Advanced Probability Theory
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Lecture 11

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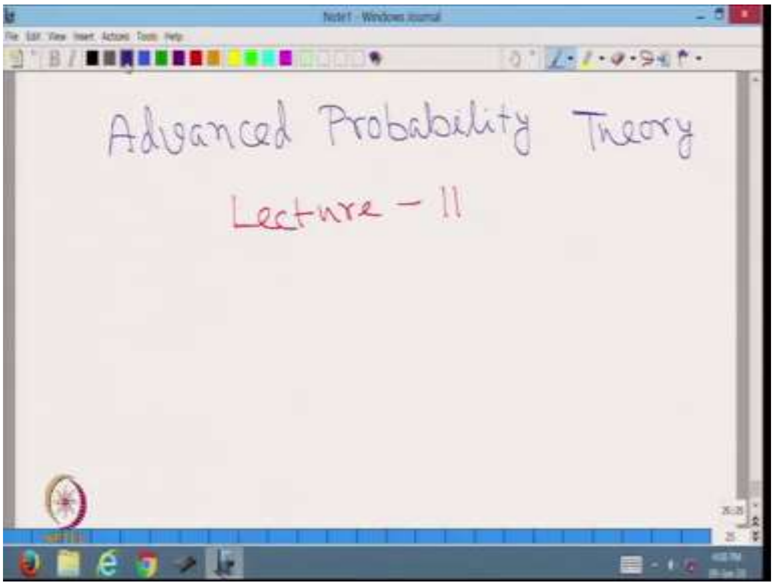

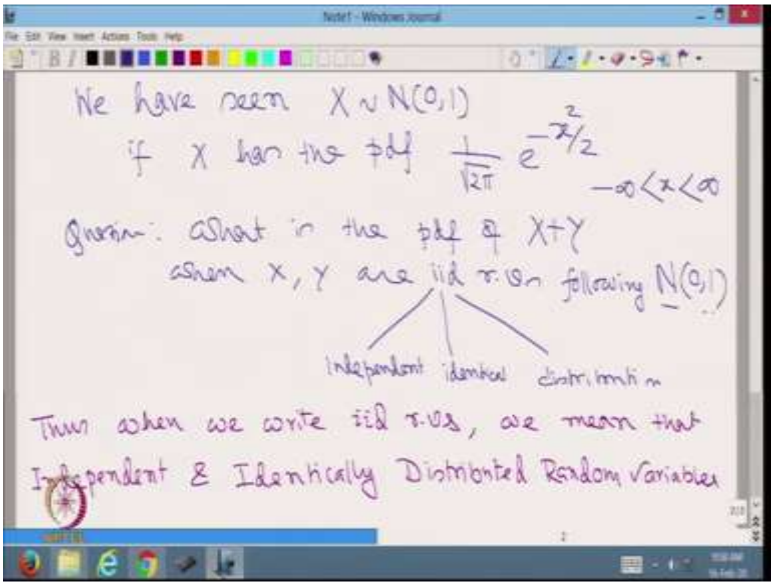

Welcome students to the MOOCs lecture series on Advanced Probability Theory, this is lecture number 11. In the last class, we have seen, X is distributed as normal 0, 1. If X has the PDF 1 over root over 2 Pi, e to the power minus x square by 2, where minus infinity less than x, less than infinity.

Question is what is the PDF of X plus Y when X and Y are independent, identically distributed random variables following normal 0,1? So this i means independent, this means identical that means both of them have the same distribution and this is the distribution. Thus, when we write iid random variables, we mean that independent and identically distributed random variables.

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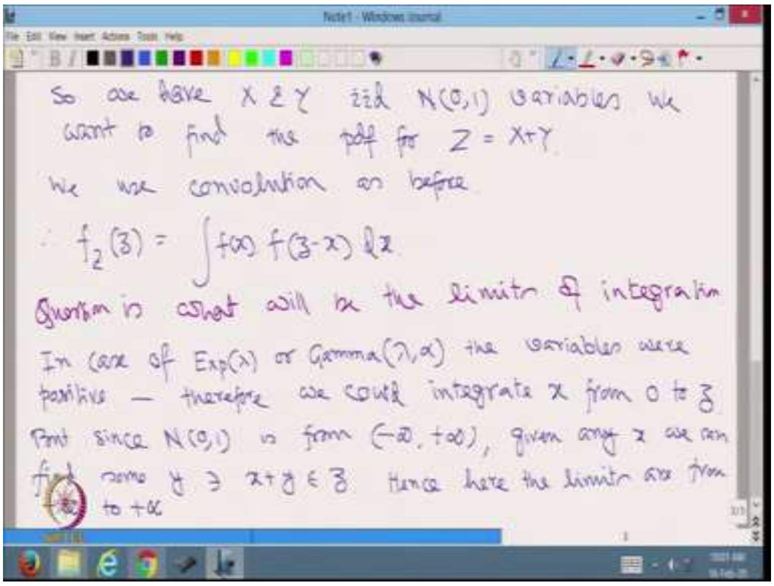

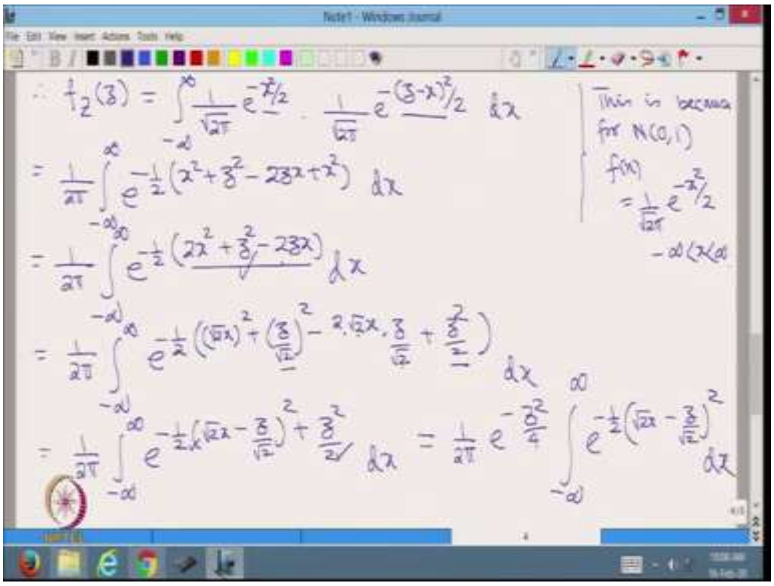

So we have X and Y iid normal 0, 1 variables. We want to find the PDF for Z is equal to X plus Y. We use convolution as before. Therefore, f Z of z is equal to integration over f x, f z minus x dx.

Question is what will be the limits of integration, in case of exponential lambda or gamma, lambda alpha the variables were positive. Therefore, we could integrate x from 0 to z. But since normal 0,1 is from minus infinity to plus infinity given any x, we can find some y such that x plus y belonging to z. Hence, here the limits are from minus infinity to plus infinity.

Therefore, f Z z is equal to minus in infinity to plus infinity 1 over root over 2 Pi, e to the power minus x square by 2 multiplied by 1 over root over 2 Pi, e to the power minus z minus x square by 2 dx. This is because for normal 0,1 a f x is equal to 1 over root over 2 Pi, e to the power minus x square by 2 minus infinity less than x, less than infinity is equal to 1 over 2 Pi.

This we take out of the integration and minus infinity to infinity e to the power minus half into x square plus z square minus 2 z x plus x square dx, this we get by expanding these 2 terms is equal to 1 over 2 Pi minus infinity to infinity e to the power minus half 2 x square plus z square minus 2 z x dx.

Now, we try to write it in the form of a square. Therefore, we write it as follows 1 over 2 Pi minus infinity to infinity e to the power minus half root 2 x square plus z by root 2 square minus 2, root over 2 x, z by root 2 plus z square by 2 dx. I hope you understand the arithmetic. So these 2 together will give me the z square, and these 2 will cancel the root 2.

And therefore will give me 2 z x is equal to 1 over 2 Pi integration minus infinity to infinity e to the power minus half into root 2 x minus z by root 2 whole square plus z square by 2 dx is equal to 1 over 2 Pi, e to the power minus z square by 4, because these 2 and these 2 together will give me z square by 4 integration minus infinity to infinity, e to the power minus half root over 2x minus z by root 2, whole square dx.

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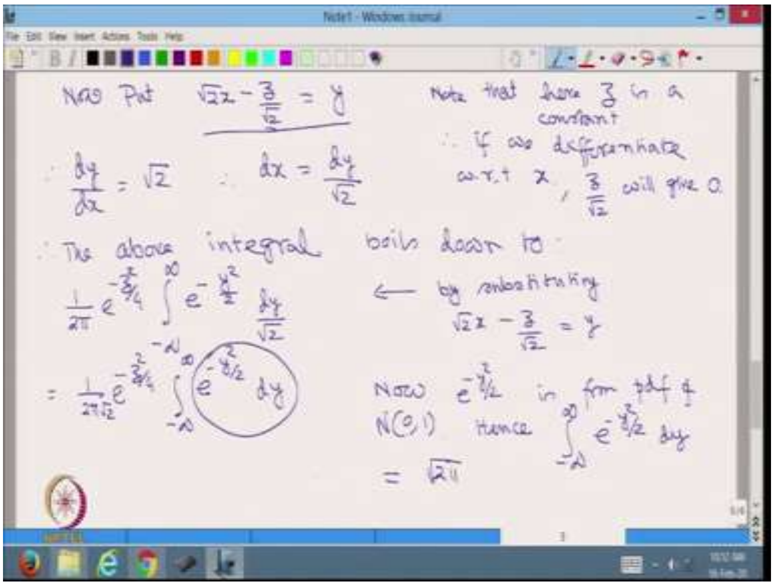

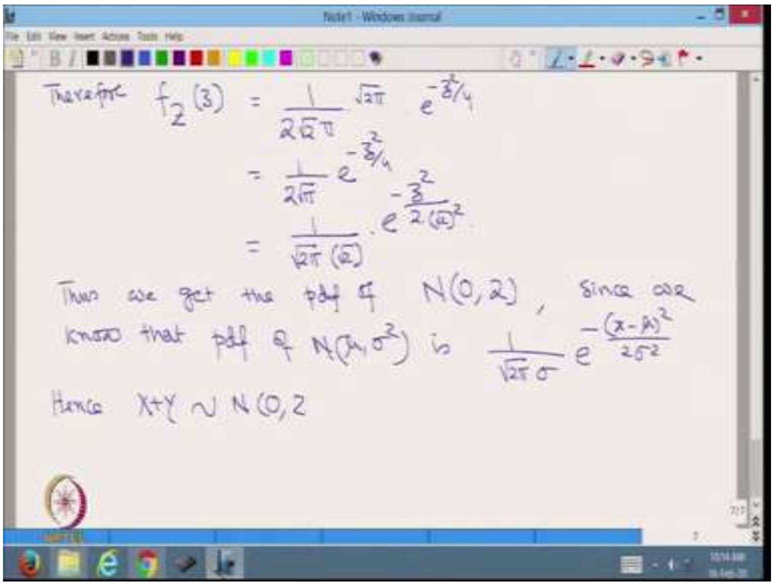

Now put root 2x minus z over root 2 is equal to y. Note that, here z is a constant. Therefore, if we differentiate with respect to x, z by root 2 will give 0. So, for this change of variable, we can see that dy dx is equal to root 2, as this term gives me 0. Therefore, dx is equal to dy upon root 2.

Now, let us go back to this equation and we write it as follows. Therefore, the above integral boils down to 1 over 2 Pi, e to the power minus z square by 4, integration minus infinity to infinity, e to the power minus y square by 2, dy upon root 2 by substituting root 2x minus z by root 2 is equal to y, is equal to 1over 2 Pi root to e to the power minus z square by 4, integration minus infinity to infinity e to the power minus y square by 2 dy.

Now, this term e to the power minus y square by 2 is from PDF of normal 0, 1. Hence, integration minus infinity to infinity e to the power minus y square by 2 dy will give us root over 2 pi. Therefore, f Z of z is equal to 1 over 2 root 2 Pi root over 2 Pi into e to the power minus z square by 4 is equal to 1 over 2 root Pi, e to the power minus z square by 4 is equal to 1 over root 2 Pi into root 2 into e to the power minus z square 2 into root 2 whole square.

Thus we get the PDF of normal with 0 comma 2. Since we know that PDF of normal Mu comma sigma square is 1 over root over 2 Pi sigma, e to the power minus x minus Mu whole square upon 2 sigma square. Hence, X plus Y is distributed as normal with 0, 2.

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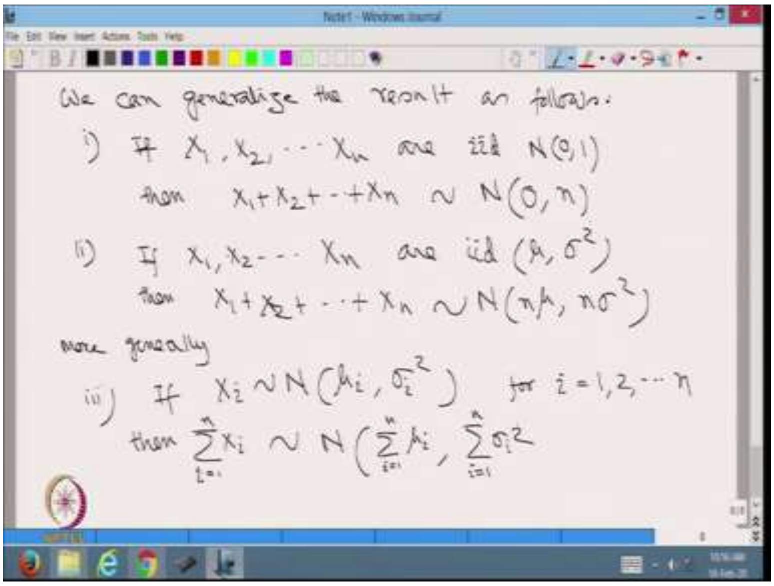

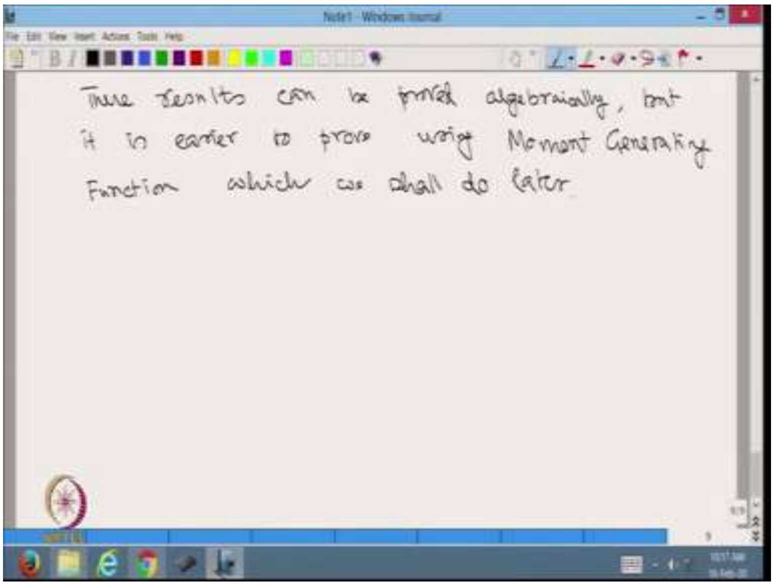

We can generalize the result as follows one, if X1 comma X2 comma Xn are iid normal 0,1 then X1 plus X2 plus Xn will be distributed as normal with 0 comma n, 2 if X1,X2, Xn are iid normal Mu comma sigma square.

Then X1 plus X2 plus Xn will be distributed as normal with N Mu and n sigma square, and more generally, if Xi is distributed as normal with Mu y comma sigma i square, for i is equal to 1, 2, n then sigma Xi, i is equal to 1 to n, will be distributed as normal with sigma mu i comma sigma square, i is equal to 1 to n. These results can be proved algebraically, but it will be easier to prove using Moment Generating Function which we shall do later.

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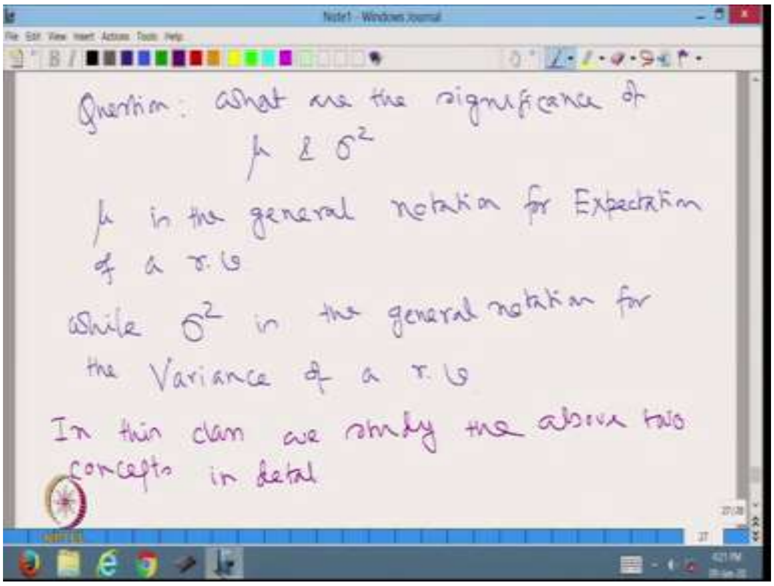

Question is, what are the significance of Mu and sigma square? Mu is the general notation for expectation of a random variable, while sigma square is the general notation for the variance of a random variable. In this class, we shall study the above two concepts in detail.

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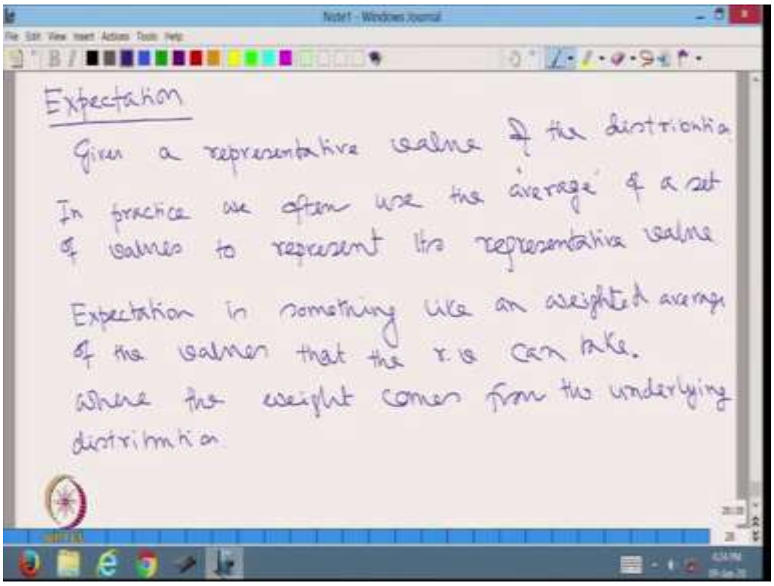

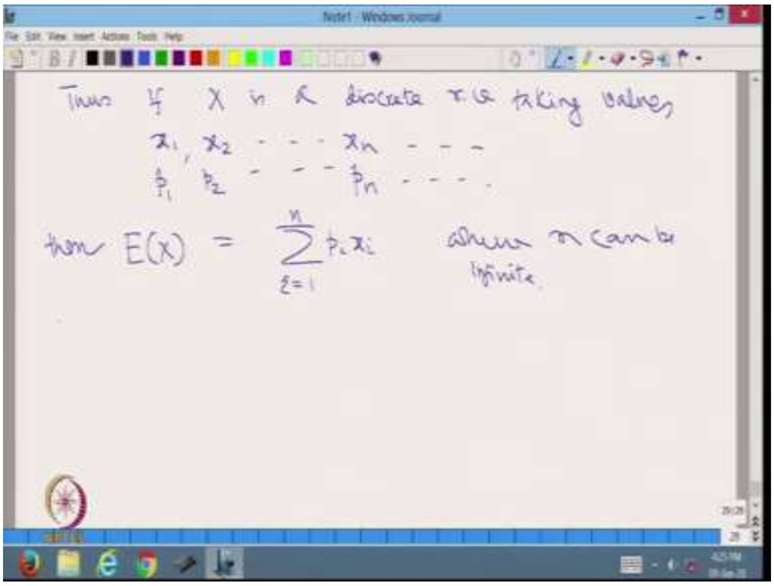

Expectation, expectation of a random variable in some sense, gives a representative value of the distribution. In practice, we often use the average of a set of values to represent its represented representative value. So expectation is something like a weighted average of the values that the random variable can take, where the weight comes from the underlying distribution.

Thus if X is a discrete random variable taking values x1, x2, xn with probabilities p1, p2, pn, then expected value of x is equal to sigma Pi xi, i is equal to 1 to n, where n can be infinite.

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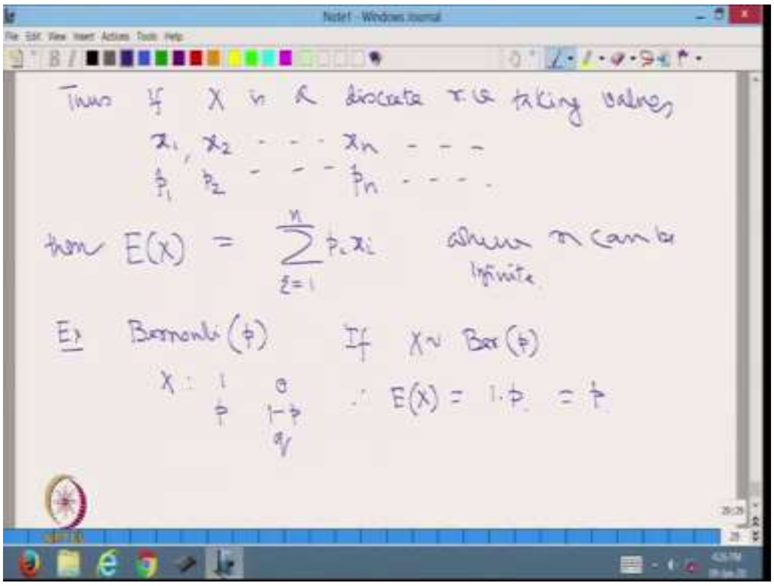

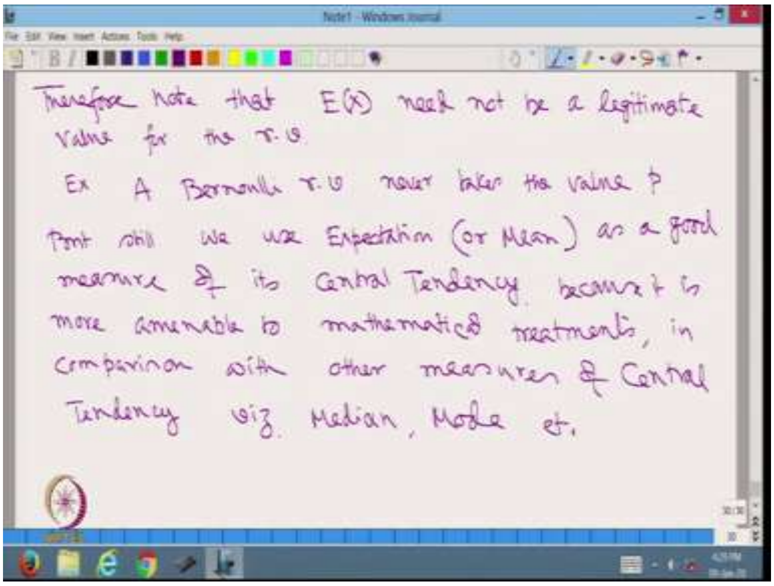

Example, Bernoulli random variable with parameter p, so we know that if x follows Bernoulli with p then X takes two values, 1 and 0 with parameters p and 1 minus p, which we often write as q. Therefore expected value of x is1 times p, it is p.

Therefore, note that expectation of X need not be a legitimate value for the random variable. For example, a Bernoulli random variable never takes value but still we use expectation or which we often called mean as a good measure of its central tendency or representative value because it is more amenable to mathematical treatments in comparison with other measures of central tendency, namely median, mode, etc.

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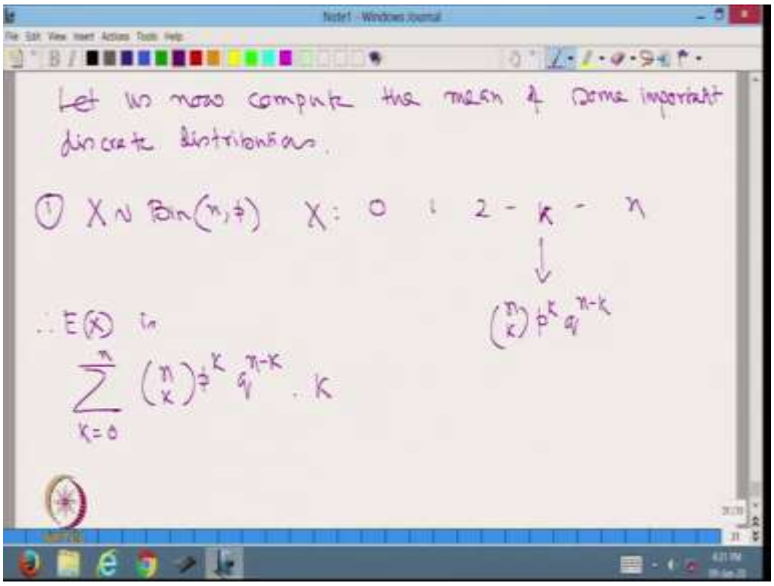

Let us now compute the mean of some important discrete distributions is from binomial n comma p, X follows binomial with n comma p. Therefore X takes the value 0, 1, 2, up to n and it takes the value k with probability n c k, p to the power k, q to the power n minus k.

Therefore, the expected value of X is sigma k is equal to 0 to n, n c k, p to the power k, q to the power n minus k multiplied by k.

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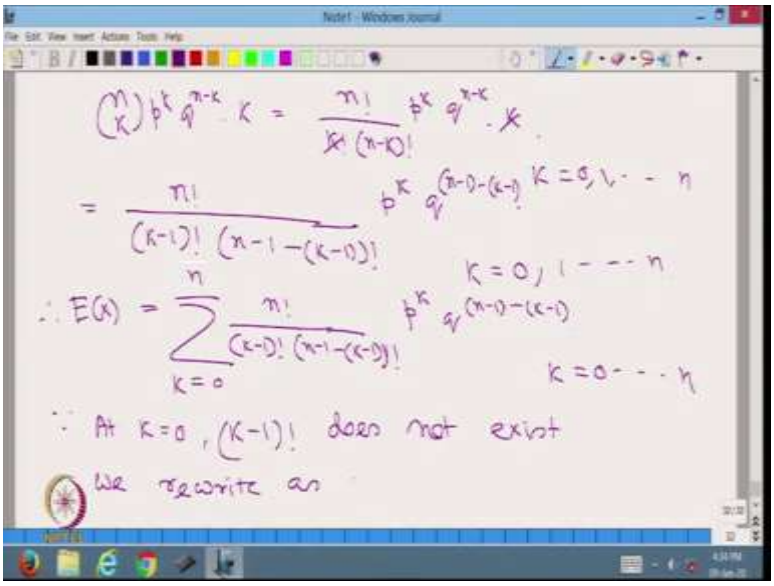

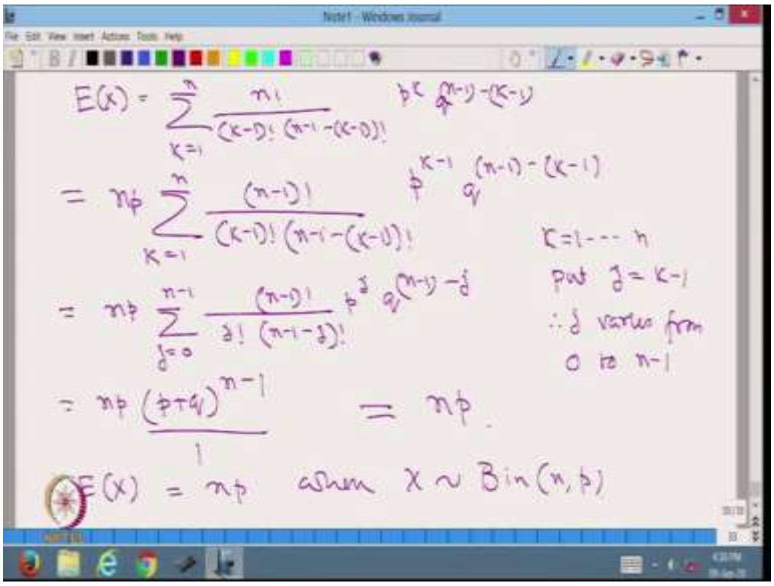

So let us explain the term n c k, p to the power k, q to the power n minus k into k is equal to factorial n upon factorial k, factorial n minus k, p to the power k, q to the power n minus k into k, k is equal to 0,1 up to n. Now this k cancels with this, therefore, we can write n factorial k minus 1 factorial, we write it as n minus 1 minus k minus 1 factorial, p to the power k, q to the power n minus 1 minus k minus 1, k is equal to 0,1 up to n.

Therefore, expected value of X is equal to sigma k is equal to 0 to n factorial k minus 1 factorial n minus 1 minus k minus 1 factorial, p to the power k, q to the n minus 1 minus k minus 1, k is equal to 0,1 up to n. Since at k is equal to 0, k minus 1 factorial does not exist, we rewrite it as the expected value of X is equal to sigma, k is equal to 1 to n, n factorial k minus 1 factorial, n minus 1 minus k minus 1 factorial, p to the power k, q to the power n minus 1 minus k minus 1 is equal to.

Let us take n out and 1 p out sigma k is equal to 1 to n, n minus 1 factorial upon k minus 1 factorial, n minus 1 minus k minus 1, factorial p to the power k minus 1, q to the power n minus 1 minus k minus 1, k is equal to 1 to n, is equal to np into, if k goes from 1 to n, put j is equal to k minus 1.

Therefore, j varies from 0 to n minus 1. Therefore, we write it as j is equal to 0 to n minus 1, n minus 1 factorial upon j factorial n minus 1 minus j factorial p to the power j, q to the power n minus 1 minus j is equal to np into p plus q whole to the power n minus 1, this term is equal to 1 therefore, we get np. Therefore, the expected value of X is equal np, when X is distributed us binomial n comma p.

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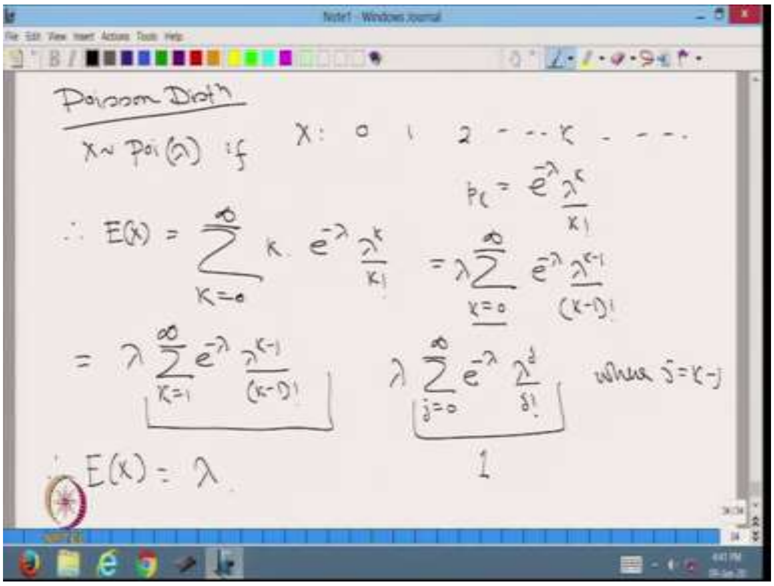

Let us now considers Poisson distribution. We know that X follows Poisson with lambda, if X takes the values 0, 1, 2, k, up to infinity with probability for X is equal to k is e to the power minus lambda, lambda to the power k upon factorial k. Therefore, expected value of X is equal to sigma, k is equal to 0 to infinity, k times e to the power minus lambda, lambda power k upon factorial k is equal to k is equal to 0 to infinity.

Let us take out lambda, e to the power minus lambda, lambda to the power k minus 1 upon factorial k minus 1, as before, since that k is equal to 0, k minus 1 factorial does not have any meaning. Therefore, we start with k is equal to 1 and if we are good enough, we can observe that this is basically a summation of lambda j is equal to 0 to infinity, e to the power minus lambda, lambda to the power j upon factorial j, where j is equal to k minus 1.

Therefore, this sum is the sum of probabilities for a Poisson distribution over this sum is 1. Therefore, the expected value of X is equal to lambda or in other words, if X is a Poisson distribution with parameter lambda than expected value of X is lambda.

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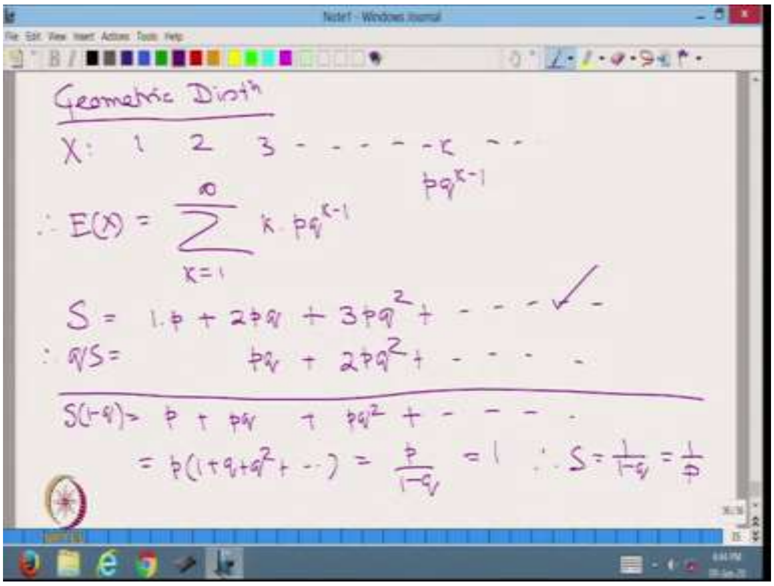

Now, let us considered geometric distribution, therefore, we know that X takes the value 1, 2, 3, k, up to infinity with probability X takes the value k is equal to p q to the power k minus 1. Therefore, the expected value of X is equal to sigma k is equal to 1 to infinity k pq k minus 1.

Question is, what is the sum? So, let us write it as, S is equal to 1 times p, q to the power 0 plus 2 times pq, plus 3 times pq square plus etc. Therefore q times S is equal to, let us write them by shifting one position to the right. It is p times q, therefore pq it is 2pq times q, therefore 2pq square, like that if we go then we can see that the difference between them is S into 1 minus q is equal to p plus pq plus pq square plus is equal to p into 1 plus q plus q square is equal to p 1 minus q is equal to 1.

Therefore yes, which is the expected value of the random variable is equal to 1 upon 1 minus q is equal to 1 upon p.

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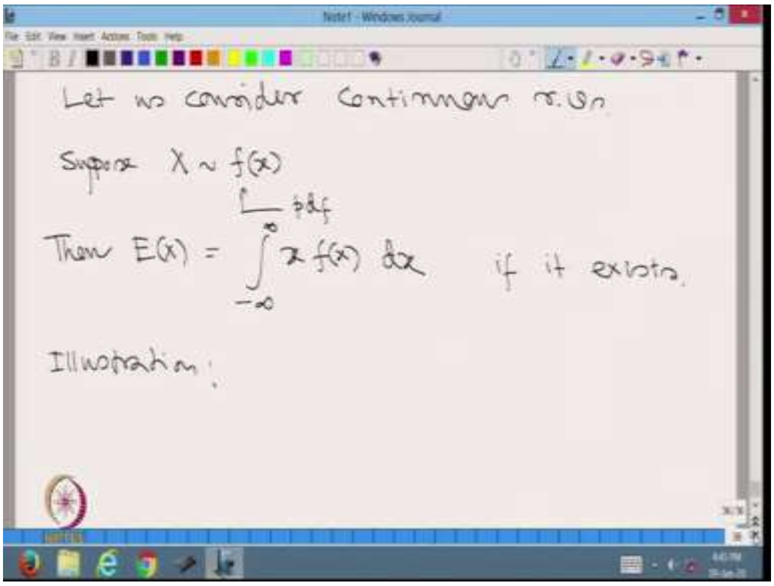

Let us now consider continuous random variables. Suppose X is a random variable with the pdf, f of x then the expected value of X is defined as integration minus infinity to infinity x fx dx, if it exists. There may be random variables for each expectation does not exist, we shall show some of them later. But let us now illustrate with several examples.

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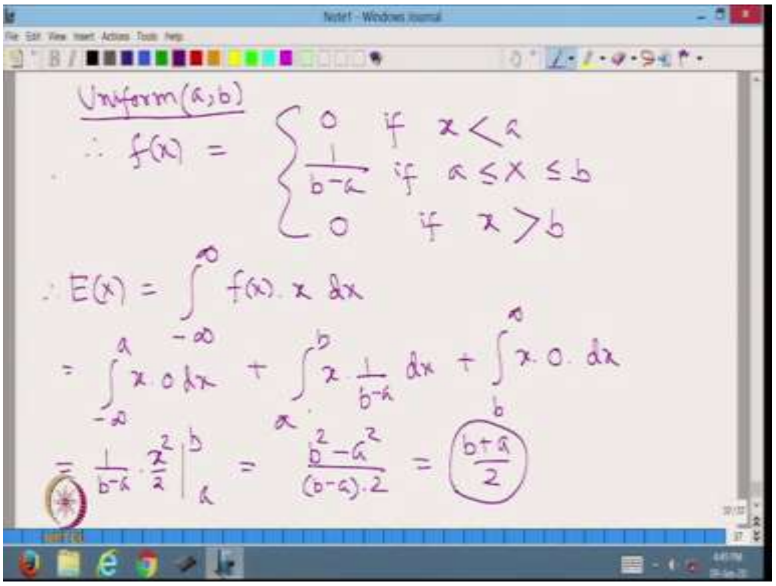

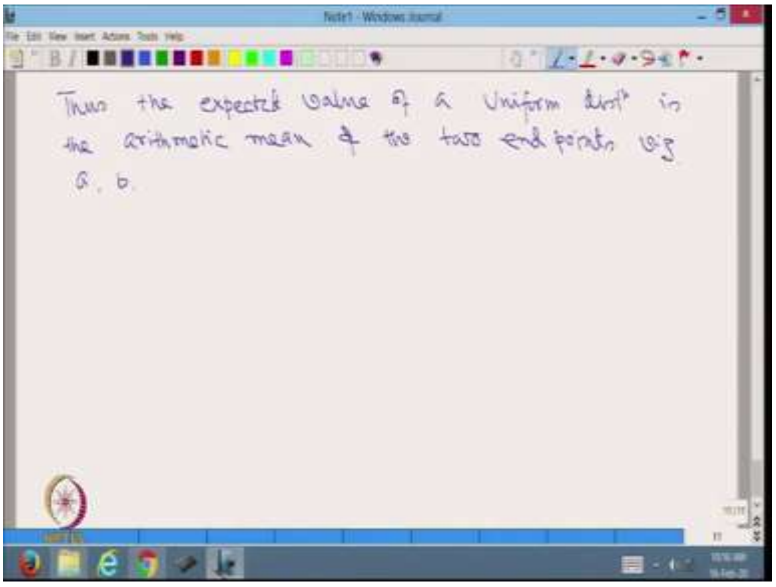

Suppose, X is a uniform random variable in the range a to b therefore, f x is equal to 0, if x less than a, it is 1 upon b minus a, if a less than equal to x less than equal to b is equal to 0, if x is greater than b.

Therefore expected value of X is equal to integration minus infinity to infinity, f of x into x dx, which is equal to minus infinity to a x into 0 dx plus a to b, x into 1 upon b minus a, dx plus integration b to infinity x into 0 dx is equal to as we can see, going to be 1 upon b minus a into x square by 2, b to a is equal to b square minus a square upon b minus a into 2 is equal to b plus a by 2. Therefore, if X is uniform in a b, its expectation is coming out to be b plus a by 2.

Thus the expected value of a uniform distribution is the arithmetic mean of the two end points namely a comma b.

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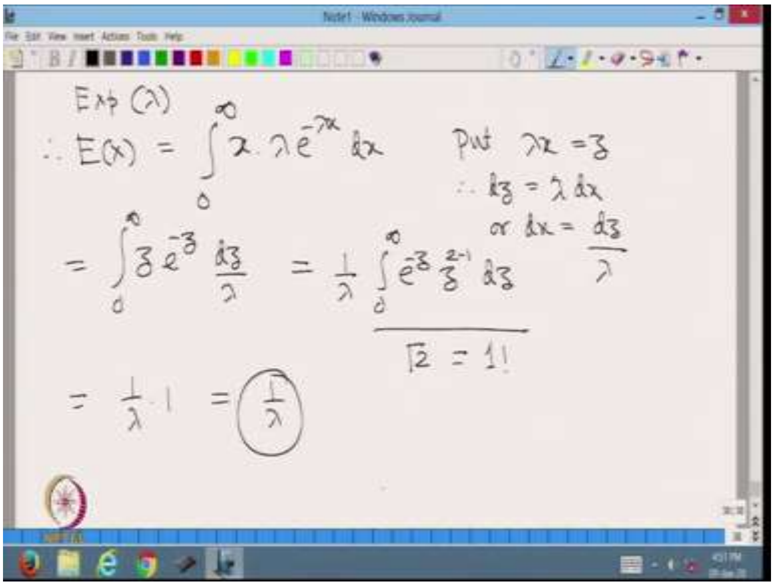

Now let us consider exponential lambda, therefore the expected value of X is going to be in a similar way, 0 to infinity x lambda e to the power minus lambda x dx, put lambda x is equal to z. Therefore, dz is equal to lambda dx or dx is equal to dz upon lambda, therefore this we can write it as integration 0 to infinity z e to the power minus z, dz upon lambda is equal to 1 upon lambda 0 to infinity e to the power minus z, z to the power 2 minus 1 dz.

Therefore this is coming out to be the gamma 2 and which we have observed that, which is equal to 1 factorial. Therefore, the expected value of X is going to be 1 upon lambda multiplied by 1 is equal to 1 upon lambda.

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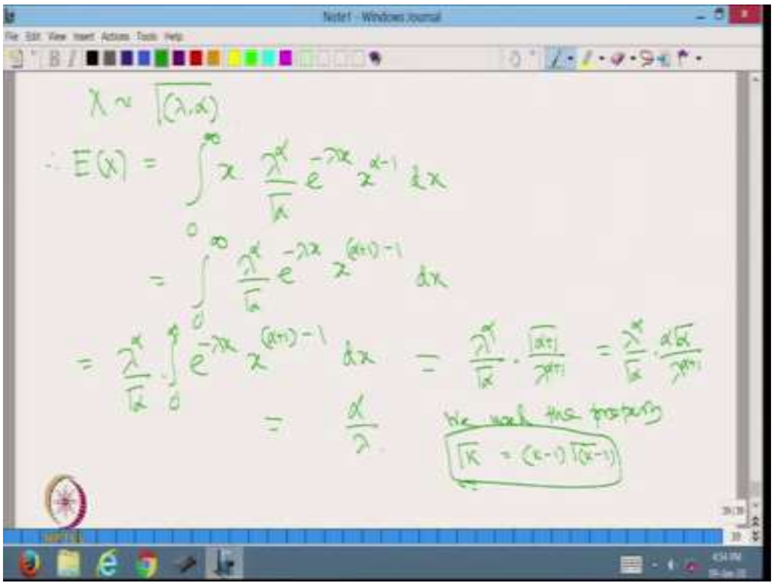

Considered X, which follows gamma lambda alpha. Therefore, the expected value of X is equal to integration 0 to infinity x lambda power alpha upon gamma alpha, e to the power minus lambda x, x to the power alpha minus 1 dx is equal to integration 0 to infinity, lambda power alpha upon gamma alpha, e to the power minus lambda x, x to the power alpha plus 1 minus 1 dx.

Therefore, this is going to be lambda power alpha upon gamma alpha, if we take out then what we have is 0 to infinity, e to the power minus lambda x, x to the power alpha plus 1 minus 1 dx and we know that this integration will lead to gamma alpha plus 1 upon lambda power alpha plus 1 is equal to lambda power alpha upon gamma alpha.

And we know from the properties of gamma integral that gamma alpha plus 1 is equal to alpha gamma alpha divided by lambda power alpha plus 1 is equal to alpha over lambda. So, we have used one property that gamma k is equal to k minus 1 gamma k minus 1.

If someone is not familiar with this result, I suggest that you check a book of first year mathematics you should be able to understand that.

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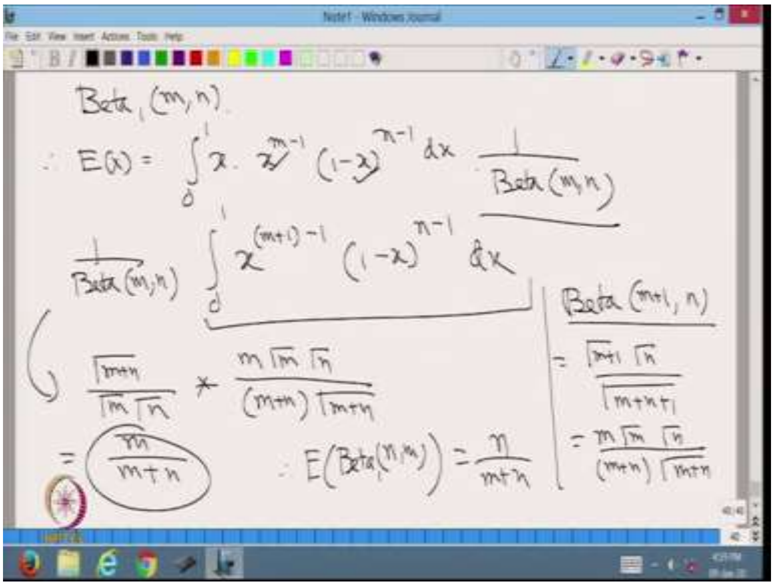

Beta 1 m comma n, therefore, the expected value of X is equal to integration 0 to 1 x, x to the power m minus 1, 1 minus x to the power n minus 1 dx 1 upon beta m comma n, let us take this outside, so, we get one upon beta m comma n integration 0 to 1, x to the power m plus 1 minus 1, 1 minus x to the power n minus 1 dx.

So, this expression is nothing but beta m plus 1 comma n, therefore this is gamma m plus 1 gamma n upon gamma m plus n plus 1, we are computing that is equal to m times gamma m gamma n into m plus n times gamma m plus n. So, beta m plus 1 comma n reduces to this, therefore this whole expression is becoming gamma m plus n upon gamma m gamma n multiplied by beta m plus 1 n, which is equal to m gamma m gamma n upon m plus n gamma m plus n after cancellation we have m upon m plus n.

Therefore, the expected value depends upon the two parameters of m and n and we can easily see that the expected value of beta 1 n comma m is going to be n upon m plus n. Therefore, what is the power of x and what is the power of 1 minus x, these two together gives us the expected value of beta 1 m n.

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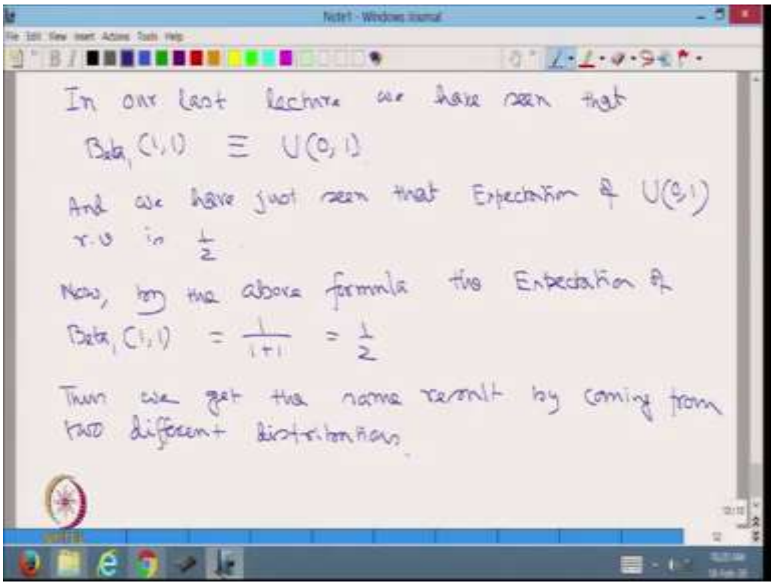

In our last lecture we have seen that beta 1, 1 comma 1 is equivalent to uniform 0 1. And we have just seen that expectation of uniform 01 random variable is half. Now, by the above formula the expectation of beta 1, 1 comma 1 is equal to 1 upon 1 plus 1 is equal to half. Thus, we get the same result by coming from two different distributions.

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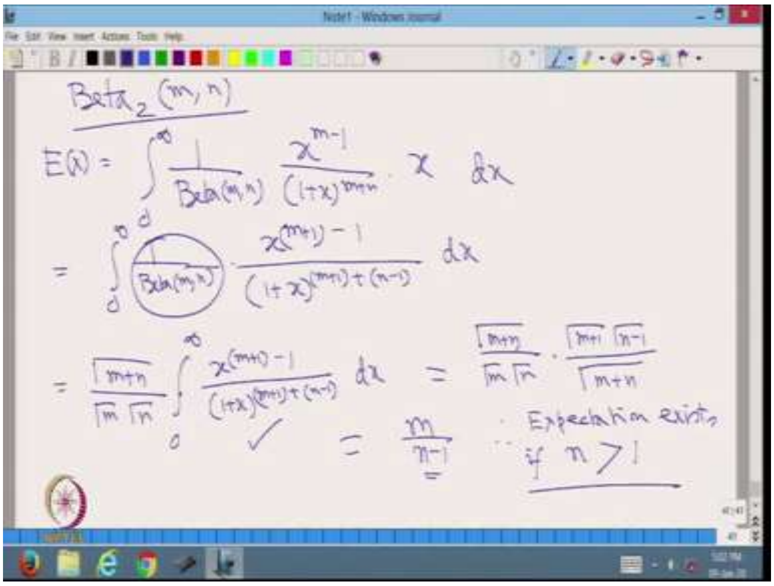

What about beta 2 m n. Since we know the pdf, we go straight into that expectation of X is equal to integration 0 to infinity, 1 upon beta m n, x to the power m minus 1 upon 1 plus x to the power m plus n multiplied by x dx, is equal to 0 to infinity 1 upon beta m n multiplied by x to the power m plus 1 minus 1 upon 1 plus x, m plus 1 plus n minus 1 dx.

Therefore, we can take out these out of the integration and we can write it as gamma m plus n upon gamma m gamma n, multiplied by the integration 0 to infinity x to the power m plus 1 minus 1 upon 1 plus x whole to the power m plus 1 plus n minus 1.

And if we are clever enough, we can understand that this is going to give me beta m plus 1 comma n minus 1. Therefore, we can write it as gamma m plus n upon gamma m gamma n multiplied by gamma m plus 1 gamma n minus 1 upon gamma m plus n, in a similar way if we cancel we get m upon n minus 1.

Therefore, this expectation will exists if n greater than 1 otherwise this is undefined and therefore, expectation does not exist. So, as I said towards the beginning, that there will be cases when for certain variables we do not have that expectation. It is one example, when the value of n is less than equal to 1.

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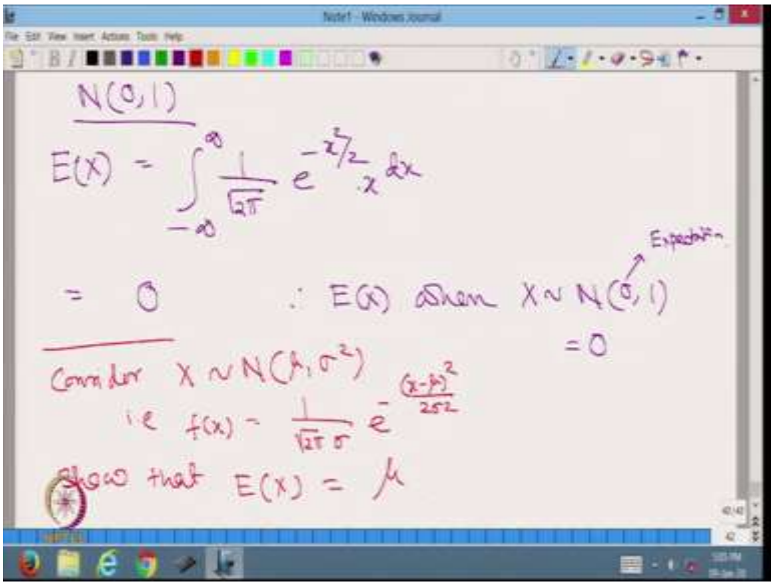

Let us now considered normal 0 1 expected value of X is equal to integration minus infinity to infinity, 1 over root over 2 Pi, e to the power minus x squared by 2 dx multiplied by x is equal to, because it is an odd function of x, we do not need to do anything, we can simply see that this integration is going to be value 0. Therefore, expected value of X, when X is normal 0 1 is equal to 0, that this gives the expectation.

Consider now normal Mu sigma square, that is f of x is equal to 1 over root over 2 Pi sigma e to the power minus x minus Mu whole square upon 2 sigma square and show that the expected value of X is equal to Mu, I leave that as an exercise. Therefore, so far we have seen that expectation of different standard random variables. Now, as I said that expectation does not exist all the time I give you some more examples.

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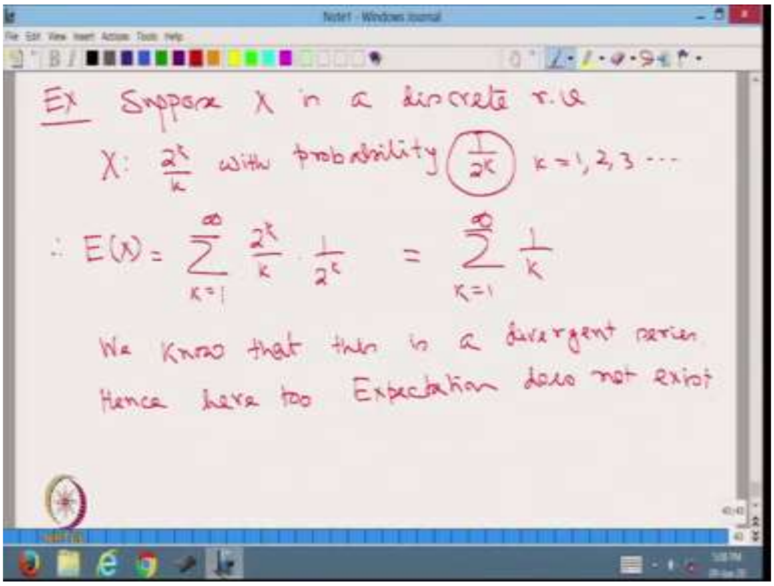

Suppose, X is a discrete random variable such that X takes the value 2 to the power k upon k with probability 1 upon 2 to the power k, k is equal to 1, 2, 3 like that, therefore expected value of X is going to be sigma k is equal to 1 to infinity, 2 to the power k upon k multiplied by 1 upon 2 the power k is equal to sigma 1 by k, k is equal to 1 to infinity.

We know that this is a divergent series. Hence here to expectation does not exist. So, this is another example, where we find that the expected value of the random variable does not exist, although there is no problem in definition of that pdf, because we know that this series sum to 1.

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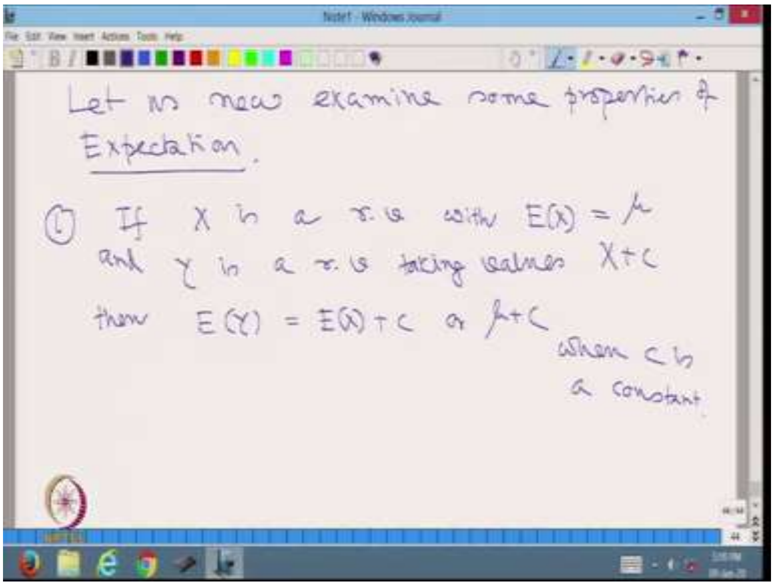

Okay friends, let us now examine some properties of expectation, one, if X is a random variable with expected value of X is equal to Mu and y is a random variable taking values X plus C then the expected value of Y is equal to expected value of X plus C or mu plus C when C is a constant.

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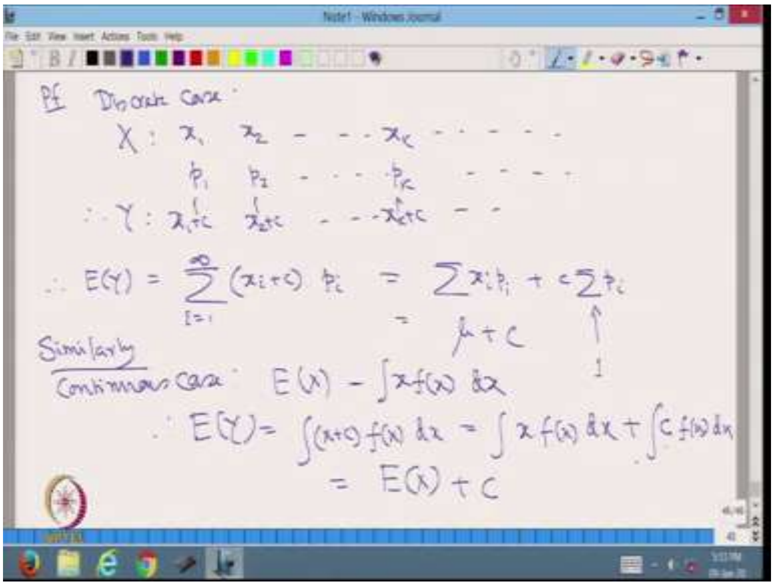

Proof discrete case, suppose X takes the values x1, x2, xk with probabilities p1, p2, pk, Therefore, Y takes is the values x1 plus c, x2 plus c, xk plus c with the same probabilities. Therefore, the expected value of Y is equal to sigma i is equal to 1 to infinity, xi plus c multiplied by pi is equal to sigma xi times pi plus c times sigma pi is equal to mu plus C as this is 1.

Similarly, for continuous case, expected value of X is equal to x fx dx, therefore expected value of Y is equal to x plus c fx dx is equal to x times fx dx plus c times f x dx is equal to expected value of X plus C. Since, C will come out of this integration as it is a constant and fx, dx will integrate to 1.

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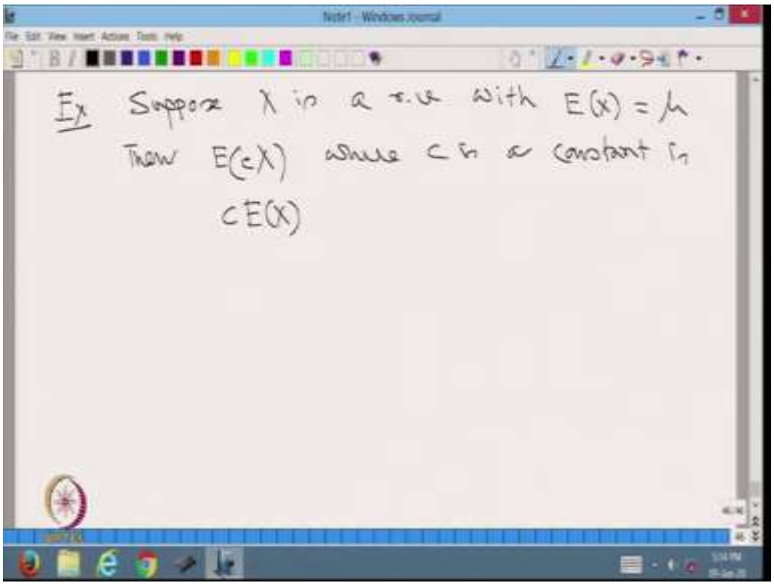

Suppose X is a random variable with the expected value of X is equal to Mu. Then expected value of c times X, where C is a constant is C times expected value of X, I leave this as an exercise, you should be able to do it in a very similar way, we have done for the summation case.

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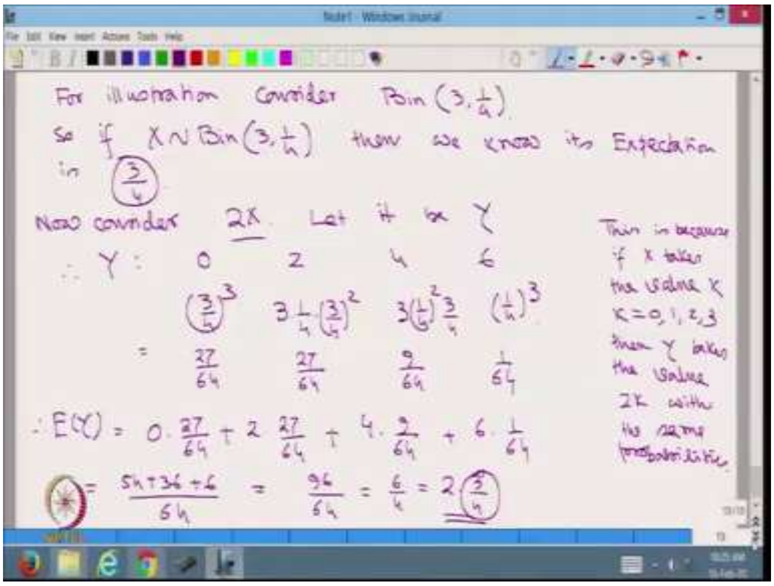

For illustration consider binomial with 3 comma 1 by 4. So, if X is distributed as binomial, 3 comma 1 by 4, then we know its expectation is 3 by 4. Now consider 2x, let it be Y. Therefore, Y takes values 0, 2, 4 and 6 with probabilities 3 by 4, whole cube 3 into 1 by 4 into 3 by 4 whole square, 3 into 1 by 4 whole square into 3 by 4 and 1 by 4 whole cube.

This is because if x takes the value k, k is equal to 0, 1, 2, 3 then Y takes the value 2k with the same probabilities. So, these are is equal to 27 by 64, 27 by 64, 9 upon 64 and 1 upon 64. Therefore, expected value of Y is equal to 0 into 27 upon 64 plus 2 into 27 upon 64 plus 4 into 9 upon 64 plus 6 into 1 upon 64 is equal to 54 plus 36 plus 6 upon 64 is equal to 96 upon 64 is equal to 6 by 4 is equal to 2 into 3 by 4.

Thus we see that for Y is equal to 2x, the expectation is 2 times the expectation of X. So this examples show us that if we have the pdf, we do not only get the expected value of the random variable, we can also get expected value of different functions of a random variable, if we can apply the definitions properly.

Okay friend, I stop here today. In the next class, I shall start with the concept of variance of a random variable. Thank you.