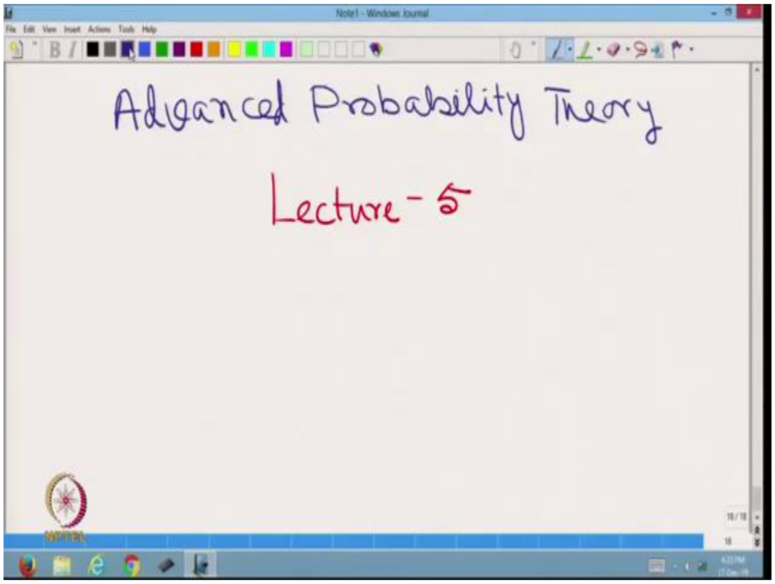
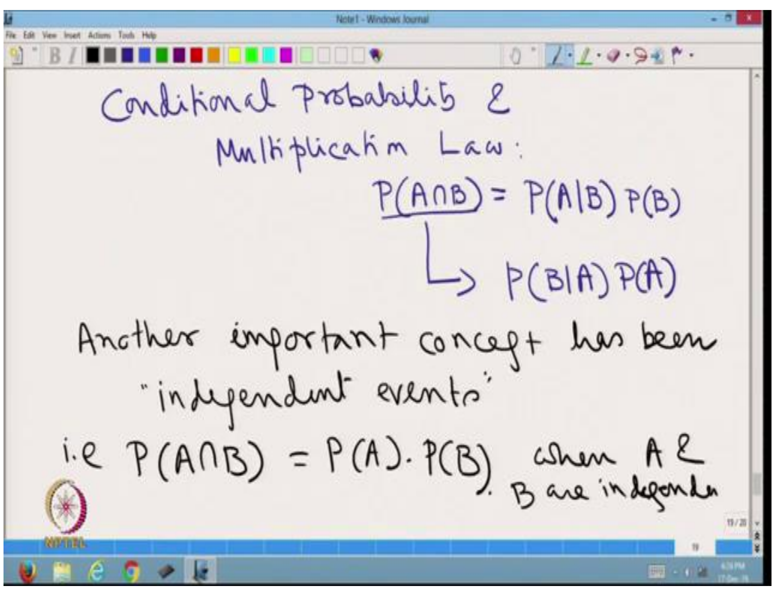
Advanced Probability Theory
Professor Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture 5

(Refer Slide Time: 00:24)



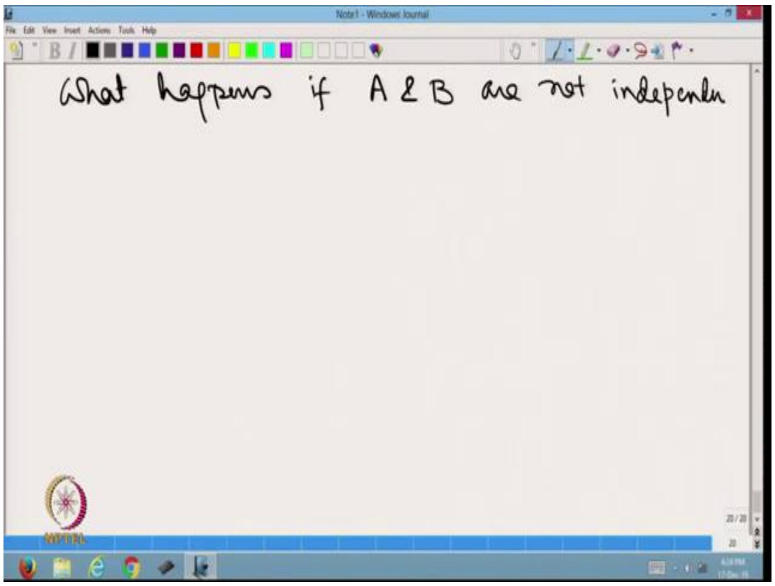
Welcome students, to the MOOC's lecture series on Advanced Probability Theory, this is lecture number 5.

(Refer Slide Time: 00:28)



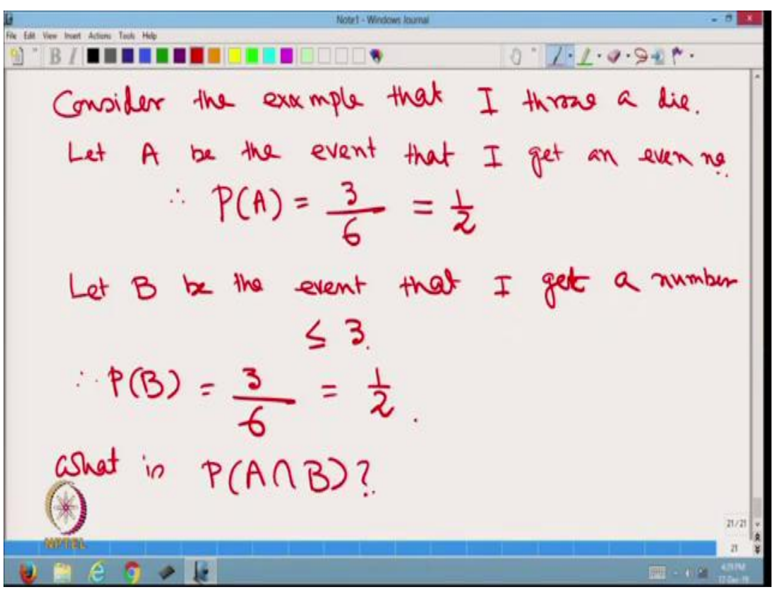
In the last class, we have seen Conditional Probability and Multiplication Law that is, probability of A intersection B is equal to probability of A given B into probability of B, in a very symmetric way also, we can write it as probability of B given A into probability of A. So, depending upon the situation, we try to obtain the probability of the joint occurrence of A intersection B. We shall see many problems later. Another thing we have seen independent events, that is probability of A intersection B is equal to probability of A into probability of B. When A and B are independent.

(Refer Slide Time: 02:31)



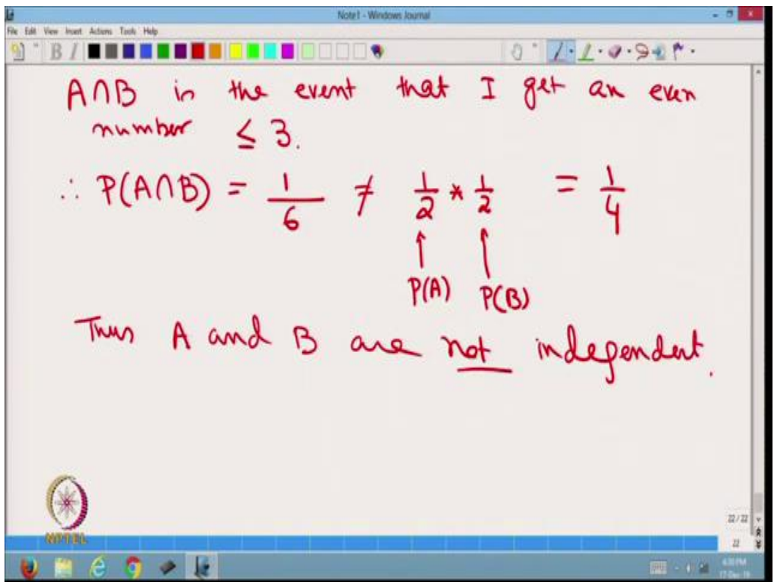
What happens if A and B are not independent?

(Refer Slide Time: 02:50)



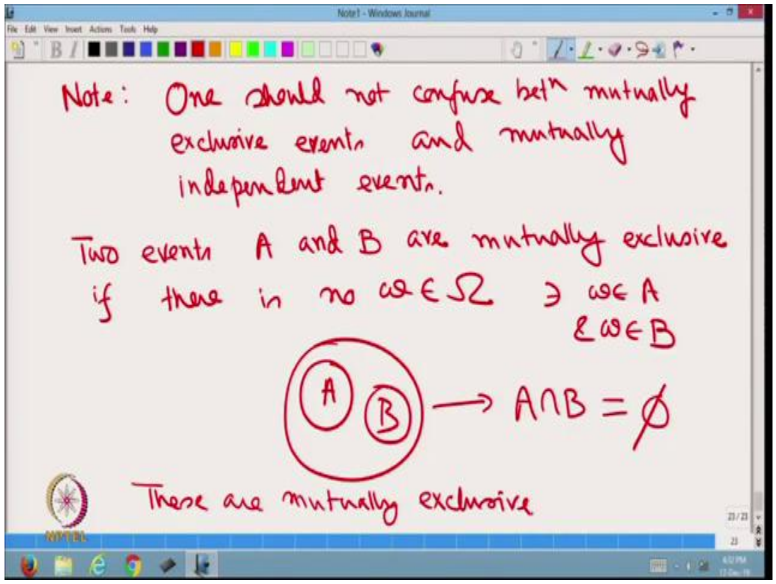
Considered the example that I throw a die, let A be the event that I get an even number, therefore probability of A is equal to the number of ways I can get even number is 3 because if the outcome is 2 or 4 or 6 out of 6 possible outcomes therefore, this is half. Let B be the event that I get a number less than equal to 3. Therefore, probability of B is equal to number of ways of getting a value less than equal to 3 is 3, because I can get 1 or 2 or 3, which are my favorable outcomes divided by the total number of possible outcomes, which is equal to half, what is probability of A intersection B?

(Refer Slide Time: 04:36)



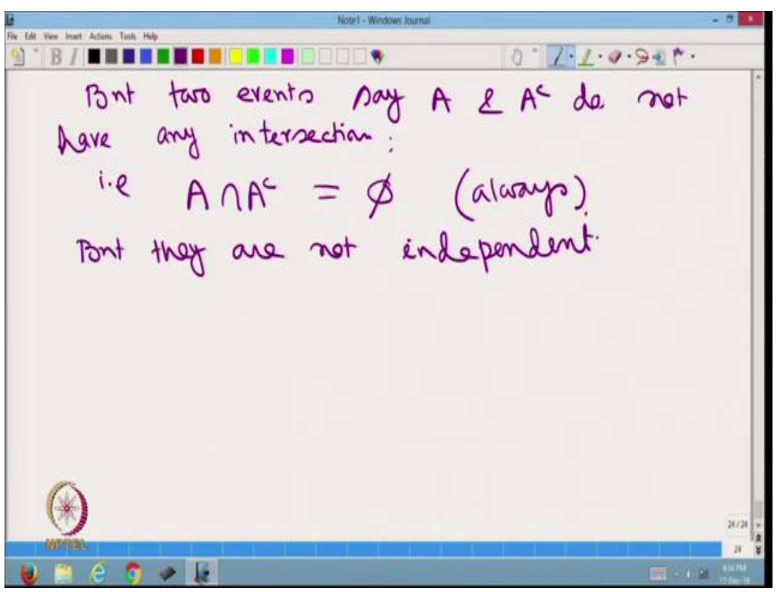
Now, A intersection B is the event that I get an even number less than equal to 3, therefore, probability of A intersection B is equal to obtaining one even number less than equal to 3, that can happen only in one way. That is, if I get 2 divided by possible outcomes is equal to 1 by 6. Is it same as the product of the two individual events? No, because the product of these two probabilities is half into half. Which is equal to 1 by 4, thus A and B are not independent.

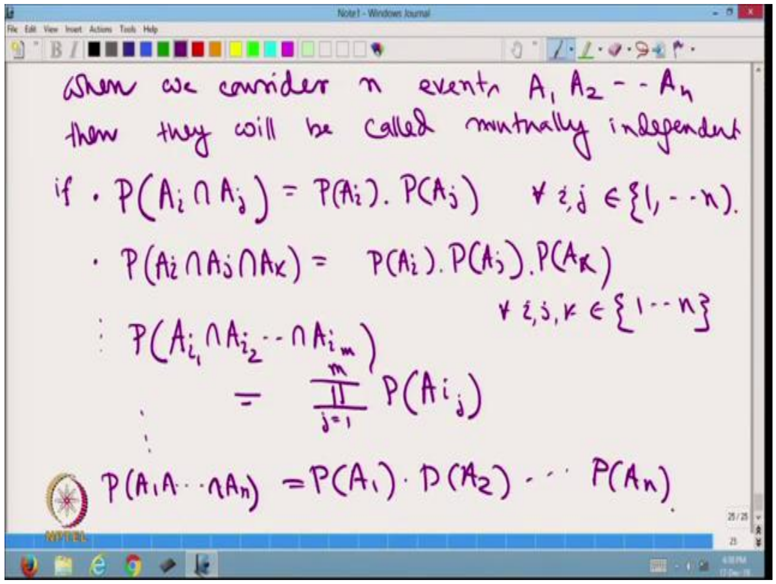
(Refer Slide Time: 05:57)



One should not confuse between mutually exclusive event and mutually independent events. Two events A and B mutually are exclusive if there is no omega belonging to omega, such that omega belongs to A and omega belongs to B, say for example, if this is my A and this is my B then or their intercession is phi. Therefore, these are mutually exclusive.

(Refer Slide Time: 07:54)



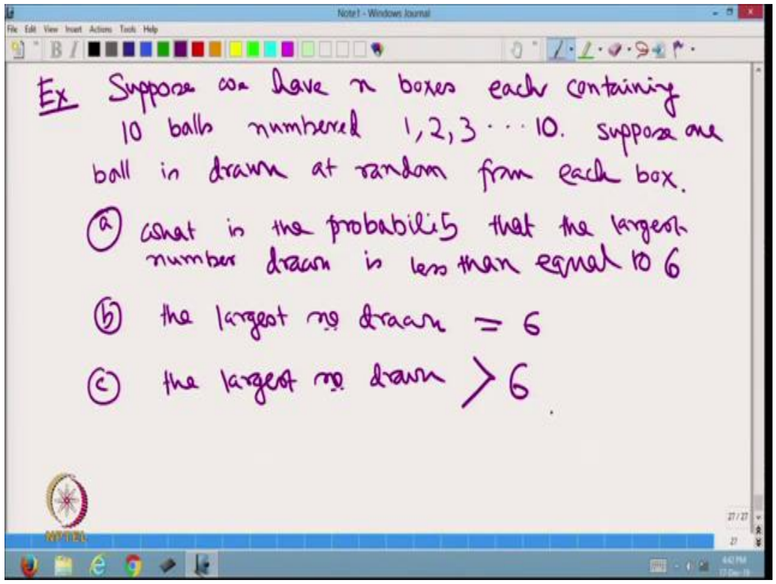


But two events say A and A compliment any intersection, that is A intersected with A compliment is equal to phi, always. But they are not independent when we considered n events A1, A2, An then they will be called mutually independent. If probability Ai intersection Aj is equal to probability of Ai into probability of Aj for all ij belonging to 1 to n.

Secondly, for all triplets probability Ai intersection Aj intersection Ak is going to be probability of Ai into probability of Aj interpretability of Ak for all i, j, k belonging to 1 to n or in fact, for any k of events that should happen, therefore we can write probability of Ai1 intersection, Ai2, intersection Aim is equal to probability of j is equal to 1 to m probability of Aij.

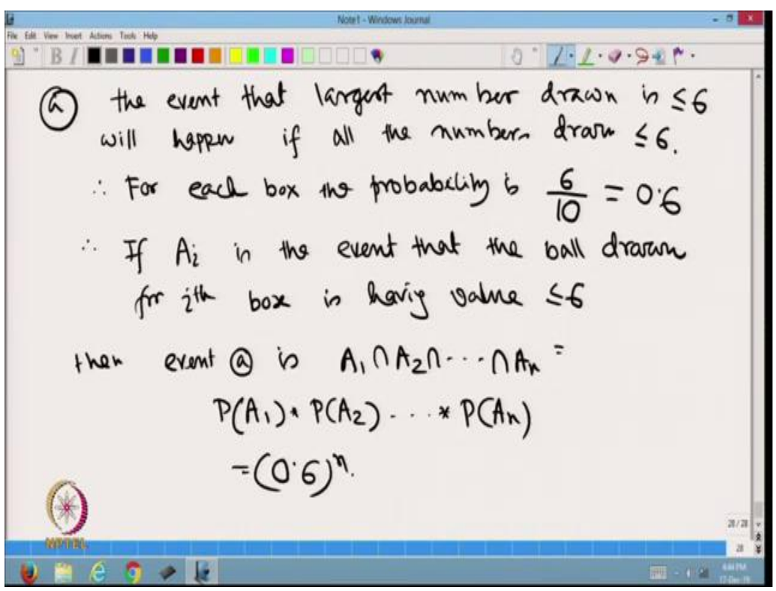
Where each subscript belongs to 1 to n and finally, probability of A1, A2, An is equal to probability of A1 into probability of A2 to probability of An, thus we can see that mutual independence is a very, very strong relationship which needs to satisfy all these different equalities.

(Refer Slide Time: 11:37)



Suppose, we have n boxes each containing 10 balls numbered 1, 2, 3 up to 10 suppose, one ball is drawn at random from each box, what is the probability that the largest number drawn is less than equal to 6, b the largest number drawn is equal to 6 and c, the largest number drawn is greater than 6. So, these are three different events.

(Refer Slide Time: 13:47)

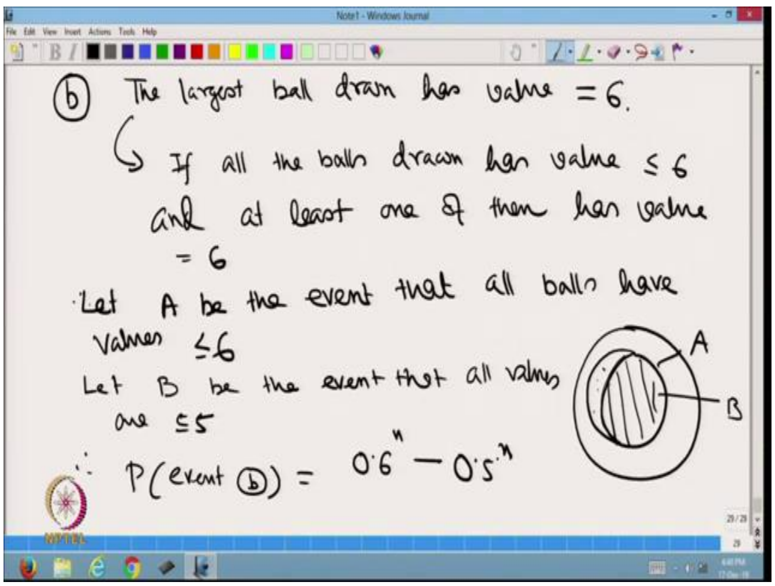


Suppose we want to compute the probabilities, a the event large largest number drawn is less than equal to 6 will happen if all the numbers drawn are less than equal to 6.

Therefore, for each box the probability is 6 upon 10. There are 10 balls and I am looking at a number less than equal to 6 therefore, it is coming out to be 0.6.

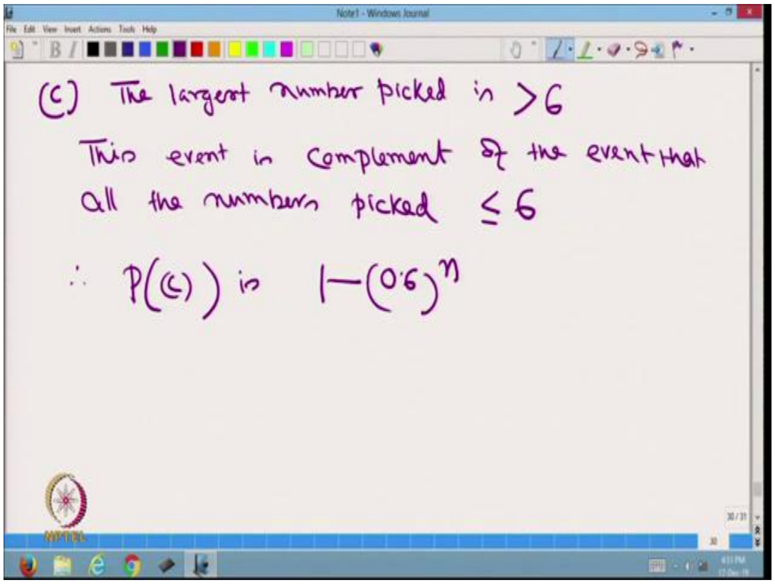
Therefore, if Ai is the event that the ball drawn from i th box is having value less than equal to 6, then the event A is A1 intersection, A2 intersection, An is equal to, because they are now going to be independent, this is going to be A1, A2 into probability of An which is going to be 0.6 n.

(Refer Slide Time: 16:00)



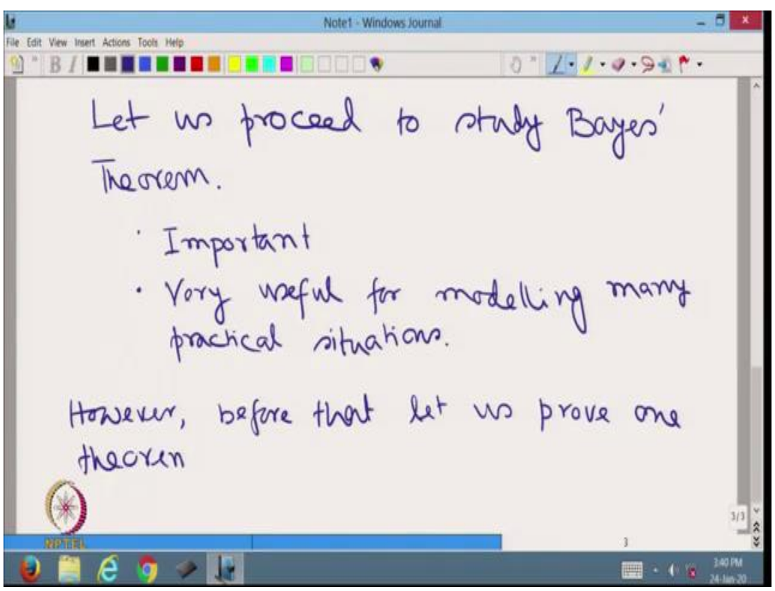
Event b is, the largest ball drawn has value equal to 6. Therefore this event can happen if all the balls drawn has value less than equal to 6 and at least one of them has value equal to 6 so, let A be the event that all balls have values less than equal to 6. Suppose, this is the event all values of less than equal to 6, and of which each suppose is subset is that all values are less than equal to 5. Therefore, the probability of the event B is equal to 0.6 to the power n minus 0.5 equal to the power n.

(Refer Slide Time: 18:16)



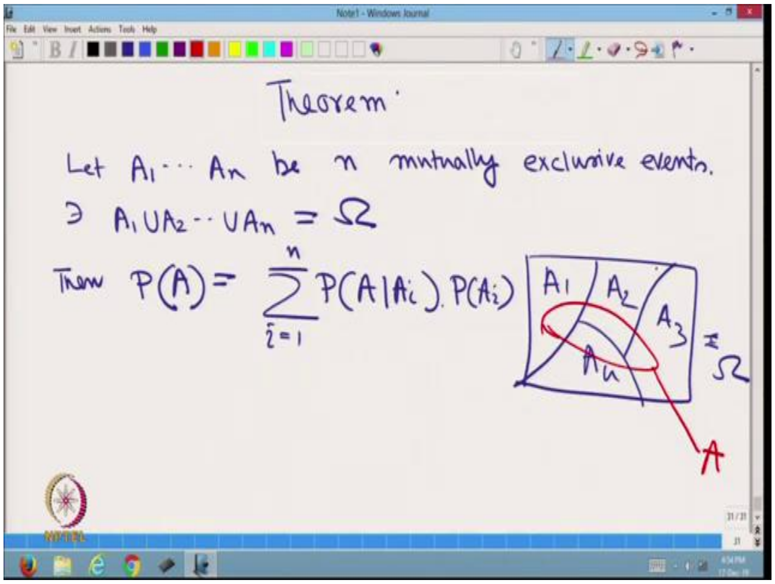
C, the event is that the largest number picked is greater than 6. This event is complement of the event that all the numbers picked are less than equal to 6 therefore, probability of the event C is 1 minus 0.6 whole to the power n. So, these are some interesting way of solving some simple problems.

(Refer Slide Time: 19:32)



Okay students, with that, let us proceed to study an important concept of probability, which is called Bayes theorem, it is important also it is very useful for modeling many practical situations however, before that let us prove one theorem.

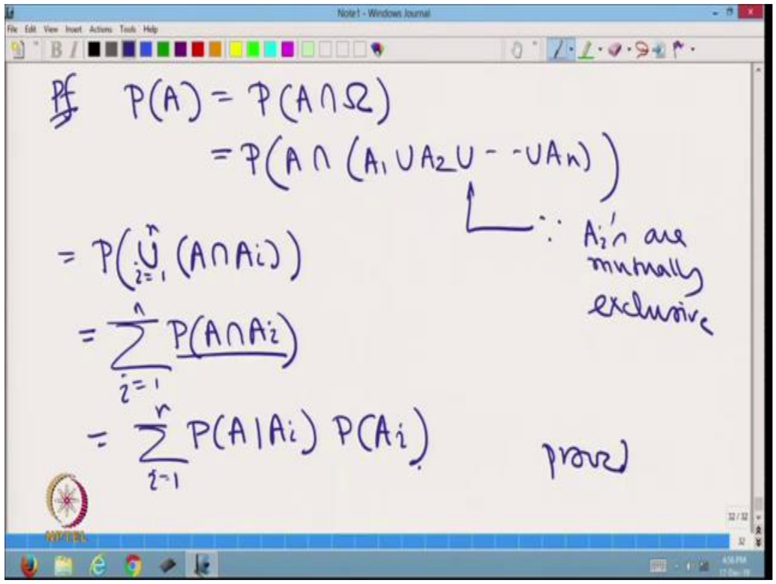
(Refer Slide Time: 20:47)



So, let A1 A2 An be n mutually exclusive events such that A1 union A2 union An is equal to omega. So, for example, if this is our omega and I partition it into four, this is A1, this is A2, this is A3, this is A4 and we can see that A1, A2, A3, A4 are mutually exclusive and their union is giving us omega, then probability of any event A is equal to sigma probability of A given Ai multiplied by probability of Ai, i is equal to 1 to n.

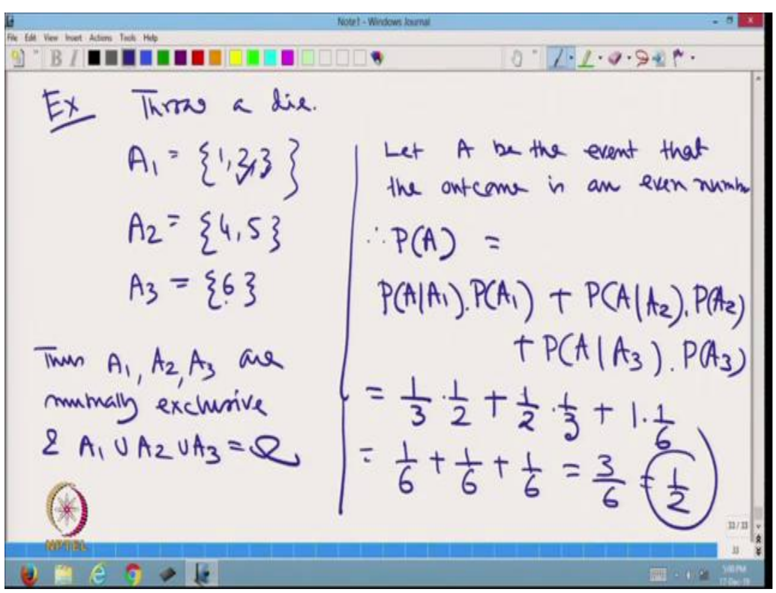
In this case with respect to this example, suppose my event is this. This is my A, then probability of A is going to be probability of A given A1 multiplied by the probability of A1, probability of A given A2 multiplied by the probability of A2 and like that others.

(Refer Slide Time: 22:48)



Probability of A is equal to probability of A intersection omega is equal to probability of A intersection A1 union A2 union An, this is possible, since Ai’s are mutually exclusive, is equal to probability of union over i equal to 1 to n A intersected with Ai is equal to sigma i equal to 1 to n probability A intersected with Ai, this is because we have already seen that these intersections are all usually disjoined. Now, this is equal to sigma i is equal to 1 to n, and probability A intersection Ai we can write it as, probability of A given Ai multiplied by probability of Ai so, proved.

(Refer Slide Time: 24:29)

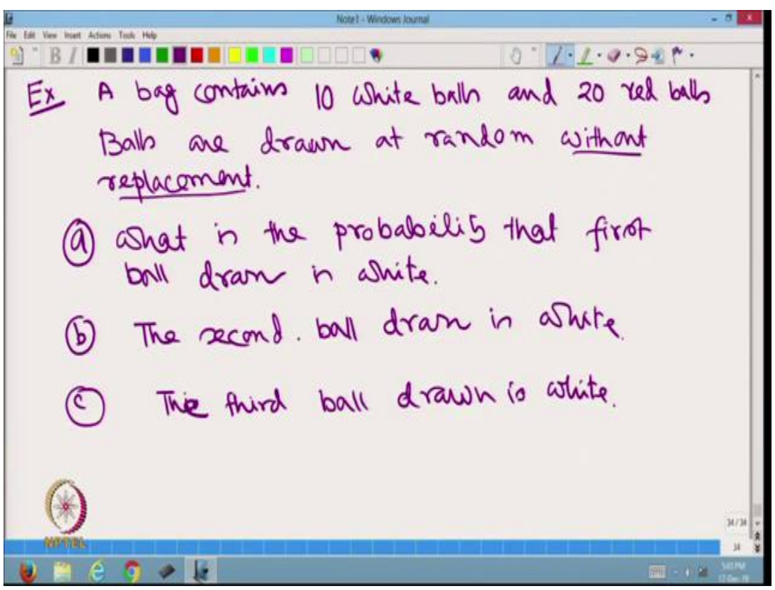


For example, we throw a die. Let A1 be 1, 2, 3. A2 is 5 and 5. A3 is equal obtaining 6 thus A1, A2, A3 are mutually exclusive. And A1 union A2 union A3 is equal to omega. Let A be the event that the outcome is an even number therefore by the above theorem, probability of A is equal to probability of A given A1 into probability of A1 plus probability of A, given A2 multiplied by probability of A2 plus probability of A given A3 multiplied by probability of A3 is equal to probability of getting an even number.

Given that A1 has occurred, that is 1 by 3, because there is only one even number out of three multiplied by probability of A1 that is half plus probability of A given A2. So, here we can get that there are two way possible values. And one of them is even so, that probability is half multiplied by probability of A2, which is going to be 1 by 3 because occurrence of 4 or 5 is 1 by 3 plus there is one number 6, which is even.

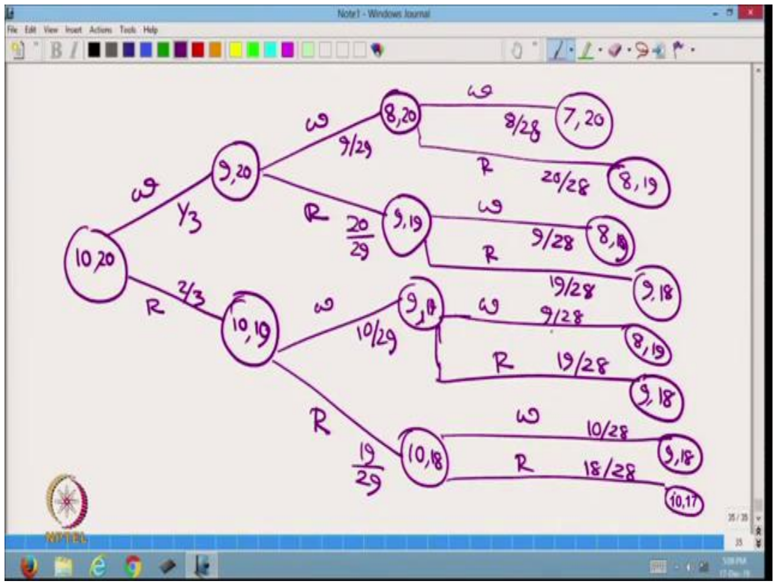
Therefore, this is going to be 1 multiplied by probability of A3, which is going to be 1 by 6 is equal to 1 by 6, plus 1 by 6, plus 1 by 6 is equal to 3 by 6 is equal to half, which is probability of an even number.

(Refer Slide Time: 27:36)



Let me give you another interesting problem. A bag contains 10 white balls and 20 red balls, balls are drawn at random without replacement a, what is the probability that first ball drawn is white? b, the second ball drawn is white. And c, the third ball drawn is white? We have seen that after the first ball, since it is without replacement, the probabilities keep changing. Then how do we compute this probability? So, we shall solve them for that.

(Refer Slide Time: 29:43)



Let me draw this diagram, here the situation is 10 white and 20 red, after the first ball is white I get 9 white and 20 red and if I draw a red ball then we get 10 white and 19 red. So, what is the probability? In this case the probability is going to be 10 upon 30 is equal to one third.

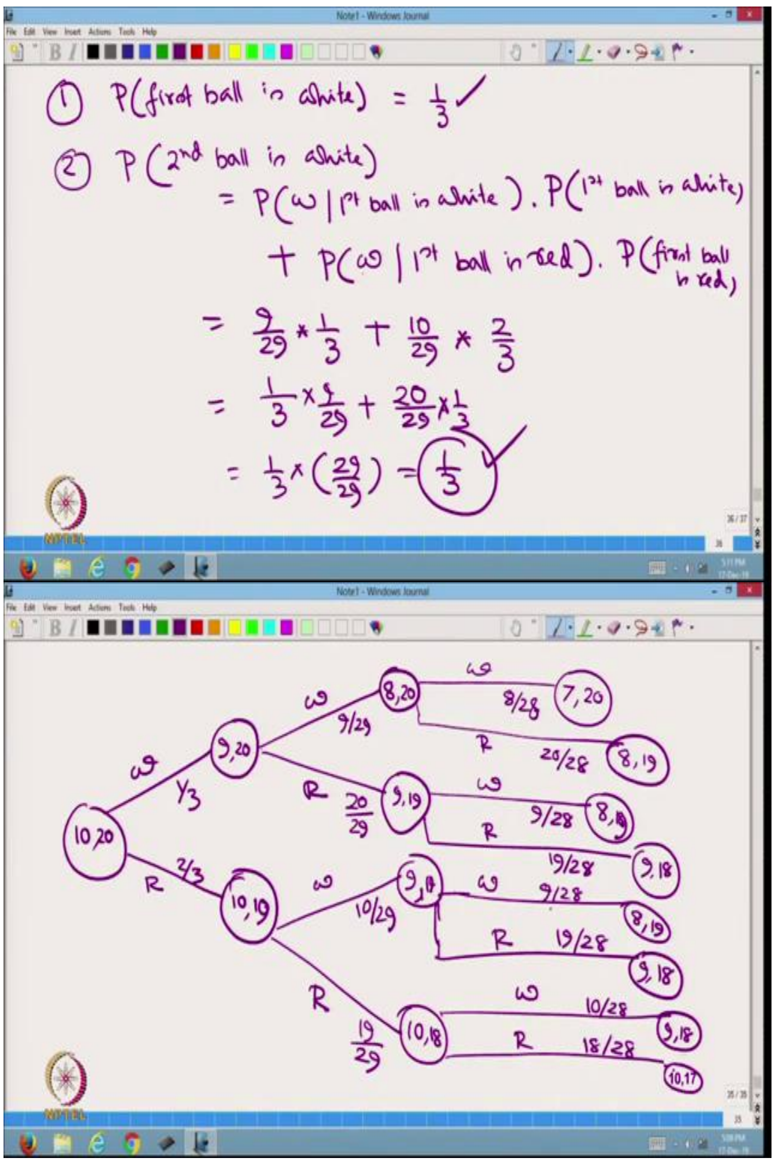
In this case the probability is going to be two third, after this we are drawing the second ball, which may be white, which may be red. And if it is a white, the configuration becomes 8 white and 20 red and this probability is going to be 9 upon 29. And if a red ball is drawn it is going to be 9 comma 19 and this probability is going to be 20 upon 29.

Similarly, from here we will get 9 comma 19 and that probability is going to be 10 upon 29. And from here we are going to be, if it is a red, then we are going to get 10 whites and 18 red and that probability is going to be 19 upon 29. Now, we are drawing the third ball then what we are going to get is 7, 20 and that probability 8 upon 28 if it is a red, we are going to get 8 comma 19. And that probability is going to be 20 upon 28.

Similarly, if this is a white, that is going to be 8 comma 19 and that probability is going to be 9 upon 28. If this is a red, this is going to be 9 comma 18 and that probability is going to be 19 upon 28. If this is a white, we are going to get 8 comma 19 and that probability is going to be 9 upon 28, this is going to be red that probability is going to be 19 upon 28 and we are going to get 9 comma 18.

And similarly from here, if it is a white then the configuration is 9 comma 18 and that probability is 10 by 28. And similarly, for red it is going to be 10 comma 17 and that probability is 18 upon 28. So, this is the picture.

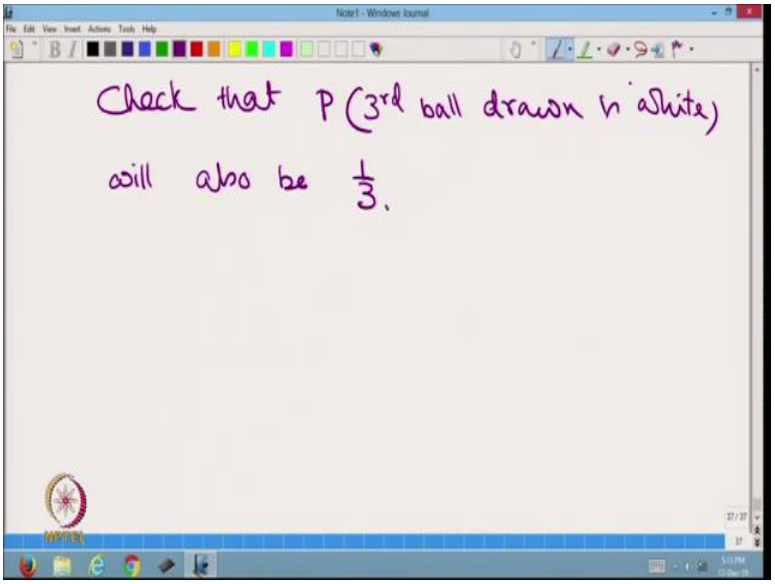
(Refer Slide Time: 33:59)



Therefore, let us now calculate probability first ball is white is equal to, as you can understand this is going to be 1 by 3, probability second ball is white, therefore its probability is by the earlier theorem, probability of white, given first ball is white multiplied by probability, first ball is white plus probability white, given that the first ball is red multiplied by probability first ball is red is equal to.

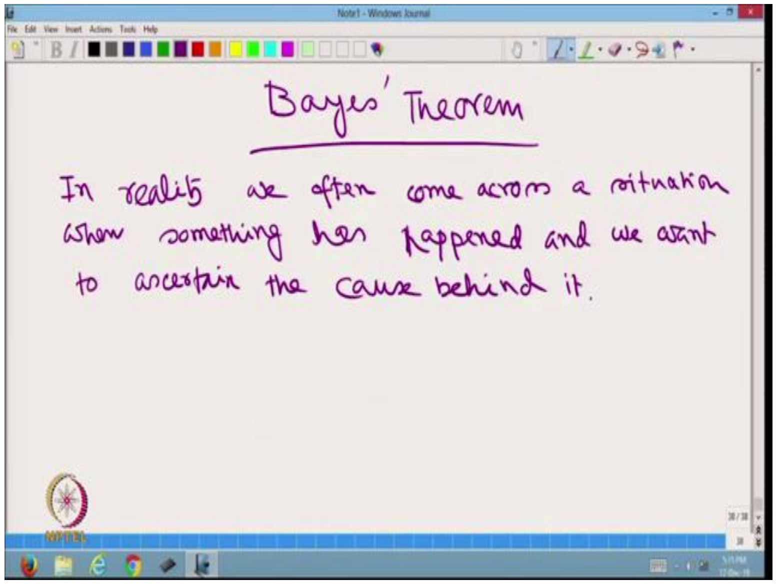
So, probability of white given the first ball is white, is equal to 9 by 29 into probability first ball is white is equal to 1 by 3, plus probability of second ball is white given first ball is red is equal to 10 by 29 multiplied by probability, the first ball is red, which is equal to 2 by 3 is equal to 1 by 3 into 9 by 29 plus 20 upon 29 into 1 by 3 is equal to 1 by 3 into 29 upon 29 is equal to 1 by 3. This is same as the probability that the first ball drawn is white.

(Refer Slide Time: 36:26)



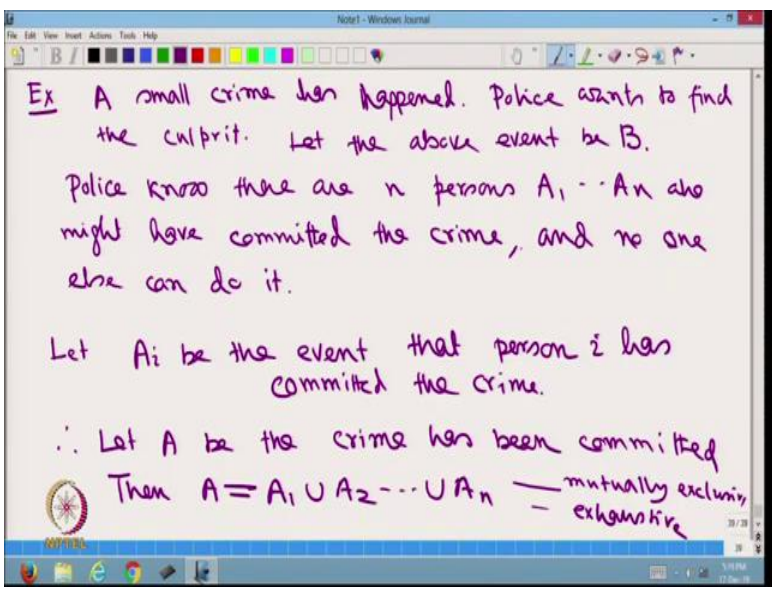
Check that probability, third ball drawn is white will also be 1 by 3, I have given you the best diagram, we have to make use of this diagram to find out that even in the third draw if you are getting a whit ball then that probability is going to be 1 by 3.

(Refer Slide Time: 37:09)



Okay friends, now let me state a very important theorem, which is called Bayes theorem, in reality we often come across a situation when something has happened and we want to ascertain the cause behind it. So, this is a very, very practical situation.

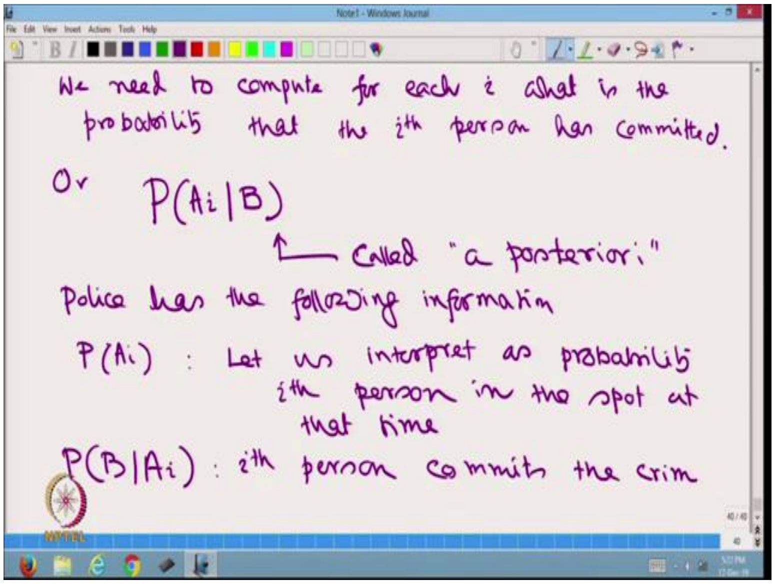
(Refer Slide Time: 38:22)



Say for example, a small crime has happened. Police wants to find the culprit, let the above event be B, that small crime has happened, the police knows there are n persons A1, A2, An who might have committed the crime and no one else can do it. Let Ai be the event that person i has committed the crime.

Therefore, let A be the event the crime has been committed then A is equal to A1 union A2 union An and these are mutually exclusive and exhaustive that means there is no other way that the crime has happened.

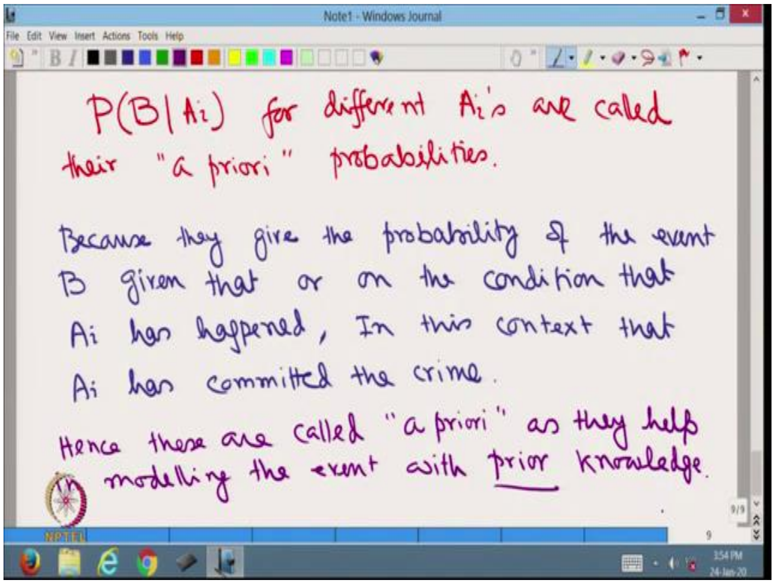
(Refer Slide Time: 41:04)



Now, we need to compute for each i what is the probability that the i th person has committed or we want to know probability of the Ai given that the crime has happened, that is B. So, we are looking at probability that Ai has committed the crime given something has happened.

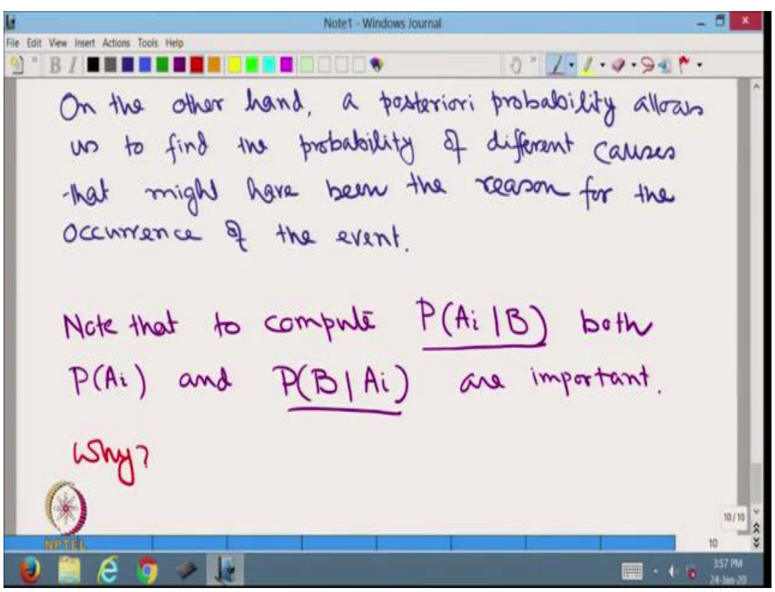
So, such probabilities are called a posteriori because we are looking at the probability after something has happened. Now, police has the following information that probability of Ai, let us interpret us probability i th person in the spot at that time and probability of B given Ai that i th person commits the crime.

(Refer Slide Time: 43:31)



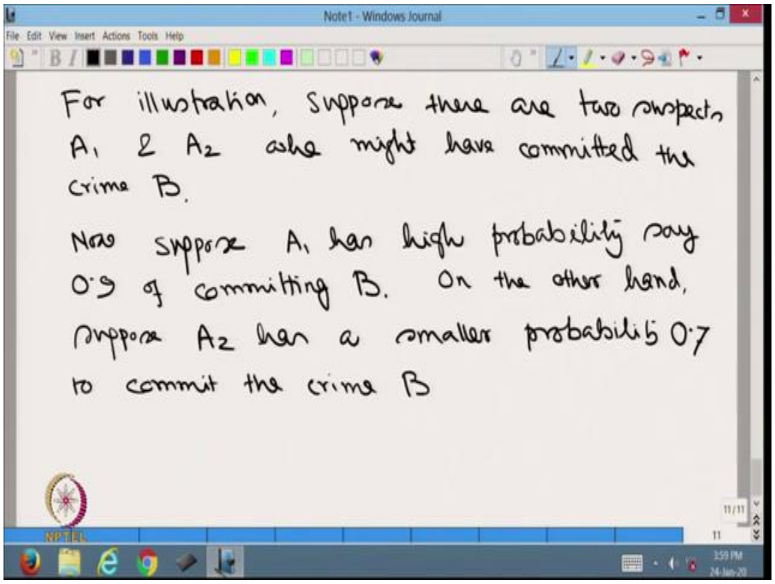
Probability B given Ai for different Ai's are called their a priori probabilities. This is because they give the probability of the event B given that or on the condition that Ai has happened or in this context that Ai has committed the crime. Hence, these are called a priori as they help in modeling the event with prior knowledge that is, knowledge that we have before the event that has happened.

(Refer Slide Time: 45:48)



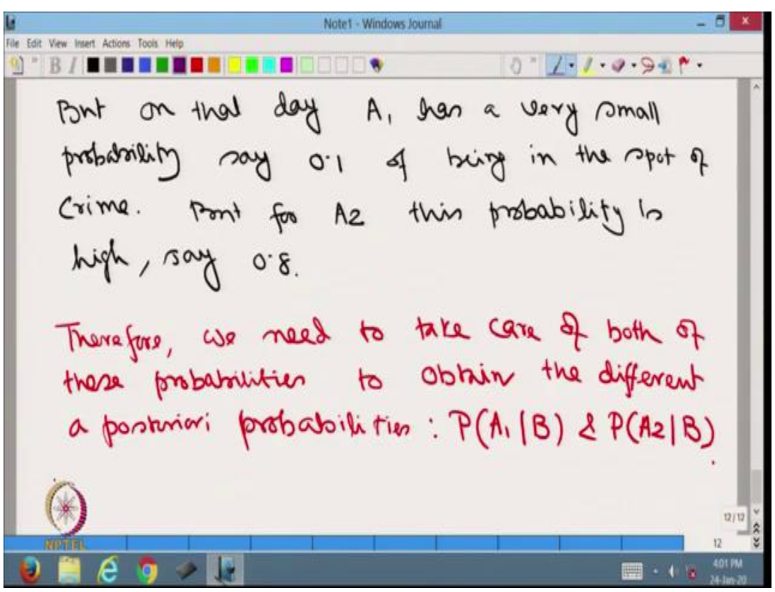
On the other hand, a posteriori probability allows us to find the probability of different causes that might have been the reason for the occurrence of the event. Note that to compute probability Ai given B that is the a posteriori probability both probability of Ai and probability of B given Ai, which is the a priori probability are important. Why?

(Refer Slide Time: 47:50)



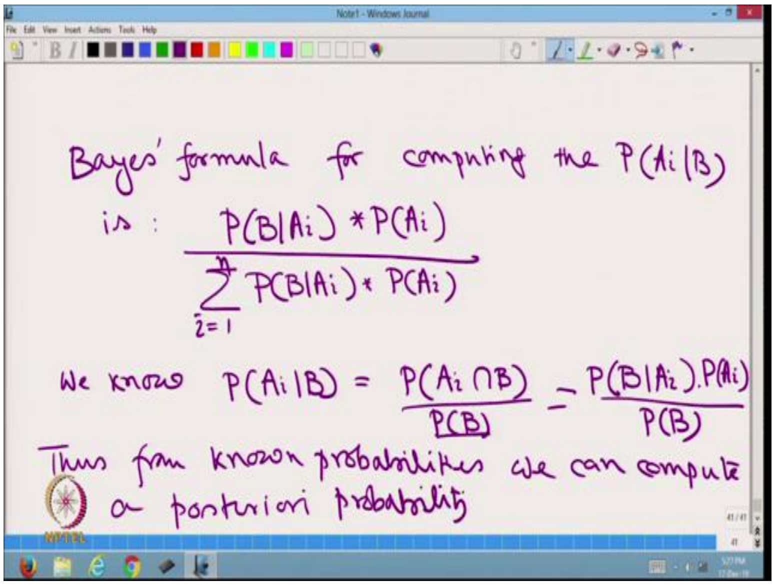
So, for illustration suppose, there are two suspects, A1 and A2 who might have committed the crime B. Now, suppose A1 has high probability say 0.9 of committing the crime B on the other hand, suppose A2 has a smaller probability say 0.7 to commit the crime B.

(Refer Slide Time: 49:40)



But on that day A1 has a very small probability say 0.1 of being in the spot of crime, but for A2 this probability high say 0.8. Therefore, we need to take care of both of them these probabilities to obtain the different a posteriori probabilities, probability of A1 given B and probability A2 given B. Hope the concept is clear.

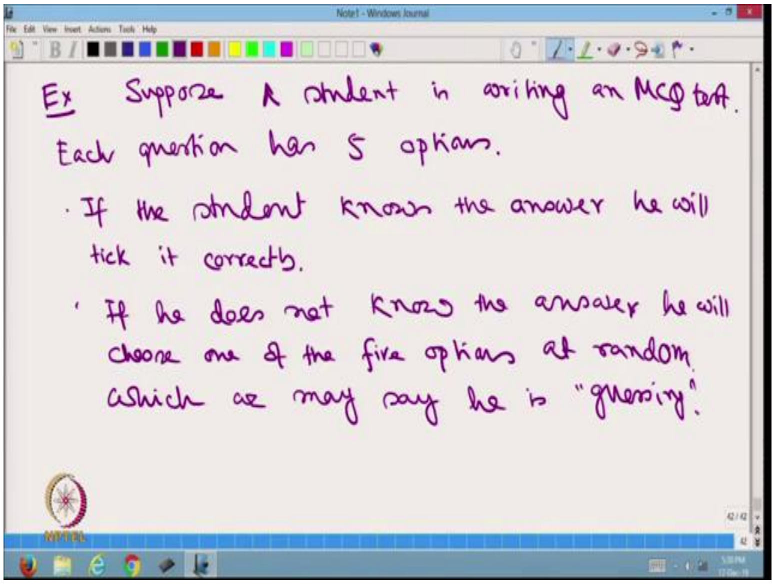
(Refer Slide Time: 51:51)



Then Bayes formula for computing the probability of Ai given B is probability of B given Ai multiplied by probability of Ai divided by sigma probability of B given Ai into probability of Ai i is equal to 1 to n. Why? Because we know probability of Ai given B is equal to probability of Ai intersected with B divided by probability of B.

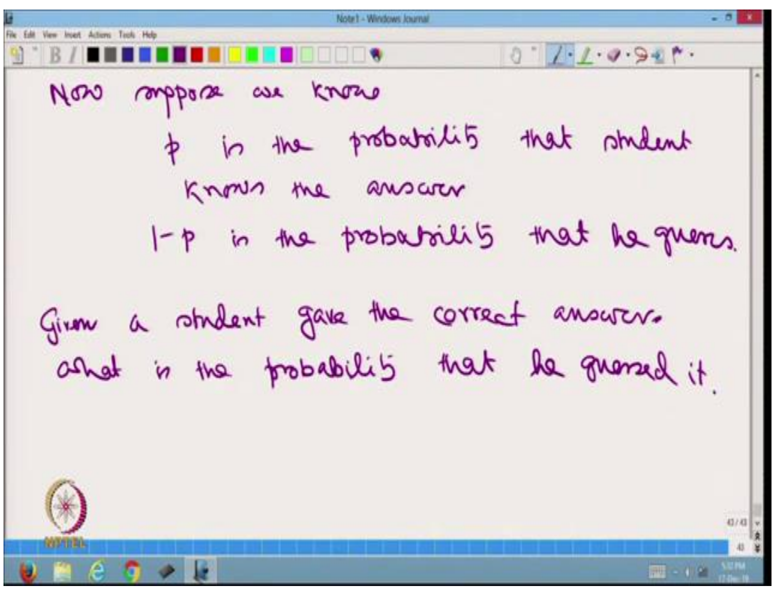
Also this we can write it as probability B given Ai multiplied by probability of Ai divided by probability of B this we get by exchanging the role of Ai and B and probability of B we know that because Ai's are disjoint event. Therefore, sometime back we have seen this formula that probability of B is equal to probability of B given Ai into probability of Ai that we are assigning from i is equal to 1 to n thus, from known probabilities we can compute a posteriori probability.

(Refer Slide Time: 54:22)



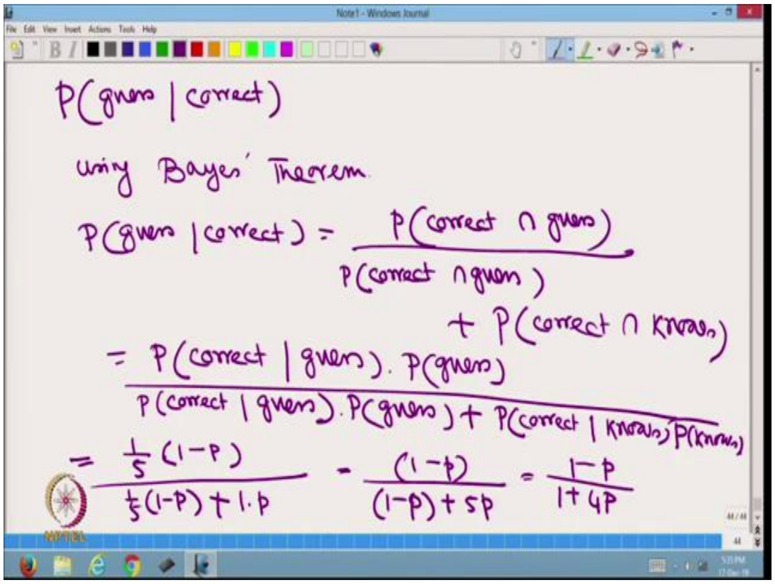
Let me give an example. Suppose, a student is writing an MCQ test, each question has 5 options. So, if the student knows the answer he will tick it correctly. If he does not know the answer he will choose one of the five options at random, which we may say he is guessing.

(Refer Slide Time: 56:17)



Now, suppose we know p is the probability that student knows the answer and 1 minus p is the probability that he guesses. Now, the question is given a student gave the correct answer, what is the probability that he guessed it? I hope you understand the question.

(Refer Slide Time: 57:45)

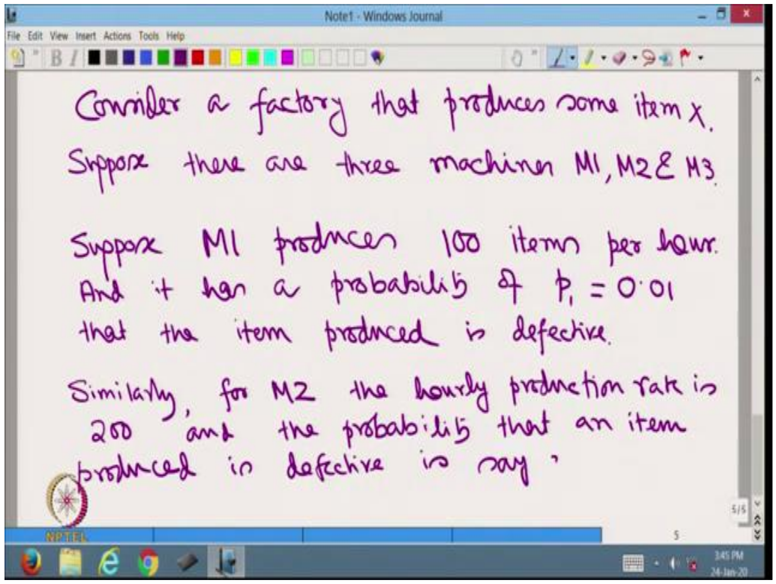


Therefore, if you look at it, we are looking at probability of guess given correct answer. Therefore, using Bayes theorem probability of guess given correct is equal to probability of correct and he guessed divided by probability of correct. And guess plus probability of correct and knows is equal to probability of correct given he guessed multiplied by probability that he has guessed divided by probability of correct given he guessed.

Multiplied by probability of guess plus probability of correct given he knows, multiplied by probability of knows is equal to, if it is a random, then probability of correct given guest is going to be 1 by 5. And probability that he guessed is 1 minus P divided by same thing 1 by 5 into 1 minus P plus if he knows the answer, then will answer it correctly. So, that probability is 1, probability he knows the answer that is P. Therefore, the answer is going to be 1 minus P upon 1 minus P plus 5P is equal to 1 minus P upon 1 plus 4P.

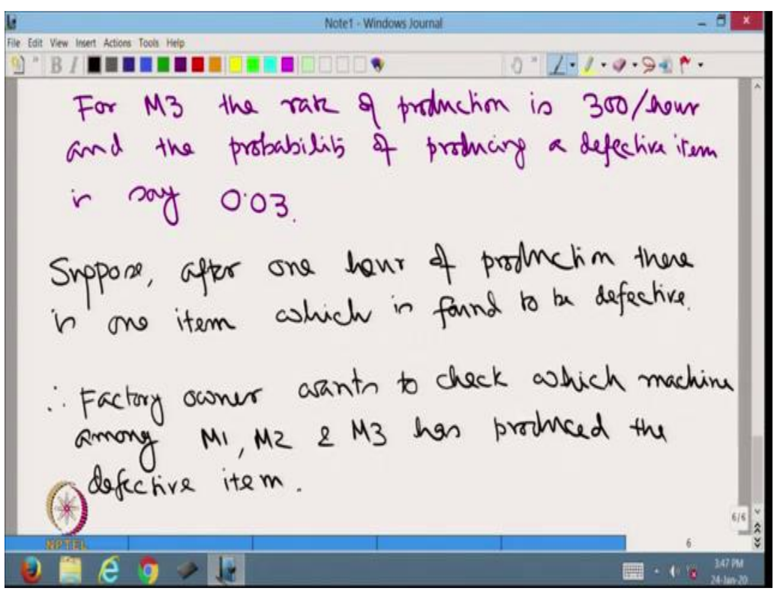
Okay students. I hope you understand the significance of Bayes theorem. It allows us to obtain the cause behind an event, when something has happened.

(Refer Slide Time: 60:48)



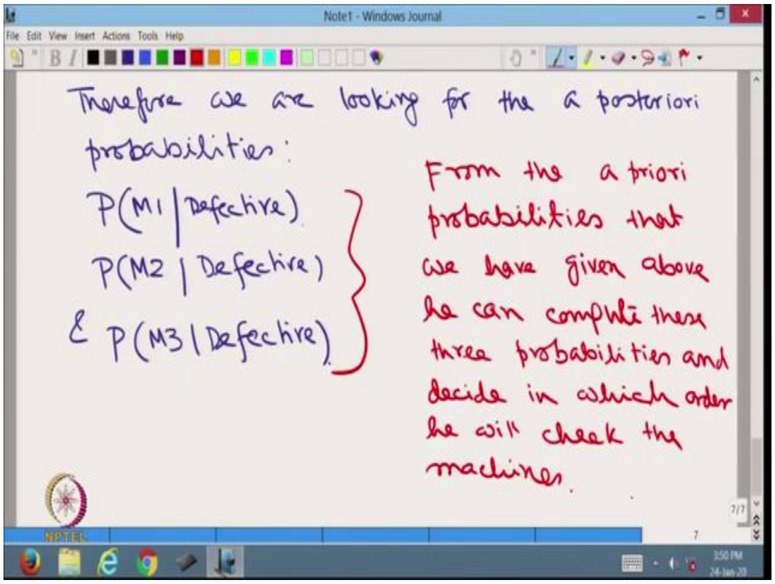
For example, consider a factory that produces some item say x. Suppose there are three machines M1, M2 and M3. Suppose, M1 produces 100 items, per hour and it has a probability of P1 which is say equal to 0.01 that the item produced is defective. Similarly, for M2 the hourly production rate is 200 and the probability that an item produced is defective is say 0.05.

(Refer Slide Time: 63:19)



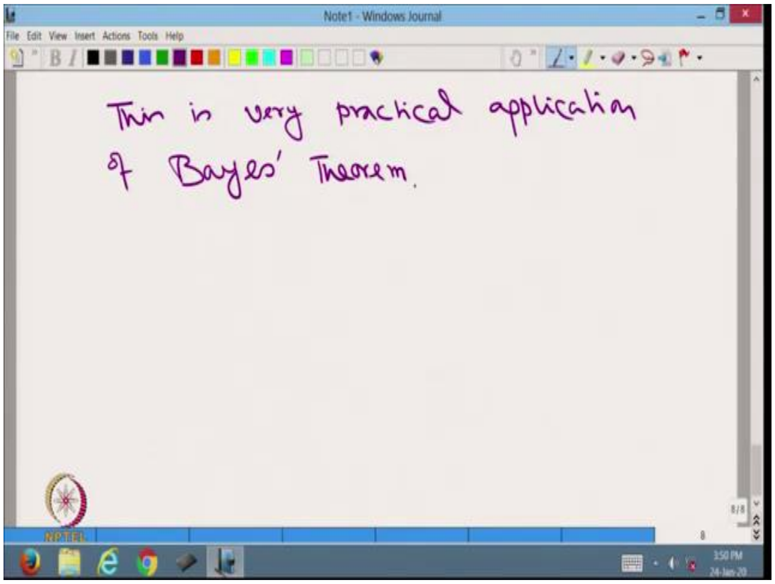
For M3 the rate of production is 300 per hour and the probability of producing a defective item is say 0.03. Now, suppose after one hour of production there is one item which is found to be defective, therefore factory owner wants to check which machine among M1, M2 and M3 has produced the defective item. It is a very natural question for the factory owners.

(Refer Slide Time: 65:36)



Therefore, we are looking for the a posterior probabilities, probability M1 given defective, probability M2 given defective and probability M3 given defective as you can understand that from the a priori probabilities that we have given above he can compute these three probabilities and decide in which order he will check the machine.

(Refer Slide Time: 67:35)



So, I hope you understood. So, this is a very practical application of Bayes theorem. Okay students, I stopped here today, the next class I shall start with random variables, which is most fundamental concept with respect to probability. Thank you.