Advanced Probability Theory
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Lecture 28

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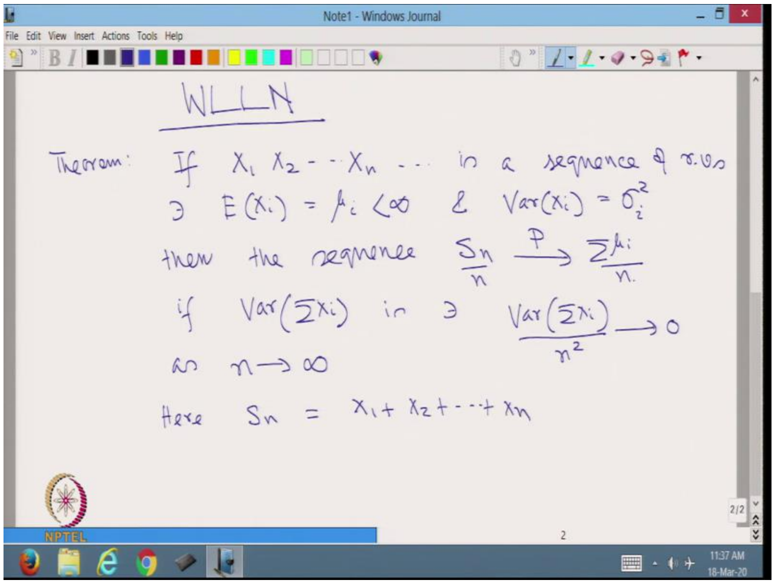

Welcome students to the mock series of lectures on Advanced Probability Theory, this is lecture number 28, if you remember in the last class we were discussing weak law of large numbers which states that, if X1 X2 Xn is a sequence of random variables such that expectation of Xi is equal to mu i which is finite and variance of Xi is equal to sigma square i then the sequence Sn upon n convergence in probability to sigma mu i upon n if variance of sigma Xi is such that variance of sigma Xi upon n square goes to 0 as n goes to infinity, here Sn is equal to X1 plus X2 plus here Sn is equal to X1 plus X2 plus Xn.

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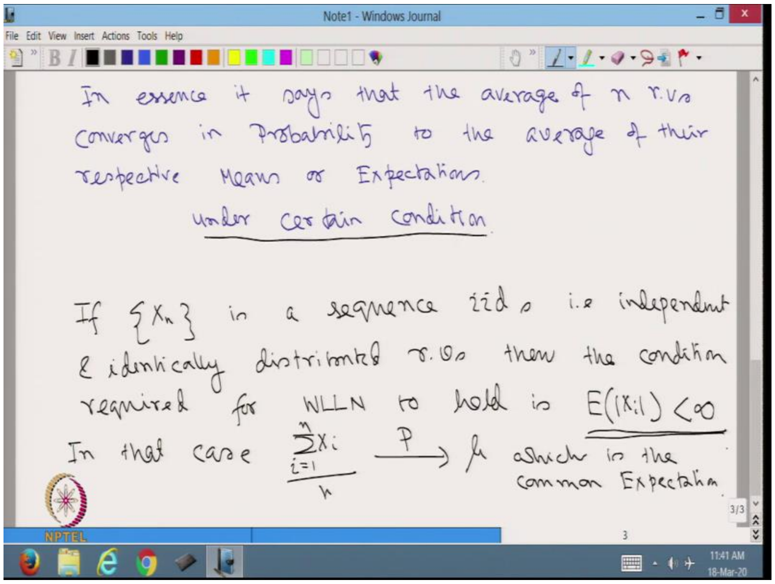

So, in essence it says that the average of the n random variables converges in probability to the average of their respective means or expectations under certain condition. Now, if Xn is a sequence of iid's that is independent and identically distributed random variable then the condition required for weak law of large numbers to hold is expected value of modulus of Xi is finite. And in that case sigma Xi i is equal to 1 to n divided by n converges in probability to mu which is the common expectation, in essence what we are saying that this one condition there that expected value of mod Xi is finite that takes care of the rather more stringent condition that we need for a general scenario when X1 X2 Xn are any arbitrary sequence of random variables.

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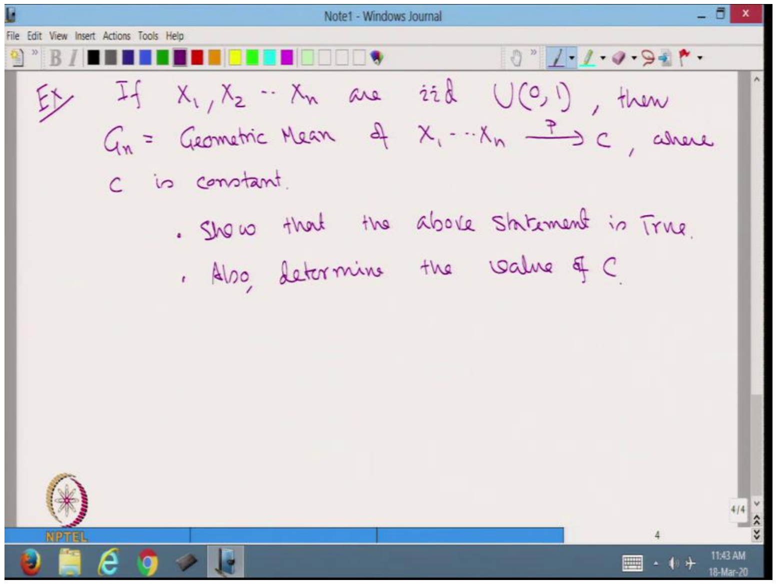

So, before we move further, let us consider one example, if X1, X2, Xn are iid uniform 0, 1, then Gn which is the geometric mean of X1, X2, Xn converges in probability to a constant C, show that the above statement is correct, true and also determine the value of C.

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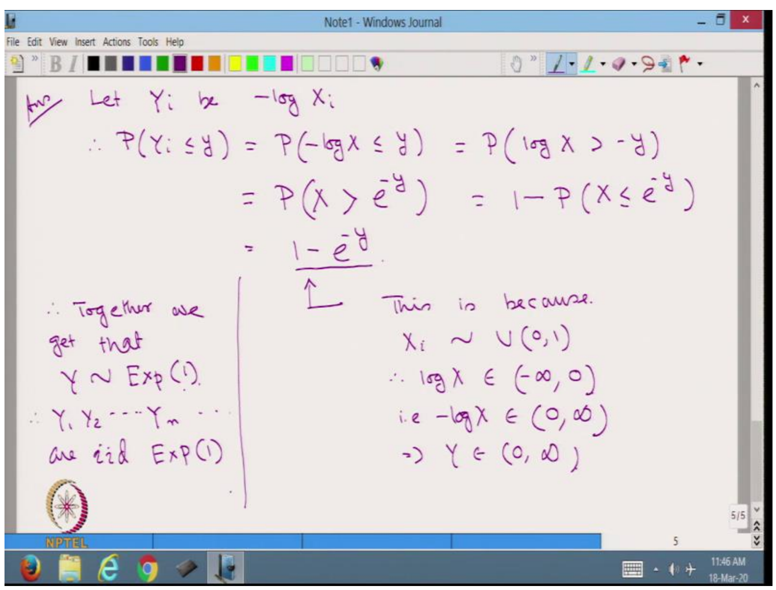

Answer, let Yi be minus log of Xi, therefore probability Yi less than equal to y is equal to probability minus of log X less than equal to y is equal to probability log X greater than minus y is equal to probability X greater than e to the power minus y is equal to 1 minus probability X less than equal to e to the power minus y is equal to 1 minus e to the power minus y. This is because X is or Xi is uniform 0, 1, therefore log X belongs to minus infinity to 0 that is minus log X belongs 0 to infinity implies Y belongs to 0 to infinity. Therefore, together we get that Y is distributed as exponential with parameter 1, as this is the CDF of exponential 1. Therefore, Y1, Y2, Yn are iid exponential 1.

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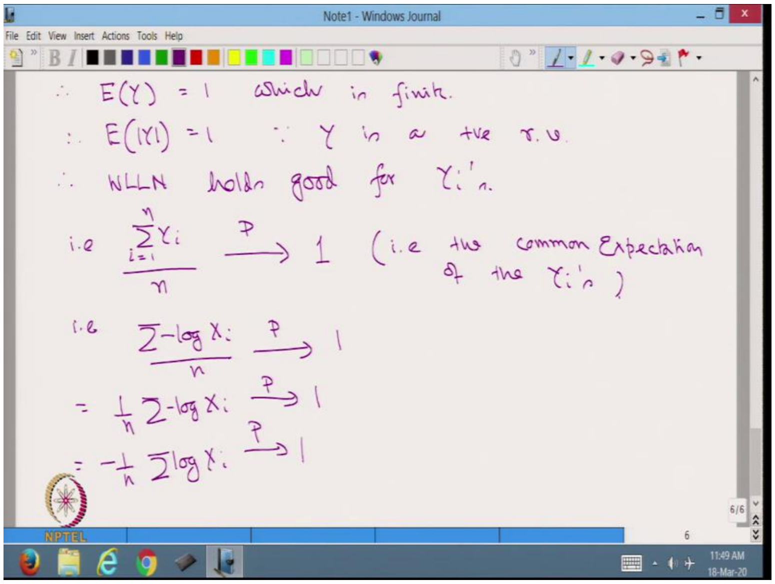

Therefore expected value of Y is equal to 1, which is finite, therefore expected value of mod Y is equal to 1, since Y is a positive random variable, therefore weak law of large numbers holds good for Yi's that is sigma Yi i is equal to 1 to n divided by n converges in probability to 1, that is the common mean, common expectation of the Yi's, that is sigma minus log Xi upon n converges in probability to 1, implies 1 by n sigma minus log Xi converges in probability to 1, implies minus 1 by n sigma log Xi converges in probability to 1.

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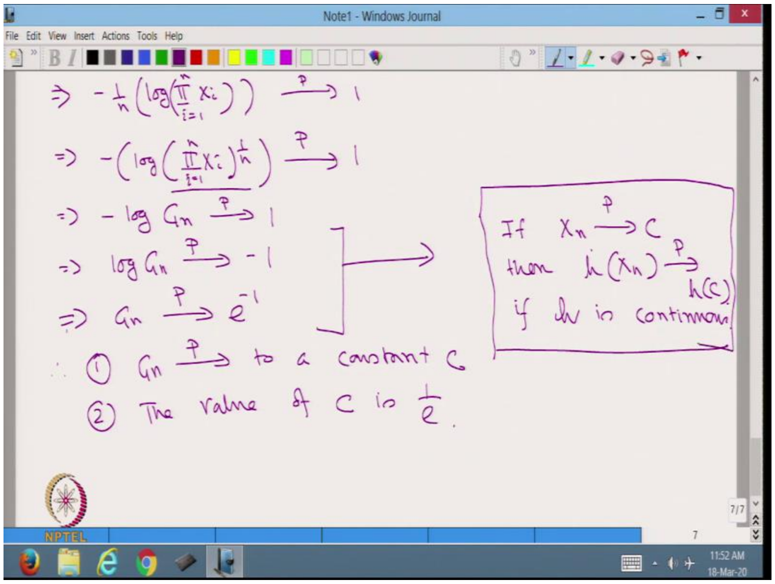

Implies minus 1 by n log of product of Xi i is equal to 1 to n converges in probability to 1, implies minus log of product of Xi i is equal to 1 to n to the power 1 by n converges in probability to 1, implies minus log of the geometric mean of X1, X2, Xn converges in probability to 1 implies log Gn converges in probability to minus 1 implies Gn converges in probability to e to the power minus 1, therefore one, Gn converges in probability to a constant C, two, the values of C is 1 upon e. Now, I have used a result at this point, the result is that if Xn converges in probability to a constant C then h of Xn converges in probability to h of C, if h is continuous. So, this is one result which I have used, so let me prove it.

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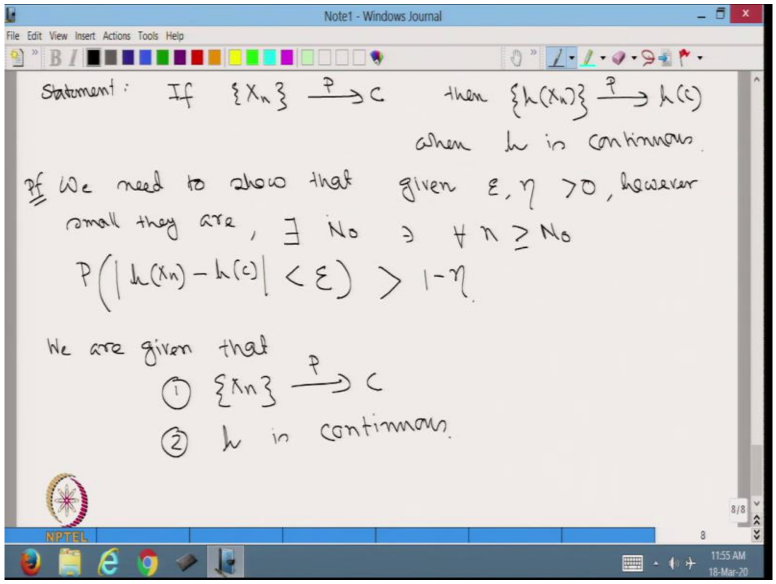

So, statement, if Xn converges in probability to C then the sequence h of Xn converges in probability to h of C when h is continuous. Proof, we need to show that given epsilon and eta greater than 0, however small they are, there exist N naught such that for all n greater than equal to N naught probability of modulus of h Xn minus h of c less than epsilon is greater than 1 minus eta. We are given that one, Xn converges in probability to C and two, h is continuous

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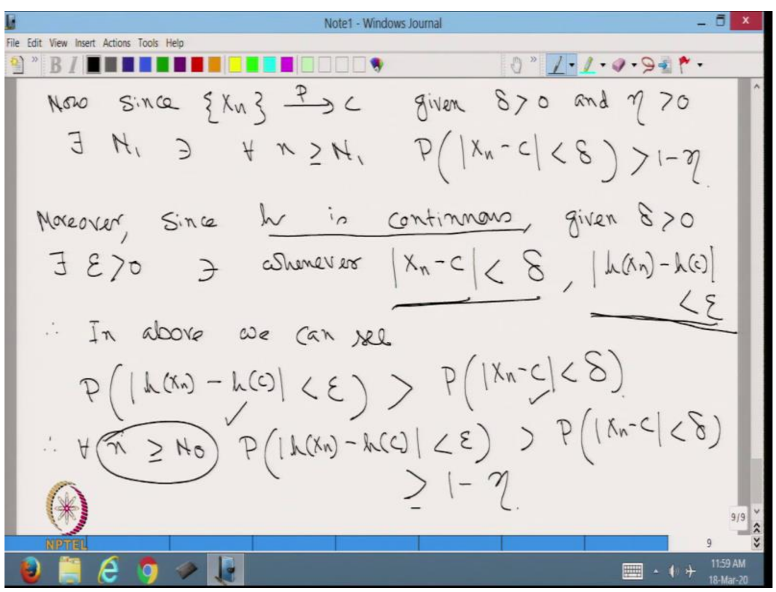

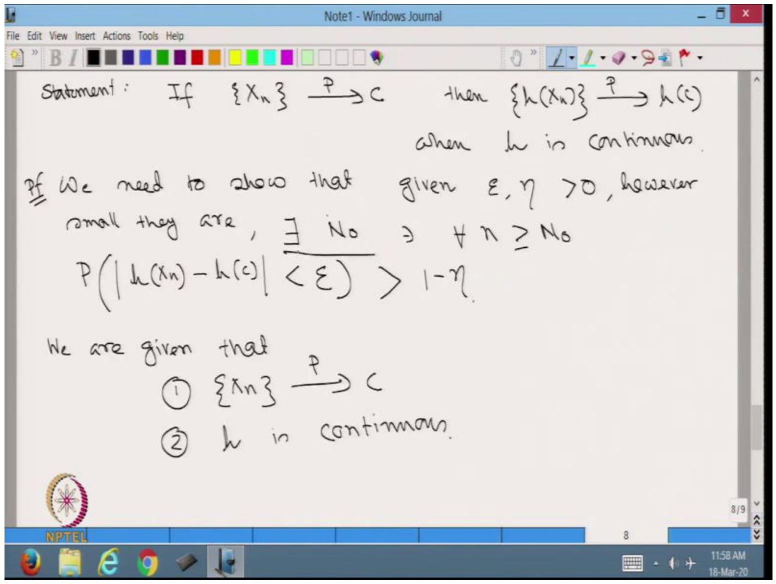

Now, since Xn converges in probability to C, given delta greater than 0 and eta greater than 0 there exist N1 such that for all n greater than equal to N1 probability modulus of Xn minus C less than delta is greater than 1 minus eta. Moreover, since h is continuous, given delta greater than 0 there exist epsilon greater than 0 such that whenever modulus of Xn minus C is less than delta modulus of h of Xn minus h of C is less than epsilon. So, because h is continuous we get a neighbourhood around C in which we get this outcome, we get this result.

Therefore, in above we can see probability modulus of h of Xn minus h of C less than epsilon is greater than probability modulus of Xn minus C less than delta, because this event implies this event therefore this has a higher probability, then this event therefore for all n greater than equal to N naught let us go back this is the N naught now I am talking about probability modulus of h of Xn minus h of C less than epsilon is greater than probability modulus of Xn minus C less than delta which is greater than equal to 1 minus eta, thus we get one N naught such that this property.

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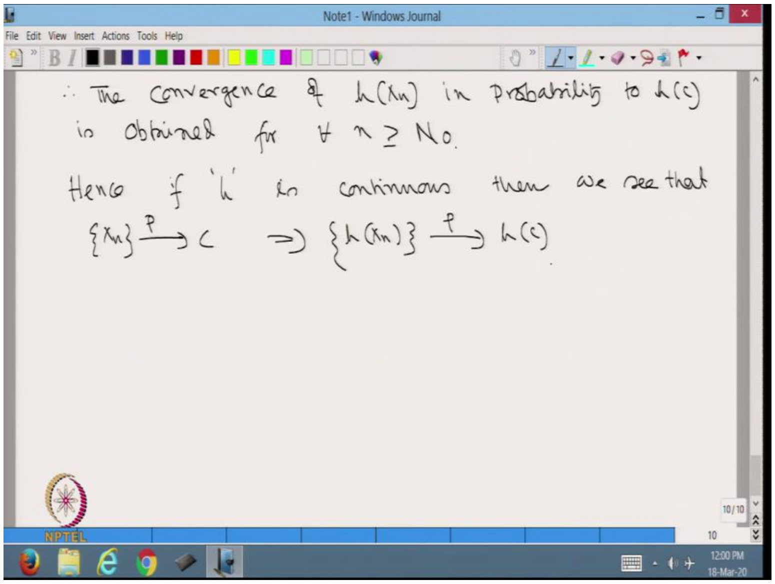

Therefore, the convergence of h of Xn in probability to h of C is obtained for all n greater than equal to N naught. Hence, if h is continuous then we see that Xn converging in probability to C implies h of Xn converges in probability into h of C

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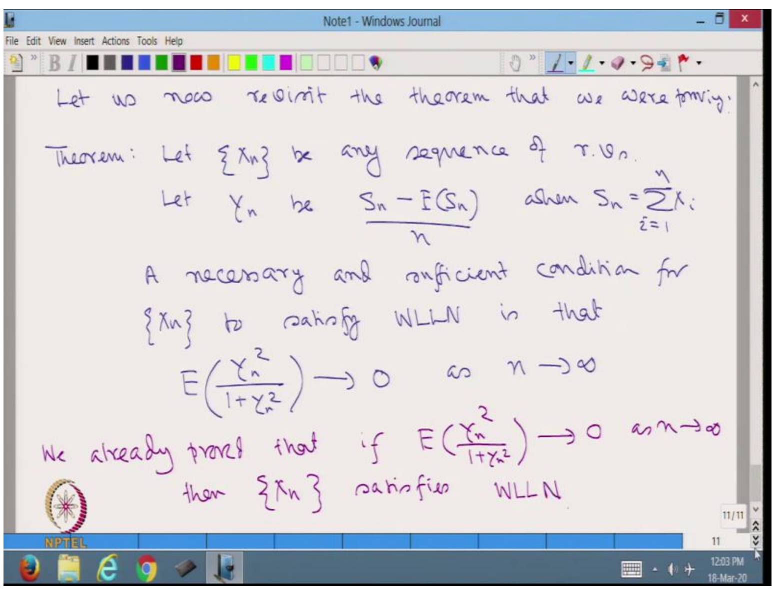

So, let us now revisit the theorem that we were proving, the theorem was that let Xn be any sequence of random variables, let Yn be Sn minus expected value of Sn divided by n when Sn is equal to sigma Xi i is equal to 1 to n, a necessary and sufficient condition for Xn to satisfy weak law of large numbers is that expected value of Yn square upon 1 plus Yn square goes to 0 as n goes to infinity. Note that we already proved in the last class, if expected value of Yn square upon 1 plus 1 square goes to 0 as n goes infinity then the sequence of random variables Xn satisfies weak law of large number.

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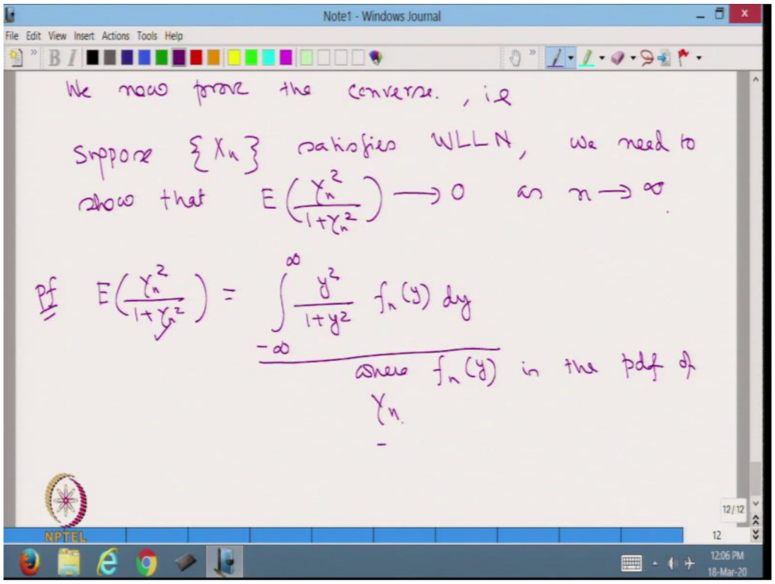

We now prove the converse, suppose Xn satisfies weak law of large numbers, we need to show that expected value of Yn square upon 1 plus Yn square goes to 0 as n goes to infinity. Proof expected value of Yn square upon 1 plus Yn square is equal to integration minus infinity to infinity y square upon 1 plus y square fn y dy, where fn y is the pdf of Yn, this we know because we are computing the expectation of a function of a random variable and we have done this many times before.

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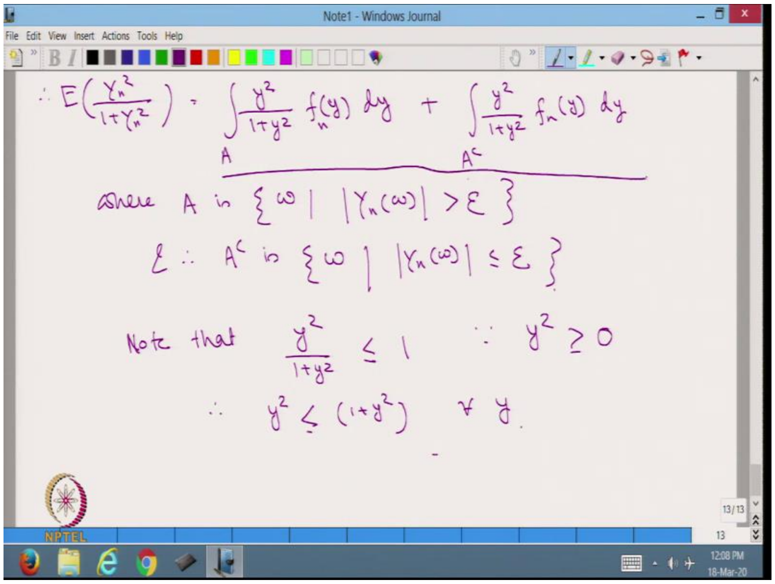

Therefore, expected value of Yn square upon 1 plus Yn square is equal to integration over A y square upon 1 plus y square f of fn of y dy plus integration of y square upon 1 plus y square fn y dy on A compliment, where A is the set of omega such that modulus of Yn omega is greater than epsilon and therefore A complement is the set of omega such that modulus of Yn omega is less than equal to epsilon. Now, note that y square upon 1 plus y square is less than equal to 1, since y square is greater than equal to 0. Therefore, y square is less than equal to 1 plus y square for all y. Now, we are going to plug in this expression.

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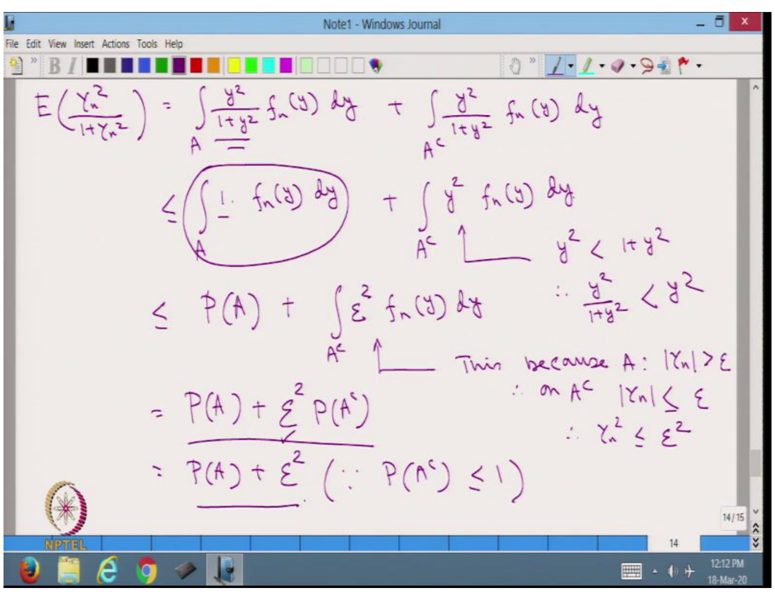

Therefore, let us rewrite expectation of Yn square upon 1 plus Yn square is equal to integration over A y square upon 1 plus y square fn y dy plus integration over A compliment y square upon 1 plus y square y fn y dy, since this is less than 1 we can write it as this is less than equal to A into 1 fn y dy plus integration over A compliment y square into fn y dy, this is because y square is less than 1 plus y square therefore y square upon 1 plus y square is less than y square, less than equal to probability of A because if we take 1 out this gives the probability of A plus integration over A compliment epsilon square into fn y dy.

This is because A is equal to such that modulus of Yn is greater than epsilon, therefore on A complement modulus of Yn is less than equal to epsilon, therefore Yn square is less than equal to epsilon square is equal to probability of A plus epsilon square into probability of A compliment because if we take epsilon square out we get probability of A compliment is equal to probability of A plus epsilon square since probability of A compliment is less than equal to 1.

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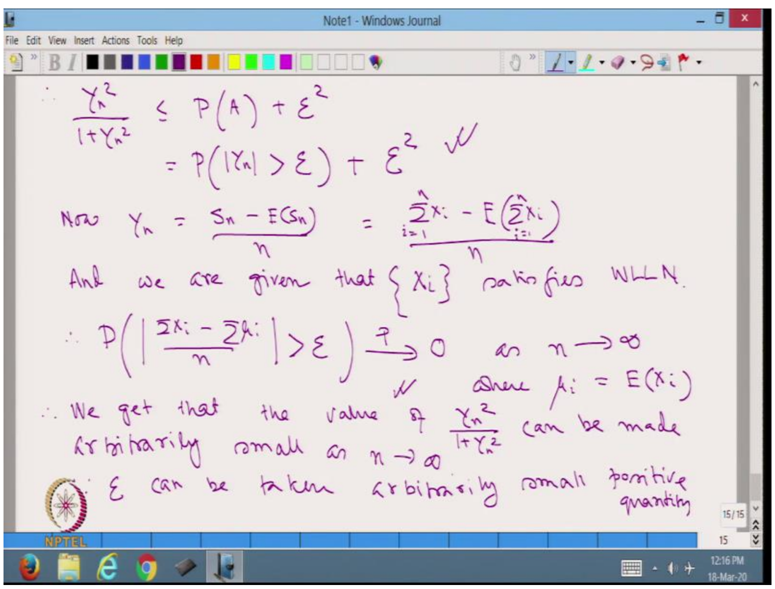

Therefore, Yn square upon 1 plus Yn square is less than equal to probability of A plus epsilon square is equal to probability modulus of Yn greater than epsilon plus epsilon square. Now, Yn is equal to Sn minus expected value of Sn divided by n is equal to sigma Xi i is equal to 1 to n minus expected value of sigma Xi i is equal to 1 to n divided by n and we are given that Xi satisfies weak law of large numbers. Therefore, probability modulus of sigma Xi minus sigma mu i divided n greater than epsilon converges to converges in probability to 0 as n goes to infinity, where mu i is equal to the expected value Xi. Therefore, from this result and this result we get that the value of Yn square upon 1 plus Yn square can be made arbitrarily small as n goes to infinity. Since epsilon can be taken arbitrarily small positive quantity.

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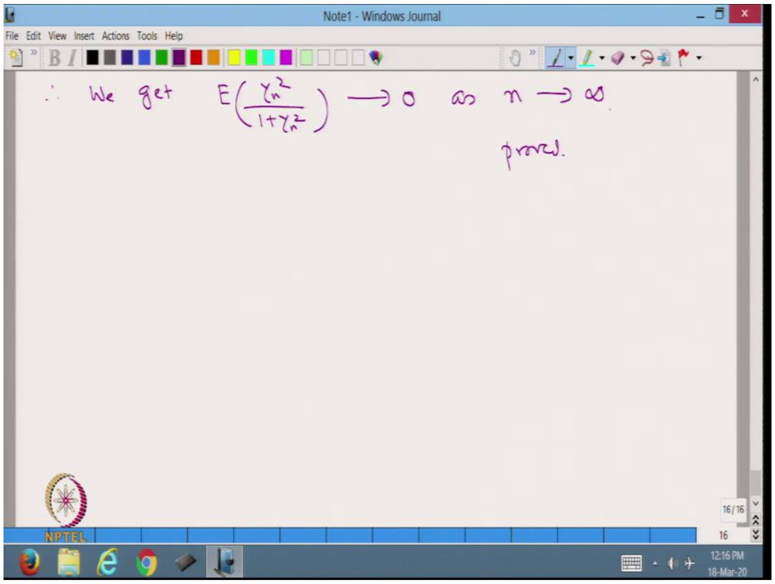

Therefore, we get expected value of Yn square upon 1 plus Yn square goes to 0 as n goes to infinity, proved.

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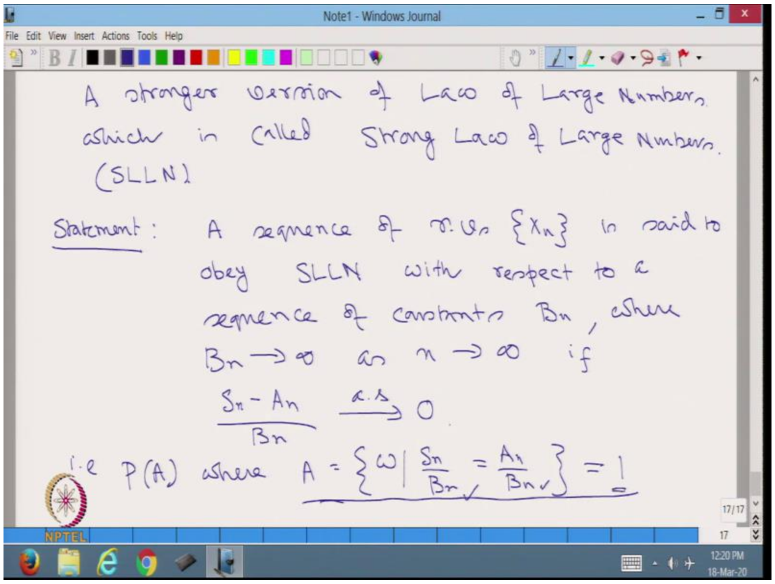

Now, let us consider a stronger version of law of large numbers which is called strong law of large numbers or SLLN. So, statement a sequence of random variables Xn is said to obey strong law of large numbers with respect to a sequence of constants Bn, where Bn goes to infinity as n goes to infinity if Sn minus An upon Bn converges almost surely to 0, that is probability of A where A is equal to the set of omega on which Sn upon Bn is equal to An upon Bn is equal to 1. That means the measure of the set on which Sn upon Bn equals An upon Bn that has the probability 1.

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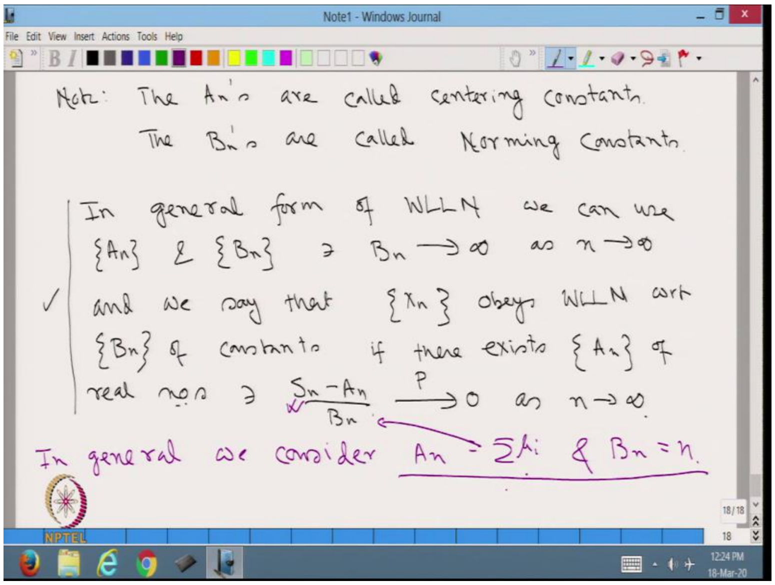

Note, the An’s are called centering constants and the Bn’s are called Norming constant. In a general form of weak law of large numbers, we can use this sequence of numbers An’s and sequence of numbers Bn’s such that Bn is going to infinity as n goes to infinity and we say that Xn obeys weak law of large numbers with respect to the sequence Bn of constants if there exists a sequence An of real numbers such that Sn minus An upon Bn converges in probability to 0 as n goes to infinity. Note that this is a general form of weak law of large numbers, in general we consider An is equal to sigma over mu i and Bn is equal to n. And put this values here you will get that we have defined weak law of large numbers in this general form with this particular value of An and Bn.

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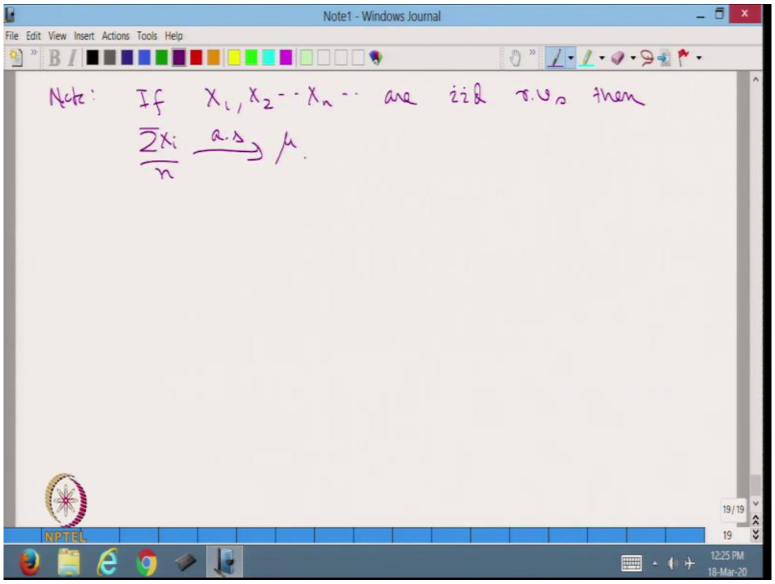

Note, if X1, X2, Xn are iid random variables then sigma Xi upon n converges almost surely to mu. So, these results can be proved with some more knowledge of mathematics and measure theory, but that is not within the gamut of this course, so I am not proving this result.

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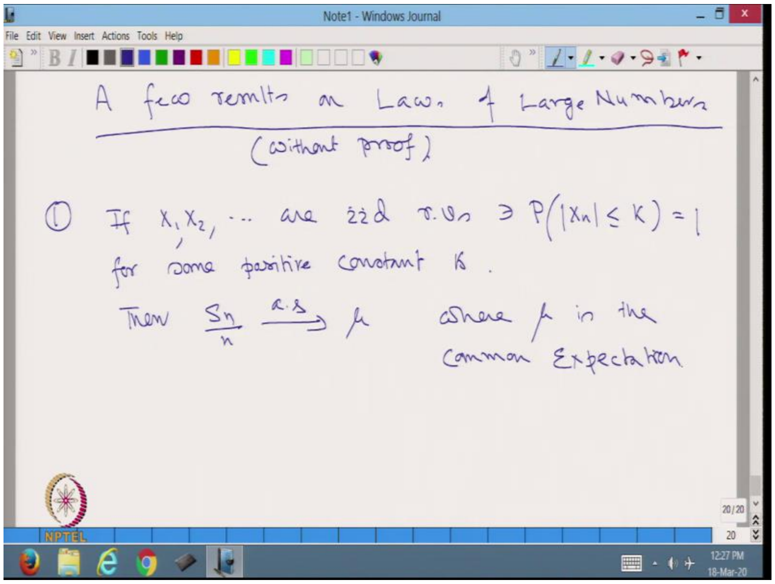

So, let us take this results and I stop this lecture by stating a few more important results on law of large numbers. One, if X1, X2 are iid random variables, such that probability modulus of Xn is less than equal to k is equal to 1 for some positive constant k, then Sn upon n converges almost surely to mu, where mu is the common expectation.

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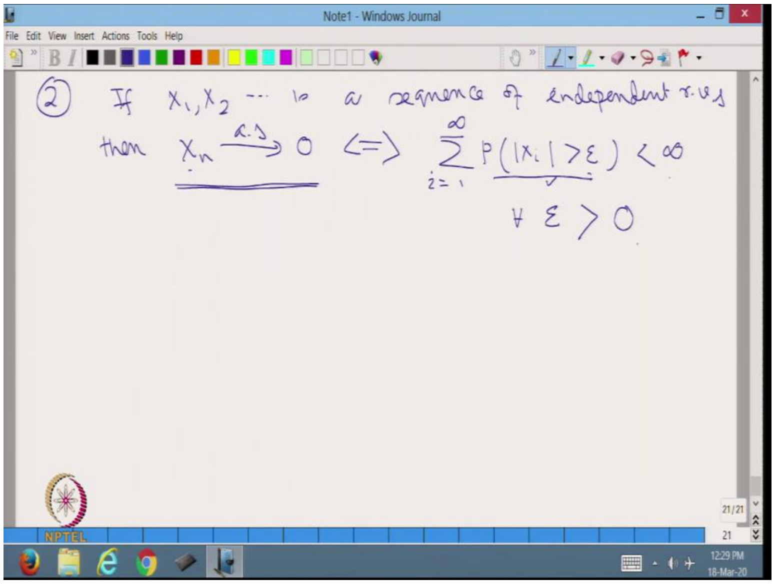

Two, if X1, X2, etcetera is a sequence of independent random variable, then Xn converges almost surely to 0 implies and implied by sigma i is equal to 1 to infinity probability modulus of Xi greater than epsilon is finite for all epsilon greater than 0, it is very easy to visualize as the probability that Xn is taking the value of 0 is going to 1, therefore whatever positive epsilon we take the modulus, the absolute value of Xi to be greater than that has to be finite, because if that is infinite then Xn cannot converge to 0 with probability 1.

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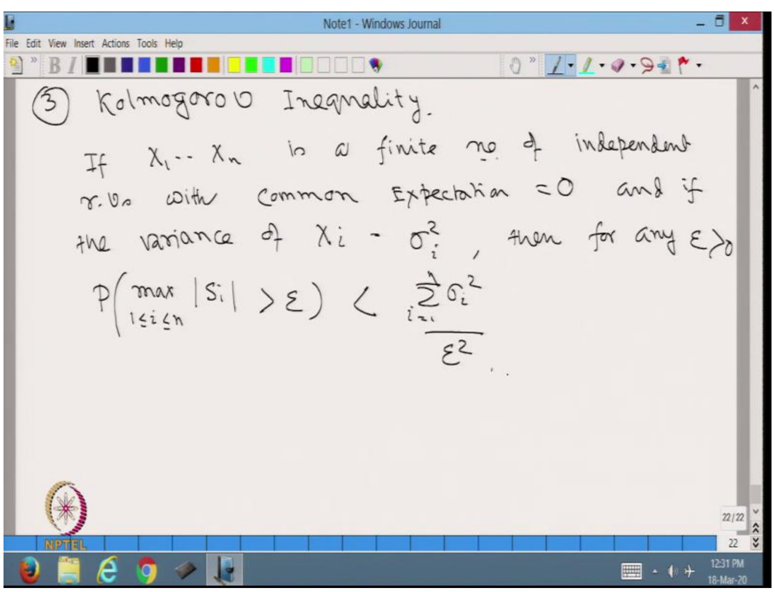

Three, this is called Kolmogorov inequality, which states that if X1, X2, Xn is a finite number of independent random variables with common expectation is equal to 0 and if the variance of Xi is equal to sigma square i, then for any epsilon greater than 0 probability maximum over 1 less than equal to i less than equal to n modulus of Si greater than epsilon is less than sigma summation of sigma i square i is equal to 1 to n divided by epsilon square.

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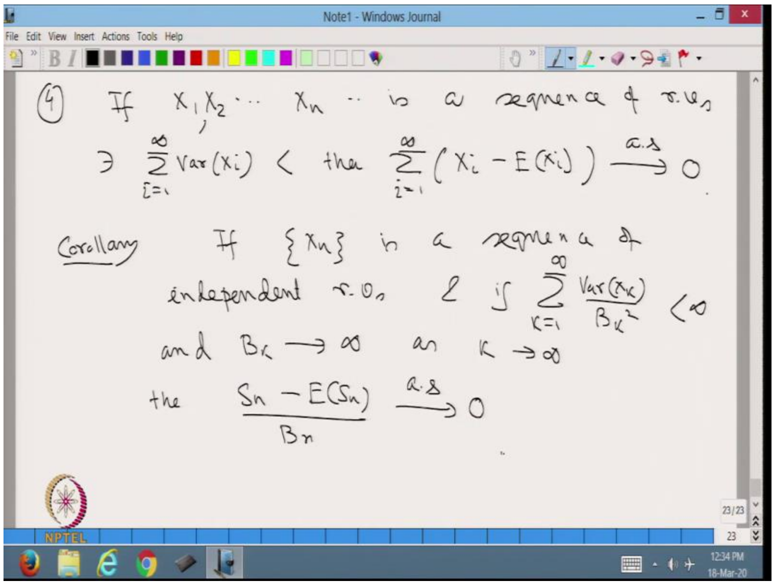

Four, if X1, X2, etcetera, Xn is a sequence of random variables such that sigma variance of Xi i is equal to 1 to infinity is finite, then sigma i is equal to 1 to infinity Xi minus expected value of Xi converges almost surely to 0. A corollary to the above is that, if Xn is a sequence of independent random variables and if sigma variance of Xk upon Bk square k is equal to 1 to infinity is finite and Bk goes to infinity as k goes to infinity, then Sn minus expected value of Sn divided by Bn converges almost surely to 0.

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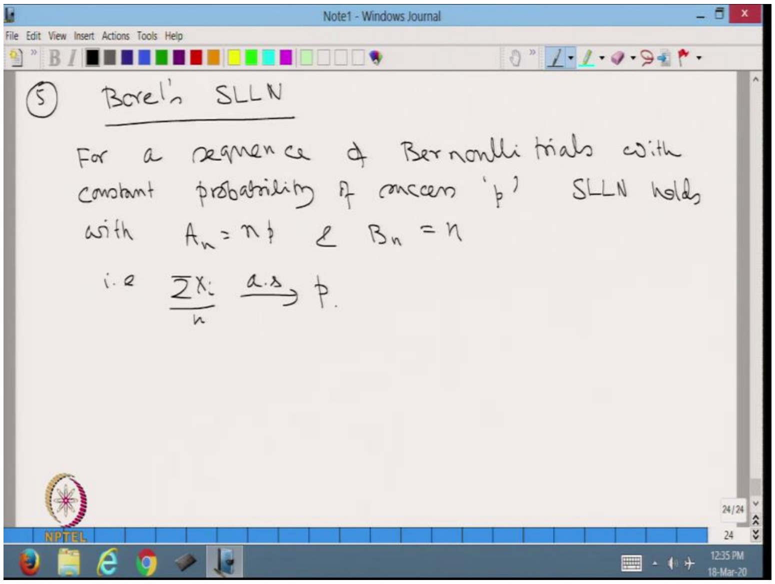

Five, Borel's strong law of large numbers. For a sequence of Bernoulli trials with constant probability of success p, the strong law of large numbers holds with An is equal to np and Bn is equal to n that is sigma Xi upon n converges almost surely to p.

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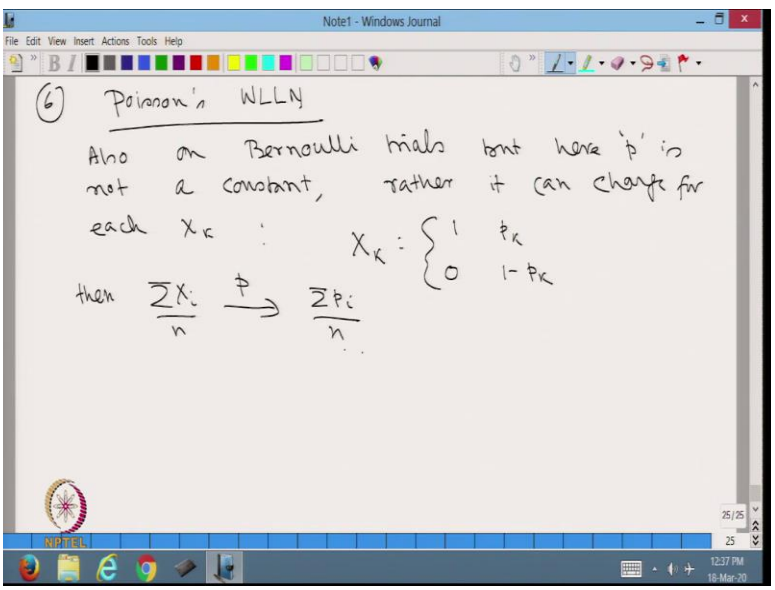

Number six, Poisson's weak law of large numbers also on Bernoulli trials but here p is not a constant rather it can change for each Xk such that Xk takes the value 1 with probability k and 0 with 1 minus pk 1 with probability pk and 0 with 1 minus pk, then sigma Xi upon n converges in probability to sigma pi upon n.

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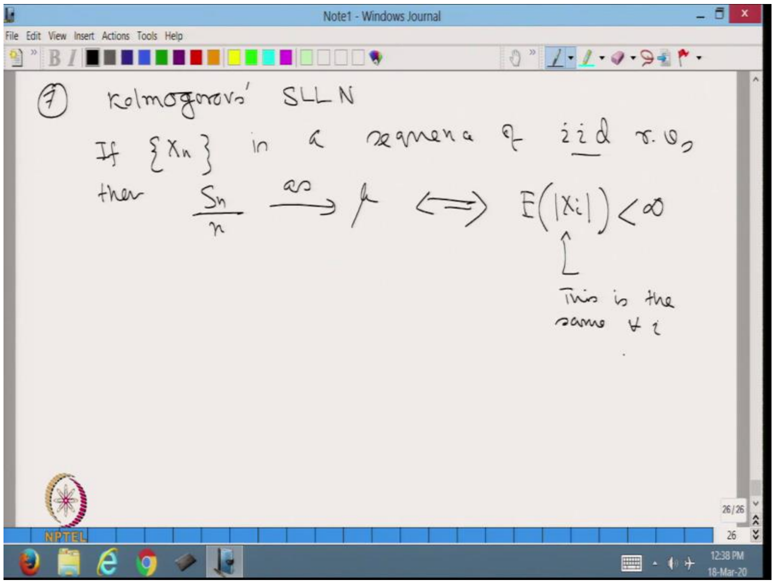

And finally Kolmogorov strong law of large numbers which say that if Xn is a sequence of iid random variables then Sn upon n converges almost surely to the common mean mu implies and implied by expected value of modulus of Xi is finite, note that this is the same for all i. Okay friends I stop here today, so I have given you a lot of results consigning convergence of sequence of random variables when the number of such variables is large and there are several theorems stating different results, they can be proved mathematically using measure theory and analysis. But as I said that in this course, you are not going into the details of those proof, but it is better for our applications to remember this results with that I conclude my talk on law of large numbers from the next class we shall start the most fundamental result of probability or one of the most fundamental results of probability mainly central limit theorems. Okay friends, thank you.