Advanced Probability Theory
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Lecture-25

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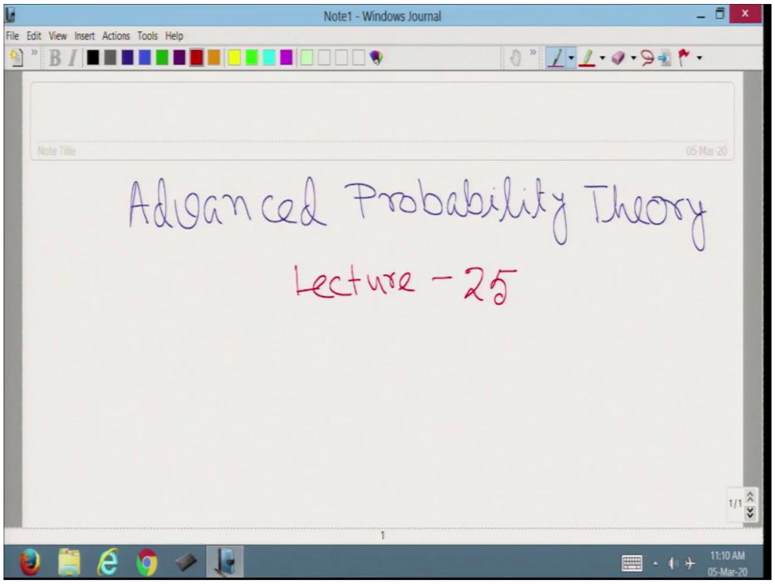

Welcome students to the MOOCs series of lectures on advanced probability theory. This is lecture number 25.

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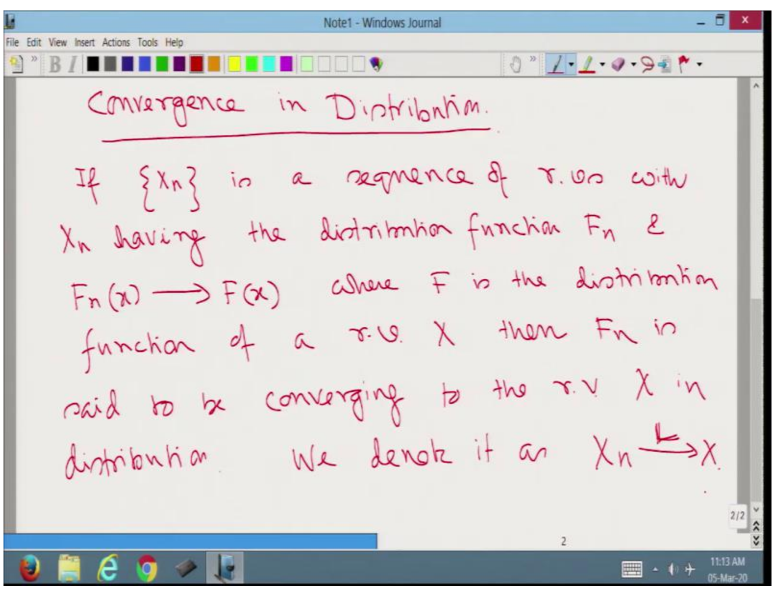

If you remember, we were discussing convergence in distribution. That is if xn is a sequence of random variables with xn having the distribution function Fn and Fn x converges to Fx, where F is the distribution function of a random variable x then Fn is said to be converging to the random variable x in distribution and we denote it as xn converges to x with this script L, which means convergence in law and that is how we typically denote convergence distribution.

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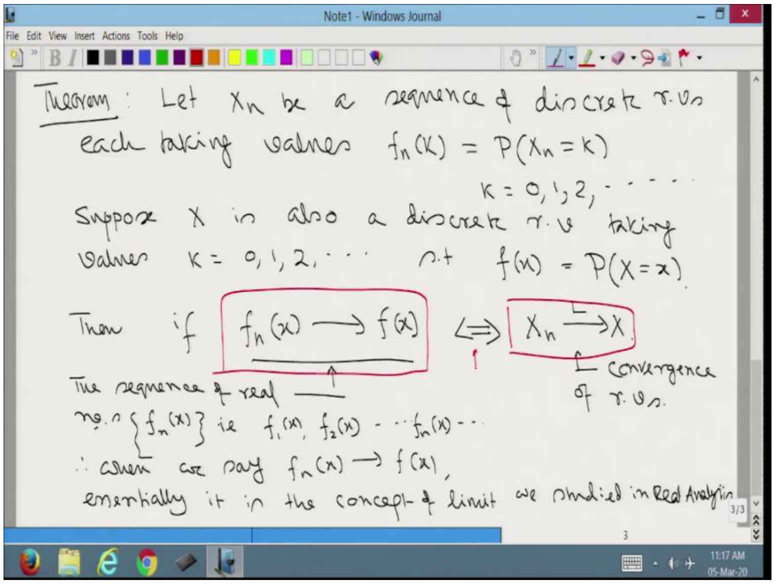

We are working out on the following theorem. Let xn be a sequnce of discrete random variables each taking values Fn k is equal to probability xn is equal to k, when k is equal to 0,1,2 up to infinity and suppose x is also a discrete random variable taking values, k is equal to 0,1,2 etc, such that F at x is equal to probability x is equal to x, then if, Fn x converges to Fx.

That implies and implied by xn convergence converges in law to the random variable x. Note that this is convergence of distribution, convergence of random variables, whereas, this is the sequence of reals Fn x that is F1x, F2x, Fnx this sequence and when we are saying Fnx converges to Fx. Essentially it is the concept of limit we studied in real analysis.

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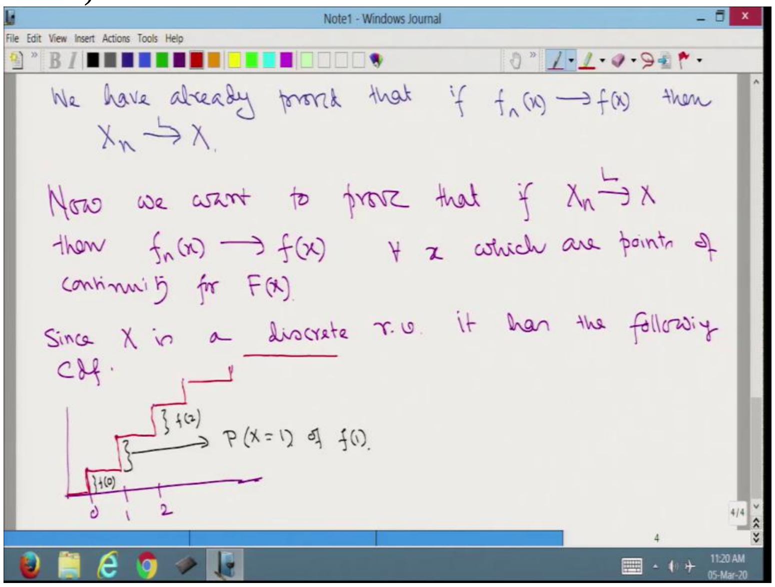

So, the distinction between this and this should be clear and this means if and only if we have already proved that if Fn x converges to Fx, then Xn converging in law or distribution to the random variable X. Now, we want to prove that if Xn converges to x in distribution then Fnx converges to Fx for all x which are points of continuity for Fx.

Now, since x is a discrete random variable it has the following Cdf 0,1,2 etc so the distribution function is going to be a step function something like this below 0, it is 0 at 0 it goes to some height, what is going to be that height? I am telling you it is coming to 1 at 1, it is also having a jump thus we are getting a step function of this type where this jump is probability x is equal to 1 or F1. Similarly, this is F2 this is F0 and ultimately it will be 1 this must be clear to you.

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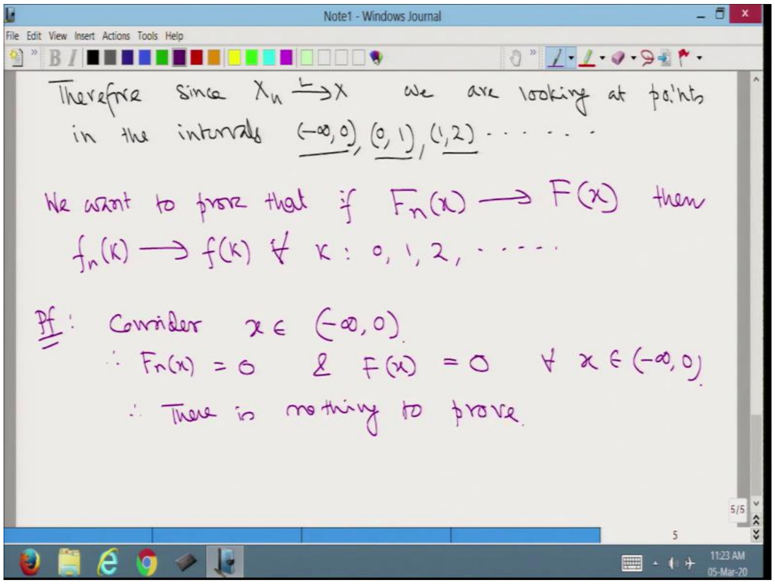

Therefore, since xn convergence, convergence in distribution to xl x, we are looking at points in the intervals minus infinity to 0 at 0 there is a jump, therefore, 0 to 1 open intervals at 1 there is a jump and therefore, we are looking at the this set of intervals or we are looking at points, which is in the union of these intervals.

We want to prove that if Fnx converges to Fx, then fnk converges to fk for all k in 0, 1, 2 etc proof. Consider x belonging to minus infinity to 0. Therefore, Fnx is equal to 0 and Fx is equal to 0 for all x belonging to minus infinity to 0. Therefore, there is nothing to prove.

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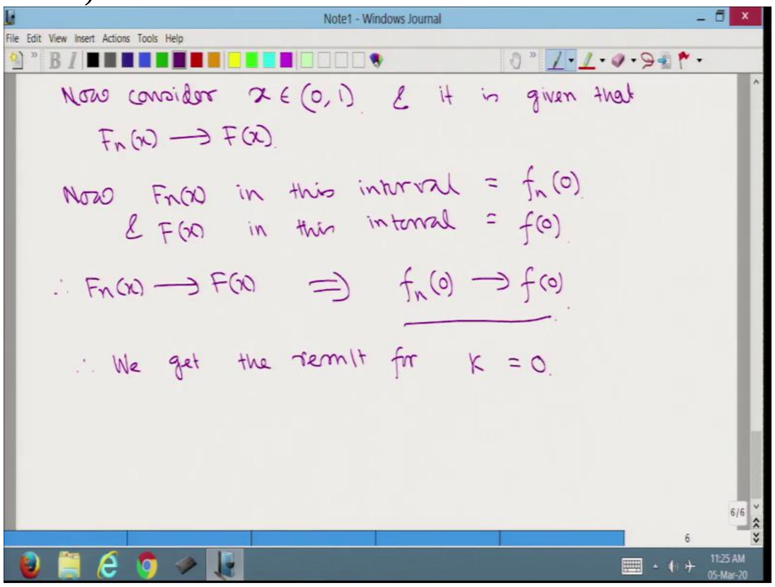

Now, consider x belonging to 0 comma 1. And it has given that Fnx converges to a Fx, now Fnx in this interval is equal to Fn0 and Fx in this interval is equal to F0. Therefore, Fnx converging to Fx implies Fn0 converges to F at 0. Therefore, we get the result for K is equal to 0.

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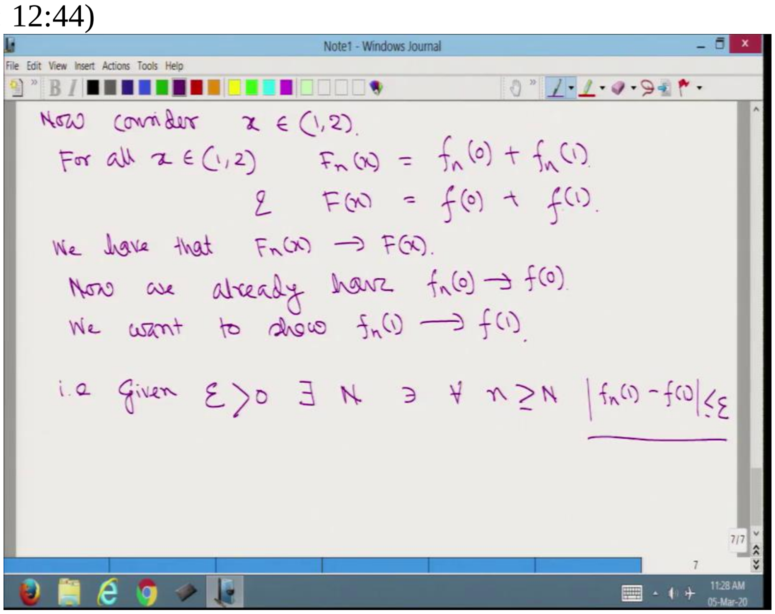

Now, consider x belonging to 1 comma 2 for all x belonging to 1 comma 2 Fnx is equal to fn0 plus fn1 and F of x is equal to small f0 plus small f 1, we have that Fnx converges to Fx. Now, we already have, fn0 converges to f0 we want to show f1 converges to f1 that is given epsilon greater than 0, there exists in such that for all n greater than equal to n modulus of fn1 minus f1 is less than equal to epsilon. This is what we need to prove.

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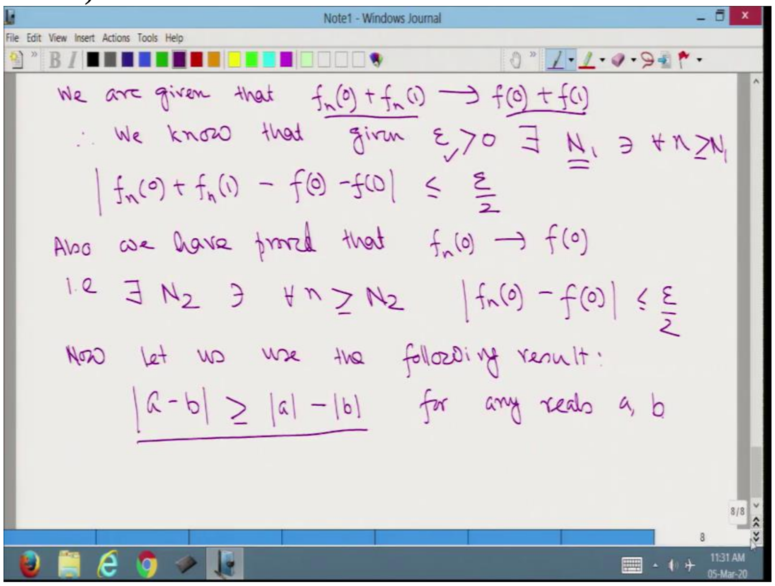

We are given that fn0 plus fn1 converges to f0 plus f1. Therefore, we know that given epsilon greater than 0, they are exist n1 such that for all n greater than equal to n1 modulus of Fn0 plus fn1 minus f0 minus f1 is less than equal to epsilon by 2, because this is converging to this given any epsilon, we can find such an n1 also we have proved that Fn0 converges to F0 that is there exist n2 for the same epsilon such that for all n greater than equal to n2 modulus of fn0 minus f0 is less than equal to epsilon by 2.

Now, let us use the following result modulus of a minus b is greater than or equal to modulus of a minus modulus of b for any reals a comma b. This we know from our school level mathematics.

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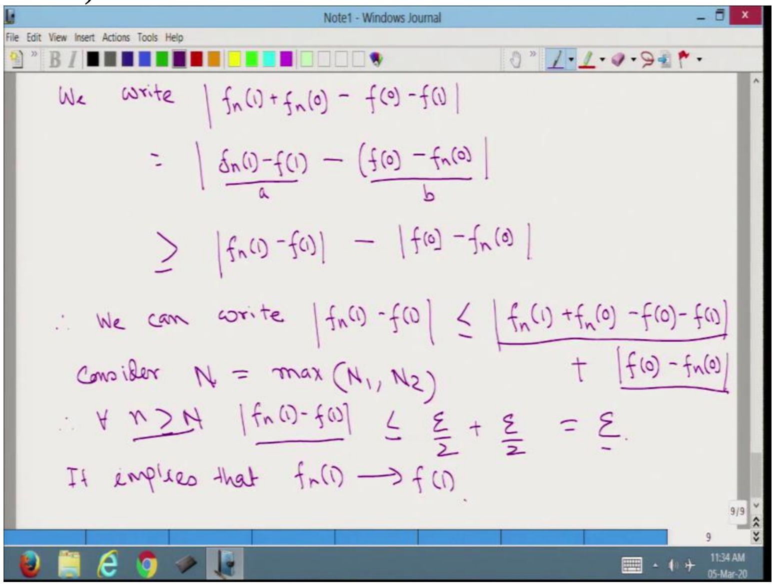

So, we write modules of fn1 plus fn0 minus f0 minus f1 is equal to modulus of fn 1 minus f1 minus f0 minus fn0. So, this is let us call a and this let us call b. Therefore, this is greater than equal to modulus of fn1 minus f1 minus modulus of f0 minus fn0 therefore, we can write, modulus of fn1 minus f1.

This quantity is less than or equal to, modulus of fn1 plus fn 0 minus f0, minus f1 plus modulus of f0 minus fn0, considered n is equal to maximum of n1 comma n2, where n1 and n2 we have just defined. Therefore, for all n greater than equal to n modules of Fn 1 minus F1 is less than equal to epsilon by 2 plus epsilon by 2, which is equal to epsilon.

Because we have already shown that this will be less than epsilon by 2 and this is going to be less than epsilon by 2. Therefore, together their sum is epsilon. Therefore, for all n greater than equal to n modules of Fn1 minus F1 is less than equal to epsilon, what does it mean? It implies that fn1 converges to f1.

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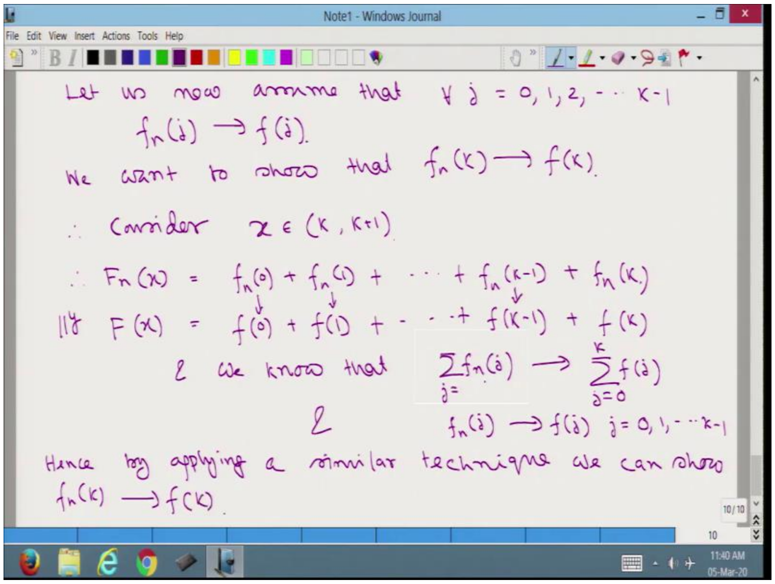

Let us now assume that for all j is equal to 0, 1, 2 up to K minus 1, fnj converges to fj we want to show that fnk converges to fk. Therefore, consider x belonging to K to K plus 1. Therefore, Fnx is equal to fn0 plus fn1 up to fnk minus 1 plus f and K. Similarly, fx is equal to f 0 plus f1 plus fk minus 1 plus fk.

And we know that fn0 converges to f0, fn1 converges to f1, and fnk minus 1 converges to fk minus 1 and we also know that sigma fnj, j is equal 0 to k converges to sigma j is equal to 0 to k, fj and fnj converges to fj for j is equal to 0, 1, 2 up to k minus 1 hence by applying a similar technique we can show fnk converges to fk. I leave this as an exercise, you just have to imitate what I did for 1 and in a similar way you can do it for k.

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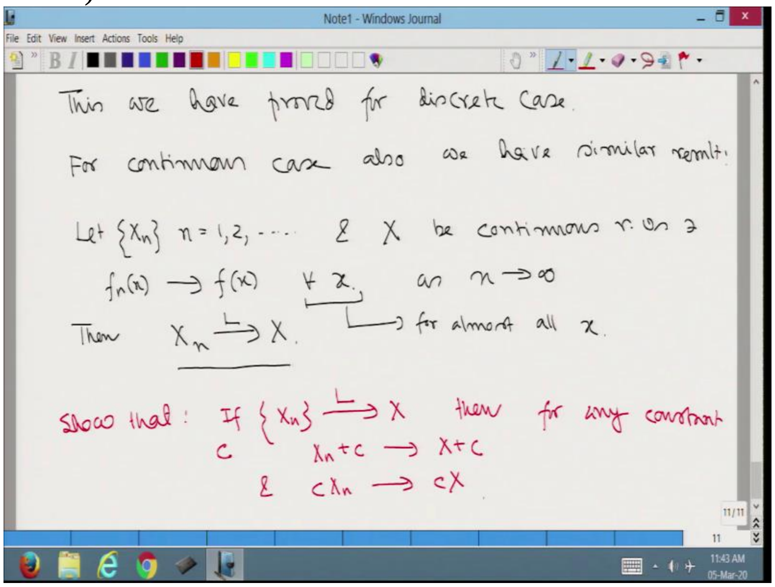

So, this we have proved for discrete case, for continuous case also we have a similar result which says that let xn n is equal to 1, 2, 3 etc and x be continuous random variables such that fnx converges to fx for all x as n goes to infinity, then xn converges in distribution to x. Actually, it is not for all x, it is actually for almost all x, but that concept still I have not given.

So, let us assume that this is true for all x. And in that case, we can show that the sequence xn of random variables will be converging to the random variable x, I am not going to give the proof because that needs more mathematical treatment, but we can accept that result, show that if xn is a sequence of random variables, which is converging in distribution to x then for any constant C, xn plus c converges to x plus c and c times xn converges to c times x. You can prove them very easily from the definition I leave that as exercise.

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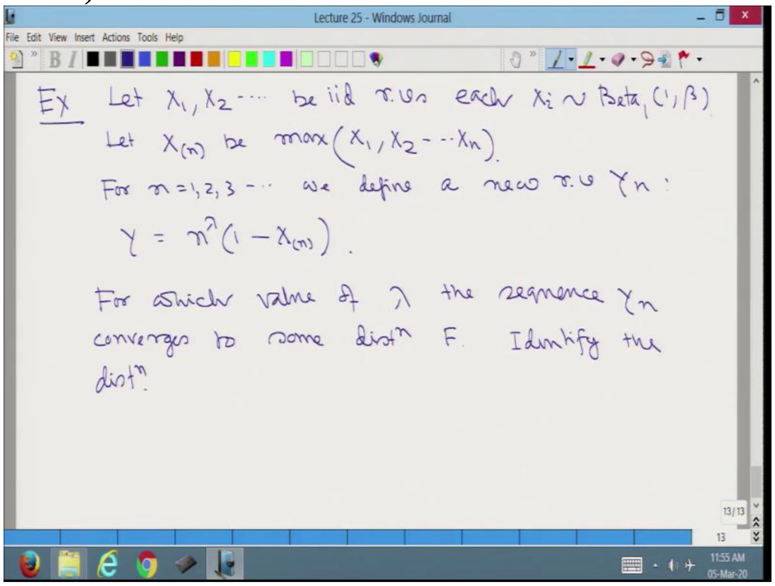

So, let me give you an example very interesting example to understand the convergence in distribution. So, let x1, x2, xn etc be iid random variables each xi distributed as beta 1 distribution with parameter 1 comma say beta let xn be maximum of x1, x2, xn for all n equal to 1, 2, 3 etc.

We define a new random variable yn as follows that yn is equal to n to the power lambda 1 minus xn. Question is for which value of lambda the sequence yn converges to some distribution f and identify the distribution.

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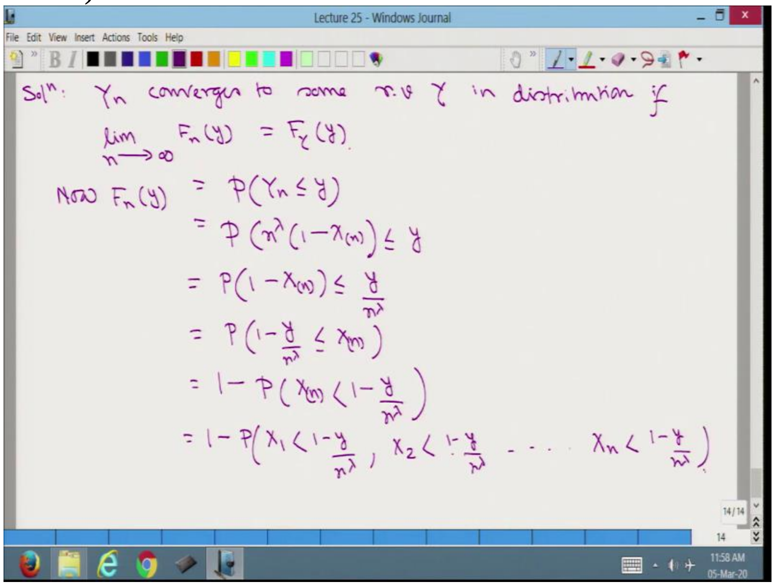

Solution yn converges to some random variable y in distribution if limit n going to infinity, fny is equal to fy of y now, fny is equal to probability yn less than equal to y is equal to probability n to the power lambda 1 minus xn less than equal to y is equal to probability 1 minus xn less than equal to y upon n to the power lambda which is is equal to probability 1 minus y upon n to the power lambda less than equal to xn by transposing the variables is equal to 1 minus probability xn less than 1 minus y upon n to the power lambda is equal to 1 minus probability x1 less than 1 minus y upon n to the power lambda x2 less than 1 minus y upon n to the power lambda etc up to xn less than 1 minus y upon n to the power lambda.

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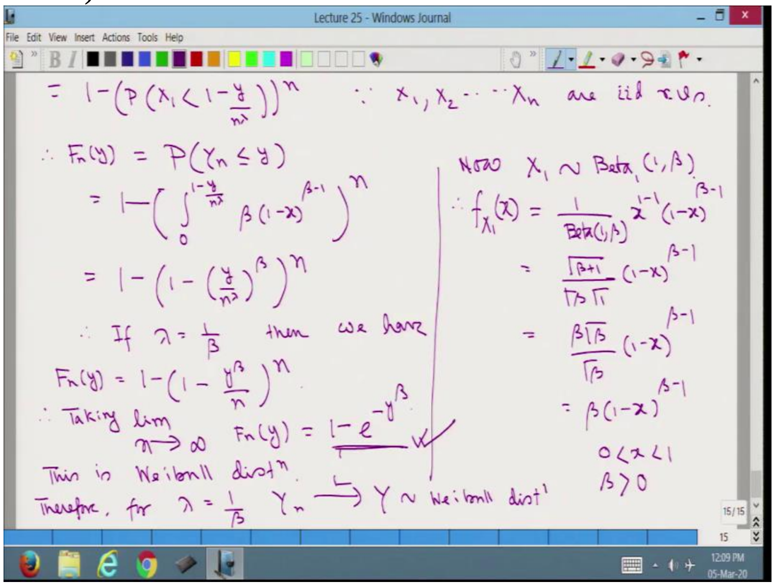

Which is equal to 1 minus probability x1 less than 1 minus y upon n to the power lambda whole to the power n since x1, x2 etc up to xn are iid random variables. Now, x1 is distributed as beta 1 with parameter 1 comma beta therefore fx1 at a point x is equal to 1 upon beta 1 comma beta x to the power 1 minus 1, 1 minus x to the power beta minus 1 is equal to gamma beta plus 1 upon gamma beta gamma 1, 1 minus x to the power beta minus 1 is equal to beta gamma beta upon gamma beta 1 minus x to the power beta minus 1 is equal to beta 1 minus x beta minus 1 for 0 less than x less than 1 and beta greater than 0.

Therefore, fny is equal to probability yn less than equal to y is equal to 1 minus integration of 0 to 1 minus y upon n to the power lambda beta 1 minus x whole to the power beta minus 1, this whole power n is equal to, I would like you to work out on this to, to check that 1 minus 1 minus y upon n to the power lambda whole to the power beta whole to the power n you please do the integration to come to this point.

Therefore, if lambda is equal to 1 upon beta then we have fny is equal to 1 minus 1 minus y to the power beta upon n whole to the power n therefore, taking limit n going to infinity, we have fny is equal to 1 minus e to the power minus y to the power beta. Can you recognize this distribution we have already seen that this is weibnll distribution. Therefore, for lambda is equal to 1 upon beta yn converges distribution to y which is weibnll distribution. Okay friends, this is about convergence in distribution, this is not a very strong convergence.

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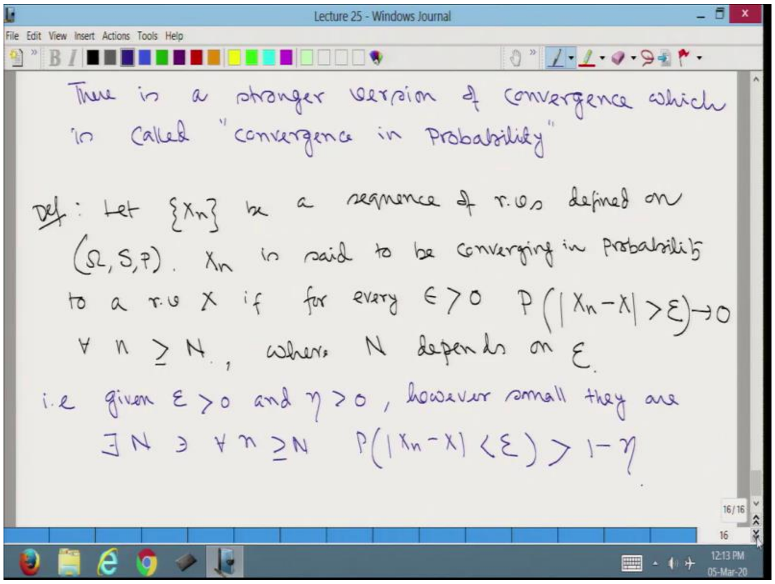

So, there is a stronger version of convergence which is called convergence in probability, let definition let Xn be a sequence of random variables defined on some probability space omega S and P. Xn is said to be converging in probability to a random variable X, if for every epsilon greater than 0 probability modules of Xn minus X greater than epsilon converges to 0.

For all n, greater than equal to some n where n depends on epsilon or in other words given epsilon greater than 0 and ita greater than 0, however small they are, they are exist in such that for all n greater than equal to n probability modulus of xn minus x less than epsilon is greater than 1 minus ita.

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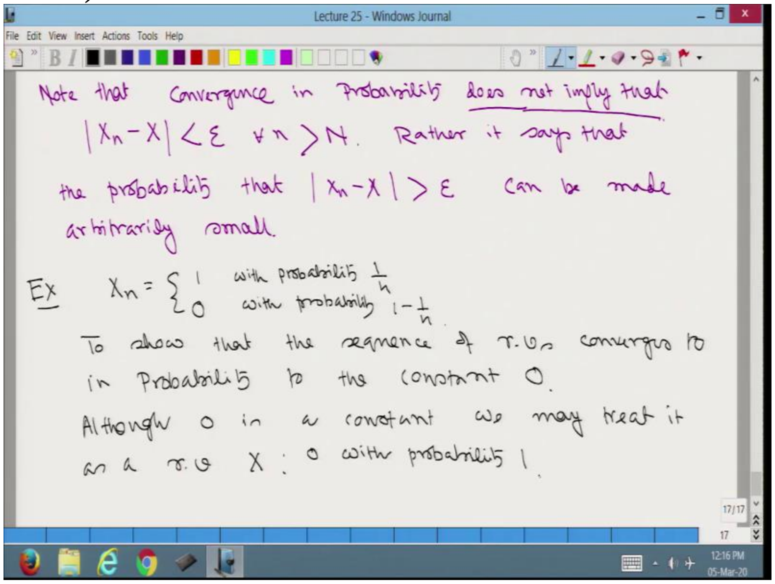

Note that convergence in probability does not imply that modulus of xn minus x is less than epsilon for all n greater than capital N. So, that is not the implication. Rather it says that the probability that modulus of xn minus x greater than epsilon can be made arbitrarily small. So, let me give you an example.

Let xn is equal to, it takes two values 1 with probability 1 by n and 0 with probability 1 minus 1 by n. To show that the sequence of random variables converges to in probability to the constant 0. So, all the 0 is a constant we may treat it as a random variable x, which takes values 0 with probability is equal to 1.

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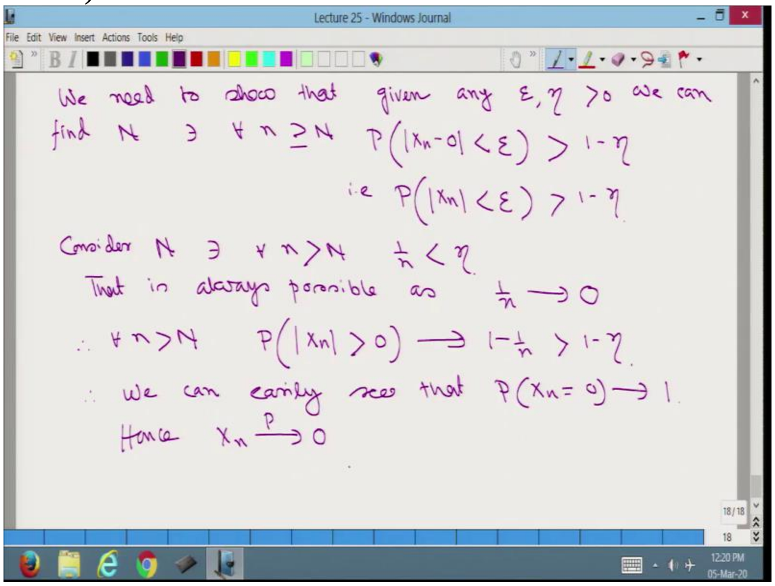

So, we need to show that given any epsilon an ita greater than 0, we can find n such that for all n greater than equal to n probability modules of xn minus 0 less than epsilon is greater than 1 minus u ita that is, probability modulus of xn less than epsilon is greater than 1 minus ita. Considers, capital N such that for all n greater than N 1 by n is less than ita, that is always possible as 1 by n goes to 0.

Therefore, for all n greater than in probability modulus of xn greater than 0 that probability goes to 1 minus 1 upon n which is greater than 1 minus ita. Therefore, we can easily see that probability xn is equal to 0 converges to 1 hence xn convergence in, convergence in probability to 0.

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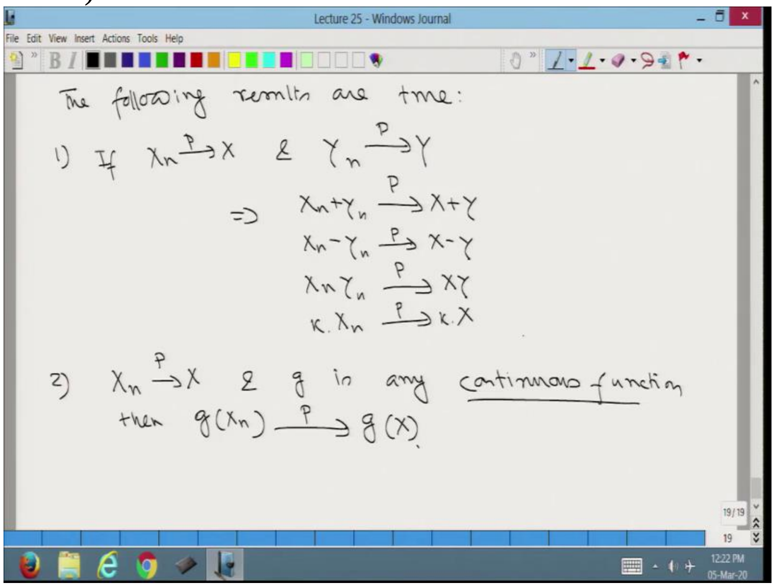

The following results are true if xn convergence in probability to X and Yn convergence in probability to y implies that Xn plus Yn convergence in probability to X plus Y, Xn minus Yn converges in probability to X minus Y, Xn Yn converges to Xy in probability k times Xn converges to k times x in probability and also if xn converges in probability to x and g is any continuous function then g of xn converges in probability to g of x. This comes from the definition of continuity and I like you to prove this results as an exercise.

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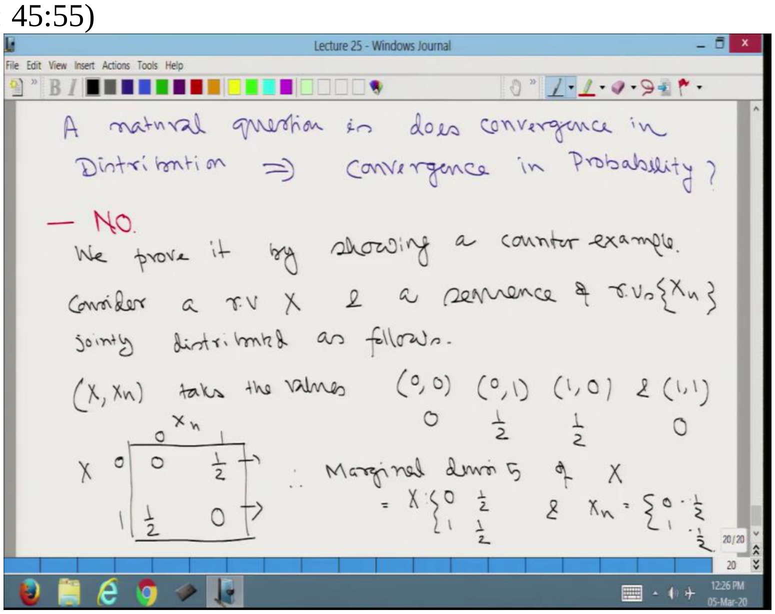

A natural question is does convergence in distribution imply convergence in probability? The answer is no and we give you a counter example. Consider a random variable x and a sequence of random variables xn jointly distributed as follows x comma xn takes the values 00,01,10 and 11 with probabilities 0, half half and 0 say something like this, xn takes the value, 0 and 1, and x takes the value 0 and 1.

So, that this has probability 0, this has probability half, this has probability half and this is 0. Therefore, marginal density of x is equal to 0 with probability half and 1 with probability half by adding these rows and for xn also it is 0 with half and 1 with half.

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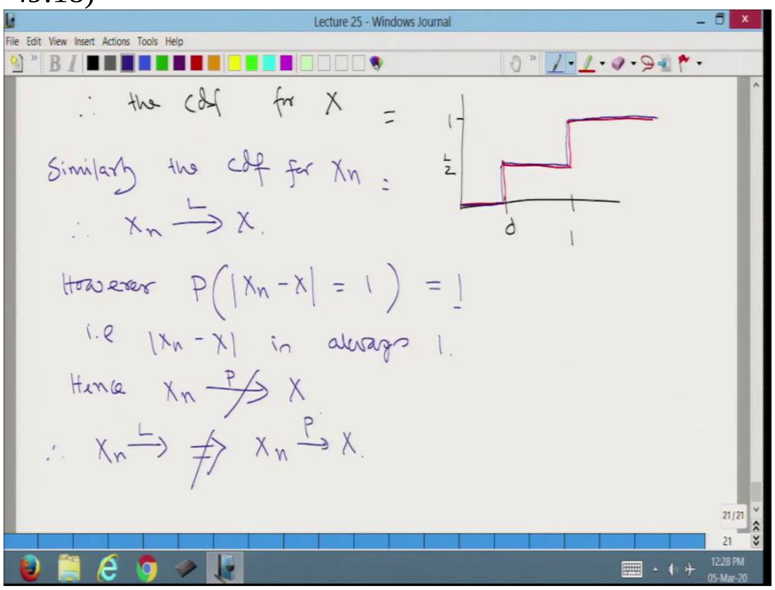

Therefore, the cumulative distribution function for x is equal to, when this is the value 1 and this is the value half and similarly the cdf for xn is also the same it takes that values 0 before 0 at 0 there is a jump at half then it continues like that till 1 then there is a jump and it continues with the value 1 till infinity.

Therefore, xn converges in distribution to x. However probability modules of xn minus x is equal to 1 is equal to 1 that is modules of xn minus x is always 1 hence xn does not converge in probability to x. Therefore, convergence in distribution does not imply convergence in probability.

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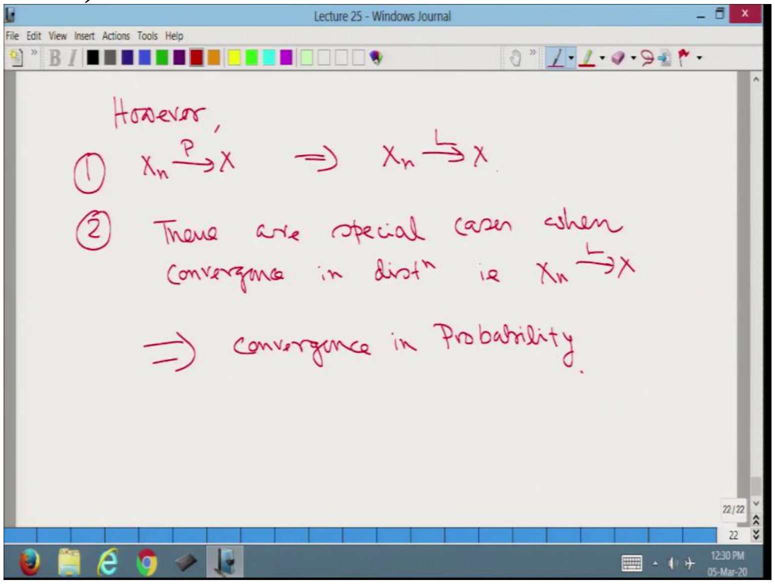

However, there are two things to know one is that convergence in probability implies convergence in distribution and two there are special cases when convergence in distribution that is xn converging to x in distribution implies convergence in probability. Okay friends, I stop here today, in the next class I shall prove this result and all also, I shall introduce you to some stronger versions or stronger modes of convergence for random variables. Okay friends thank you so much.