Advanced Probability Theory
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Lecture 19

Welcome students to the MOOCS lecture series on Advanced Probability Theory, this is lecture number 19.

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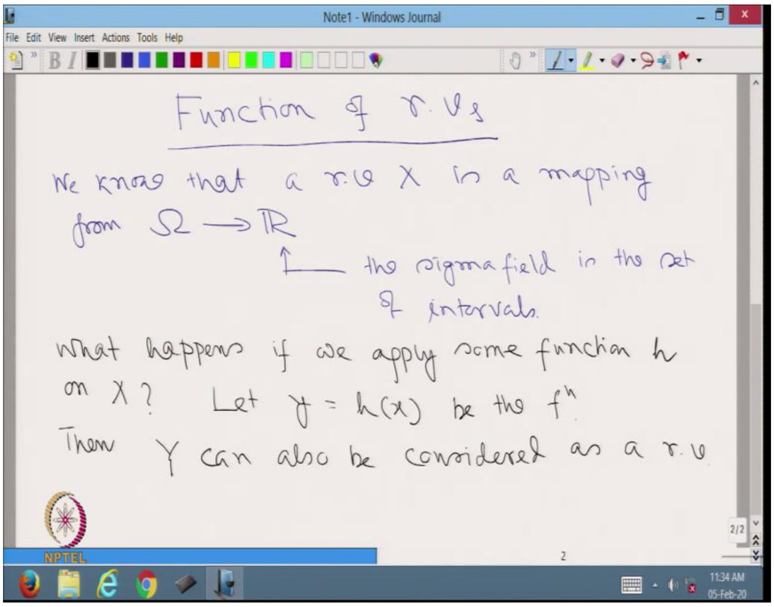

As I said at the end of the last class that from today. I shall start function of random variables. We know that a random variable X is a mapping from omega to real line on which the sigma field is the set of intervals. Now, what happens if, we apply some function h on X? Let, y is equal to hx be the function. Then Y can also be considered as a random variable.

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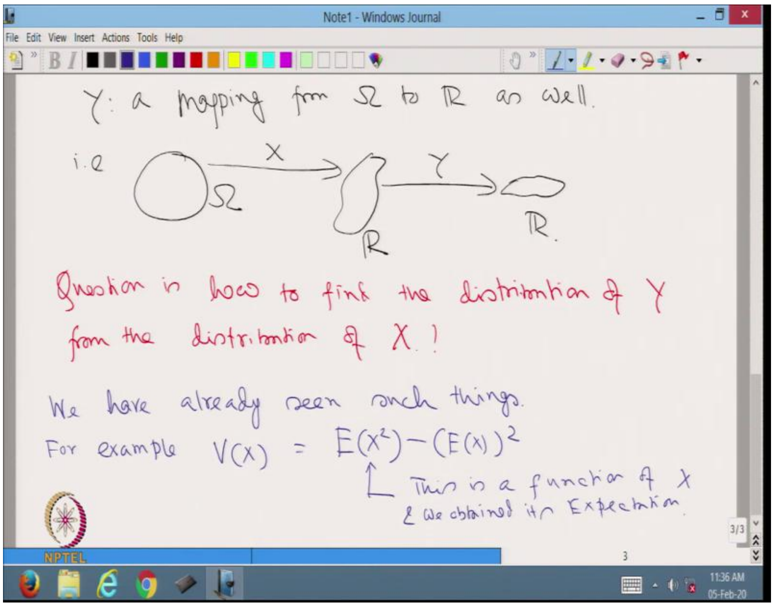

Because, Y is a mapping from omega to R as well. That is, we have omega from there we get X which maps to R and from there we have the mapping Y that is also on R. Question is, how to find the distribution of Y from the distribution of X? That is the question.

We have already seen such things. For example, variance of X is equal to the expected value of X square minus expected value of X whole square. Now, obviously this is a function of x and we obtained its expectation. This type of problem we have already seen where from that distribution of X we are trying to find the expectation of X square or we can think of getting the PDF of x square that we have also seen.

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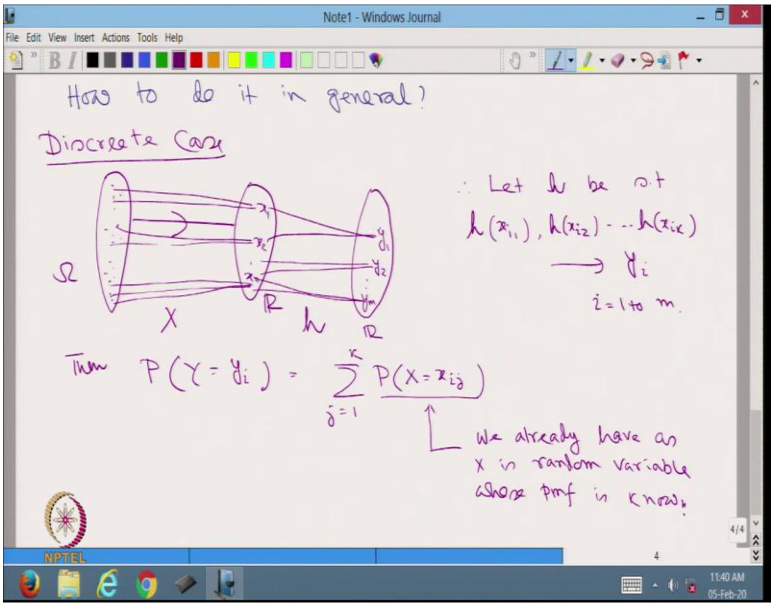

How to do it in general? So, consider discrete case. Suppose, this is omega and X is such that all these points are mapping to say x1 some other set of omegas are mapping to x2 and some other set of points are mapping to xn and that is how x behaves, what we are saying, we are now, looking at the mapping of Y and some points from here are mapping to say y1, some other points are mapping to say y2 and some other points say mapping to ym.

And suppose, this is the function h. Therefore, let h be such that h of xi1, comma h of xi2, h of xik they all map to yi, i is equal to 1 to m. Then, probability Y is equal to yi is equal to sigma, probability X is equal to xij, j is equal to 1 to k and do we know that, these probabilities we already have as X is a random variable whose Pmf is known.

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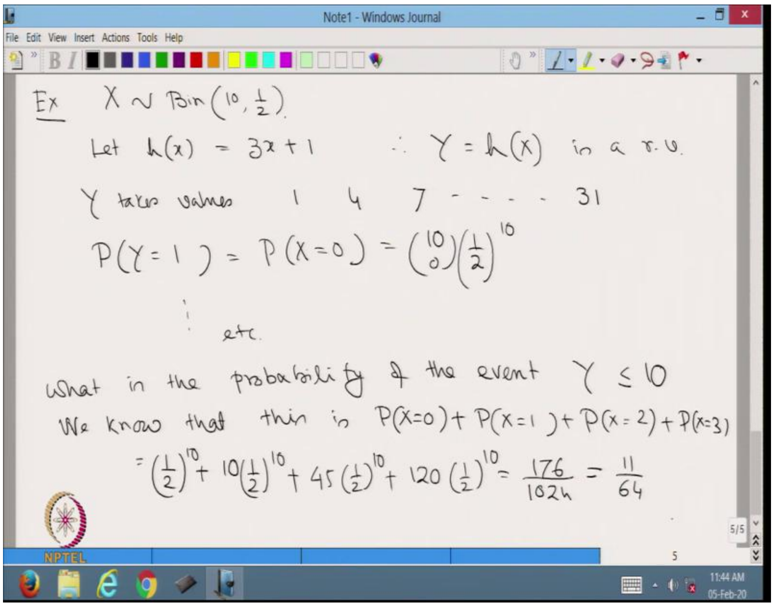

So, let us give an example, consider X to binomial 10, comma half. Let h of x is equal to 3x plus 1 therefore, Y is equal to hx is a random variable where Y takes values 1 4 7 up to 31 with the probabilities that we can understand that probability Y is equal to 1 is equal to probability X is equal to 0 is equal to 10c0 half to the power 10 etc. So, what is the probability of the event Y less than equal to 10?

We know that this is probability X is equal to 0 plus probability X is equal to 1 plus probability X is equal to 2 plus probability X is equal to 3 if x takes value beyond 3 then Y will be greater than equal to 10 is equal to half to the power 10 plus 10 half to the power 10 plus 10c2 which is going to be 45 half to the power 10 plus 10c3 which is going to be 120 into half to the power 10 is equal to 176 upon 1024 which is equal to 11 upon 64.

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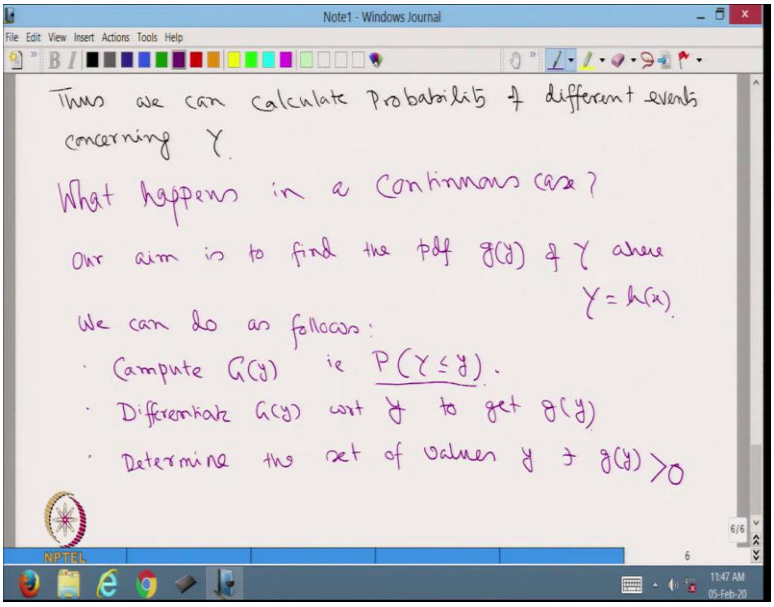

Thus, we can calculate probability of different events concerning Y. What happens in a continuous case? So, our aim is to find the pdf gy of Y where Y is equal to h of x, one way of doing it is first compute G of y that is probability Y less than equal to y. Differentiate Gy with respect to y to get gy. Determine the set of values of y such that, gy is greater than 0. But, the main question is how to get probability capital Y is less than equal to small y?

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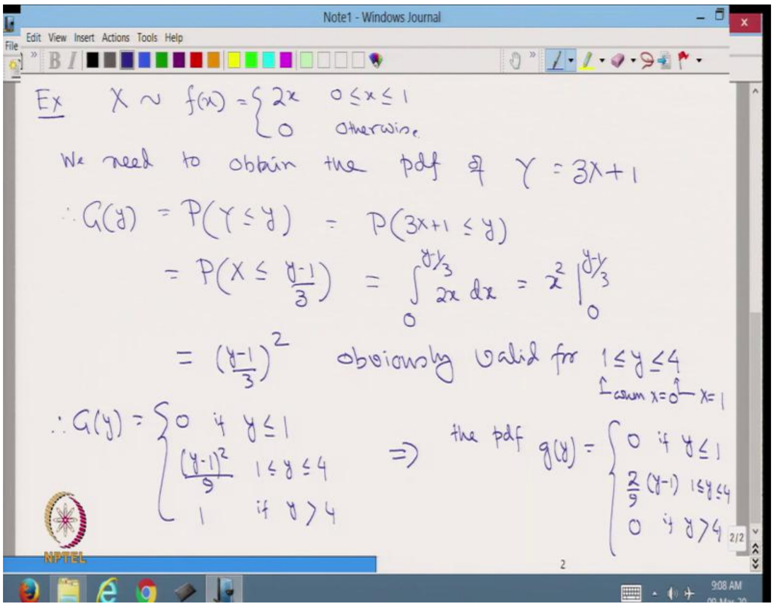

Let us start with an example, X is distributed as follows that f of x is equal to 2x when 0 less than equal to x less than equal to 1 and it is 0 otherwise. Therefore, we need to obtain the pdf of Y is equal to 3x plus 1. Therefore, G of y is equal to probability Y less than equal to small y is equal to probability 3x plus 1 less than equal to small y is equal to probability X less than equal to y minus 1 upon 3 is equal to integration 0 to y minus 1 by 3 2x dx is equal to x square from 0 to y minus 1 upon 3 is equal to y minus 1 upon 3 whole square, obviously valid for 1 less than equal to y less than equal to 4.

So, y takes the value 1 when X is equal to 0 and it takes the value 4 when X is equal to 1. Therefore, G of y is equal to 0 if y less than equal to 1, y minus 1 whole square by 9 if 1 less than equal to y less than equal to 4 and is equal to 1 if y greater than 4, implies that the pdf gy is going to be 0 if y less than equal to 1 is equal to 2 by 9 y minus 1 for 1 less than equal to y less than equal to 4 and is equal to 0 if y greater than 4.

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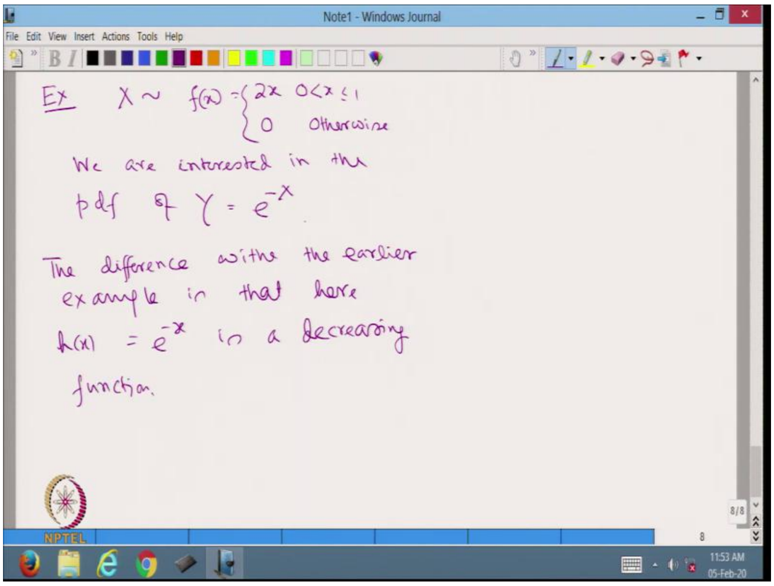

Let us consider another example, the same X which is fx is equal to 2x for 0 less than x less than equal to 1 and it is 0 otherwise. And we are interested in the pdf of Y is equal to e to the power minus X. The difference with the earlier example is that here, hx is equal to e to the power minus x is a decreasing function.

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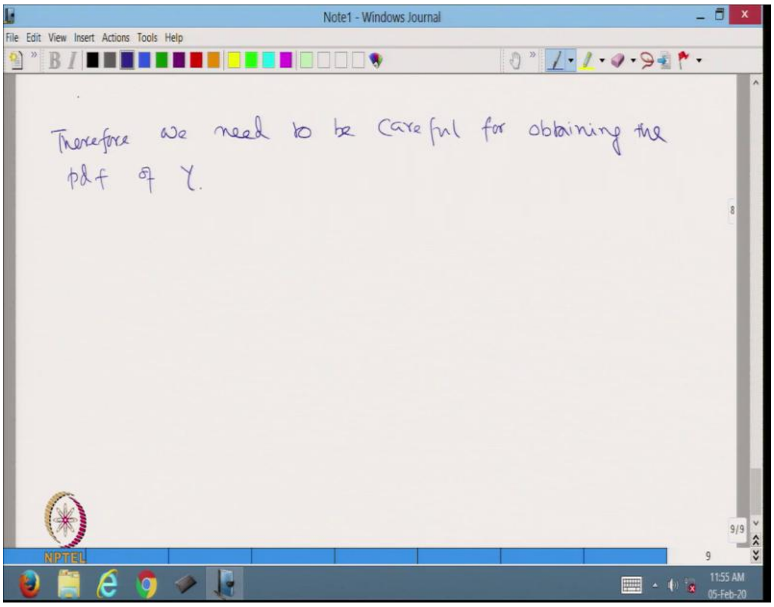

Therefore, we need to be careful for obtaining the pdf of Y.

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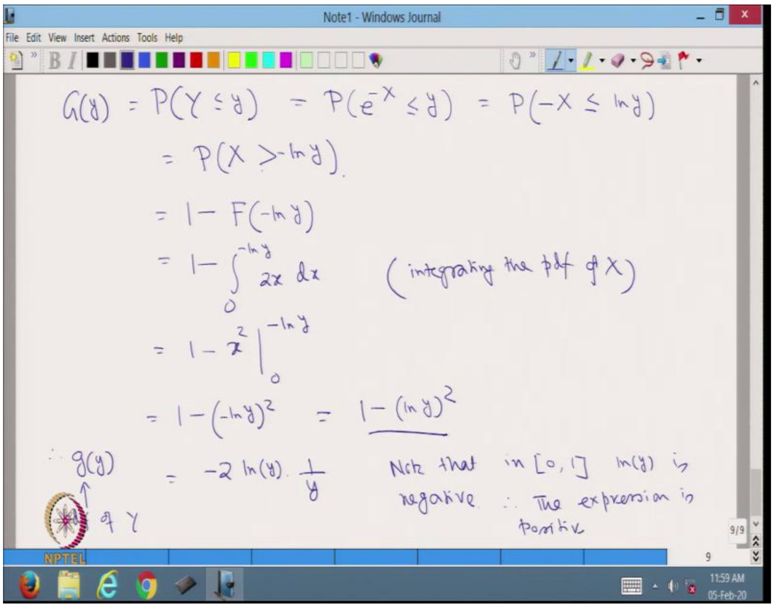

So, G of y is equal to probability Y less than equal to y is equal to probability e to the power minus X less than equal to y is equal to probability minus x less than equal to log y is equal to probability X greater than minus log y is equal to 1 minus F of minus log y is equal to 1 minus integration 0 to minus log y of 2x dx this is the pdf of X is equal to 1 minus x square from 0 to minus log of y is equal to 1 minus, minus log y whole square is equal to 1 minus log y square. Therefore, g of y this is the pdf of Y is equal to minus 2 log y multiplied by 1 upon y. Note that, in 0 to 1 log y is negative. Therefore, the expression is positive.

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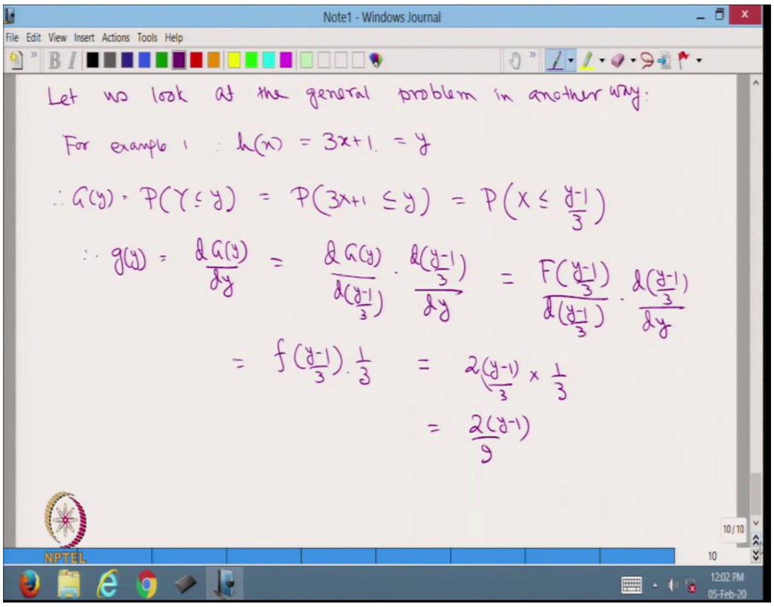

Let us look at the general problem in another way. Say, for example 1, h of x is equal to 3x plus 1 is equal to y. Therefore, Gy is equal to probability y less than equal to Y is equal to probability 3x plus 1 less than equal to y is equal to probability X less than equal to y minus 1 upon 3.

Therefore, Gy is equal to dGy dy is equal to dGy dy minus 1 upon 3 multiplied by dy minus 1 upon 3 dy which is is equal to F at y minus 1 upon 3 d y minus 1 upon 3 into d y minus 1 upon 3 dy is equal to F at y minus 1 upon 3 multiplied by 1 upon 3 is equal to 2 into y minus 1 upon 3 multiplied 1 upon 3 is equal to 2 into y minus 1 upon 9.

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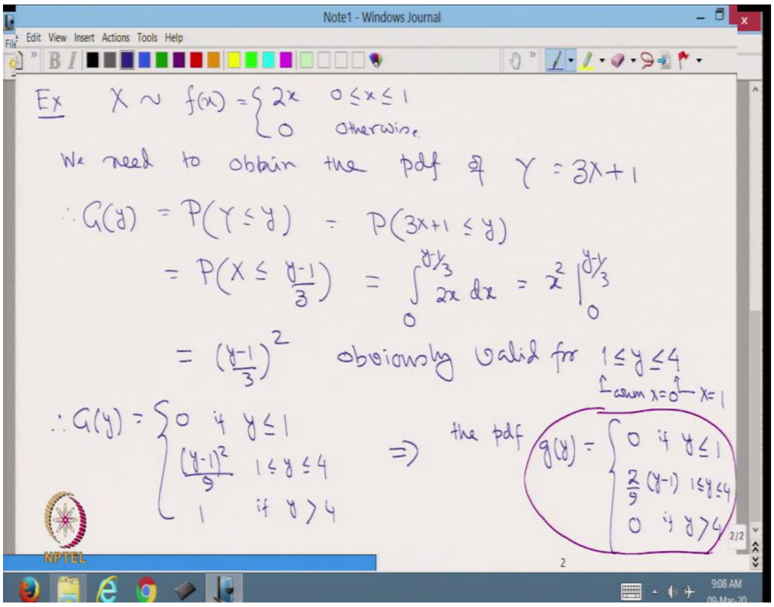

So, let us see, what we have got all your and we find that we have got the same pdf there as well.

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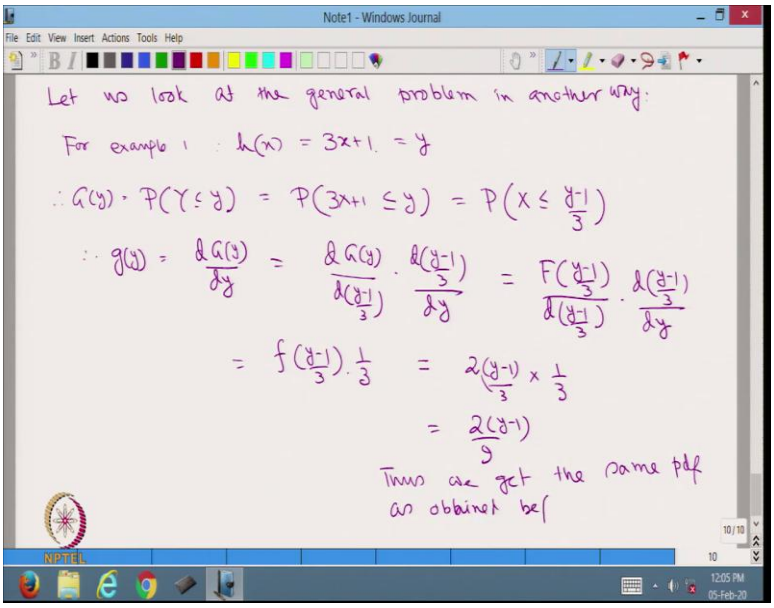

Thus, we get the same pdf as obtained before.

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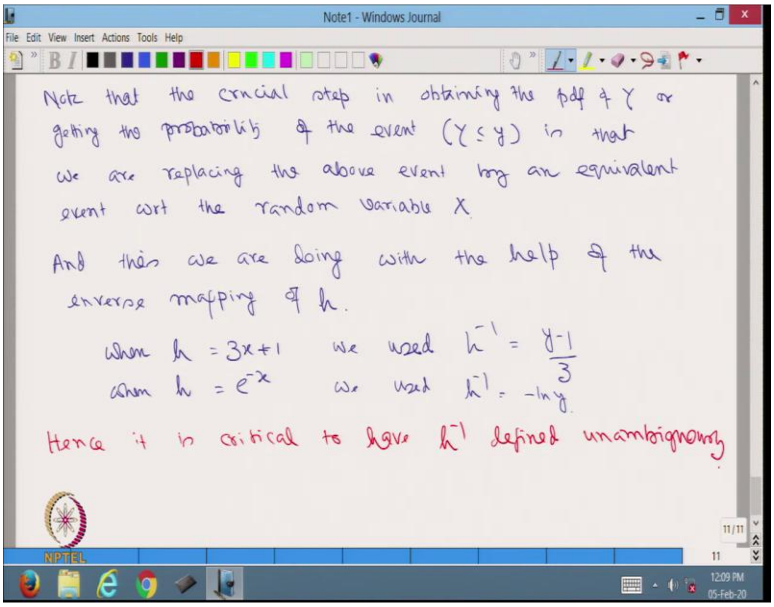

Note that, the crucial step in obtaining the pdf of Y or getting the probability of the event Y less than equal to y is that we are replacing the above event by an equivalent event with respect to the random variable X. And this we are doing with the help of the inverse mapping of h. So, when h is equal to 3x plus 1 we used h inverse is equal to y minus 1 upon 3, when h is equal to e to the power minus x we used h inverse is equal to minus log of y. Hence, it is critical to have h inverse defined unambiguously.

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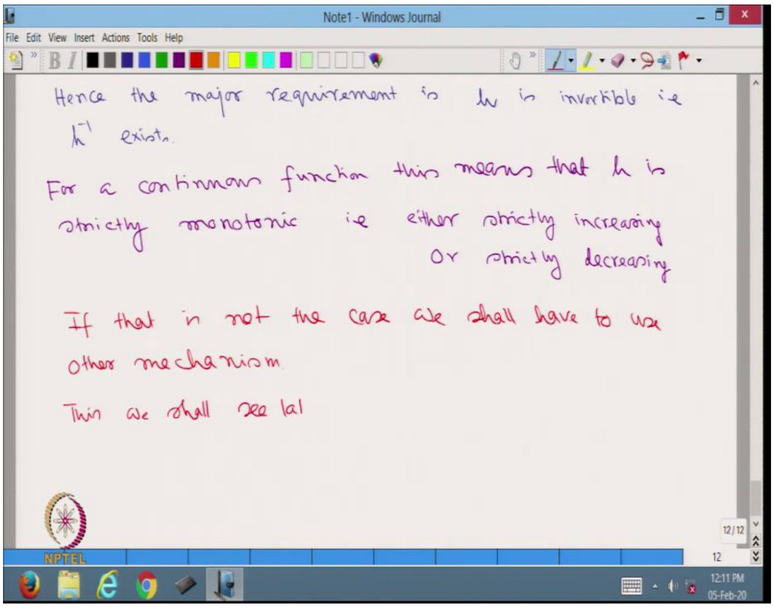

Hence, the measures requirement is h is invertible that is h inverse exists for a continuous function, this means that h is strictly monotonic that is either strictly increasing or strictly decreasing. If that is not the case we shall have to use other mechanism as we will see later.

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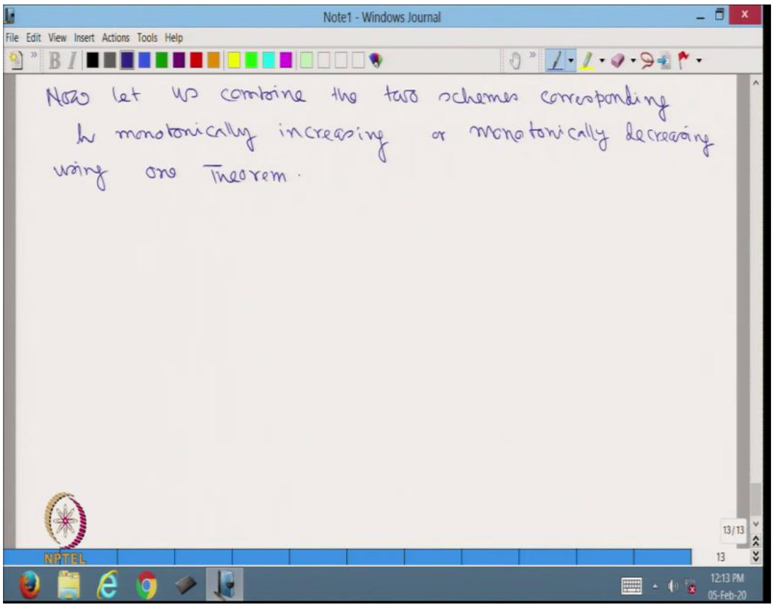

Now, let us combine the two schemes corresponding to h monotonically increasing or monotonically decreasing using one theorem.

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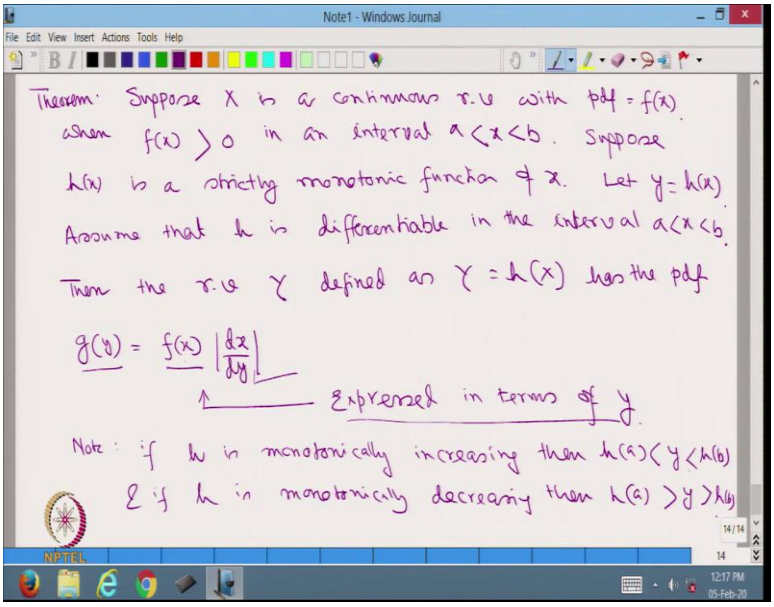

Theorem, suppose X is a continuous random variable with pdf equal to f of x when, f of x is greater than 0 in an interval a less then x less then b. Suppose, hx is a strictly monotonic function of x. Let, y is equal to h of x. Assume that h is differentiable in the interval a less than x less than b. Then the random variable y defined as Y is equal to h of x has the pdf, g of y is equal to fx multiplied by dx dy absolute value of that one the whole thing expressed in terms of y. Note, if h is monotonically increasing then h a less than y less than hb and if, h is monotonically decreasing then ha is greater than y greater than hb.

So, that is the theorem we are obtaining the pdf of Y by looking at the pdf of X but, this we are expressing in terms of Y that means, we are using h inverse here and we are looking at the derivative of X with respect to Y and that also is express to Y.

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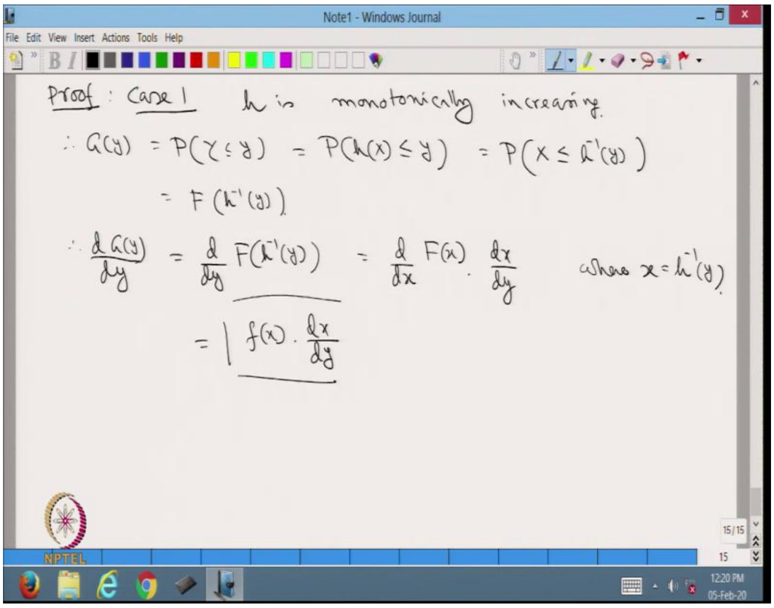

Proof, case 1 h is strictly increasing or monotonically increasing. Therefore, G of y is equal to probability Y less than equal to y is equal to probability h of X less than equal to y is equal to probability X less than equal to h inverse y is equal to F at h inverse y therefore, dGy dy is equal to d dy of f of h inverse y is equal to d dx of Fx multiplied by dx dy where x is equal to h inverse y is equal to f x into dx upon dy.

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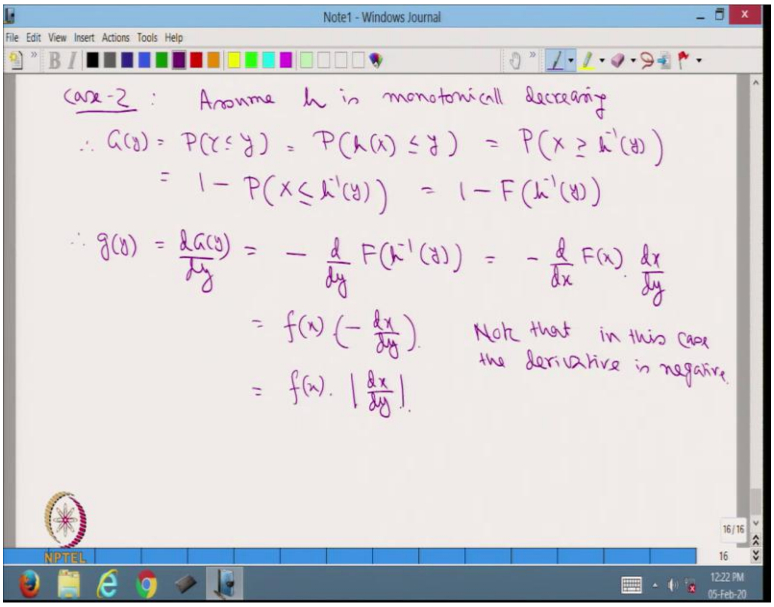

Case 2, h is monotonically decreasing. Therefore, G of y is equal to probability Y less than equal to y is equal to probability h of X less than equal to y is equal to probability X greater than equal to h inverse y is equal to 1 minus probability X less than H inverse y in a continuous case we can write with equality also is equal to 1 minus F of h inverse y.

Therefore, g of y is equal to dGy dy is equal to minus d dy of F of h inverse y is equal to minus d dx of Fx into dx dy is equal to fx into minus dx dy. Note that, in this case the derivative is negative. Therefore, we can write it as a fx times modulus of dx dy.

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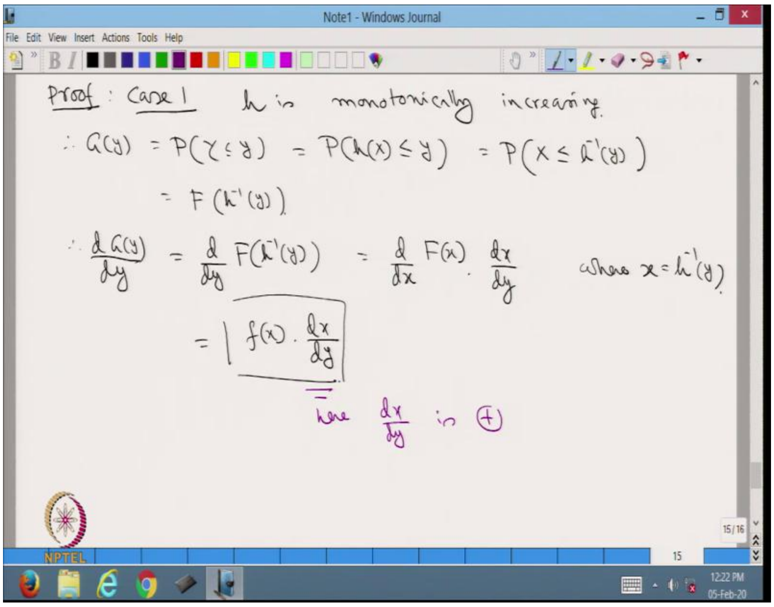

Note, in the earlier case here dx dy is positive.

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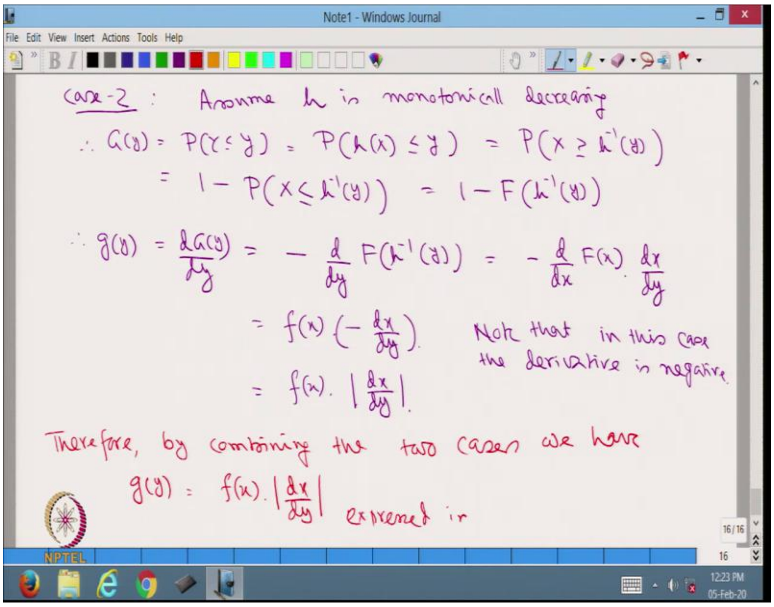

Therefore, by combining the two cases we have g of y is equal to fx modulus of dx dy expressed in y.

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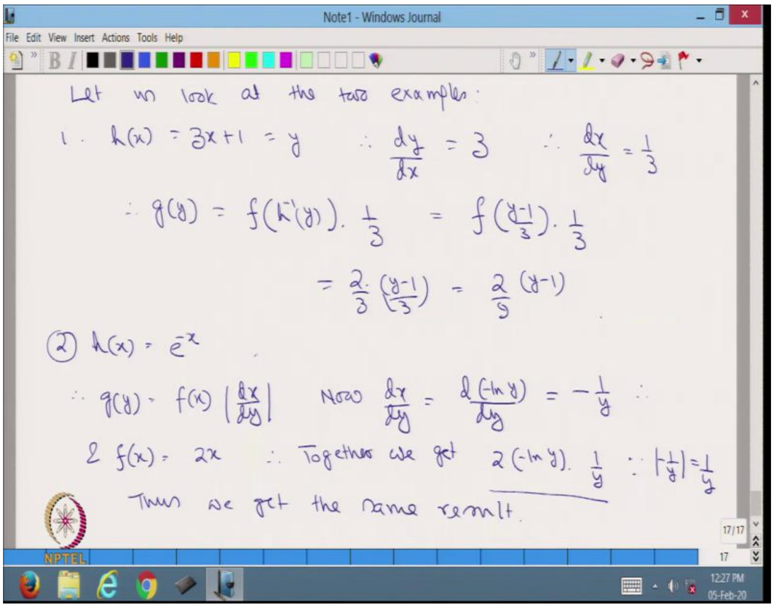

So, let us look at the two examples. 1, h of x is equal to 3x plus 1 is equal to y. Therefore, dy dx is equal to 3, therefore, dx dy is equal to 1 upon 3, therefore, g of y is equal to f at h inverse y into 1 by 3 is equal to f at y minus 1 upon 3 into 1 by 3 is equal to 2 by 3 into y minus 1 upon 3 is equal to 2 into y minus 1 by 9.

In a similar way case 2, h of x is equal to e to the power minus x g of y is equal to fx into modulus of dx dy. Now, dx dy is equal to d minus log y dy is equal to minus 1 upon y and f of x is equal to 2 x therefore, together we get 2 into minus log value into 1 upon y because modulus of minus 1 upon y is equal to 1 upon y. Thus, we get the same result.

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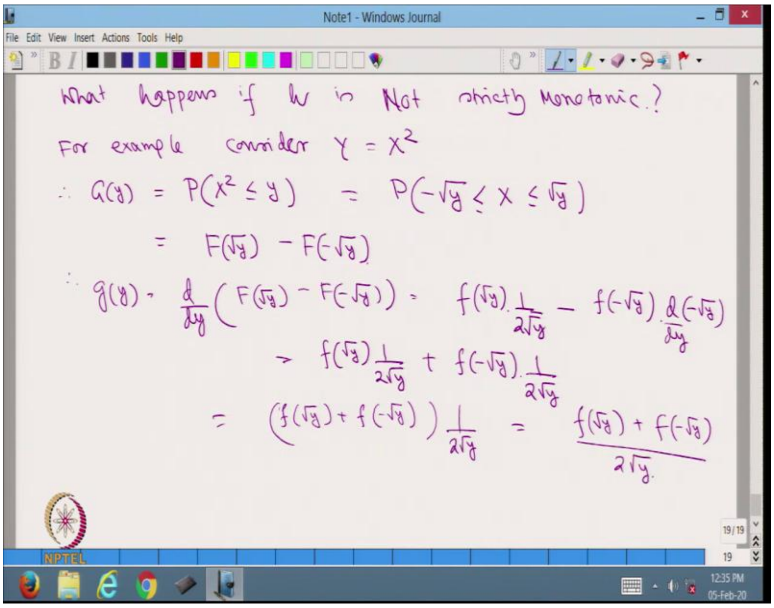

What happens if h is not strictly monotonic? That is the question. For example, consider Y is equal to X square. Therefore, G of y is equal to probability X square less than equal to y is equal to probability minus root to y less than equal to X less than equal to root y is equal to F of root y minus F of minus root y.

Therefore, g of y is equal to d dy of F of root y minus F of minus root y is equal to f root y multiplied by half root y minus f at minus root y into d minus root y dy is equal to f of root y 1 upon 2 root y this minus minus makes it plus f at minus root y upon 1 upon 2 root y is equal to f of root y plus f of minus root y into 1 upon 2 root y is equal to f of root y plus F of minus root y upon 2 root y.

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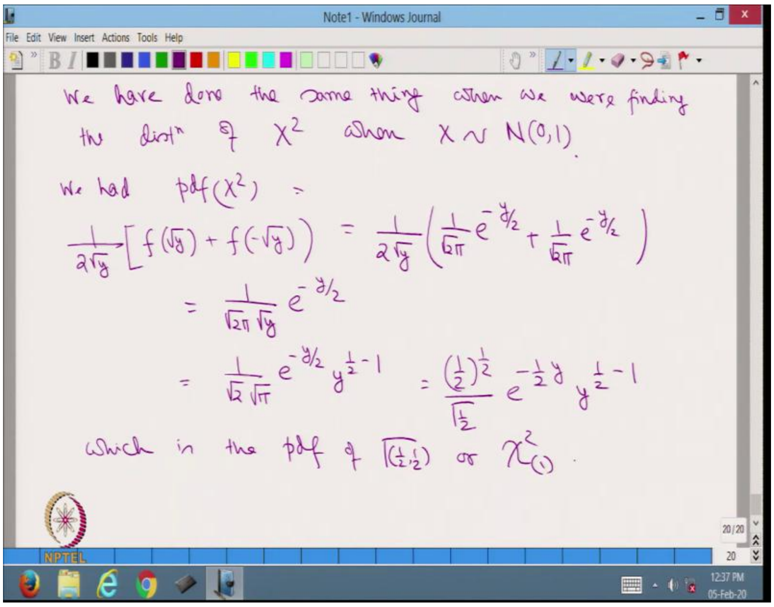

if you remember, we have done the same thing when we were finding the distribution of X square when X is distributed as normal 0, 1 we had pdf of x square is equal to 1 over 2 root y into f of root y plus f of minus root y is equal to 1 over 2 root y into 1 over root over 2 pi e to the power minus y by 2 plus 1 over root over 2 pi e to the power minus y by 2 is equal to 1 over root over 2 pi root over y e to the power minus y by 2 is equal to 1 over root over 2 root over pi e to the power minus y by 2 y to the power half minus 1 is equal to half to the power half gamma half e to the power minus half y y to the power half minus 1 which is the pdf of gamma half, half or Chi square with 1 degree of freedom.

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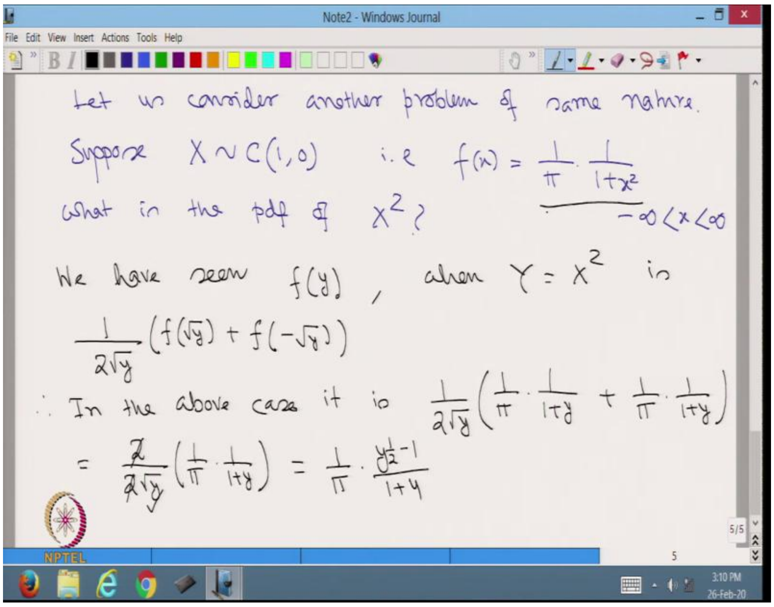

Let us consider another problem of same nature. Suppose, X is distributed as Cauchy with 1, comma 0 that is f of x is equal to 1 over pi into 1 over 1 plus x square minus infinity less than x less than infinity. What is the pdf of X square?

We have already seen f of y, when Y is equal to x square is 1 over 2 times root y into f at root y plus f at minus root y. Therefore, in the above case it is 1 over 2 root y into 1 over pi into 1 upon 1 plus y because root y square is y plus 1 by pi into 1 upon 1 plus y is equal to 2 upon 2 root y multiplied by 1 over pi into 1 upon 1 plus y is equal to 2 and 2 gets cancelled. Therefore, we get 1 over pi into y to the power half minus 1 we transfer it like that upon 1 plus y.

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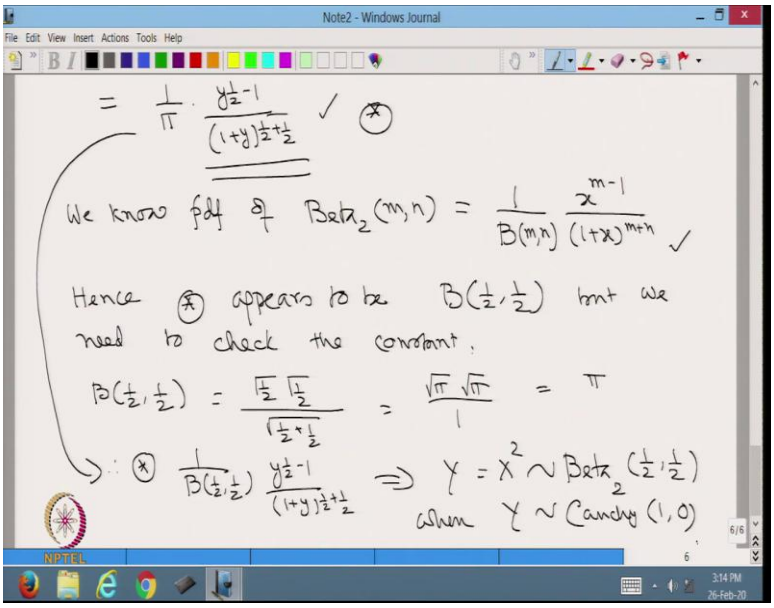

Is equal to 1 upon pi y to the power half minus 1 upon 1 plus y whole to the power half plus half so this part looks like beta 2 distribution. We know pdf of beta 2 with m, comma n is equal to 1 upon beta m, comma n x to the power m minus 1 upon 1 plus x whole to the power m plus n thus, we can notice a very similar pattern.

Hence, let us call it star, star appears to be beta with half and half but we need to check the constant. Now, beta half, comma half is equal to gamma half gamma upon gamma half plus half is equal to root pi root pi upon 1 is equal to pi. Therefore, star is 1 upon beta half comma half y to the power half minus 1 upon 1 plus y whole to the power half plus half implies Y is equal to X square is distributed as beta 2 with half comma half when y is Cauchy with 1 and 0.

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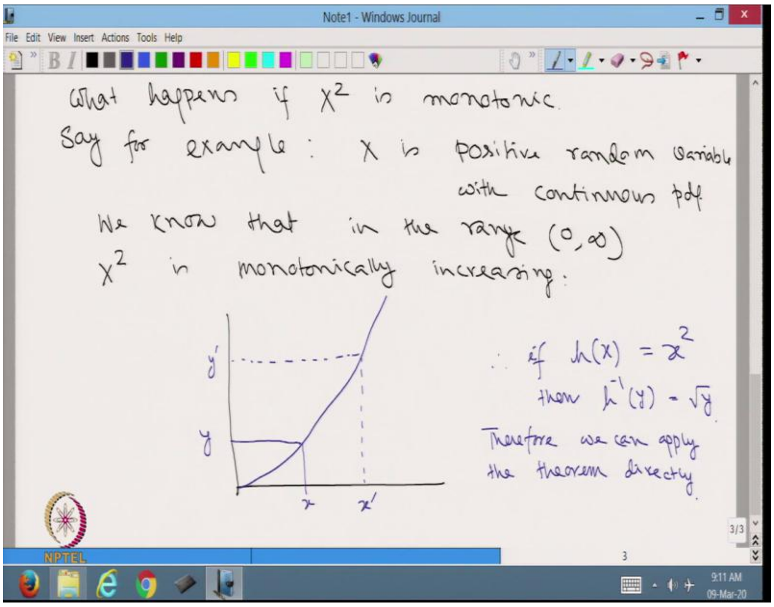

What happens if X square is monotonic? Say, for example, X is a positive random variable with continuous pdf. Now, we know that in the range 0 to infinity X square is monotonically increasing. So, if this is the graph X is monotonically increasing therefore, given any x we can find y uniquely and conversely if given any y y prime say we can get corresponding x prime uniquely. Therefore, if hx is equal to x square then h inverse of y is equal to root over y. Therefore, we can apply the theorem directly.

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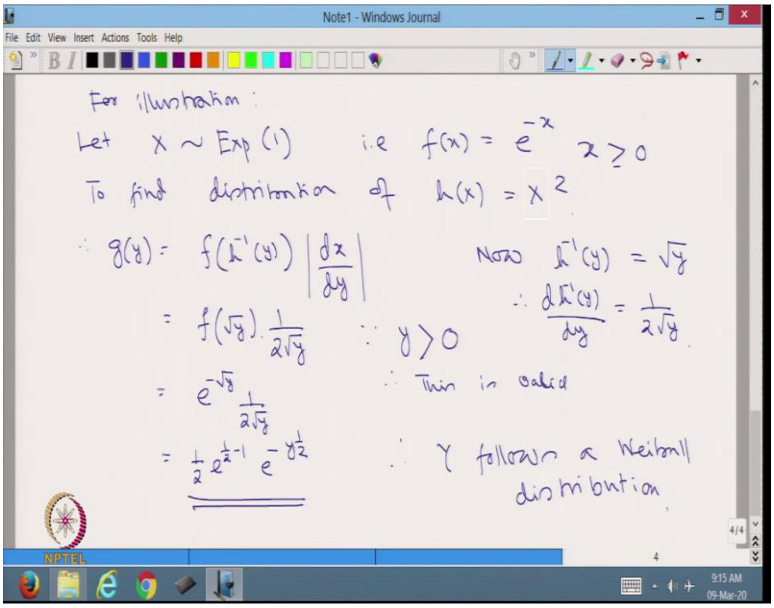

For illustration, let X distributed as exponential with parameter 1 that is f of x is equal to e to the power minus x when x is greater than equal to 0. To find distribution of h of x is equal to x square. Therefore, g of y is equal to f at h inverse y multiplied by dx dy. Now, h inverse of y is equal to root over of y therefore, dh inverse of y dy is equal to half root y.

Therefore, this equation becomes, f at root y multiplied by 1 upon 2 root over y since, y is greater than 0 therefore, this is valid is equal to e to the power minus root over y 1 upon 2 root over y is equal to half e to the power half minus 1 into e to the power minus y to the power half. Do you remember this density function, yes? Therefore, y follows a weibull distribution.

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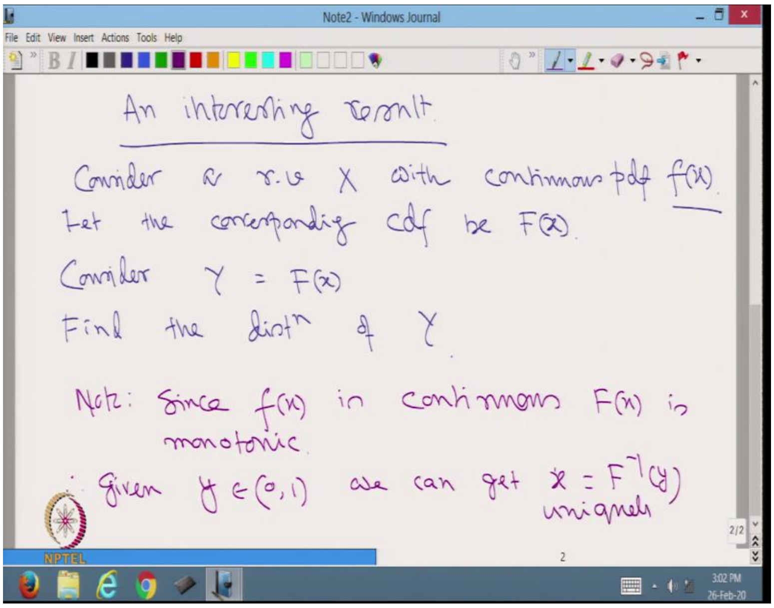

So, let us consider an interesting result which comes from the application of the above theorem, consider a random variable X with continuous pdf fx. Let the corresponding cumulative distribution function be capital Fx. Consider Y is equal to Fx find the distribution of Y.

So, that is a very interesting question because I am looking at any arbitrary continuous density random variable and Y is equal to its cumulative distribution function. Note that since, fx is continuous capital Fx is monotonic. Therefore, given y belonging to 0, 1 we can get x is equal to F inverse y uniquely.

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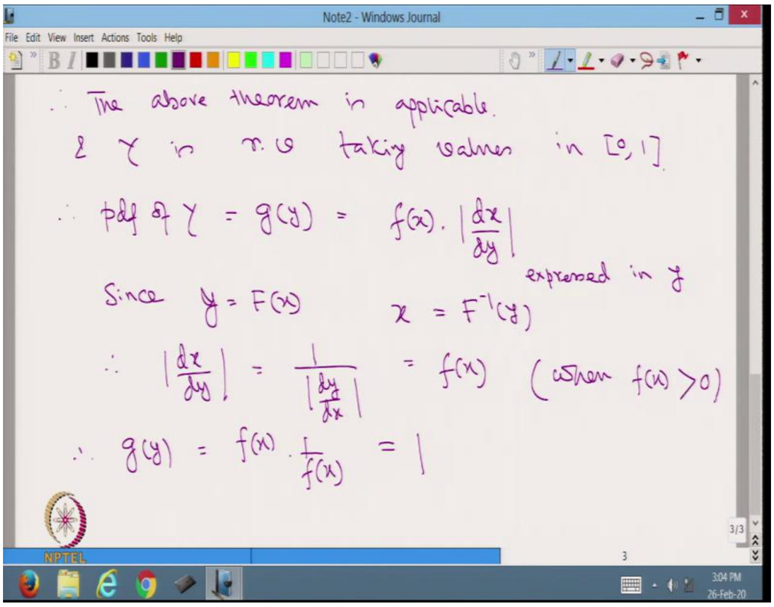

Therefore, the above theorem is applicable and y is a random variable taking values in 0, 1. Therefore, pdf of Y is equal to gy is equal to by the above theorem fx into modulus of dx dy expressed in y. But, we do not do that, we write it as follows since, y is equal to Fx, x is equal to F inverse of y. Therefore, modulus of dx dy is equal to 1 upon modulus of dy dx is equal to fx. Therefore, gy is equal to fx into 1 upon fx is equal to 1. Obviously, we can write when fx greater than 0.

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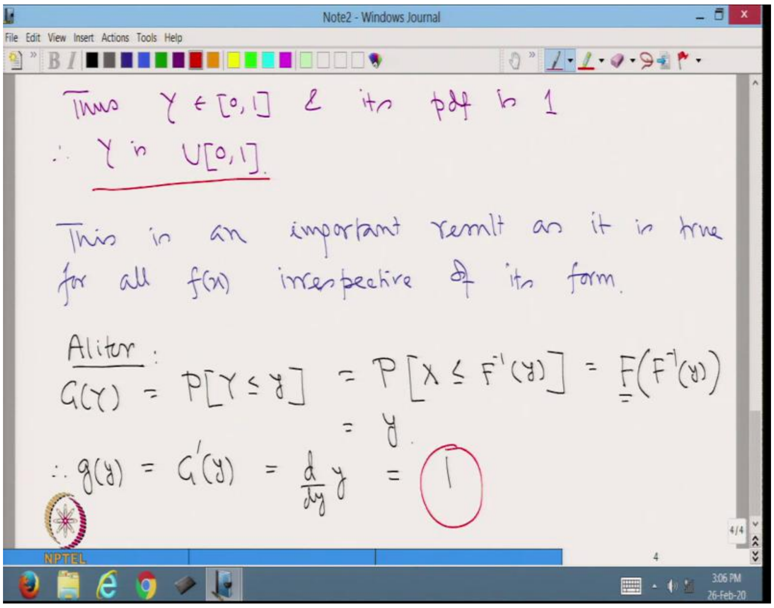

Thus, Y belongs to 0, 1 and its pdf is 1. Therefore, Y is uniformly distributed in the interval 0, 1. This is an important result as it is true for all fx irrespective of its form. Now, some of u may doubt that whether this result is correct or not so, let me prove it in a different way. G of Y is equal to say probability Y less than equal to y is equal to probability X less then equal to F inverse y is equal to F of F inverse y since, f is the CDF of X is equal to y. Therefore, g of y is equal to G Prime y is equal to d dy of y is equal to 1. So, we get the same result.

Okay, friends I stop here today. In the next class I shall look at functions of two random variables and from there we shall try to see the pdf of different interesting functions and also we shall arrive at two very important distributions namely T and F. Okay friends. Thank you so much.