Advanced Probability Theory
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Lecture 04

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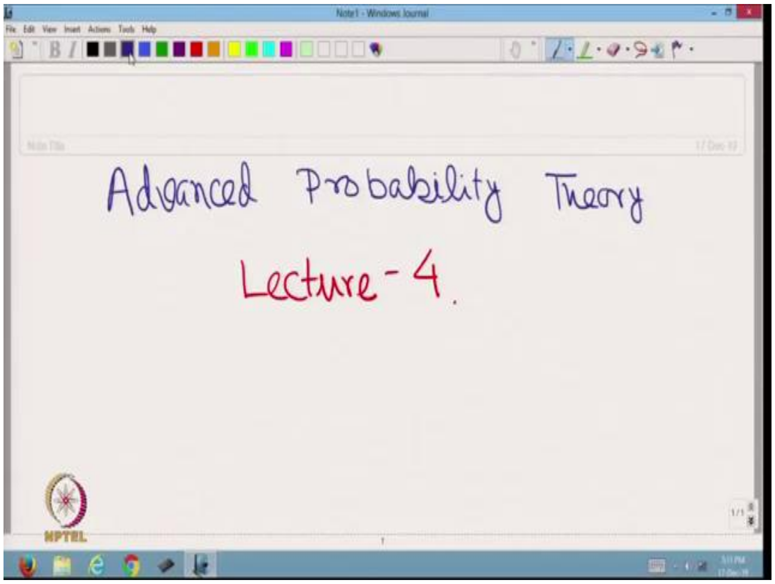

Welcome students to the fourth lecture on Advanced Probability Theory. So, in the first week, we have seen certain basics of probability and certain actions on probability. As I said, at the end of the last class that today, first I will solve one or two complicated problems, then we will go for conditional probability.

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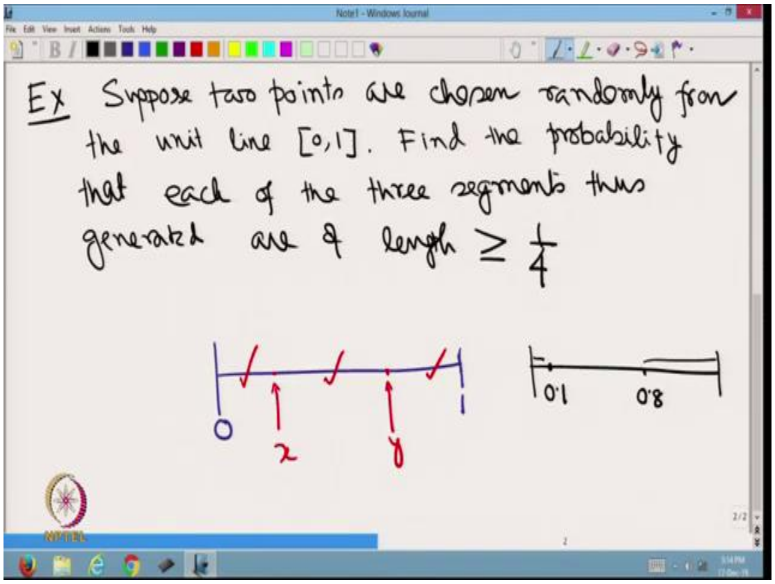

So, let us first start with a problem. Suppose two points are chosen randomly from the unit line 0 to 1. Find the probability that each of the three segments thus generated are of length greater than equal to 1 by 4. So, let us first explain the problem, suppose, on real line, this is the interval 0 to 1, we need to randomly put two points.

Suppose, these are the two points, let me call them x and y, we want the probability that these three segments are each one of length greater than equal to 1 by 4. So, for example, this would not happen in this case suppose, my first point is 0.1 and second point is 0.8 then we can see that this segment and this segment are of length less than 1 by 4. So, I hope you understand the problem.

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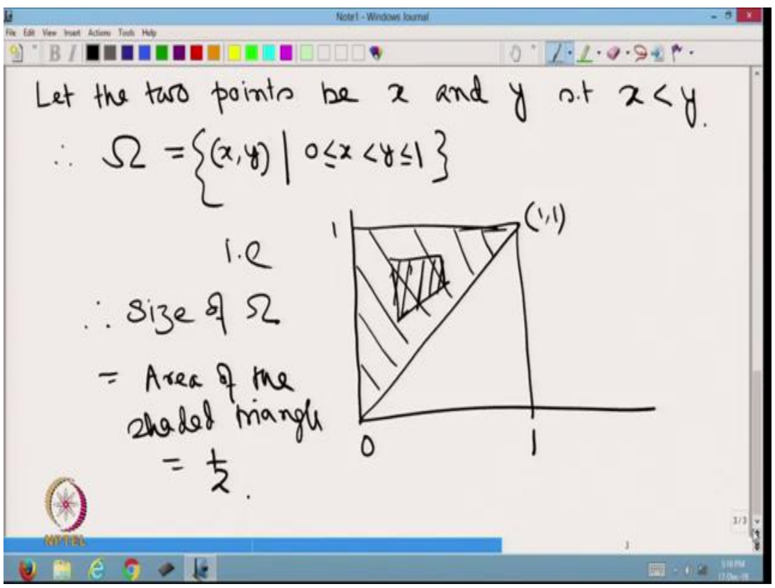

Now, let us see how to solve this. So, let the two points be x and y, such that x is less than y, therefore, what is the Omega? The omega is all those pairs x, y such that 0 less than equal to x, less than y less than equal to 1 that is if we consider the unit square then, y has to be greater than x or in other words, my omega is going to be this triangle.

Therefore, size of omega is equal to area of this shaded triangle is equal to half. Therefore, we will look at the favorable set of points, suppose this is this triangle or this shape then will compute its area, which will be divided by half. And that is going to give us the probability of the event.

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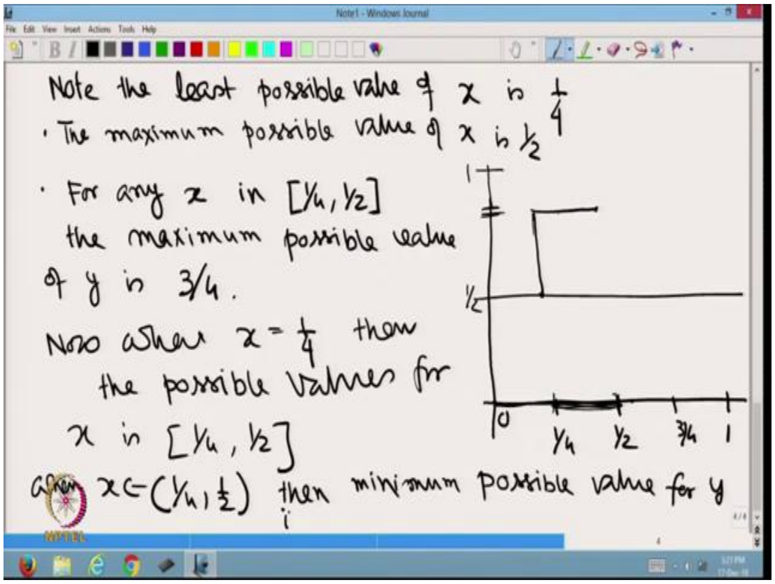

Now, we need to obtain this area and we need to obtain the exact size of that one, note the least possible value for x is 1 by 4, because if x is less than 1 by 4, then the length of the first segment is going to be less than 1 by 4. So, let us call it 0, half, 1 and this is 1 by 4 and this is 3 by 4.

Therefore, the possible values of x between 1 by 4 to half. Now, the maximum possible value of x is half, because if x is greater than half, then we cannot divide the remaining part in two parts such that each one of them is going to be of length greater than 1 by 4. Thus, we could see the possible values for x has to be from this range. Also note that for any x in this range therefore, the possible values for x is going to be between 1 by 4 to half.

Now, for any x in this range, the maximum possible value of y is 3 by 4 because if y is above 3 by 4, then the third segment going to be of length less than 1 by 4. Now, when x is equal to 1 by 4 then, y can take any value from half to 3 by 4, it cannot be less than half.

Because in that case this segment is going to be of length less than 1 by 4, it cannot be more than 3 by 4, as we have discussed. Therefore, if this is my half, when x is equal to 1 by 4, the possible values for y is going to be between half to 3 by 4. When x is equal to half, the only possibility for y is 3 by 4, because otherwise one of the segment is going to be less than 1 by 4. Therefore, when this is half, the only possible point is for y is this therefore, this is the shape that we are getting.

Now, let us consider so, this we have discussed. Now, suppose x is somewhere here, what is going to be the possible values for y? Obviously, the maximum value of y can be 3 by 4, but the minimum value, the minimum possible value for y is what?

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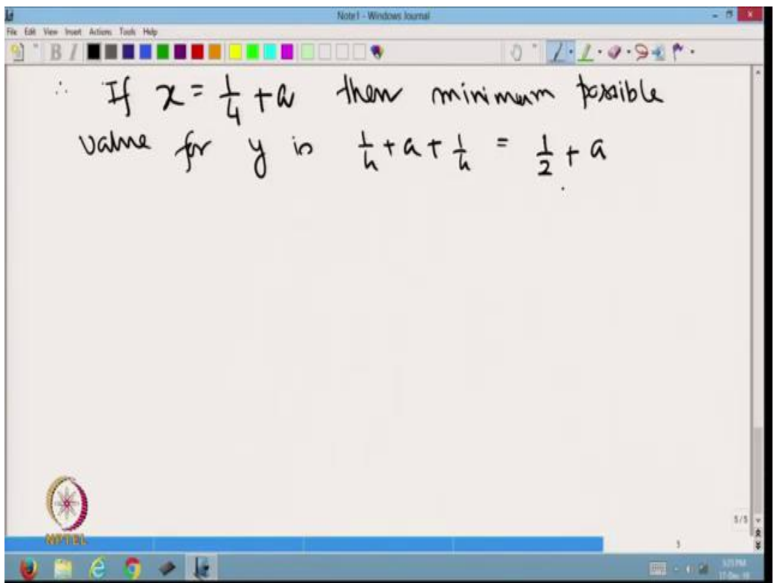

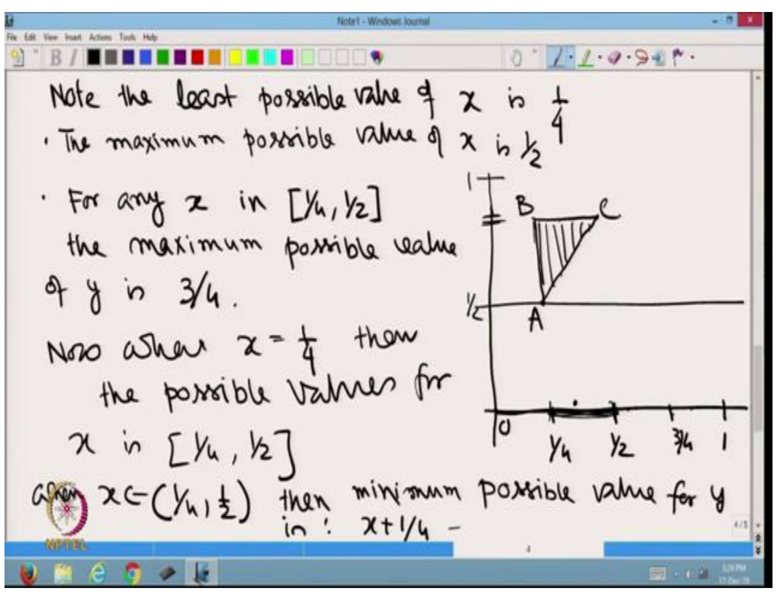

Therefore, if x is equal to 1 by 4 plus a then minimum possible value for y is 1 by 4 plus a plus 1 by 4 is equal to half plus a. Therefore, if we go back to this diagram, we can see that when this is a, the corresponding value is going to be half plus a. Therefore, when x moves from 1 by 4 to half, this is the triangle that we are getting as the possible values of x y pair such that the favorable event will happen. Hence, so, let us call that triangle ABC.

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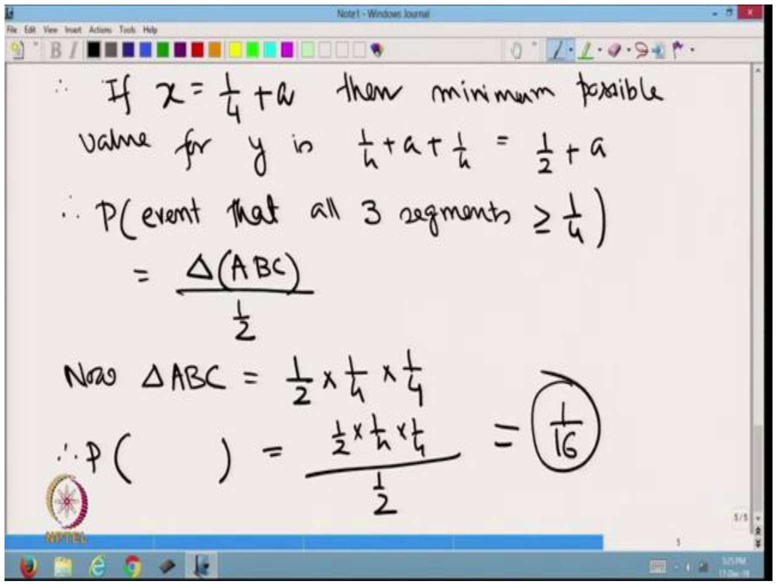

Therefore, probability of the event that all 3 segments greater than equal to 1 by 4 is equal to area of the triangle ABC divided by area of omega which is half. Now, area of triangle ABC is equal to half into base, the base is of length 1 by 4 and altitude, which is again from half to 3 by 4.

Therefore, this is 1 by 4 into 1 by 4 into 1 by 4. Therefore, probability this event is equal to half into 1 by 4 into 1 by 4 divided by half, which is the size of omega is equal to 1 by 16, it is not a very difficult problem, but if you understand the geometry, you can solve it very easily.

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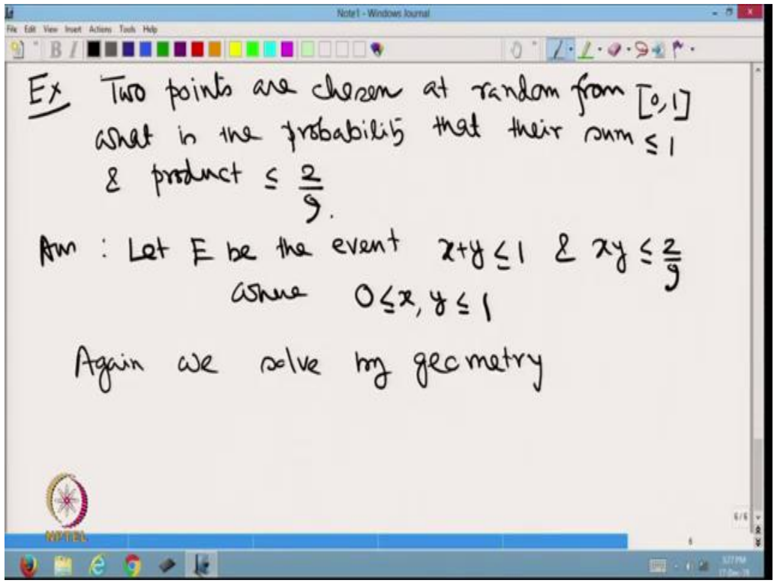

Let me solve another problem for you. This problem is as follows. Two points chosen at random from 0 1, what is the probability that their sum is less than equal to 1 and product less then equal to 2 by 9? Let E be the event that x plus y less than equal to 1 and x y is less than equal to 2 by 9 where 0 less than equal to x and y is less than equal to 1, again we solve by geometry.

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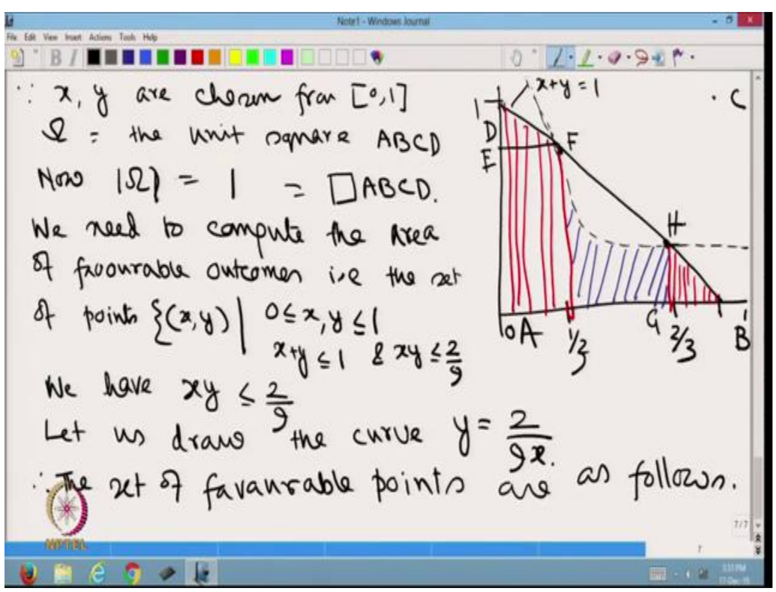

So, we have 0 and 1 therefore, since x comma y are chosen from the interval 0 comma 1 omega is equal to the unit square. Let me call it ABCD for which each of the sizes of length 1, so let us call it A, B, C, D. Now, size of omega is equal to 1, which is equal to area of the square A, B, C, D. We need to compute the area of favorable outcomes that is the set of points x y, such that 0 less than equal to x comma y less than equal to 1 x plus y less than equal to 1, x y less than equal to 2 by 9.

So, let us first draw this line, this is the line x plus y is equal to 1 therefore, we know that the points will lie only in this region. Let us consider these two points, 1 by 3 and 2 by 3 when x is equal to 1 by 3, y is 2 by 3 on this line and x is equal to 2 by 3, then y is equal to 1 by 3 on this line. Now, these two points are significant, because we have x y less than equal to 2 by 9.

So, let us draw the curve y is equal to 2 by 9 x, we know that as x is close to 0, this is going to be infinity. But when x is equal to 1 by 3, this is going to pass through the point 1 by 3, 2 by 3 also it is going to pass through this point, 2 by 3, 1 by 3 because the product along each curve is going to be 2 by 9, which is going to be something like a hyperbola. Therefore, the set of favorable points are as follows.

Let me put them in the diagram, all these points in this area satisfy both the inequalities, all this point in this triangle satisfy both the inequalities and between 1 by 3 to 2 by 3 see the curve is below x plus y is equal to 1. Therefore, the area is going to be this part therefore, we have to compute the total area to find the probability of the event. Now, let me call this E and F and let me call this G and H.

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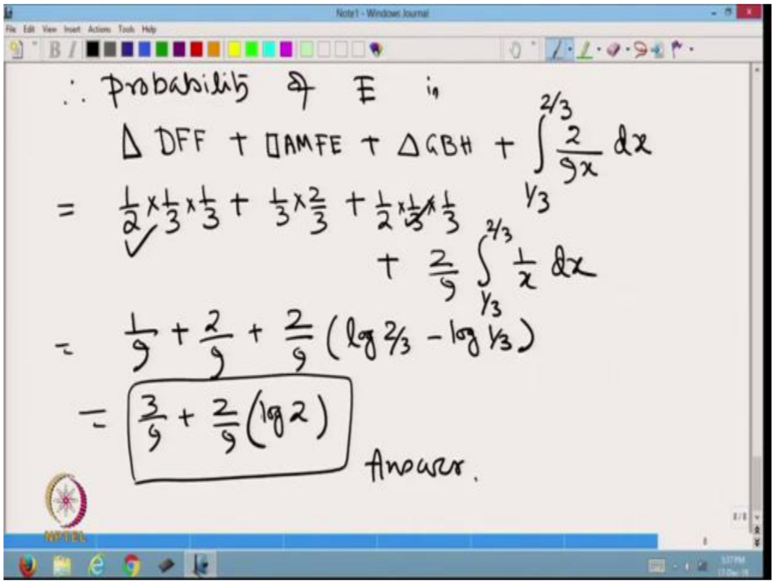

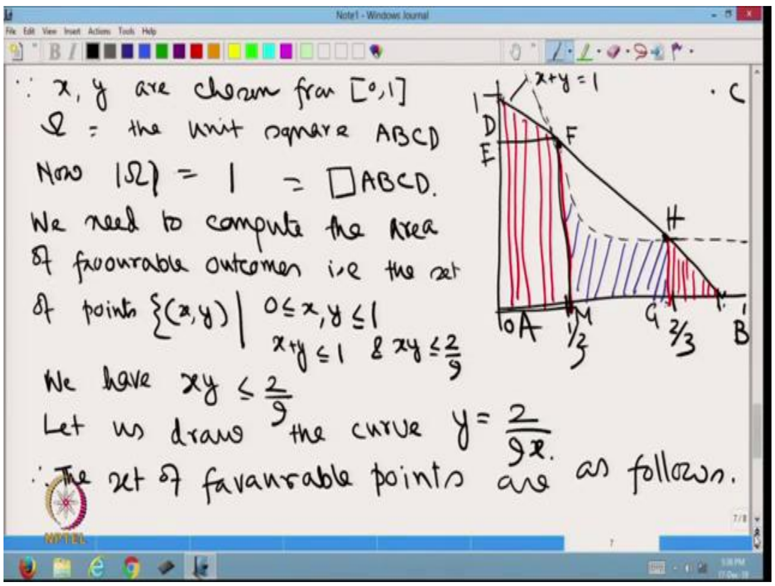

Therefore, probability E is area of the triangle DEF plus area of this rectangle AMFE plus area of the triangle GBH plus between 1 by 2 to 3 by 3, we have to integrate this therefore we are going to have integration of 1 by 3 to 2 by 3, y is equal to 2 by 9x dx. So, this is that the total sum of these areas is going to give us the probability.

Therefore, triangle DEF, this is 2 by 3, this is 1, and this is 1 by 3, therefore, this is going to be half into 1 by 3 into 1 by 3 plus this rectangle, AMFE it has one side, 1 by 3, the other side, 2 by 3, therefore, it is going to be 1 by 3 into 2 by 3 plus, this triangle is going to be 2 by 3 to 1, this is 1 by 3 and at 2 by 3 the height is also 1 by 3.

Therefore, this area is also going to be half into 1 by 3 into 1 by 3 plus 2 by 9 integration of 1 by 3 to 2 by 3, 1 by x dx is equal to this plus this will give me 1 by 9 plus this will give us, 2 by 9 plus 2 by 9 integration of 1 by x is log x.

So, it is going to be log 2 by 3 minus log 1 by 3 is equal to 3 by 9 plus 2 by 9 into this is going to be log of 2. So, this is the answers. Friends, so that gives you some idea of how to solve these problems.

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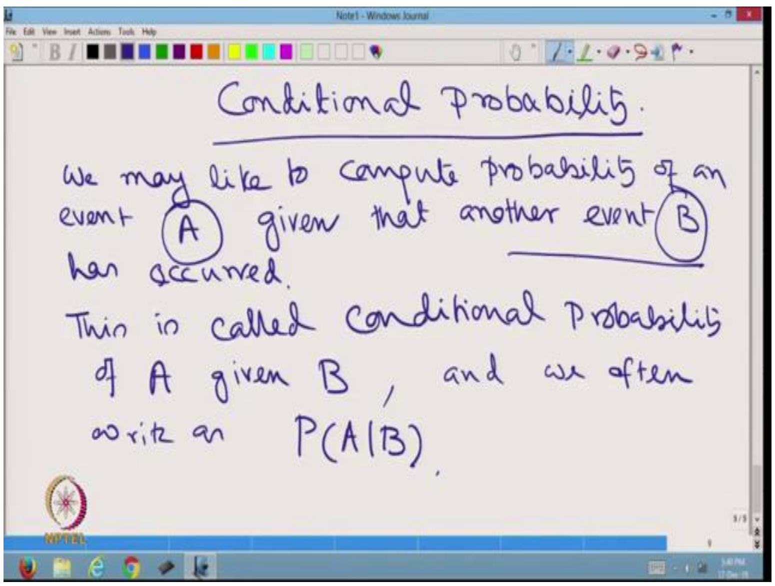

Now with that, now let me move forward, let us start the topic Conditional Probability. In practice, we often face a situation where when we need to compute probability under certain constraints, these constraints come in the form of certain event that is, we may like to compute probability of an event A given that another event B has occurred.

That means, you are trying to find the probability of the event A given that the event B has happened, under this condition, what is going to be the probability of the event A that we call as the conditional probability of A given B and we often write as probability of A given B.

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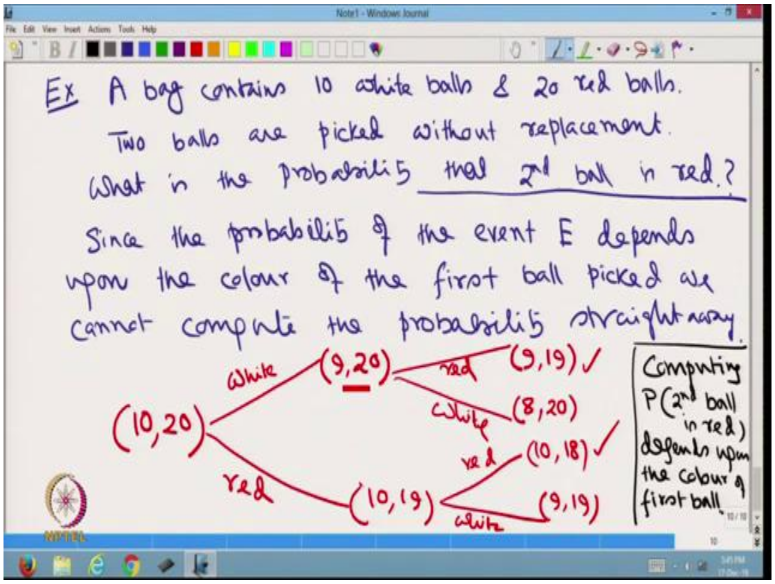

Example, a bag contains 10 white balls and 20 red balls, 2 balls are picked without replacement and we ask, what is the probability that second ball is red? Suppose we need to compute this probability, we can see that we cannot compute it straightaway since, the probability of the event E, what is E?

That the second ball is red, depends upon the color of the first ball picked, we cannot compute the probability straight away because suppose, we started with 10 comma 20 that is initial content of the box, if a white is picked then, then the configuration is going to be 9 comma 20. On the other hand, if the first ball picked is red, the configuration is going to be 10 comma 19.

So, in this case, we can pick a red ball or a white ball and from here also, we can pick a red ball and white ball. If I pick a red ball, then the configuration is going to be 9 comma 19. If we pick a white ball then the configuration is going to be 8 comma 20. If we pick a red ball, this is going to be 10 comma 18.

If we pick a white ball then it is going to be 9 comma 19. Therefore, probability of picking up a red ball from here will be different, if we come along this route or if we come along this route therefore computing probability second ball is red depends upon the color of first ball. I hope you understand the problem therefore, we can compute the probability that the second ball is going to be red based upon what happened in the first pick.

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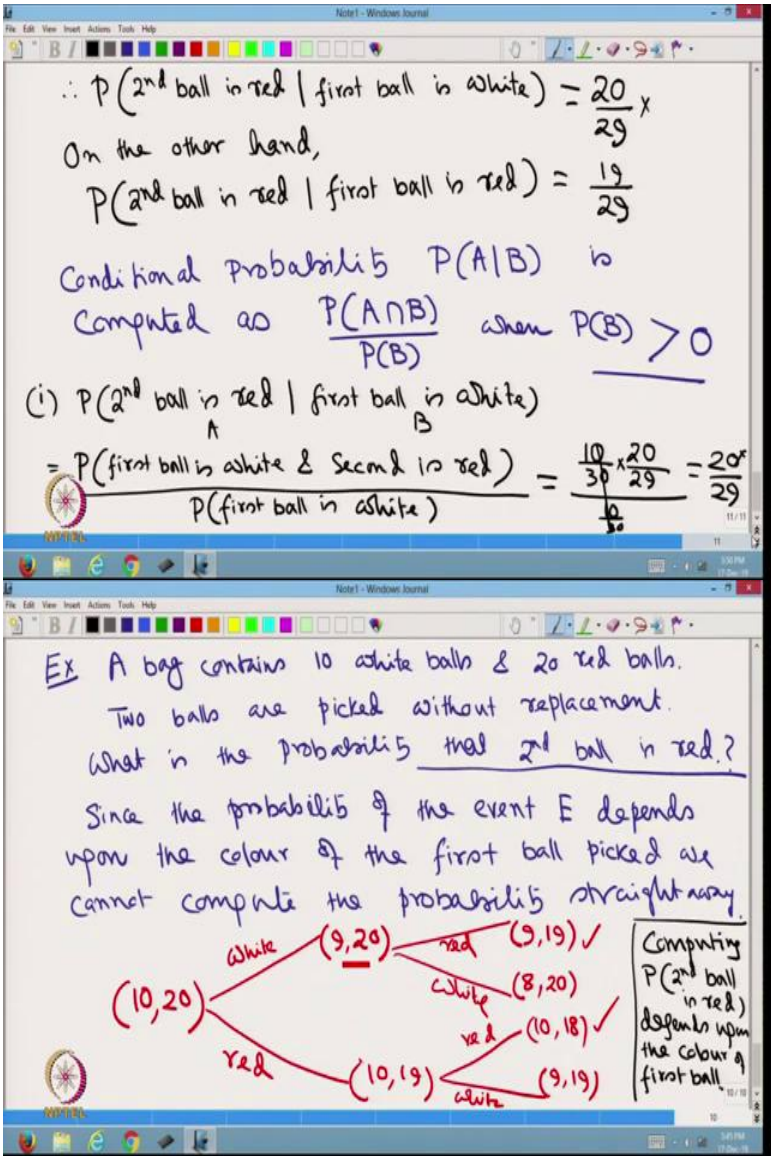

Therefore, probability, second ball is red, given first ball is white is equal to 20 upon 29 because there are 29 balls of which 20 of them are red therefore, this probability is going to be 20 upon 29 on the other hand probability second ball is red given first ball is red is going to be given that the first ball is red, there are 29 balls of which 19 or red therefore, this probability is going to be 19 upon 29.

So, this is what is Conditional Probability, conditional probability, probability of A given B is computed as probability of A intersession B divided by probability of B when probability of B is greater than 0, therefore the conditional probability of A given B will happen if the B has to be an event with positive probability, and then we will look at the joint occurrence of the two events A and B.

So, let us look at this example, therefore, 1 probability second ball is red given first ball is white, if this is my A and this is my B is equal to probability both first ball is red, sorry first ball is white and second ball is red divided upon probability first ball is white, which is equal to first ball is white and second ball is red.

This probability, probability first ball is white is 10 upon 30 then the second ball is red, that probabilities 20 upon 29 divided by probability first ball is white is equal to 10 by 30 and when it cancels, we get 20 upon 29. Therefore, we verified this formula, I suggest that you verify for the second case.

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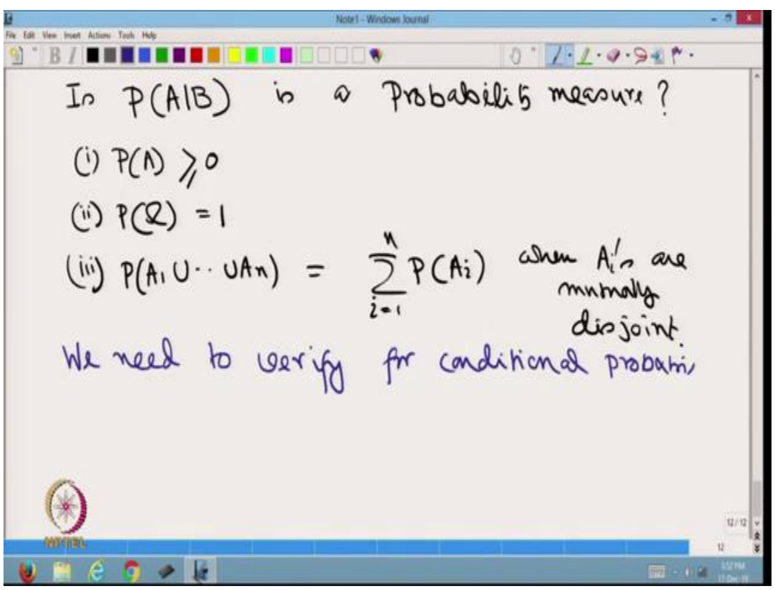

Let us move forward for some theoretical discussion. The first question is, is it a probability measure? We know that for a probability measure there are three properties and thirdly, we know that probability of A1 union An is equal to sigma probability of Ai, i is equal to 1 to n, when Ai is mutually disjoint. We need to show similar things with respect to conditional probability.

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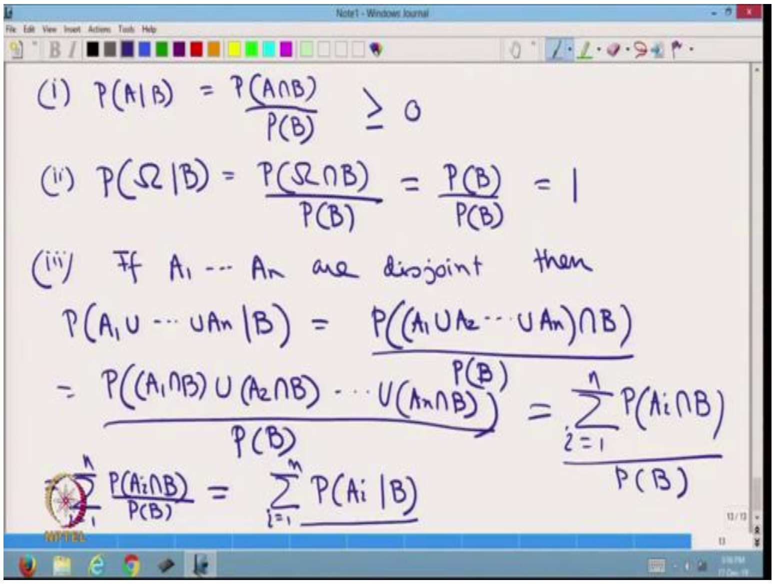

So, one probability A given B is equal to probability of A intersection B upon probability of B. And therefore, it has to be greater than equal to 0. Probability of omega given B is equal to probability of omega intersected with B divided by probability of B. Now, omega intersected with B is equal to B therefore, this is going to be probability of B upon probability of B is equal to 1.

And thirdly, if you want A1, A2, An are disjoined then probability A1 union, A2 union, An given B is equal to probability A1 union, A2 union, An intersected with B divided by probability of B is equal to probability of A1 intersected with B union A2 intersected with B union An intersected with B whole thing divided by probability of B.

Now since, A1, A2 and An are disjoint therefore, the probability of their union is going to be since A1, A2…An are disjoint. Therefore, A1 intersected with B, A2 intersected with B and An intersected with B, these events are also going to be mutually disjoint therefore, probability of their union is going to be the sum of their probabilities.

Therefore, we can write it as sigma i equal to 1 to n probability Ai interested with B divided by probability of B which we can write it as sigma i is equal to 1 to n probability Ai intersected with B divided by probability of B is equal to sigma i equal to 1 to n probability Ai given B. That is, we get that if A1 to An are disjoint, then this conditional probability is sum of their conditional probabilities. Therefore, we see that all that three actions for probability hold here, therefore, conditional probability is also going to give us a probability measure.

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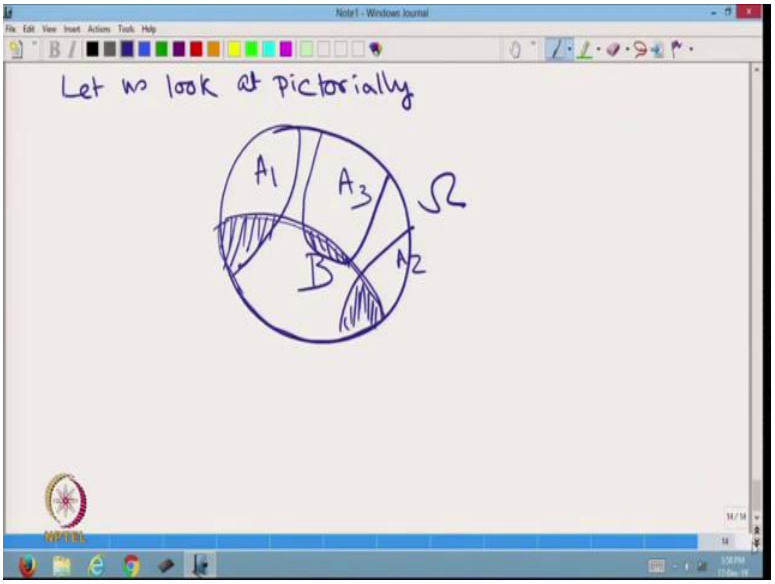

So, let us look at pictorially suppose, this is my omega, suppose this is my A and suppose this is my B therefore, when we look at probability of A given B, we are actually looking at the intersection of A intersection B which is this event and given B therefore, we look at its let it measure it with respect to the measure of B, the conditional probability of A given B.

Now, if A1, A2 and A3 suppose, three disjoint events and therefore, when you look at A1 union, A2 union, A3 given B, then actually we are looking at these three mutually disjoint areas. Therefore, what we have proved just can be visualized from here.

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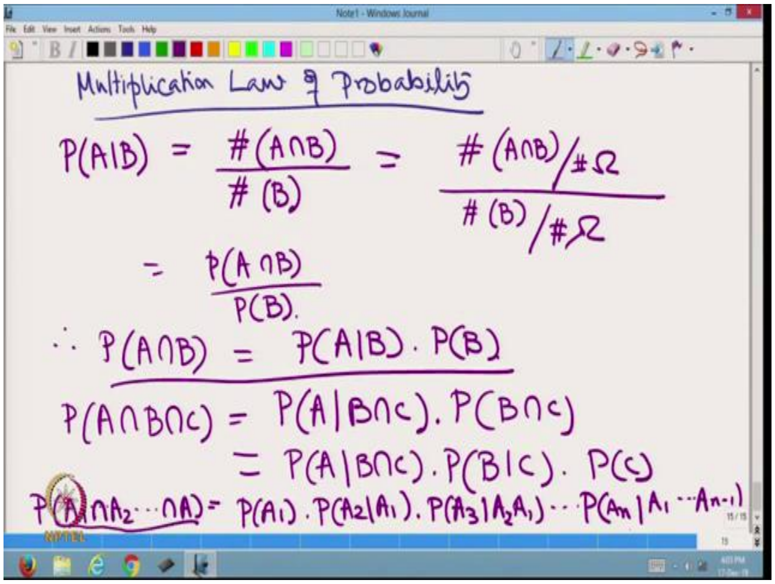

So, multiplication law of probability is an interesting and important section here. So, we get an intuitive idea, probability of A given B is equal to, if we look at it from discrete then we can easily understand the number of points in A intersection B divided by the number of points in B is equal to the number of points of A intersection B divided by the number of points in omega divided by the number of points in B divided by the number of points in omega.

Therefore, the numerator is probability A intersection B, and the denominator is probability of B therefore, probability of A intersection B is equal to probability of A given B multiplied by the probability of B. So, this is a very important result in probability, we often use it in solving problems, more generally.

Probability of A intersection B intersection C, we can write as probability of A given B intersection C multiplied by probability of B intersection C, by the same law. Now, we can again simplify it as probability of A given B intersection C into probability of B given C multiplied by the probability of C like that.

When there is an intersection n events, we can write it as, probability of A1 intersected with A2 intersected with An is equal to, we can write it in a reverse way. Probability of A1 multiplied by probability of A2 given A1 multiplied by probability of A3 given A2 and A1 up to probability of An given A1, A2 up to A minus 1. So, that is that general formula for computing the simultaneous occurrence of n different events A1, A2, An.

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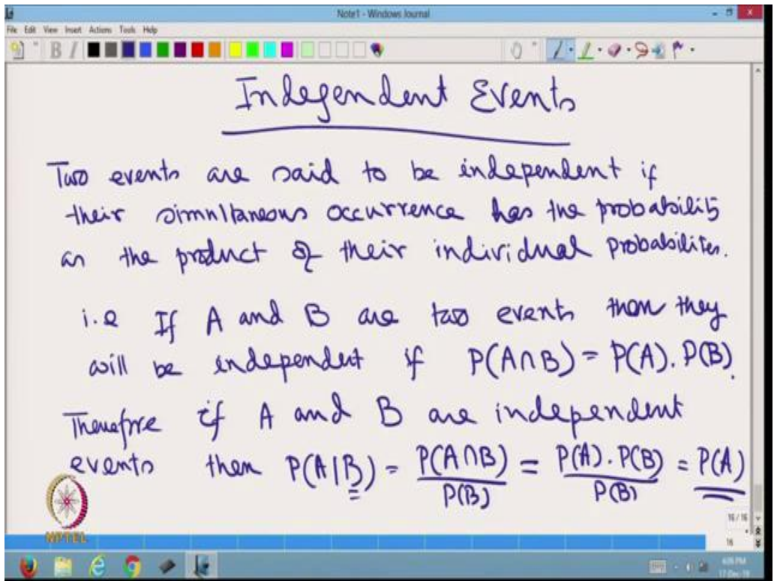

Now, a very interesting concept with respect to probability is the concept of independent events. Two events are said to be independent if their simultaneous occurrence has the probability as the product of their individual probabilities that is, if A and B are two events then they will be independent.

If probability of A intersection B is equal to probability of A multiplied by probability of B. Therefore, if A and B are independent events, then probability of A given B is equal to probability of A intersection B is equal to. Therefore, we can see that the conditional probability of A given B is same as its unconditional probability of A therefore, in other words, we can say that occurrence or non occurrence of B does not have any effect on the probability of the event A.

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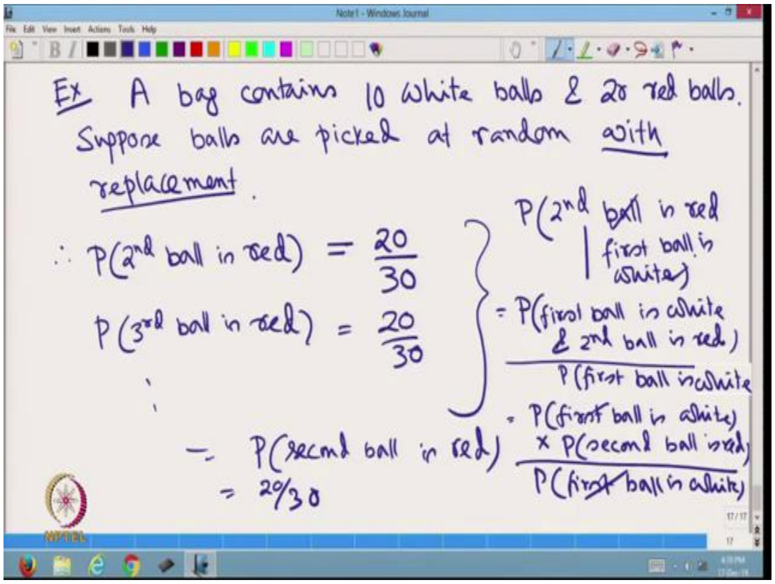

For example, considered the earlier problem a bag contents 10 white balls and 20 red balls suppose, balls are picked at random with replacement, what does it mean? It means that once a ball is picked its color is noted and it is put back into the bag. Therefore, probability second ball is red is equal to when you are picking the second ball, again we have 10 white balls and 20 red balls irrespective of what was picked in the first time.

Therefore, this is going to be again 20 upon 30 again the ball is put back therefore, probability third ball is red is equal to again 20 upon 30, like that we can say that if it is with replacement, then there is no effect of the color of the previous ball on the probability of the next ball to be picked. Therefore, probability second ball is red given, first ball is white is equal to probability first ball is white and second ball is red divided by first ball is white.

Is same as probability first ball is white multiplied by probability second ball is red divided by probability first ball is white. Therefore, it cancels is equal to probability second ball is red is equal to 20 by 30. Therefore, these two events are independent, okay friends I stopped here today in the next class, I shall continue with the conditional probability and also we shall see a very important theorem in this regard which is called Bayes theorem. Thank you.