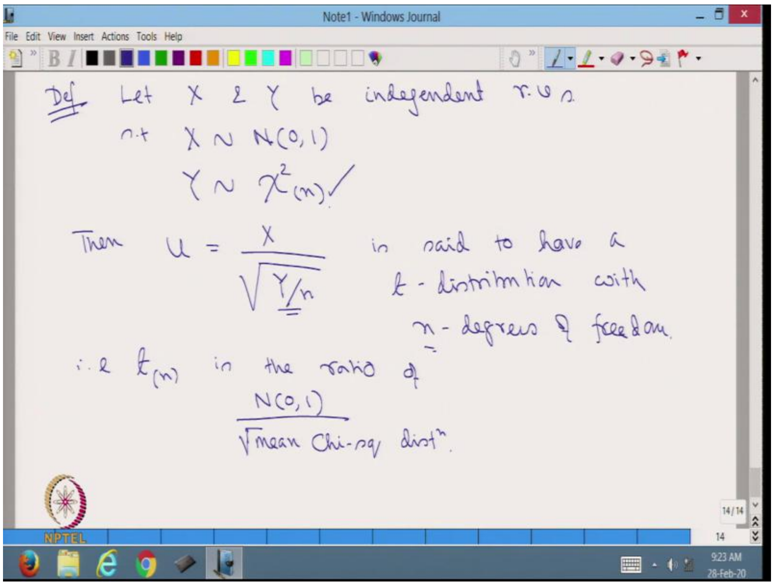
Now, log of 1 plus v square upon v square plus u square is equal to log of 1 upon v square plus 1 upon 1 plus u square upon v square. Therefore, taking limit v going to infinity, we get is equal to log of 1 upon 1 is equal to log of 1 is equal to 0. Therefore, this term goes to 0 and we get this is is equal to 1 upon pi square into u square minus 1 into minus 2 log of mod of u with a minus sign is equal to 2 log mod of u upon pi square into u square minus 1 when minus infinity less than u less than infinity. So, we get another new density function although this is not very commonly used function.

(Refer Slide Time: 35:50)



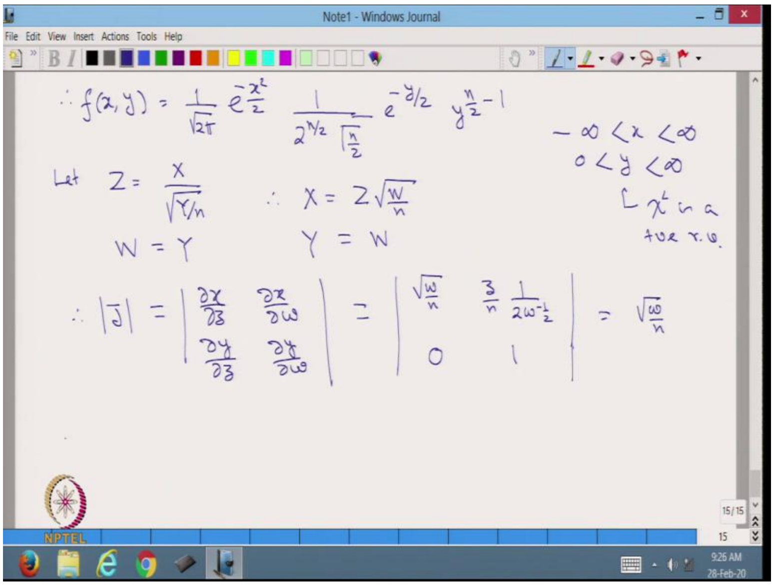
Let me now define T - distribution two variants of it. One is called Fisher's t - distribution which is defined mathematically and the second one is called Student's t - distribution which is defined statistically from samples. This is important to know if some of you are going to use it in testing statistical hypothesis but for this class we are sticking to Fisher's t - distribution.

(Refer Slide Time: 37:16)



So, definition, let X and Y be independent random variables such that X is distributed as normal 0, 1 and Y is distributed as chi square with n degrees freedom then u which is defined as X upon square root of Y by n is said to have a t - distribution with n degrees of freedom. This n is coming from the degrees of freedom of the chi square distribution. That is, t n is the ratio of standard normal 0, 1 and square root of mean chi square distribution. Mean is coming because we are dividing by n the degrees of freedom. So, what is the pdf?

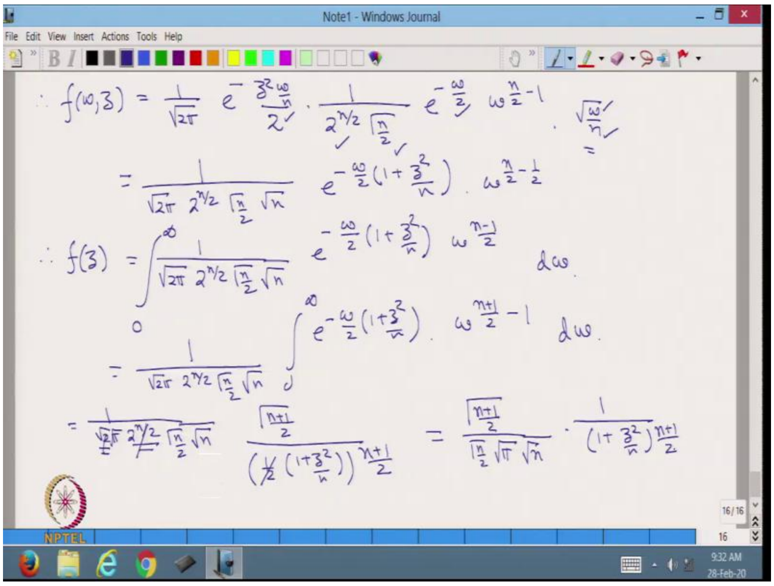
(Refer Slide Time: 39:04)

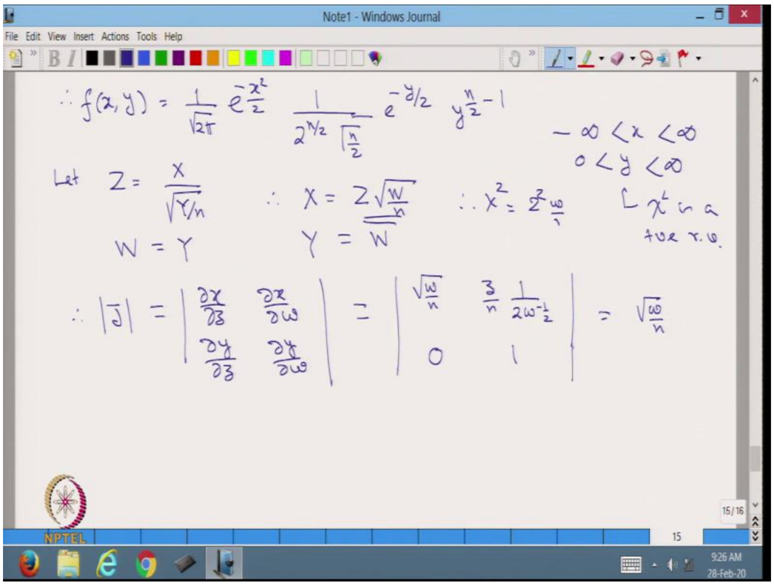


Therefore, f of x, y we can write it as 1 over root of 2 pi e to the power minus x square by 2, 1 upon 2 to the power n by 2 gamma n by 2 e to the power minus y by 2, y to the power n by 2 minus 1, when minus infinity less than x less than infinity and 0 less than y less than infinity. This is because chi square is a positive random variable.

Let Z is equal to X upon root over Y by n and let W is equal to Y. Therefore, X is equal to Z into root over W by n and Y is equal to W. Therefore, the Jacobian is equal to del x del z del x del w del y del z del y del w is equal to root of w by n z by and n half w to the power minus half 0 and 1 is equal to root over w by n.

(Refer Slide Time: 41:04)





Therefore, as before f of w and z is equal to 1 over root 2 pi e to the power minus z square w by n by 2. This is because X is equal to Z w by n. Therefore, X square is equal to z square w by n. So, I am using that multiplied by 1 upon 2 to the power n by 2 gamma n by 2 e to the power minus w by 2 w to the power n by 2 minus 1 multiplied by root over w by n.

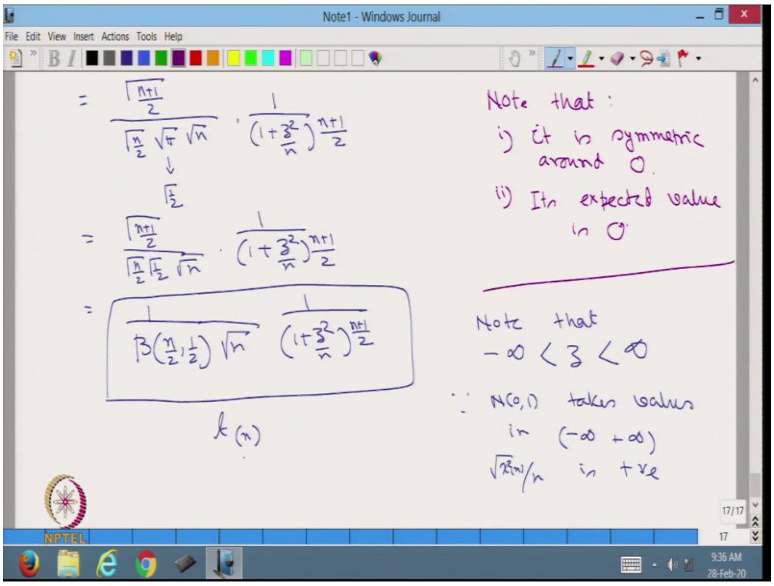
This is from the Jacobian is equal to, let us take out all the constants, 1 over root over 2 pi 2 to the power n by 2 gamma n by 2 and root over n multiplied by e to the power minus w by 2 into 1 plus z square by n into w to the power n by 2 minus half. Because we get w power half from there. Therefore, f of z is equal to we are integrating out w therefore, taking out the constant, 1 over root over 2 pi 2 to the power n by 2 gamma n by 2 root over n e to the power minus w by 2 1 plus z square by n w to the power n minus 1 by 2.

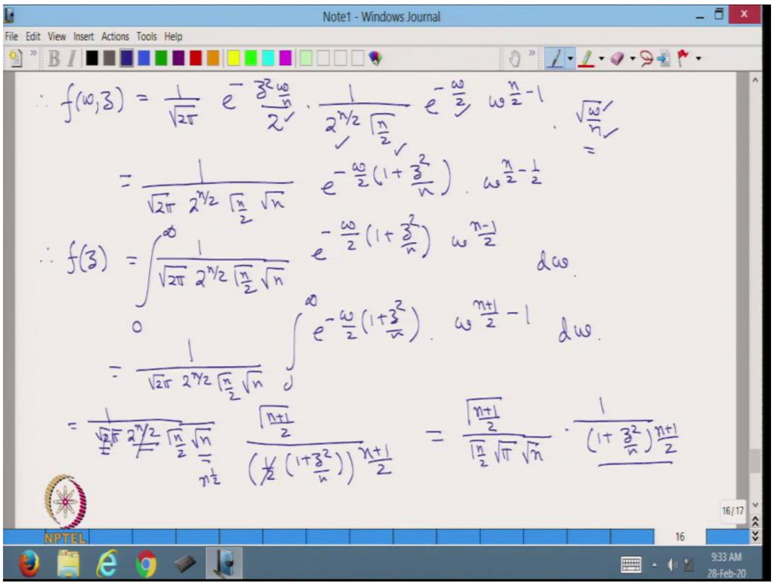
And I am integrating it from 0 to infinity with respect to dw is equal to all the constant terms come out. 1 over root over 2 pi 2 to the power n by 2 gamma n by 2 root over n integration 0 to infinity e to the power minus w by 2 1 plus z square by n w to the power n plus 1 by 2 minus 1. We are writing it in this form dw is equal to 1 over root over 2 pi 2 to the power n by 2 gamma n by 2 root over n.

Now, this is again our familiar gamma integral. Therefore, we can write it as gamma n plus 1 by 2 upon half 1 plus z square by n whole to the power n plus 1 by 2. Fairly complicated but we can make it slightly simpler is equal to now here it is root 2 to the power n by 2 so together it is 2 to the power n plus 1 by 2. And here it is half to the power n plus 1 by 2. So, these cancels.

Therefore, we can see that it is coming out to be gamma n plus 1 by 2 upon gamma n by 2 and root over pi into root over n into 1 upon 1 plus z square by n whole to the power n plus 1 by 2. So, let me write it again.

(Refer Slide Time: 45:46)

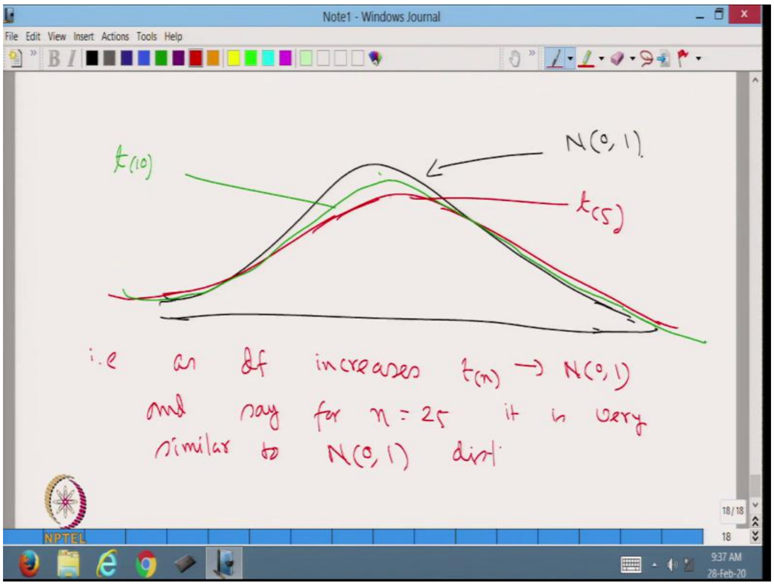




So, gamma n plus 1 by 2 upon gamma n by 2 root over pi root over n. Now, we know that root over pi is equal to gamma half multiplied by 1 upon 1 plus z square by n to the power n plus 1 by 2. Is equal to gamma n plus 1 by 2 upon gamma n by 2 gamma half square root of n into 1 upon 1 plus z square by n to the power n plus 1 by 2. Is equal to 1 upon beta n by 2, comma half root over n 1 upon 1 plus z square by n whole to the power n plus 1 by 2.

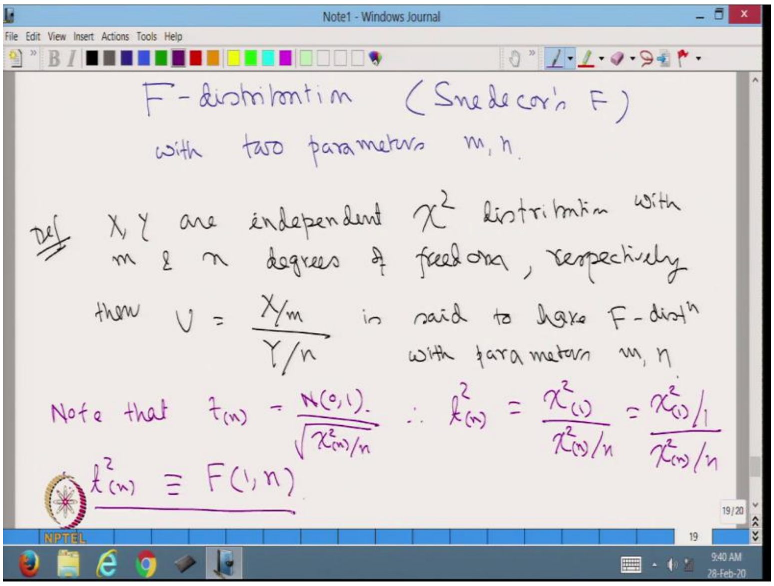
Note that minus infinity less than z less than infinity. Because normal 0, 1 takes values in minus infinity to plus infinity. And root over chi square n by n is positive. So, that is the density function for t with n degrees of freedom. Note that one, it is symmetric around 0 and two is therefore, its expected values is 0. One can compute the other moments from the first definition, but let me explain one diagram.

(Refer Slide Time: 48:25)



Suppose this is a normal 0, 1 diagram. Normal 0, 1 density. Then, t will look slightly flatter than that and as the degrees of freedom increases, it will go like this. Therefore, this is say, t with say, 10 degrees of freedom. But suppose, this is say, t with 5 degrees of freedom, that is, as df increases t n converges toward normal 0, 1 and say, for n is equal to 25, it is very similar to standard normal distribution.

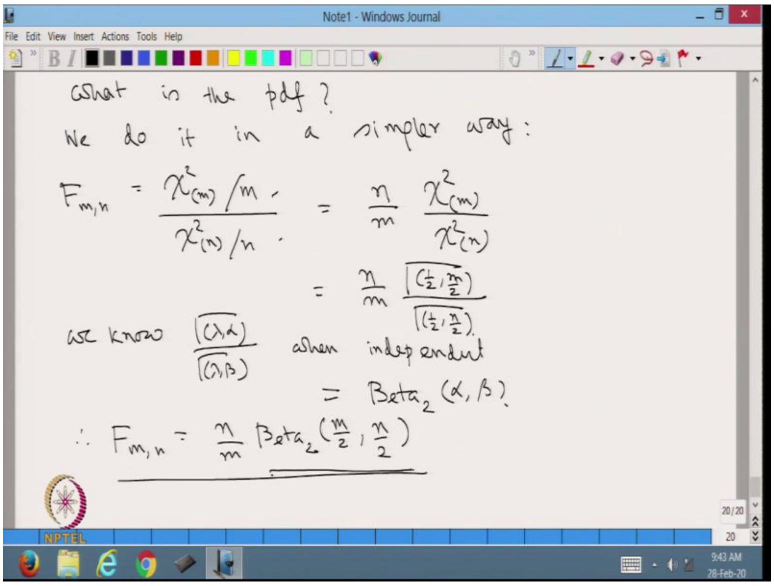
(Refer Slide Time: 50:03)



So, let us conclude the talk with one more distribution namely F - distribution which is called Snedecor's F with two parameters m, comma n. So, definition, if X and Y are independent, chi square distribution with m and n degrees of freedom respectively, then U is equal to X by m upon Y by n is said to have F - distribution with parameters m and n. Note that t n is equal to normal 0, 1 upon square root of chi square n by n.

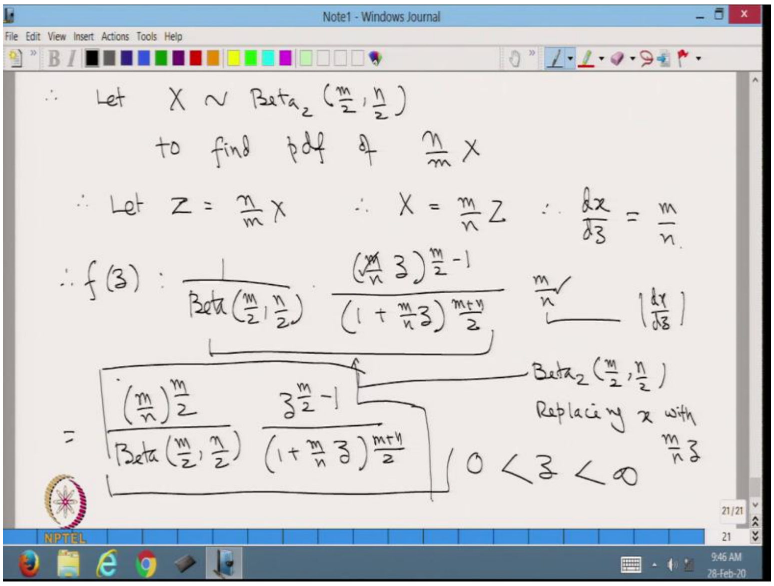
Therefore, t square of n is equal to chi square 1 because we know normal 0, 1 square is equal to chi square 1 divided by chi square n upon n is equal to chi square 1 by 1 divided by chi square n by n. Therefore, t square n is same as F with 1, comma n. So, that is the relationship between t and f.

(Refer Slide Time: 52:29)



So, what is the pdf? We do it as follows in a simpler way. So, F m, n is equal to chi square m by m upon chi square n by n is equal to n by m chi square m divided by chi square n is equal to n by m gamma with half, comma m by 2 upon gamma with half n by 2. Now, we know that gamma lambda comma alpha upon gamma lambda, comma beta when independent becomes beta 2 with alpha, comma beta. Therefore, F m, n is equal to n by m times beta 2 distribution with parameter m by 2 and n by 2. Thus, from 2 variables, we convert it into 1 variable.

(Refer Slide Time: 54:20)



Therefore, let X is distributed as beta 2 with m by 2, comma n by 2 to find pdf of n by m X. Therefore, let Z is equal to n by m X. Therefore, X is equal to m by n Z, therefore dx dz is equal to m by n. Therefore, f z from the theory of single variable function is 1 upon beta m by 2, comma n by 2 multiplied by m by n Z to the power m by 2 minus 1 upon 1 plus m by n Z to the power m plus n by 2 multiplied by m by n.

So, this is coming from beta 2 with m by 2, comma n by 2 replacing x with m by n Z, and this is coming from dx dz is equal to as you can understand it is going to be m by n to the power m by 2 because I am taking this term and this term upon beta m by 2, comma n by 2 z to the power m by 2 minus 1 upon 1 plus m by n Z to the power m plus n by 2, when 0 less than Z less than infinity. So, that is the pdf of F - m n distribution.

Again one can think of computing its expectation variance etc the way we have done in earlier cases. Okay friends, I stop here today. I hope you have understood how to compute the distribution of a function of a random variable when the distribution of the original random variable is known and over the last few classes, we have seen such development of pdf's from single variable and two variables. So, with that I stop on functions of random variable. From the next class, I shall start with a very interesting topic which is called ordered statistics. Okay then. Thank you.