Advanced Probability Theory
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Lecture 01

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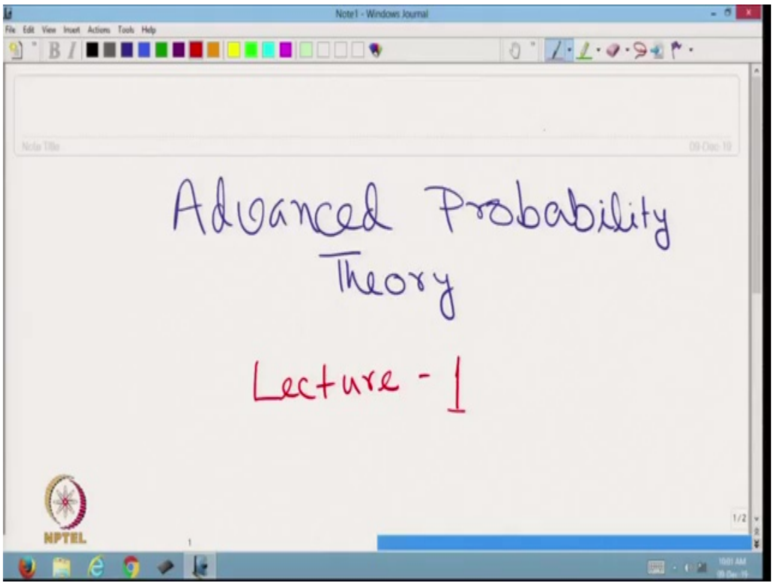

Welcome students to the first lecture on advanced probability theory. As I said in my introduction, that probability is something that we often use very loosely in our daily conversation. Say for example, today there is high probability of having a rain. How is that probability computed, like any mathematical science, there should be a solid mathematical theory behind it and you can compute probability of an event, if and only if that particular event has occurred a large number of times, so that we can compute its probability.

Therefore, such use of the term probability in my opinion is not at all scientific, it is highly subjective. And therefore, for any scientific calculation, we need to understand how the probability needs to be calculated. And to do that we need to have a firm grip on the theory of probability. So, with that small introduction, let me now come to the subject.

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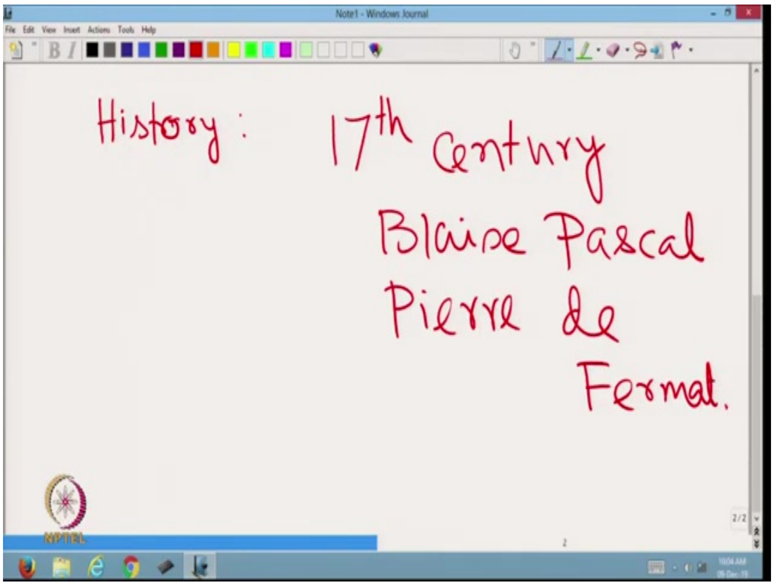

So history, probability some of you may know that is something that people are talking about for centuries. In fact, in the mid 17^{th} century, people have started talking about probability, say people like Blaise Pascal, Fermat. They started using the concept of probability.

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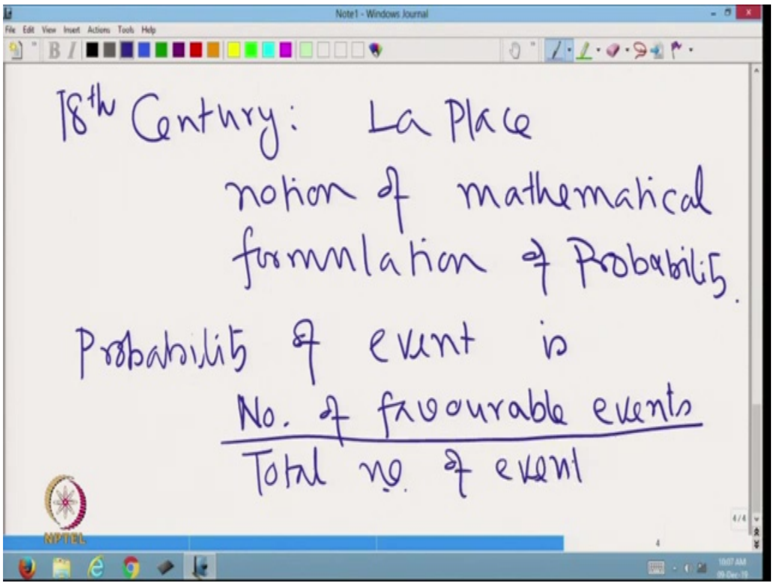

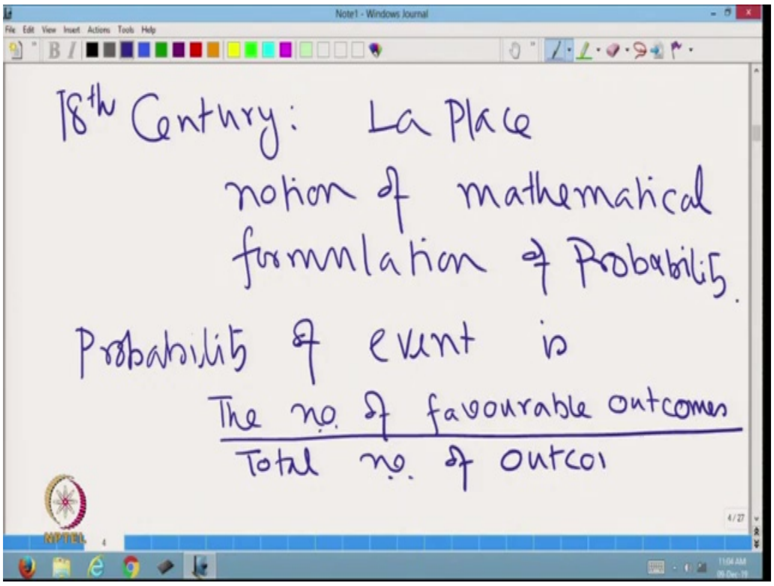

Later in 18^{th} century La Place brought the notion of mathematical formulation of probability. And he suggested that probability of an event is the $\frac{The\ no.of\ favourable\ events}{Total\ no.of\ events}$.

But, such a definition, probability of the event is defined as the $\frac{The\ no.of\ favourable\ outcomes}{Total\ no.of\ outcomes}$

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But such a definition almost invariably needs to define certain things such as:

What is event?

What is total number of events?

What is favorable events?

And therefore, we need to define these terms before we go into the more basic fundamentals of probability.

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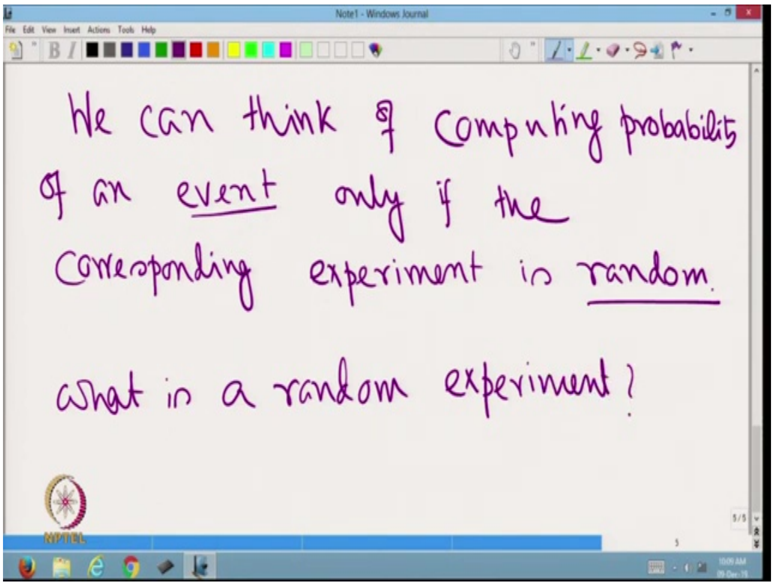

So, we can think of computing probability of an event only if the corresponding experiment is random. So, what is a random experiment?

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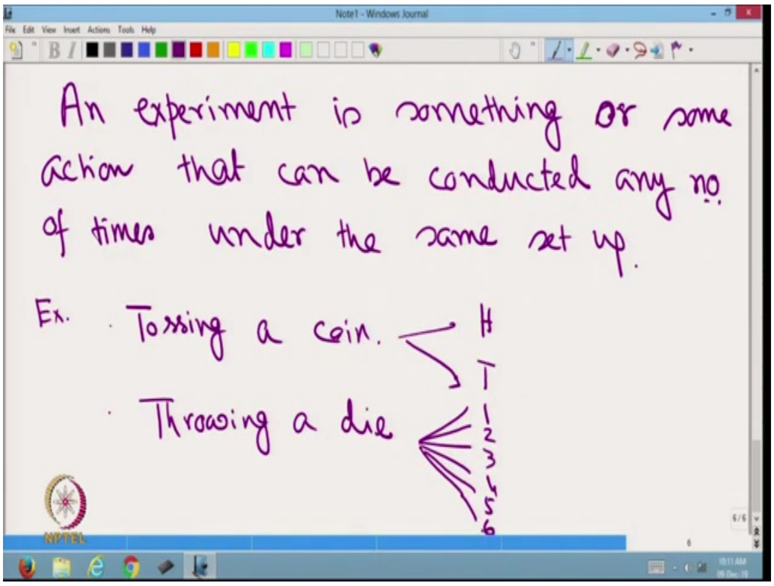

So, let us start with that first, an experiment is something or some action that can be conducted any number of times under the same setup for example, simplest one is tossing a coin. If you have a coin, you can toss it any arbitrary number of times and you can get the output to be head or tail.

Similarly, throwing a die we can keep on throwing a die any number of times and we would expect one of the possible 6 outcomes 1, 2, 3, 4, 5 and 6. There can be many other examples say for example, there is a deck of cards of 52 cards, all of us know that there are 4 suits clubs, hearts, spades and diamonds and under each suit there are 13 from Ace to King. So, there are 52 possible outcomes. If we take a card from complete deck.

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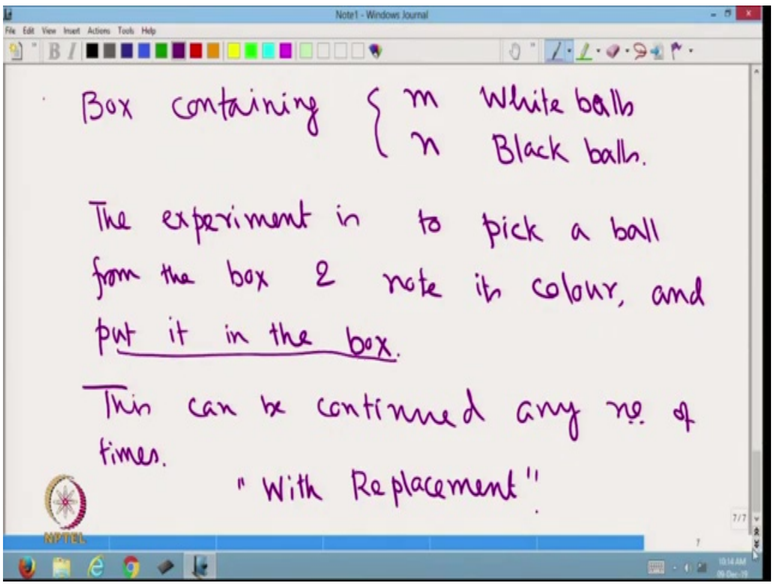

Some other standard example are, suppose a box contain say m white balls and n black balls and our job is or the experiment is to pick a ball from the box and note its color and put it in the back in the box. As you can understand, this can be continued any number of times, this experiment is called “with replacement”, why it is called with replacement? Because the ball that is picked up from the box is going to be put back into the box again. Hence it is called with replacement.

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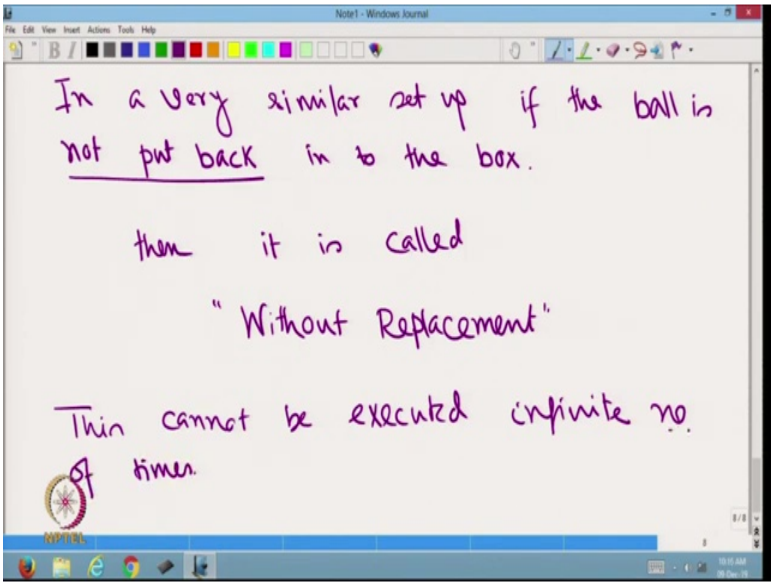

In a very similar setup if the ball is not put back into the box then it is called “without replacement”. Quite obviously, this cannot be executed infinite number of times because the box will be empty after all the balls are taken out of the box, but still without replacement sampling or picking up the ball from the box is often very important for our studies, as we will see later.

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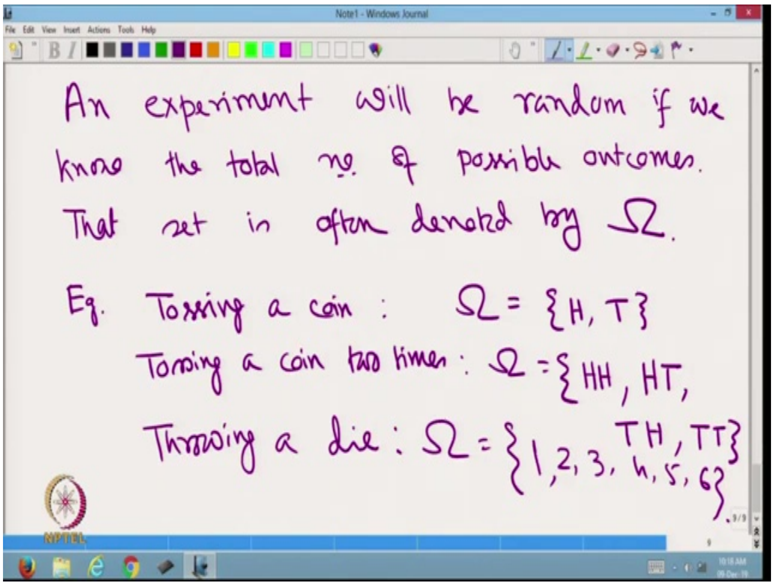

An experiment will be random. If we know the total number of possible outcomes and that set is often denoted by Ω for example,

$$Tossing\ a\ coin : \Omega = \{H, T\}$$

Tossing a coin 2 times. Then Ω will be, if you are thinking it can be HH the outcomes of the 2 events. It can be

$$Tossing\ a\ coin\ two\ times : \Omega = \{HH, HT, TH, TT\}$$

and throwing a die will have omega is equal to 1, 2, 3, 4, 5, 6. So, I hope you understand what is the Ω or the total number of possible outcomes is.

$$Throwing\ a\ die : \Omega = \{1,2,3,4,5,6\}$$

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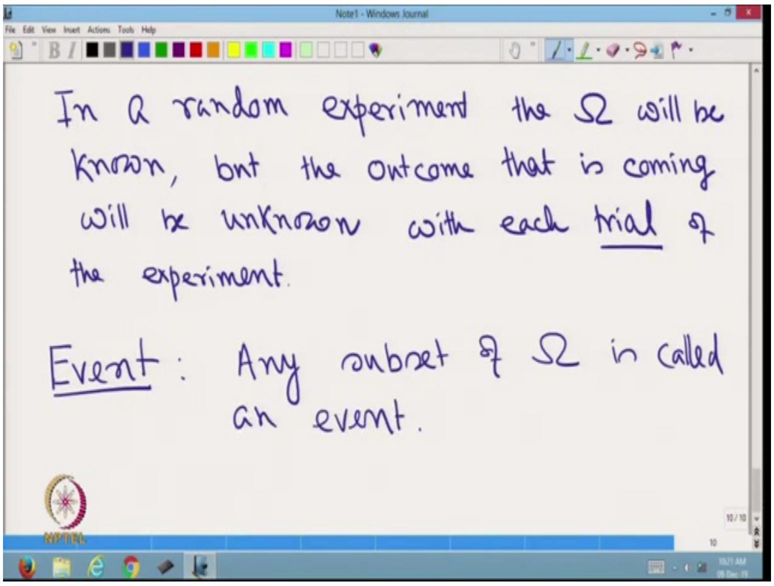

Now in a random experiment the Ω will be known but the outcome that is coming will be unknown with each trial of that experiment, now that what is a trial, each time an experiment is carried out, we call it a trial. So, if I toss a coin 3 times, then we mean there are 3 trials but suppose I take a pair of coin I toss them together and then what I am getting HH, HT, TH or TT and that itself is one trial and that if I do 5 times, we will get 5 pairs of outcome. Event: any subset of Ω is called an event.

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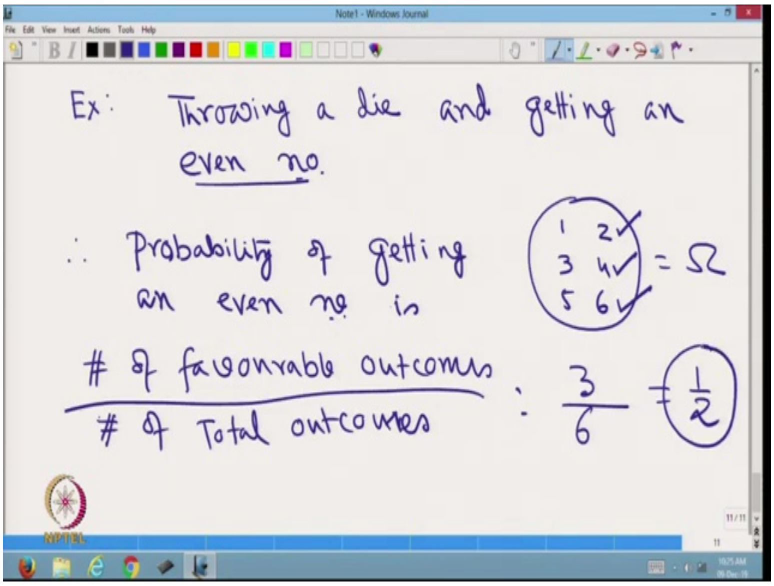

For example, say throwing a die and getting an even numbers. So, if we consider the Ω is equal to 1, 2, 3, 4, 5, 6 these are the 6 points of the Ω and the event is getting an even number, so that event will occur if the outcome is 2 or 4 or 6. So, this event can occur in 3 different ways. And there are 3 other ways when the event will not occur.

Therefore, by Laplace formulation, probability of getting and even numbered is

.

$$\frac{\# \; of \; favourable \; outcomes}{\# \; of \; total \; outcomes}$$

Alright or write it at number of total outcomes. Now, in this case, the favorable outcome is 3 if it is 2, if it is 4 and if it is 6 and the total number of outcomes is cardinality of Ω, which is equal to 6 and therefore, $\frac{1}{2}$ thus probability of getting an even number will be computed as $\frac{1}{2}$.

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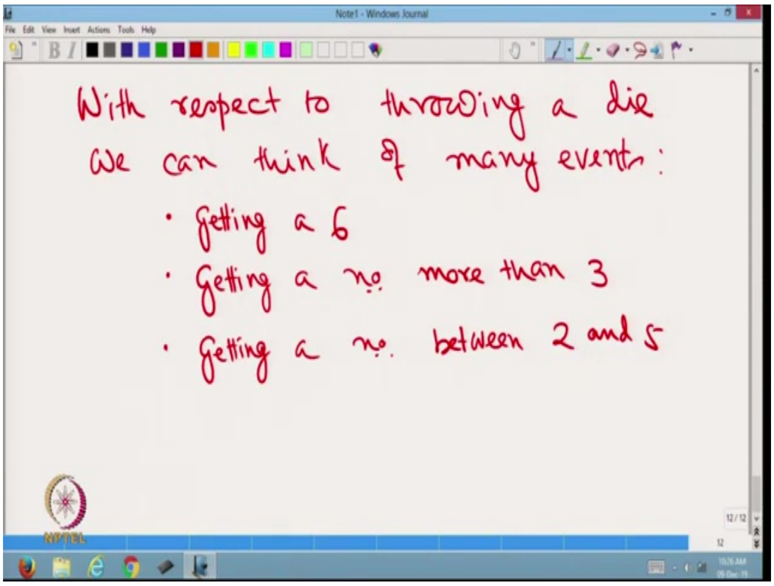

Similarly, with respect to throwing a die, we can think of many events. For example, getting a 6 that is an event, getting a numbers more than 3 that is an event, getting a numbers between 2 and 5 that can be an event or in other words, the way we take a subset of Ω , we can define a particular event.

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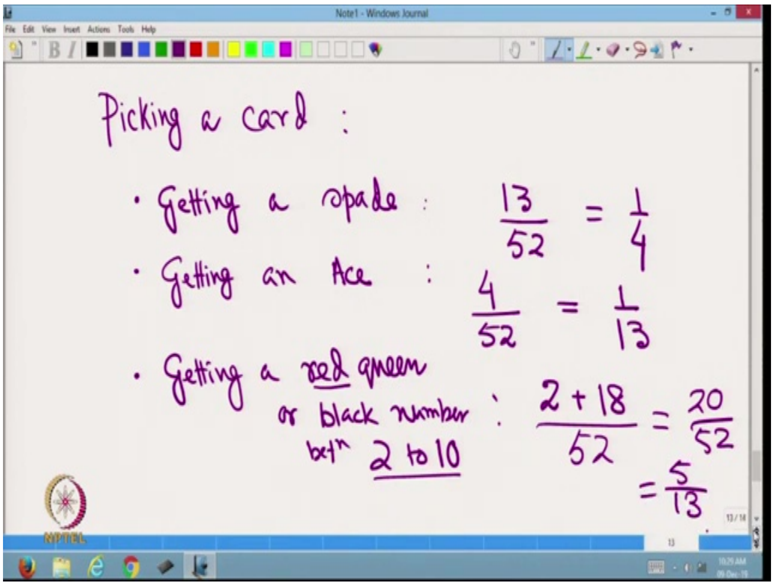

Let us take the example of picking a card possible events are getting spade, what will be the probability?

There are 13 cards of the suit spade in a complete deck which has 52 elements. Therefore, this probability is going to be $\frac{1}{4}$, getting an ace there are 4 aces corresponding to the 4 suits.

Therefore, number of favorable outcomes is 4 total number of outcomes is 52. Therefore, this probability is going to be $\frac{1}{13}$, getting a red queen or black numbers between 2 to 10.

Now, we know that there are 2 queens of red colored, queen of hearts and queen of diamonds and 2 to 10 there are 9 numbers and this 9 can be black, if it is coming from card or if it is coming for the clubs or coming from the spade. So, there are 18 ways of getting a black number between 2 to 10. Therefore, number of favorable outcomes is 2 plus 18 is equal to 20, total number of outcomes is 52. So, it is $\frac{20}{52} = \frac{5}{13}$. So, now you can understand how we are defining an event and how we are computing its probability.

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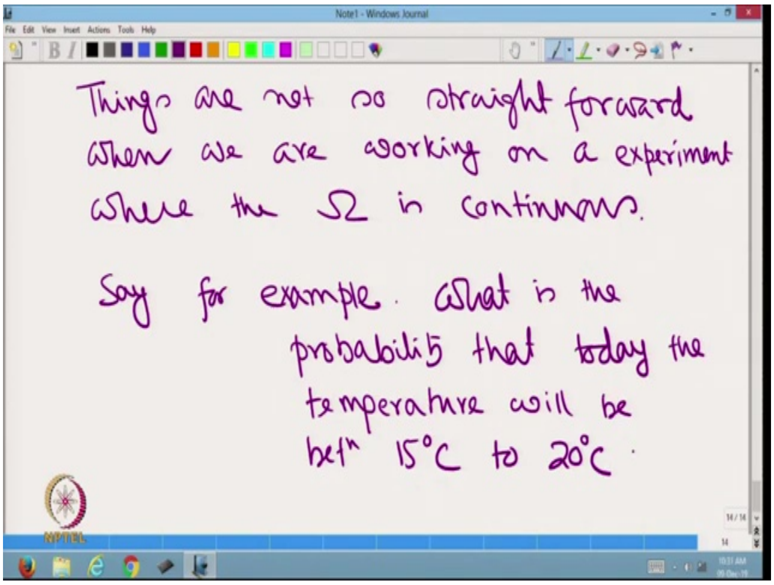

However, things are not so straightforward when we are working on an experiment where the Ω is continuous say for example, what is the probability that today the temperature will be between $15\ °C\ to\ 20°C$?

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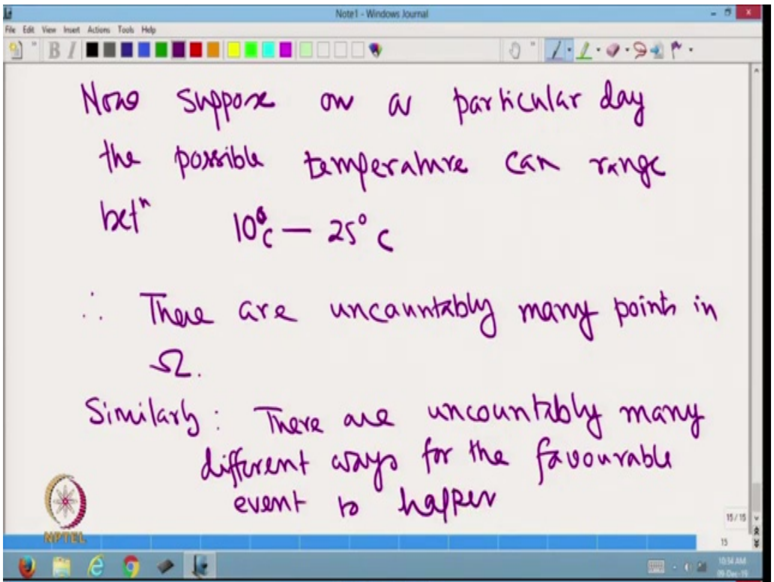

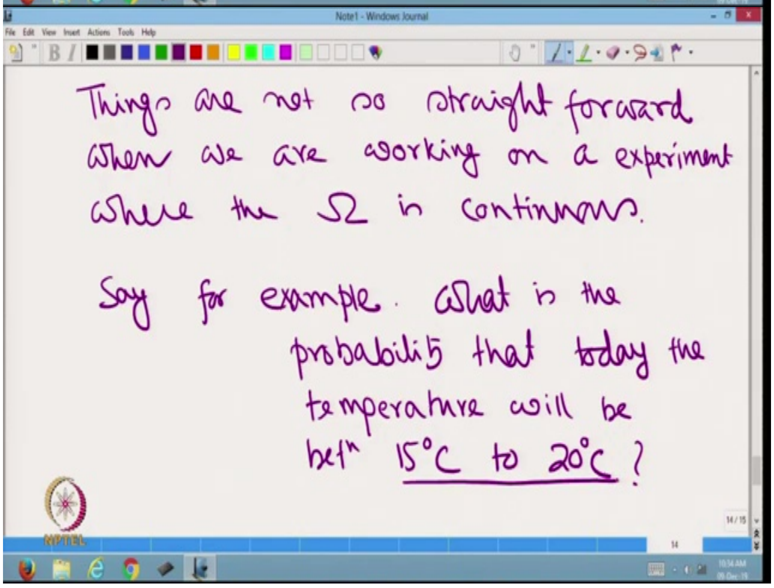

Now, suppose on a particular day, the possible temperature can range between say $10°\text{C}\ to\ 25°\text{C}$. Therefore, there are uncountably many points in Ω , because on the real line between 10 to 25 there are uncountably many real numbers and each one of them can happen to be the temperature of a particular day. Similarly, there are uncountably many different ways for the favorable event to happen, because our favorable event is that our temperature will be between $15°\text{C}\ to\ 20°\text{C}$.

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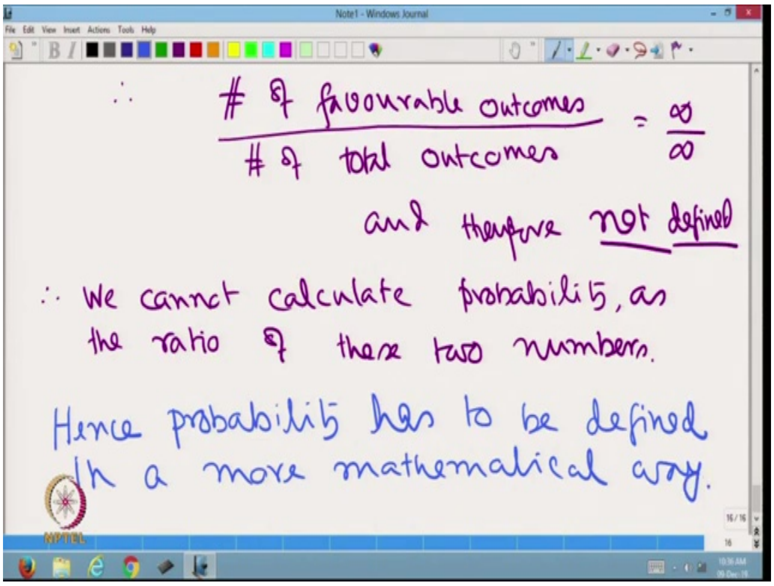

Therefore, both the number of favorable events,

$$\frac{\#\ of\ favourable\ outcomes}{\#\ of\ total\ outcomes} = \frac{\infty}{\infty}$$

and therefore not defined and if it is not defined, therefore, we cannot calculate probability as the ratio of these two numbers. Hence probability has to be defined in a more mathematical way.

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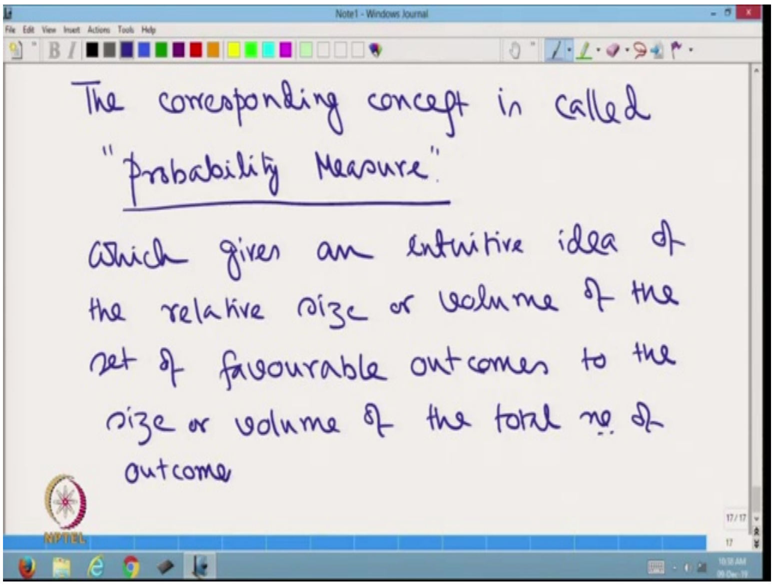

The corresponding concept is called probability measure.

Say for example, measure can be something that gives an idea of the relative size or volume of the set of favorable outcomes to the size or volume of the total number of outcomes.

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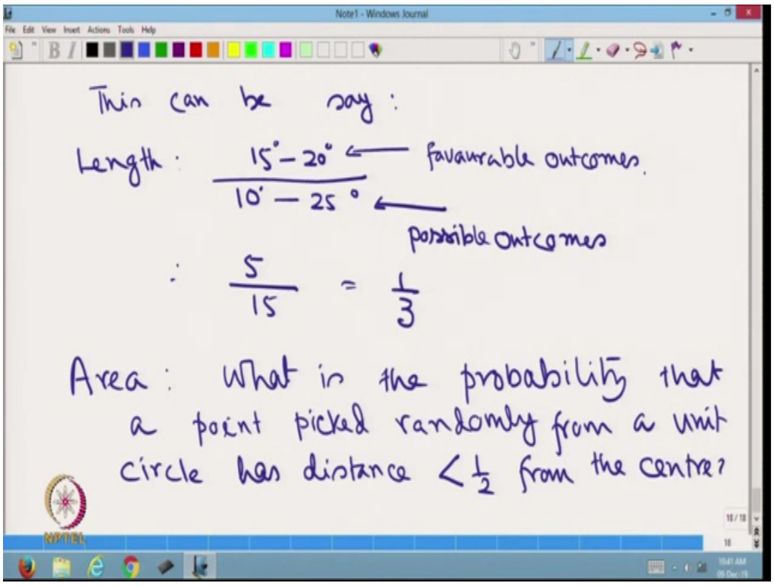

This can be say length for example, $\frac{15°C\ to\ 20°C}{10°C\ to\ 25°C}$. So, this is the possible outcomes and this is favorable outcomes. Therefore, the length is say 5 here the length is 15. So, we could say the probability is something like $\frac{1}{3}$ in a similar way, say sometimes we can use area. Suppose, we are looking at event, what is the probability that a point picked randomly from a unit circle has distance $< \frac{1}{2}$ from the center.

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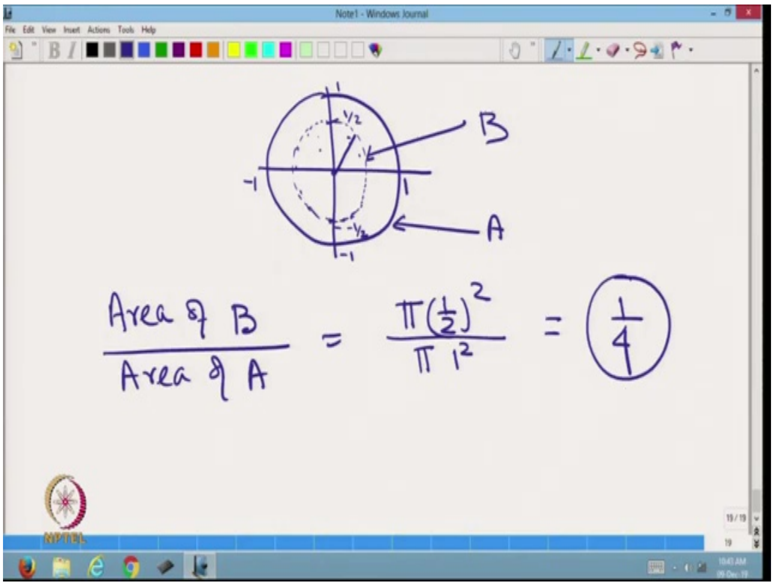

So, let us consider a unit circle, the radius is 1 and suppose this is the center and consider this circle of radius half therefore we shall say a favorable outcome will be, if we choose the point from inside the circle, this circle therefore its distance from the origin will be $< \frac{1}{2}$.

Therefore, area of consider this is B and let us call it A, B upon area of A, that can give us the probability of that event to occur.

That is, the randomly picked up point is at a distance $< \frac{1}{2}$ from the origin.

Now, $\frac{Area\ of\ B}{Area\ of\ A} = \frac{\pi \left(\frac{1}{2}\right)^2}{\pi\ (1)^2}$

Therefore, this probability is $\frac{1}{4}$. I hope I could make it clear how the probability is calculated not just merely counting the number of favorable outcomes to the total number of outcomes. Rather, we look at the subset that is depicting the event and we will compare its measure in comparison with the total measure of Ω and that is what we are trying to drive at.

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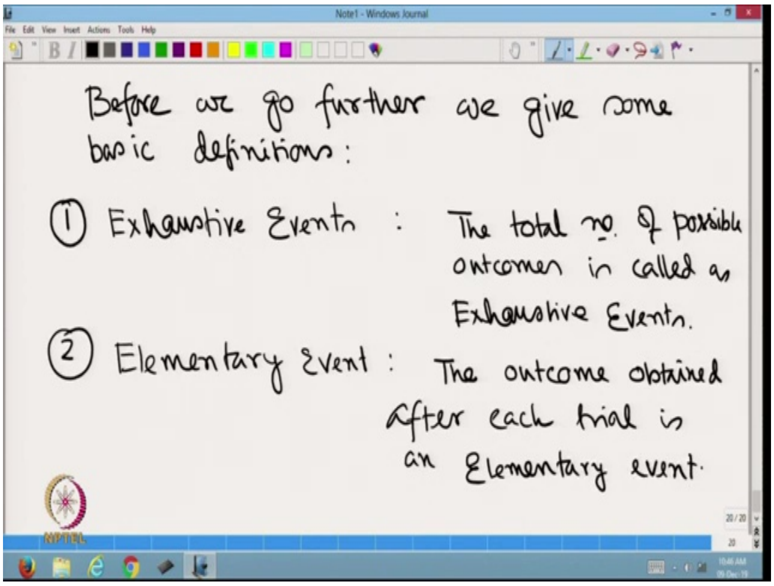

Before we go further, we give some basic definitions. Exhaustive events the total number of possible outcome is called as exhaustive events. Therefor all the members of omega together gives us the set of exhaustive events. Elementary event, the outcome obtained after each trial is an elementary event.

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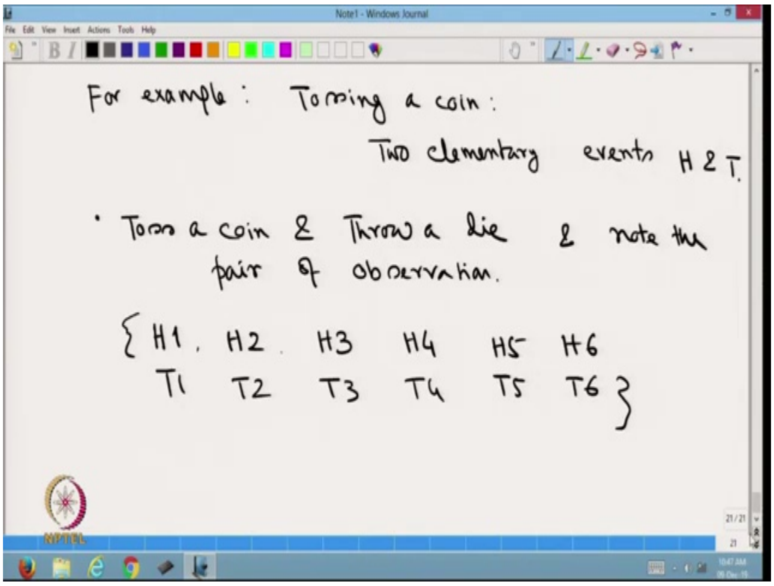

For example tossing a coin 2 elementary events H and T. Suppose my experiment is toss a coin and throw a die and not the pair of observation. Then there are 12 elementary events H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5 and T6. So, this set of 12 pairs is my set of elementary events and together they are the exhaustive event.

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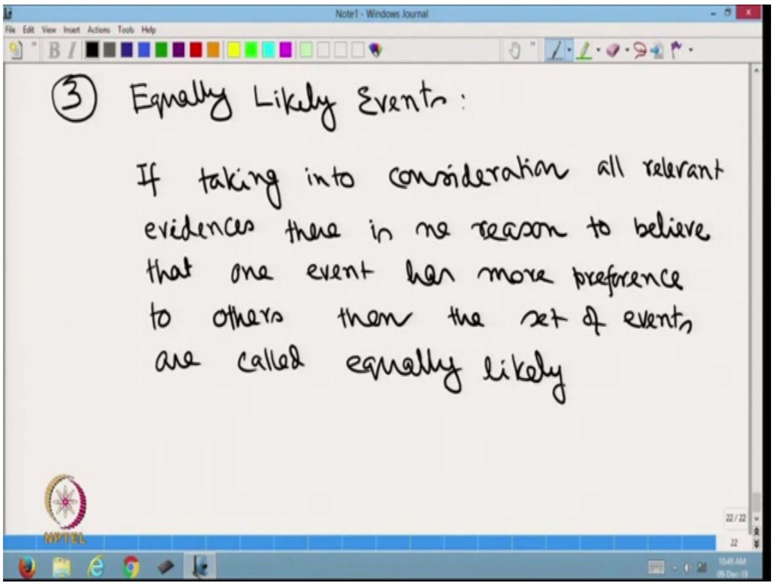

Equally likely events, if taking into consideration all relevant evidences there is no reason to believe that one event has more preference to other then the set of events are called equally likely.

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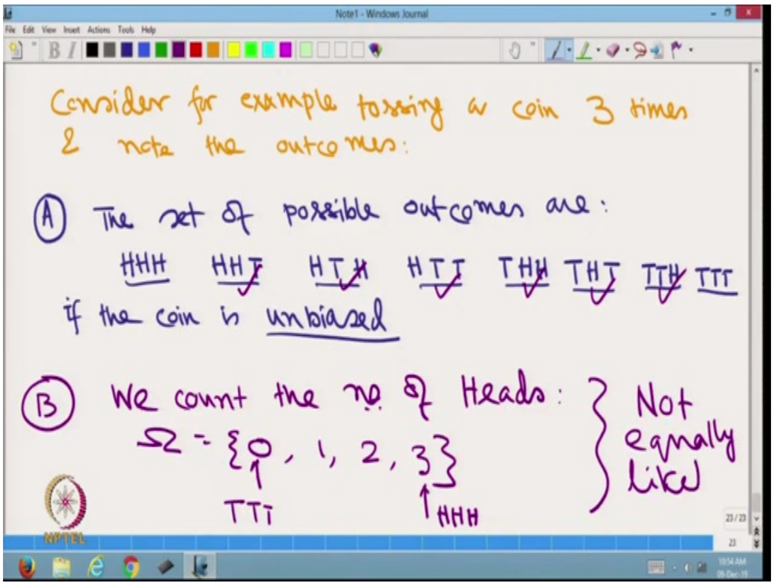

Consider for example, tossing a coin 3 times and note the outcomes. So, let us look at it first from the perspective of the following, the set of possible outcomes are HHH, HHT, HTH, HTT, THH, THT, TTH and TTT. If the coin is unbiased that is both head or tail have equal chance to occur, then all these 8 possible observations may be considered equally likely. On the other hand suppose under the same experimental setup, we count the number of heads then the set of possible outcomes are 0, 1, 2 and 3.

However, we cannot say that they are equally likely, because 0 can happen only if the outcome is TTT, the 3 can happen if only the possible outcome is HHH, because all 3 are heads therefore, count of head is 3 here none of them is head therefore count of head is 0 and therefore as you can understand that 2 can happen for this, this and this. On the other hand, one can happen for this, this and this. Thus, we can understand that there are more chance for 1 or 2 happening instead of 0 and 3 hence not equally likely.

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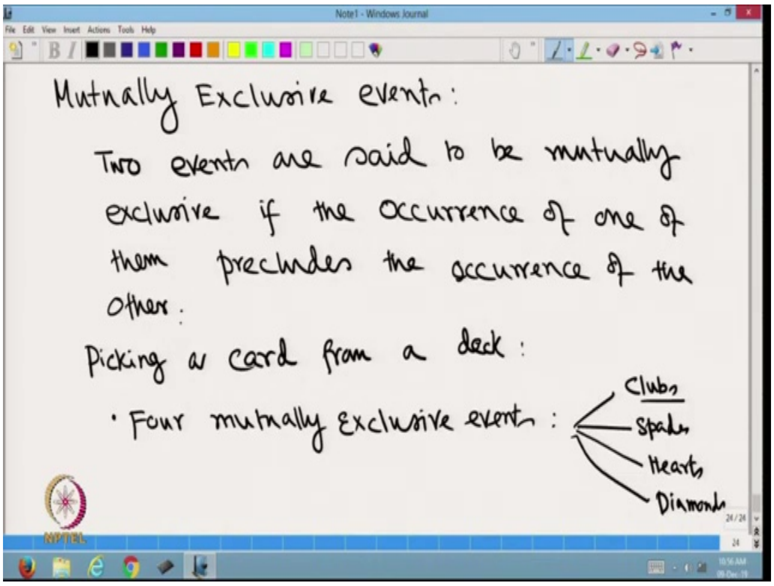

Mutually exclusive event, two events are said to be mutually exclusive if the occurrence of one of them precludes the occurrence of the other. For example, picking a card from a deck there can be 4 mutually exclusive events, getting a club, getting a spade, getting a hearts and getting diamonds. So, if I get a club, we know that we cannot get a spade or a heart or a diamond and similarly for other events.

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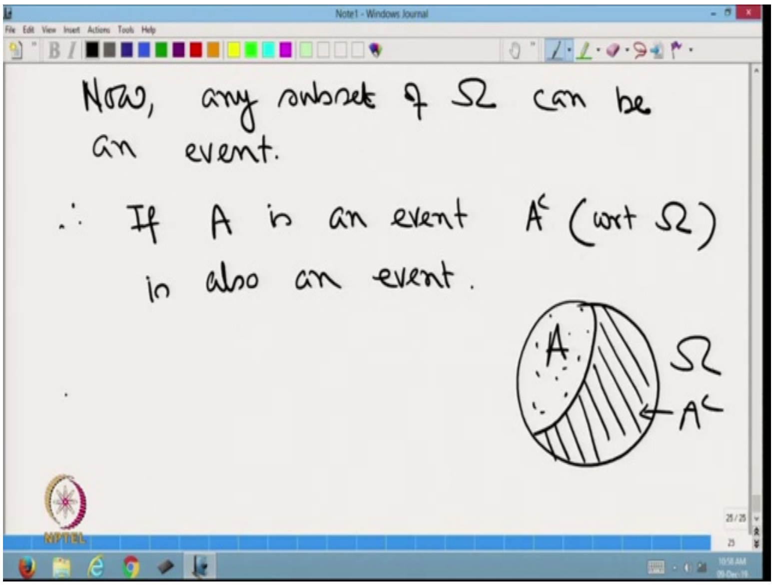

Now any subset of Ω can be an event. Therefore, if A is an event A^c (with respect to Ω) is also an event because A^c is also a subset of Ω or in other words this is my Ω and this is A, so which consists of this elementary points then A^c is also an event.

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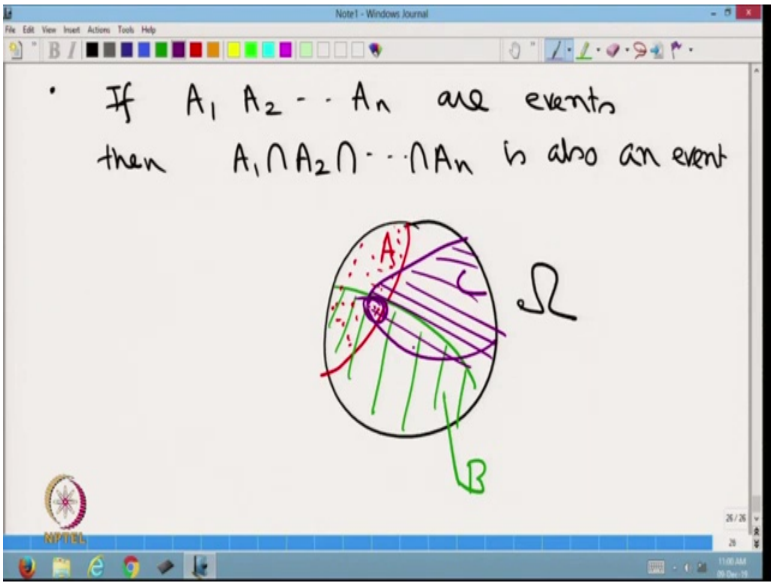

If $A_1, A_2 \ldots, A_n$ are events then $A_1 \cap A_2 \cap \ldots \cap A_n$ is also an event. For example, suppose this is my Ω and this is the event A, this is the event B, this is my event A, this is my event B and this is the event C, then consider this part which is A intersection B intersection C and therefore, this can be an event because it is a subset of Ω .

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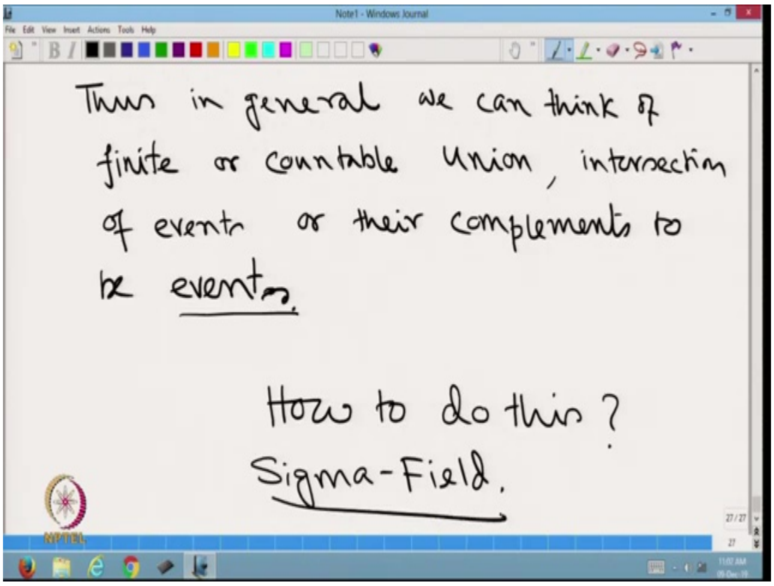

Thus in general we can think of finite or countable union intersection of events or their complements to be events. And whenever we talk about events, we need to think about assigning probability to all possible events that we are looking at, under the random experiment that we are pursuing for studying our Probability.

Question is how to do this?

The basic mathematical trick is said to be that of $\sigma - field$.

I am not sure how many of you are familiar with the term $\sigma - field$, but this is the basic mathematical notion that we shall need along with something which is called a probability measure that is helpful for us for defining probability. Okay students, I stopped here now, in the next class, I shall start with these mathematical concepts for our study of probability theory.

Okay friends,

thank you.