Advanced Probability Theory
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Lecture 22

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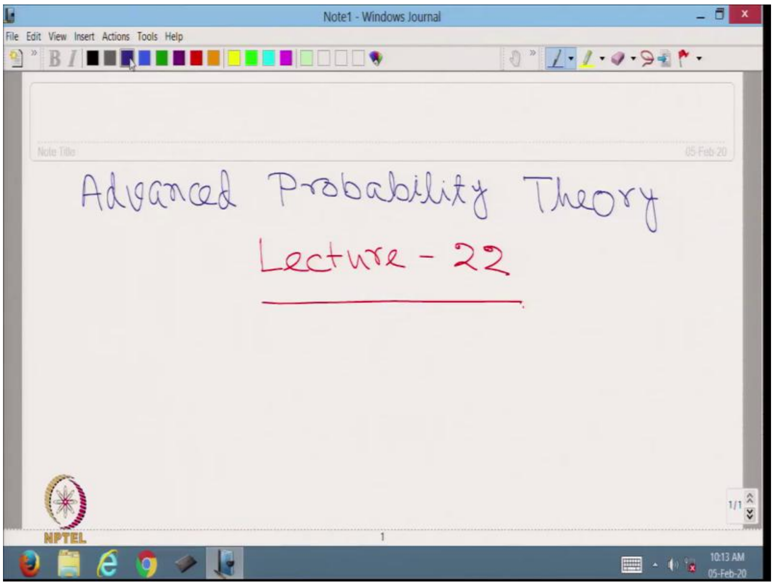

Welcome students to the MOOCS lecture series on Advanced Probability Theory. This is lecture number 22.

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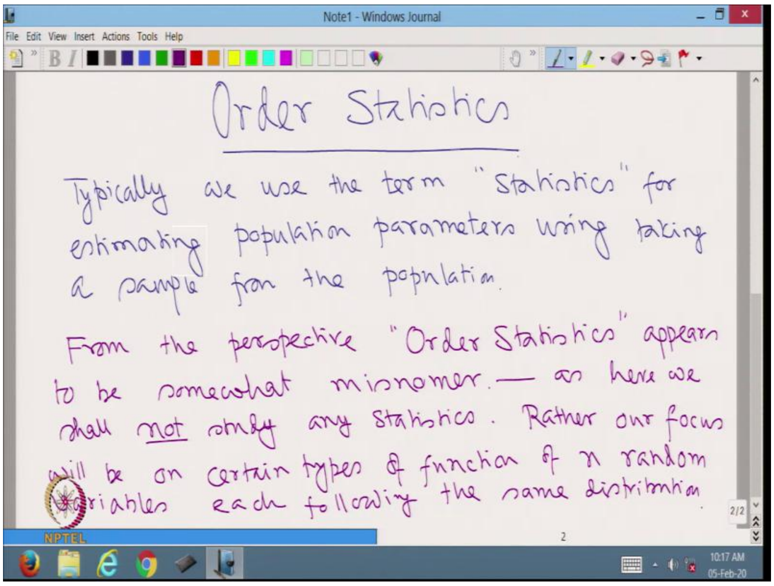

As I said at the end of the last class, that today we shall start with Order Statistics. Typically, we use the term "statistics" for estimating population parameters using taking a sample from the population. From that perspective "Order Statistics" appears to be somewhat misnomer as here we shall not study any statistics. Rather, our focus will be on certain types of functions of n random variables, each following the same distribution. So, we are actually looking at probabilities of certain function of in random variables.

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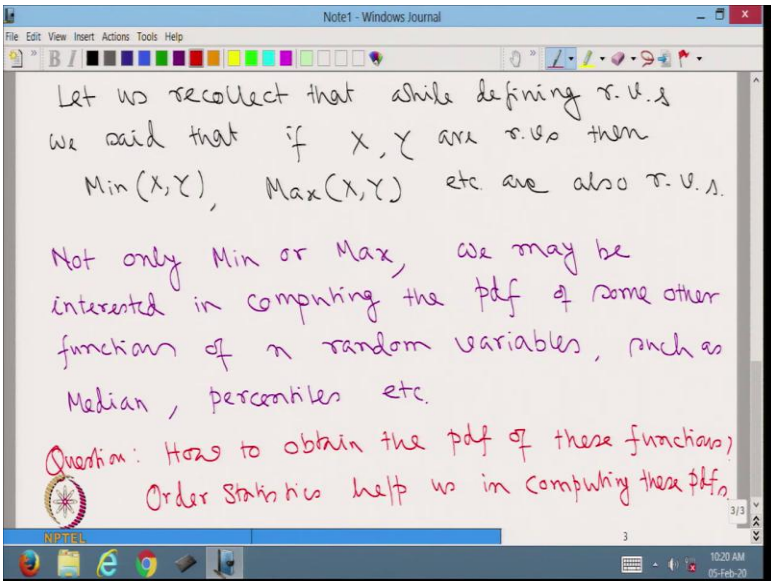

Let us recollect that, while defining random variables, we said that, if, X, Y are random variables then minimum over X, Y, maximum of X, Y etc are also random variables. Not only min or max, we may be interested in computing the pdf, Probability Density Function of some other functions of n random variables, such as say, median, percentiles, etc. Question is, how to obtain the pdf of these functions? Order Statistics help us in computing these pdf's. So, with that small introduction, let us define what is an order statistics.

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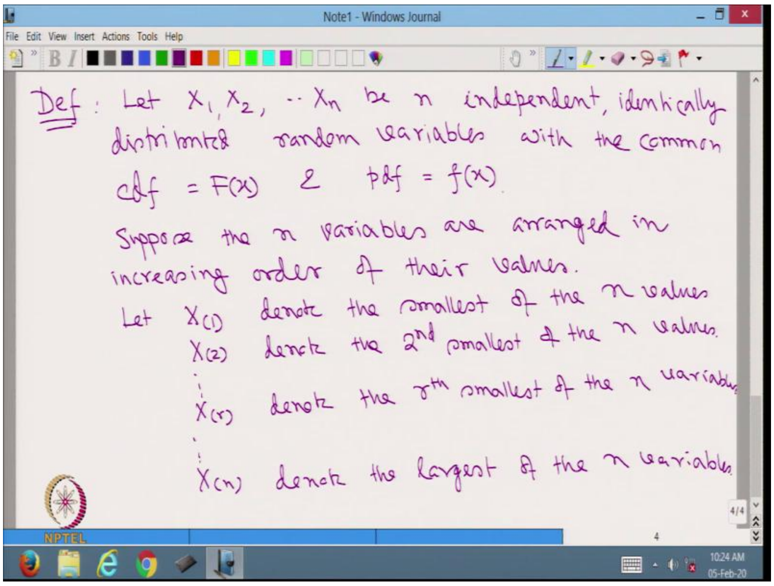

So, definition, let X1, X2, XN be n independent and identically distributed random variables with the common cdf, the cumulative distribution function is equal to F x and pdf is equal to small f x. Now, suppose, these n variables are arranged in increasing order of their values. Let X1 denote the smallest of the n values. X2 denote the second smallest of the n values. Xr denote the r'th smallest of the n values or n variables and Xn denote the largest of the n variables.

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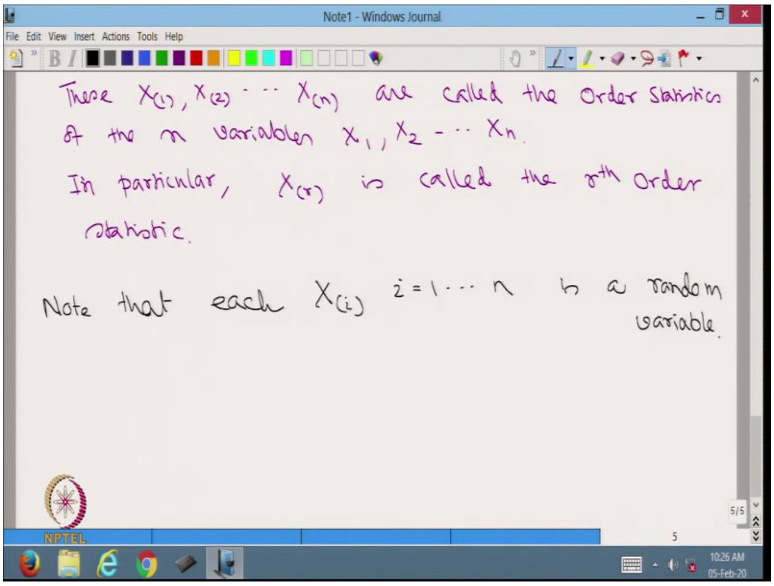

These X1, X2, Xn are called the order statistics of the n variables, X1, X2, Xn. In particular, Xr is called the rth order statistic. Note that each Xi, i is equal to 1 to n is a random variable.

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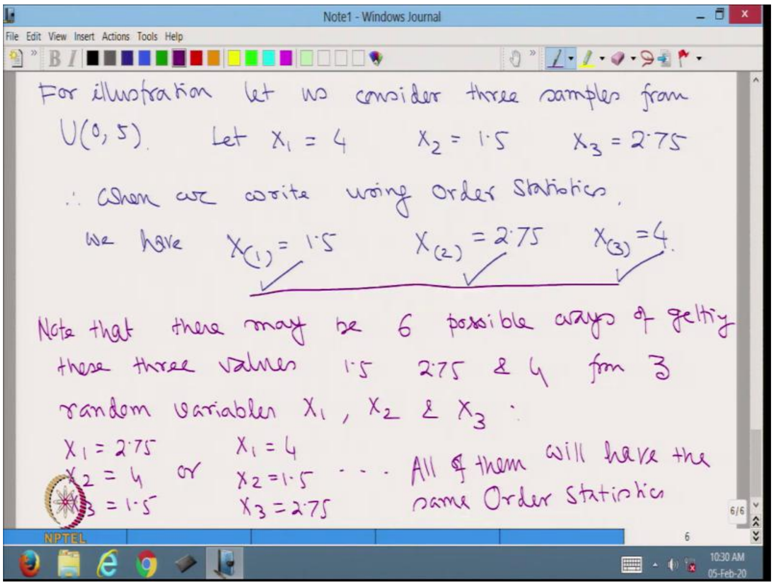

For illustration, let us consider 3 samples from uniform 0, 5. Let X1 is equal to 4, X2 is equal to 1.5 and X3 is equal to 2.75. Therefore, when we write using order statistics, we have X1 is equal to 1.5, X2 is equal to 2.75 and X3 is equal to 4. That is, the smallest of the 3 random variables is taking the value 1.5. The second smallest of the 3 variables is taking the value 2.75 and the largest of them is taking the value 4.

Note that there may be 6 possible ways of getting these 3 values 1.5, 2.75 and 4 from 3 random variables X1, X2 and X3. Namely it can be X1 is equal to 2.75, X2 is equal to 4, X3 is equal to 1.5 or it can be X1 is equal to 4, X2 is equal to 1.5, and X3 is equal to 2.75, etc. we know that there can be 6 possible permutations. All of them will have the same order statistic namely this one.

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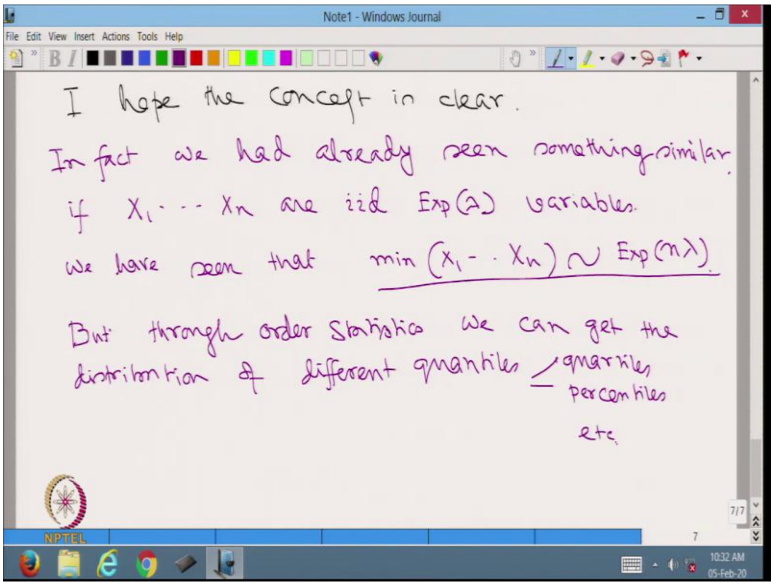

I hope that the concept is clear. In fact, we had already seen something similar if X1, X2, Xn are independent identically distributed exponential lambda variables. We have seen that minimum of X1, X2, Xn is distributed as exponential with n lambda. This we have already seen. But through order statistics, we can get the distribution of different quantiles. So, by quantile we mean it can be quartiles, percentiles, etc.

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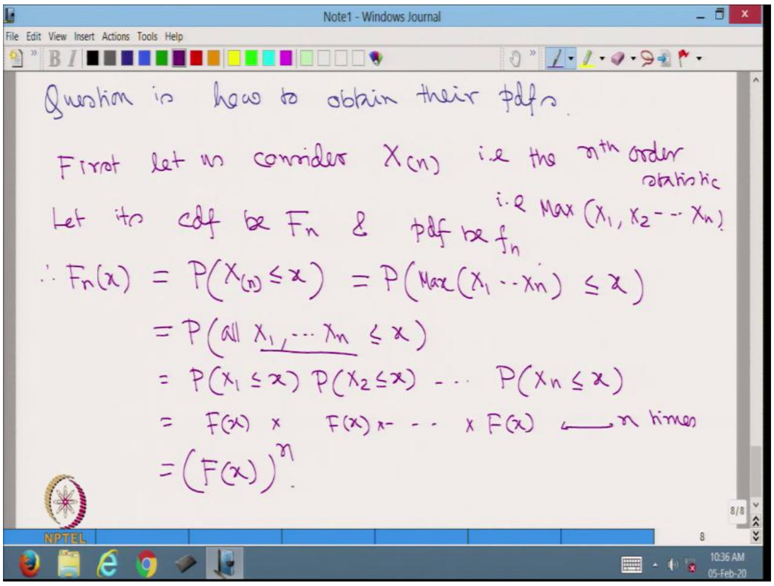

Question is how to obtain their pdf's. So, first let us consider Xn that is the nth order statistic that is maximum of X1, X 2, Xn. Let its cdf be F n and pdf be small f n. Therefore, F n x is equal to probability Xn less than equal to x is equal to probability maximum of X1, X2, Xn less than equal to x. Is equal to probability all X1, X2, Xn less than equal to x. And since they are independent and identically distributed, this is is equal to probability X1 less than equal to x probability X2 less than equal to x and probability Xn less than equal to x. Is equal to F x into F x into F x n times is equal to F x whole to the power n.

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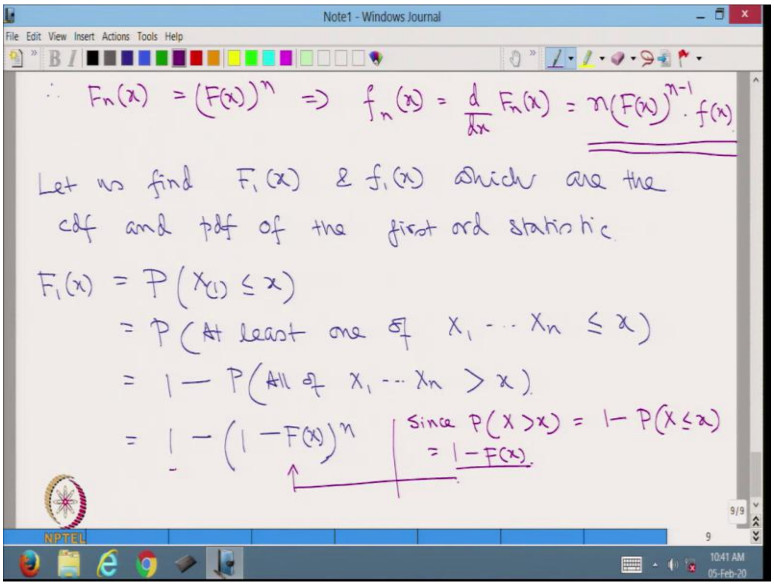

Therefore, F n x is equal to F x whole to the power n implies small f n x which is equal to d dx of F n x is equal to n into F x to the power n minus 1 times f x. So, that is the pdf of nth order statistic. Let us find F1 x and f1 x which are the cdf and pdf of the first order statistic. So, F1 x is equal to probability X1 less than equal to x is equal to probability at least one of X1, X2, Xn less than equal to x.

Is equal to 1 minus probability all of X1, X2, Xn is greater than x is equal to 1 minus 1 minus F x whole to the power n. Why? Since probability X greater than x is equal to 1 minus probability X less than equal to x. Is equal to 1 minus F x. Therefore, we get this term and now we have subtracted it from 1.

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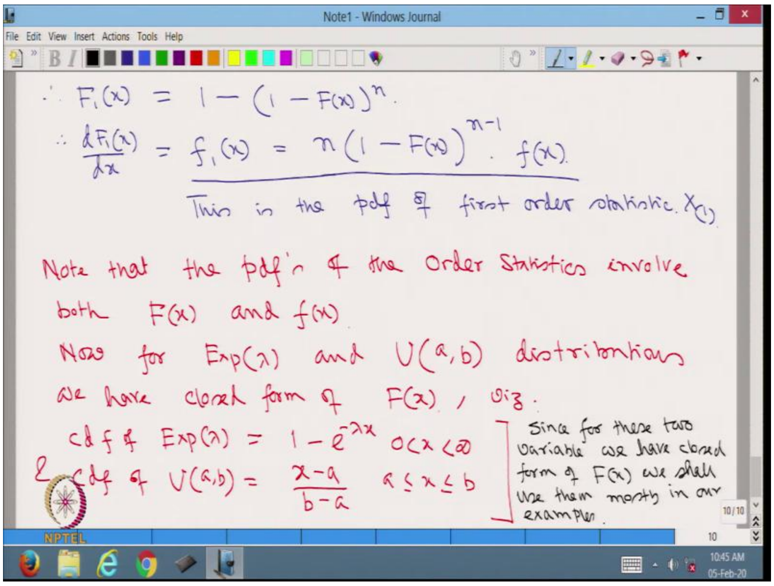

Therefore, F1 x is equal to 1 minus 1 minus F x whole to the power n. Therefore, d F1 x dx is equal to f1 x is equal to n into 1 minus F x whole to the power n minus 1 into small f x. After differentiating with respect to x. So, this is the pdf of first order statistic, X1. Note that the pdf's of the order statistics involve both F x and small f x.

Now, for exponential lambda and uniform a, b distributions we have closed form of F x, namely cdf of exponential lambda is equal to 1 minus e to the power minus lambda x, 0 less than x less than infinity. And cdf of uniform a, b is equal to x minus a upon b minus a, for a less than equal to x less than equal to b. Since, for these two variables we have closed form of F x. We shall use them mostly in our examples.

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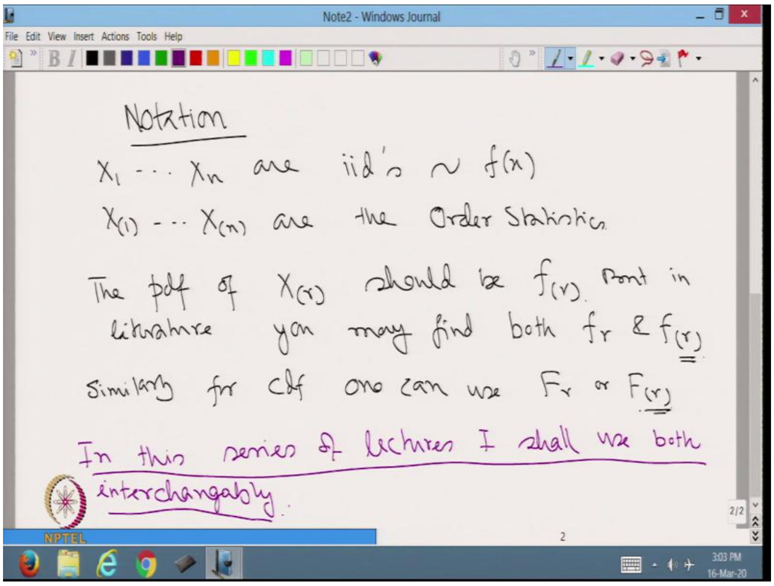

So, let me give you little bit about notation. So, X1, X2, Xn are iid's following some f x. X1, X2, Xn are the order statistics. The pdf of Xr should be f r but in literature you may find both f r and f r with the parentheses. Similarly, for cdf, one can use F r or F r with the parentheses. In this series of lectures, I shall use both interchangeably. So, that you have to remember and there should not be any confusion.

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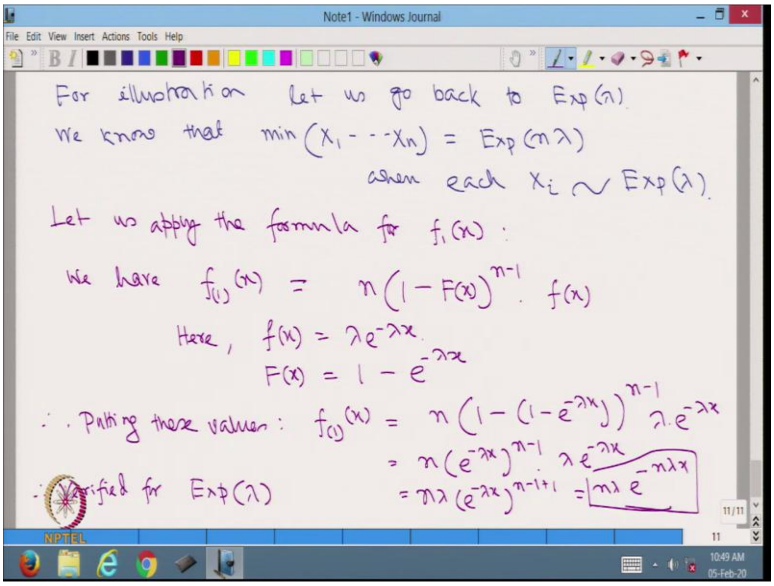

So, for illustration, let us go back to exponential lambda. We know that minimum of X1, X2, Xn is distributed as exponential with n lambda when each Xi is an exponential lambda variable. Let us apply the formula for f1 x, we have f1 x is equal to n into 1 minus F x whole to the power n minus 1 into small f x.

Here, f x is equal to lambda e to the power minus lambda x. F x is equal to 1 minus e to the power minus lambda x. Therefore, putting these values, f1 x is equal to n into 1 minus 1 minus e to the power minus lambda x whole to the power n minus 1 into lambda e to the power minus lambda x. Is equal to n into e to the power minus lambda x whole to the power n minus 1 into lambda e to the power minus lambda x. Is equal to n lambda into e to the power minus lambda x to the power n minus 1 plus 1. Is equal to n lambda e to the power minus n lambda x. Thus, verified for exponential lambda.

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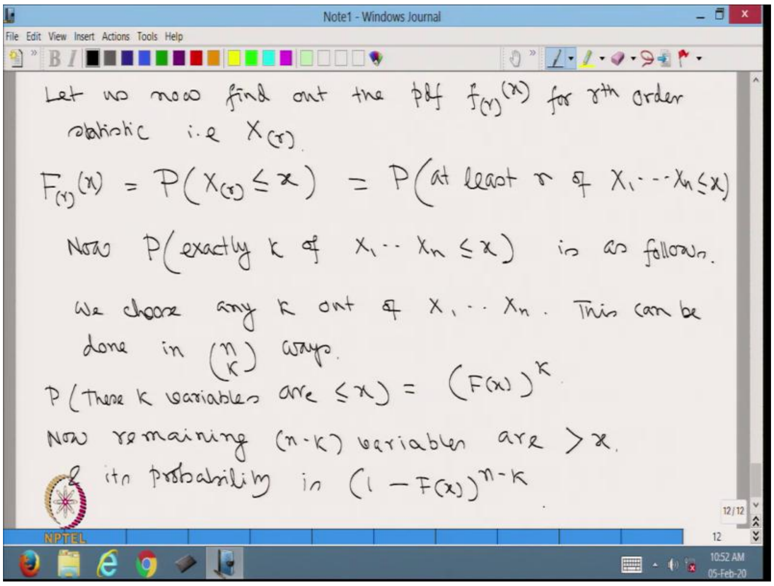

Let us now find out the pdf f r x for rth order statistic that is, Xr. F r x is equal to probability Xr less than equal to x is equal to probability at least r of X1, X2, Xn less than equal to x. Now, probability exactly k of X1, X2, Xn less than equal to x is as follows. We choose any k out of X1, X2, Xn. This can be done in n k ways.

Probability these k variables are less than equal to x is equal to F x to the power k. Now, remaining n minus k variables are greater than x and its probability is 1 minus F x whole to the power n minus k.

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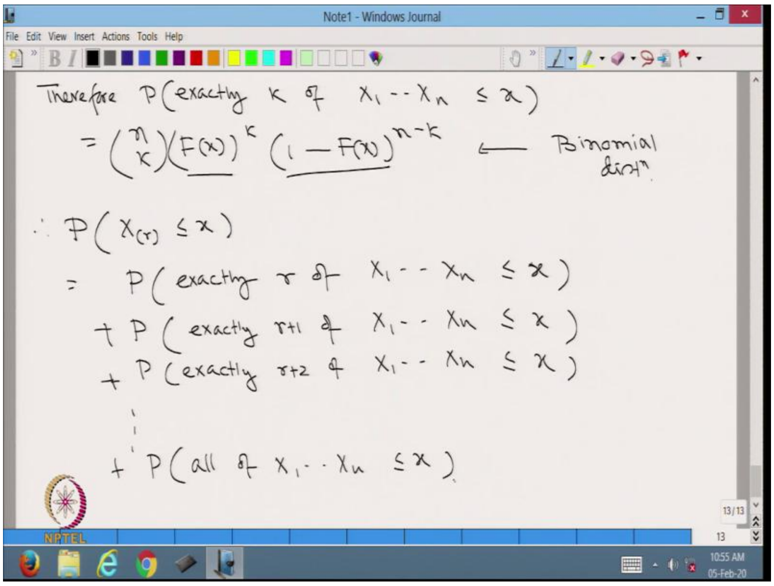

Therefore, probability exactly k of X1, X2, Xn less than equal to x is equal to n C k F x to the power k 1 minus F x to the power n minus k. It is clear that this is a binomial distribution which is very clear if we call this to be as success, this to be a failure, we get the binomial distribution.

Therefore, the probability Xr less than equal to x is equal to union of the disjoint events. Probability exactly r of X1, X2, Xn less than equal to x plus probability exactly r plus 1 of X1, X2, Xn less than equal to x plus probability exactly r plus 2 of X1, X2, Xn less than equal to x. Like that we go up to probability all of X1, X2, Xn less than equal to x.

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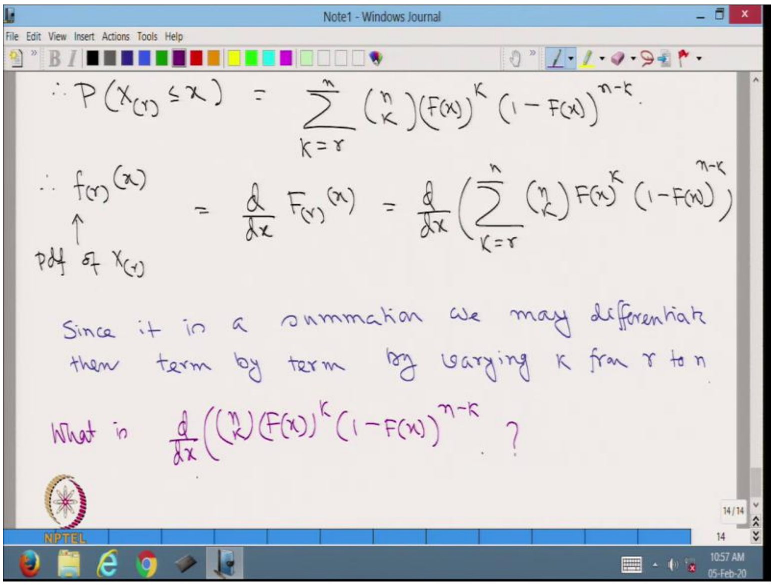

Therefore, probability Xr less than equal to x is equal to sigma over k is equal to r to n, n C k F x to the power k 1 minus F x to the power n minus k. Therefore, f r x that is, pdf of Xr is equal to d dx of F r x is equal to d dx of sigma k is equal to r to n, n C k F x to the power k 1 minus F x whole to the power n minus k.

Since, it is a summation we may differentiate them term by term by varying k from r to n. So, what is d dx of n C k F x to the power k 1 minus F x to the power n minus k? So, let us first compute this.

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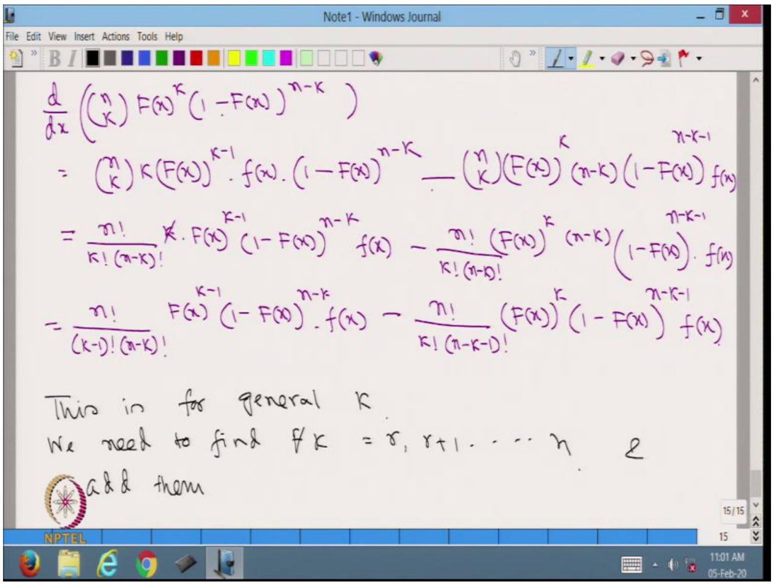

d dx of n C k F x to the power k 1 minus F x to the power n minus k is equal to n C k, k F x to the power k minus 1 into f x multiplied by 1 minus F x whole to the power n minus k minus n C k F x to the power k and n minus k 1 minus F x to the power n minus k minus 1 into f x.

Since, here F x is with a minus term therefore, we get a negative sign there. This is for general k. Let us simplify it, is equal to factorial n upon factorial k factorial n minus k into k F x to the power k minus 1 into 1 minus F x to the power n minus k f x minus factorial n upon factorial k factorial n minus k f x to the power k into n minus k into 1 minus F x whole to the power n minus k minus 1 into f x.

Is equal to after cancellation factorial n upon factorial k minus 1 into n minus k factorial F x to the power k minus 1, 1 minus F x to the power n minus k into f x minus n factorial k factorial n minus k minus 1 factorial F x to the power k 1 minus F x whole to the power n minus k minus 1 into f x. This is for general k. We need to find for all k is equal to r, r plus 1 up to n and add them.

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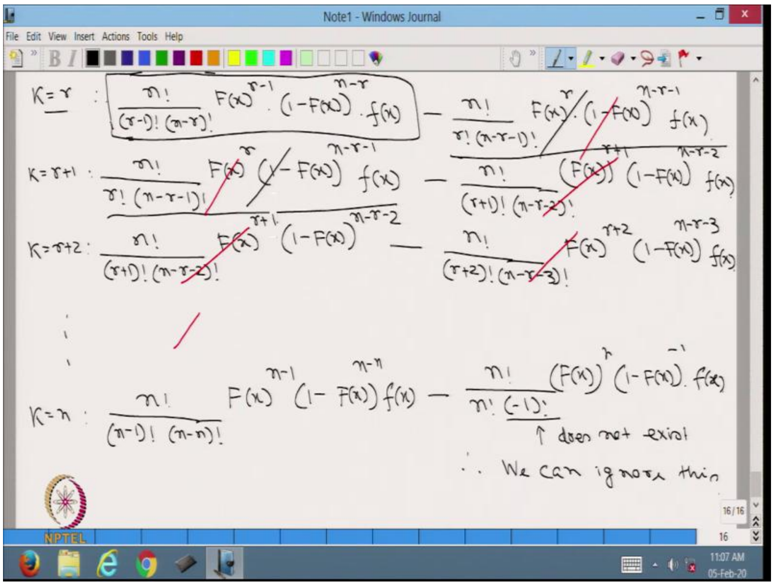

So, let us start k is equal to r. Therefore, we have factorial n upon factorial r minus 1 into factorial n minus r F x to the power r minus 1, 1 minus F x to the power n minus r into f x minus n factorial upon r factorial n minus r minus 1 factorial F x to the power r 1 minus F x to the power n minus r minus 1 into f x. k is equal to r plus 1 that gives us.

Put k is equal to r plus 1, therefore we will get n factorial upon r factorial into n minus r minus 1 factorial. We are putting r plus 1 in place of r. F x to the power r 1 minus F x to the power n minus r minus 1 f x minus n factorial upon r plus 1 factorial n minus r minus 2 factorial F x to the power r plus 1, 1 minus F x to the power n minus r minus 2 into f x.

Now, let us put k is equal to r plus 2. This we will get by putting r is equal to r plus 2 in this expression therefore, this is going to be n factorial r plus 1 factorial n minus r minus 2 factorial F x to the power r plus 1, 1 minus F x whole to the power n minus r minus 2 minus n factorial r plus 2 factorial into n minus r minus 3 factorial to F x to the power r plus 2 into 1 minus F x whole to the power n minus r minus 3 into f x.

We will go like that but in the meantime, we noticed that, this term is same as this term so, they cancel each other. In a similar way, this cancels with this therefore, this is going to cancel with this therefore, at the end of the day what we will have so, let us calculate for k is equal to n, this is going to be n factorial n minus 1 factorial into n minus n factorial which is 0 factorial to F x to the power n minus 1, 1 minus F x whole to the power n minus n small f x minus n factorial upon n factorial minus 1 factorial F x to the power n 1 minus F x to the power minus 1 into f x. Now, minus 1 factorial does not exists therefore, we can ignore this. Therefore, we are left with only 1 term that is this.

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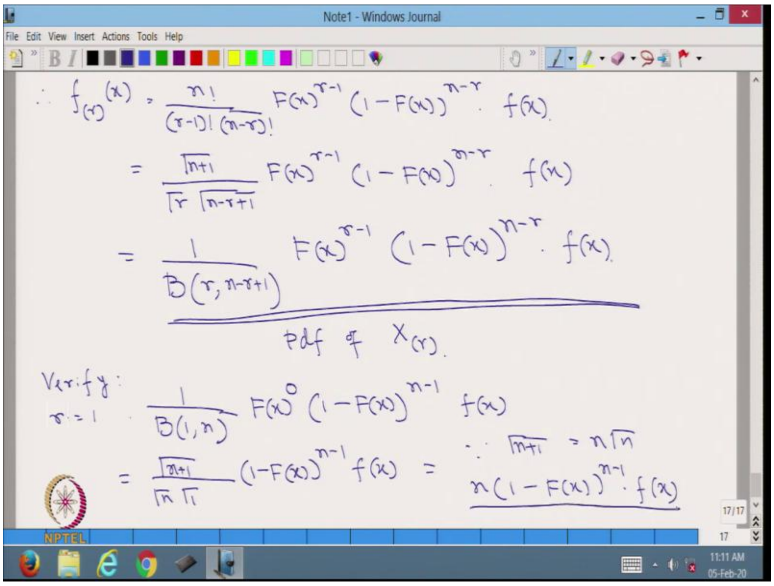

Therefore, f r of x is equal to factorial n upon factorial r minus 1 factorial n minus r F x to the power r minus 1, 1 minus F x whole to the power n minus r into f x which is is equal to gamma n plus 1. We know that gamma n plus 1 is equal to n factorial gamma r gamma n minus r plus 1 F x to the power r minus 1, 1 minus F x to the power n minus r into f x. Is equal to 1 upon beta of r, comma n minus r plus 1 into F x to the power r minus 1, 1 minus F x to the power n minus r into small f x. So, this is the pdf of rth order statistic namely, Xr.

Verify r is equal to 1 therefore, we are getting 1 upon beta of 1, comma n F x to the power 0 1 minus F x whole to the power n minus 1 f x. Is equal to gamma n plus 1 upon gamma n gamma 1, 1 minus F x to the power n minus 1 into f x. Is equal to since, gamma n plus 1 is equal to n into gamma n. Therefore, this is is equal to n into 1 minus F x whole to the power n minus 1 into f x. The same result that we got earlier.

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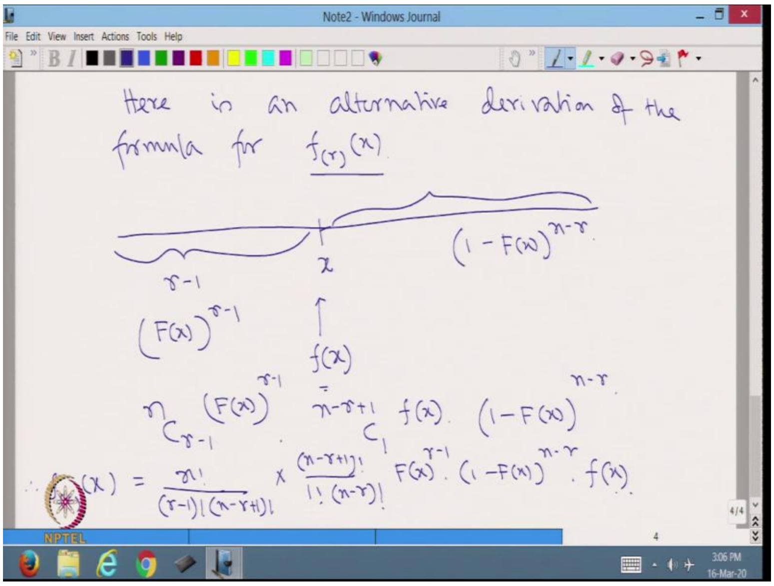

Now, some of you may find the formula to be complicated so, here is an alternative derivation of the formula for f r x. So, we have the observations on the real line and suppose this is the point x and we are looking at f r x so one observation will come at x that will have the pdf f x, r minus 1 observations will be here and each of them are less than x therefore, that should give us F x to the power r minus 1 and remaining n minus r are in this region because they are greater than x so, that should give us 1 minus F x to the power n minus r.

Now, this r minus 1 can be chosen out of n in n C r minus 1 ways so, that is being multiplied by this probability. This one has to be chosen from the remaining n minus r plus 1 and that can be done in n minus r plus 1 C1 ways that multiplied by f x and remaining all of them are going there. So, that will be 1 minus F x whole to the power n minus r.

Therefore, if f r x is equal to factorial n upon factorial r minus 1 into factorial n minus r plus 1 into let me take out the constant n minus r plus 1 factorial upon 1 factorial into n minus r factorial multiplied by F x to the power r minus 1, 1 minus F x to the power n minus r multiplied by f x.

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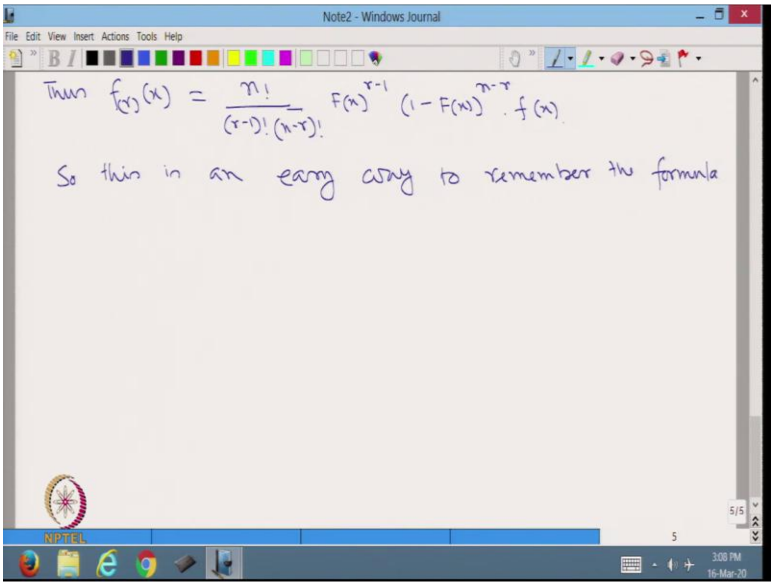

Thus, f r x is equal to factorial n upon factorial r minus 1 into n minus r factorial F x to the power r minus 1, 1 minus F x to the power n minus r into f x. So, this is an easy way to remember the formula.

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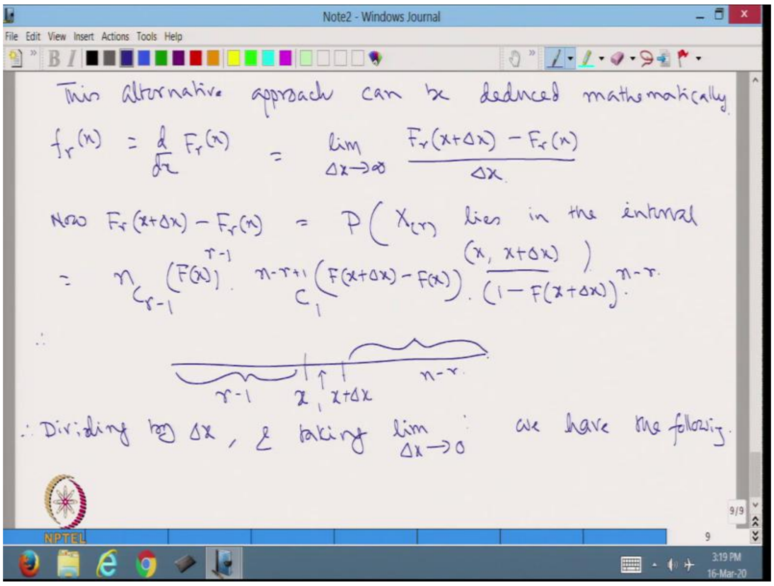

Now, this above trick can be deduced mathematically so, what is f r x? This is is equal to d dx of f r x is equal to limit delta x going to infinity F r x plus delta x minus F r x upon delta x. Now, F r x plus delta x minus F r x is equal to probability X r lies in the interval x to x plus delta x. Is equal to n C r minus 1 F x to the power r minus 1.

So, out of n r minus 1 are chosen, they are below x. Therefore, F x to the power r minus 1. From the remaining n minus r plus 1, we choose 1 and that goes between x to x plus delta x.

Therefore, we can write it as F x plus delta x minus F x multiplied by the remaining one so, that is 1 minus F of x plus delta x whole to the power n minus r. This is because suppose this is the real line.

This is x, this is x plus delta x. one of them is going to be here, r minus r are going to be here and n minus r are going to be above x plus delta x. Therefore, dividing by delta x and taking limit delta x going to 0, we have the following.

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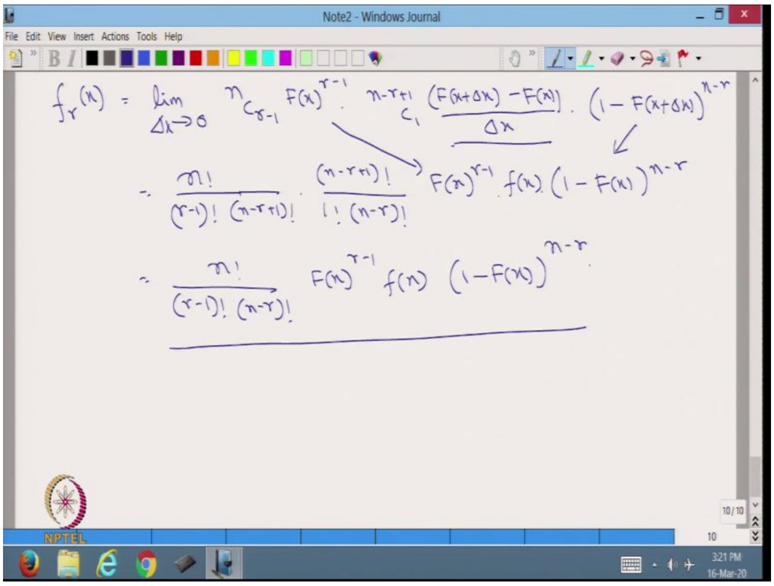

f r x is equal to limit delta x going to 0, n C r minus 1 F x to the power r minus 1 n minus r plus 1 C1 F x plus delta x minus F x divided by delta x multiplied by 1 minus F x plus delta x whole to the power n minus r. Is equal to factorial n factorial r minus 1 factorial n minus r plus 1 multiplied by n minus r plus 1 factorial upon 1 factorial n minus r factorial.

Now, this gives up F x to the power r minus 1. This gives us f x and from here we get 1 minus F x whole to the power n minus r. Is equal to n factorial upon r minus 1 factorial n minus r factorial into F x to the power r minus 1 f x 1 minus F x to the power n minus r. So, this is a mathematically justified the way we have proved the, the way we have derived the formulae. Okay friends. I stop here today. In the next class, I shall start at this point and we will solve several problems of order statistics.

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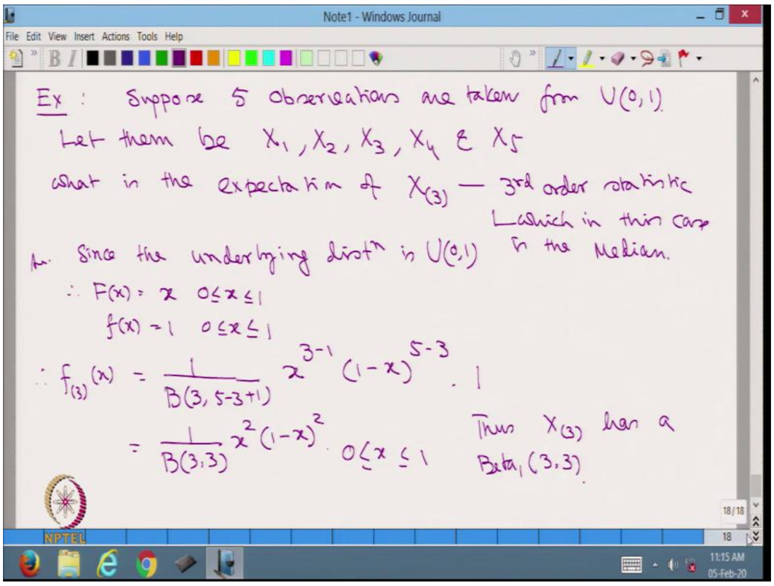

Now, let me give you an example. Suppose, 5 observations are taken from uniform 0, 1. Let them be X1, X2, X3, X4 and X5. What is the expectation of X3 that is the third order statistic which in this case is the median. Answer. Since, the underlying distribution is uniform 0, 1 therefore F x is equal to x for 0 less than equal to x less than equal to 1 and small f x is equal to 1 for 0 less than equal to x less than equal to 1.

Therefore, f 3 x is equal to 1 upon beta of 3, comma 5 minus 3 plus 1 into x to the power 3 minus 1, 1 minus x to the power 5 minus 3 into 1. Is equal to 1 upon beta of 3, comma 3 x square 1 minus x square for 0 less than equal to x less than equal to 1. Thus, X3 has a beta 1 distribution with parameter 3, comma 3.

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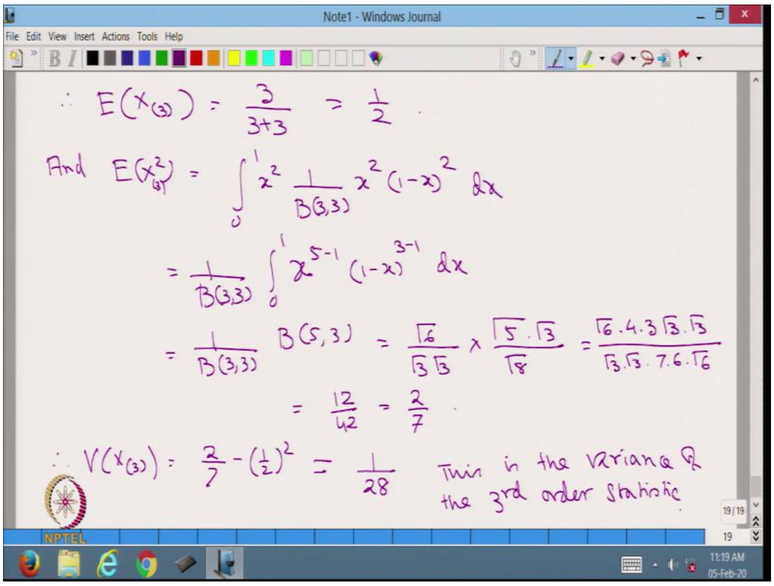

Therefore, expected value of X3 is equal to 3 upon 3 plus 3 is equal to half and expected value of x square is equal to 0 to 1 x square 1 upon beta 3, comma 3 x square into 1 minus x square dx. Is equal to 1 upon beta 3, comma 3 integration 0 to 1 x to the power 5 minus 1 into 1 minus x 3 minus 1 dx. Is equal to 1 upon beta 3, comma 3 into beta 5, comma 3 is equal to gamma 6 upon gamma 3 gamma 3 into gamma 5 into gamma 3 upon gamma 8.

Is equal to gamma 6 into 4 into 3 into gamma 3 into gamma 3 upon gamma 3 gamma 3 into 7 into 6 into gamma 6. Is equal to 12 upon 42 is equal to 2 upon 7. Therefore, variance of this is X3 square, variance of X3 is equal to 2 upon 7 minus half square is equal to 1 upon 28. So, this is the variance of the third order statistic. Okay friends. I stop here today. In the next lecture we shall do joint distribution of rth and sth order statistic and also we shall solve a few problems on order statistic. Okay friends. Thank you so much.