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Theorem: Prove that $\sqrt{2}$ is irrational.

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Theorem: There are infinite number of primes.

Proof: Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7, \ldots$ be the primes in ascending order and suppose that there is a last prime called p_n.

Consider

$$P = p_1 p_2 \cdots p_n + 1$$

$P > 1$, $\therefore$ By FTA (Fundamental theorem of arithmetic) P is divisible by some prime p, but $p_1, p_2, \ldots, p_n$ are the only prime numbers,

$\therefore$ p must be equal to one of $p_1, p_2, \ldots, p_n$.

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$\Rightarrow \quad p \mid p_1 p_2 \cdots p_n$ and $p \mid P$

$\Rightarrow \quad p \mid P - p_1 p_2 \cdots p_n$

$\Rightarrow \quad p \mid 1$

a contradiction as $p > 1$

$\Rightarrow$ There are infinite number of primes.

$\rightarrow$ Product of two or more integers of the form $4n+1$ is of the same form.

$\rightarrow$ There are infinite number of primes of the form $4n+3$.

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<u>Proof</u>! Assume that there are only finite number of primes $q_1, q_2, \ldots q_s$ of the form $4n+3$.

Consider

$$N = 4 q_1 q_2 \ldots q_s - 1$$
$$= 4(q_1 q_2 \ldots q_s - 1) + 3$$

Let $N = r_1 r_2 \ldots r_t$ be the prime factorization.

$r_k \neq 2 \quad \forall k \quad \because N$ is odd

* (Using product of two integer of the form ~~4n~~ $4n+1$ is $4n+1$) *

r_k is either of the form $4n+1$ or $4n+3$. $\therefore$ One of the r_k must be of the form $4m+3$ for N to take the form $4n+3$.

But $r_k \neq q_1, q_2, \ldots q_s$. Hence the result.

ie there are infinite number of primes of the form $4n+3$.

Arithmetic Function: An arithmectic function a function $f : \mathbb{N} \to \mathbb{C}$

e.g $\pi(n)$: the number of primes $\leq n$.

$\tau(n) = d(n)$: the number of positive divisors of n.

$\sigma(n)$: the sum of positive divisors of n.

$$\sigma(1) = 1$$

$$\sigma(2) = 1+2 = 3$$

$$\sigma(3) = 1+3 = 4$$

$$\sigma(6) = 1+2+3+6 = 12$$

$$\tau(1) = 1$$

$$\tau(2) = 2$$

$$\tau(3) = \cancel{3}\ 2$$

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Perfect Number: A number n is perfect if $\sigma(n) = 2n$.

If n is a prime number, then

$$\tau(n) = 2$$

$$\sigma(n) = n+1$$

$\sum_{d|n} f(d)$ = Sum of values of $f(d)$ as d runs over all the positive divisors of the positive integer n.

$$\therefore \quad \tau(n) = \sum_{d|n} 1$$

$$\sigma(n) = \sum_{d|n} d$$

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Theorem: If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then the positive divisors of n are precisely those integers d of the form

$$d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \quad 0 \le a_i \le k_i$$

$$i = 1, 2, \ldots r.$$

Proof: