Continued Fractions: An expression of the form

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots \, a_{n-2} + \cfrac{b_{n-1}}{a_{n-1} + \cfrac{b_n}{a_n}}}}}$$

$a_0, a_1, \ldots a_n \in \mathbb{R}$
$b_1, b_2, \ldots b_n \in \mathbb{R}$

is called continued fraction. This is finite continued fraction.

Simple Continued Fraction: A continued fraction is called a simple continued fraction if all the b_j's are 1 and all the a_j's are integers such that $a_1, a_2, \ldots \geq 1$.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \, \vdots \, \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

$$= [a_0; a_1, a_2, \ldots, a_n]$$

Def: The continued fraction made from $[a_0, a_1, \ldots a_n]$ by cutting off the expansion after the kth partial denominator a_k is called the k-th convergent of the simple continued fraction and is denoted by C_k, where

$$C_k = [a_0, a_1, \ldots a_k], \quad k \le n.$$

Theorem: Any rational number can be written as a finite simple continued fraction.

Proof: Let $\frac{a}{b}$, $b>0$ be an arbitrary rational number.

By Euclidean algorithm for a and b

$$a = ba_0 + r_1 \quad , \quad 0 \leq r_1 < b$$

$$b = r_1 a_1 + r_2 \quad , \quad 0 \leq r_2 < r_1$$

$$r_1 = r_2 a_2 + r_3 \quad , \quad 0 < r_3 < r_2$$

$$\vdots$$

$$r_{n-2} = r_{n-1} a_{n-1} + r_n \quad , \quad 0 < r_n < r_{n-1}$$

$$r_{n-1} = r_n a_n + 0$$

Each remainder r_K is a positive integer, $a_1, a_2, \ldots a_n$ are all positive.

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Rewrite the above equations:

$$\frac{a}{b} = a_0 + \frac{r_1}{b} = a_0 + \frac{1}{\frac{b}{r_1}}$$

$$\frac{b}{r_1} = a_1 + \frac{r_2}{r_1} = a_1 + \frac{1}{\frac{r_1}{r_2}}$$

$$\frac{r_1}{r_2} = a_2 + \frac{r_3}{r_2} = a_2 + \frac{1}{\frac{r_2}{r_3}}$$

$$\vdots$$

$$\frac{r_{n-1}}{r_n} = a_n$$

$$\therefore \quad \frac{a}{b} = a_0 + \frac{1}{\frac{b}{r_1}}$$

$$= a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{r_2/r_3}}}$$

$$\vdots$$

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$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{\vdots}{\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

Represent $\frac{19}{51}$ as a Continued Fraction

$$\frac{19}{51} = \frac{1}{51/19} = 0 + \frac{1}{51/19}$$

$51 = 2 \cdot 19 + 13$ or $\frac{51}{19} = 2 + \frac{13}{19}$

$19 = 1 \cdot 13 + 6$ $\qquad \frac{19}{13} = 1 + \frac{6}{13}$

$13 = 2 \cdot 6 + 1$ $\qquad \frac{13}{6} = 2 + \frac{1}{6}$

$6 = 6 \cdot 1$

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$$\frac{19}{51} = \frac{1}{51/19} = \frac{1}{2+\frac{13}{19}}$$

$$= \frac{1}{2+\frac{1}{19/13}}$$

$$= \frac{1}{2+\cfrac{1}{1+\frac{6}{13}}}$$

$$= \frac{1}{2+\cfrac{1}{1+13/6}}$$

$$= \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\frac{1}{6}}}}$$

$$= [0; 2, 1, 2, 6]$$

Note! This representation is not Unique.

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$$C_0 = 0$$

$$C_1 = [0,2] = 0 + \frac{1}{2} = \frac{1}{2}$$

$$C_2 = [0,2,1] = 0 + \frac{1}{2+1} = \frac{1}{3}$$

$$C_3 = [0,2,1,2] = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

$$= \cfrac{1}{2 + \cfrac{1}{3/2}}$$

$$= \frac{1}{2 + 2/3}$$

$$= \frac{3}{8}$$

$$C_4 = [0,2,1,2,6] = \frac{19}{51}$$

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Define

$p_0 = a_0 \qquad q_0 = 1$

$p_1 = a_1 a_0 + 1 \qquad q_1 = a_1$

$p_2 = a_2 p_1 + p_0 \qquad q_2 = a_2 q_1 + q_0$

$\vdots \qquad \vdots$

$p_K = a_K p_{K-1} + p_{K-2}$

$q_K = a_K q_{K-1} + q_{K-2}, \quad K \geq 2$

$$C_0 = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}$$

$$C_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}$$

$$C_2 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}$$

$$= \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}$$

$$= \frac{p_2}{q_2}$$

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Theorem: The kth convergent of the simple continued fraction $[a_0, a_1, \ldots a_n]$ has the value

$$C_k = \frac{p_k}{q_k} \; ; \; 0 \leq k \leq n$$

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2} \; ; \; k \geq 2$$

$$p_0 = a_0 \, , \quad q_0 = 1$$

$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

Proof: As $C_0 = \frac{p_0}{q_0}$

$$C_1 = \frac{p_1}{q_1}$$

$$C_2 = \frac{p_2}{q_2}$$

The theorem is true for $k = 0, 1, 2.$

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Assume that the theorem is true for $k = m$, $2 \leq m < n$, i.e

$$C_m = \frac{p_m}{q_m}$$

$$p_m = a_m p_{m-1} + p_{m-2}$$

$$q_m = a_m q_{m-1} + q_{m-2}$$

$$C_m = [a_0, a_1, \ldots a_m]$$

$$= \frac{p_m}{q_m} = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}} \quad -(*)$$

Replace a_m by $a_m + \frac{1}{a_{m+1}}$

L.H.S of (*) is $\cancel{C_{m+1}}$ =

$$\left[a_0, a_1, \ldots, a_m + \frac{1}{a_{m+1}}\right]$$

$$= \frac{\left(a_m + \frac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \frac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}}$$

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$$= \frac{a_m a_{m+1} p_{m-1} + p_{m-1} + a_{m+1} p_{m-2}}{a_m a_{m+1} q_{m-1} + q_{m-1} + a_{m+1} q_{m-2}}$$

$$= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}}$$

$$= \frac{a_{m+1} p_m + p_{m-1}}{a_{m+1} q_m + q_{m-1}}$$

$$= \frac{p_{m+1}}{q_{m+1}}$$

$$= c_{m+1}$$

Result