Advanced Probability Theory
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Lecture 03

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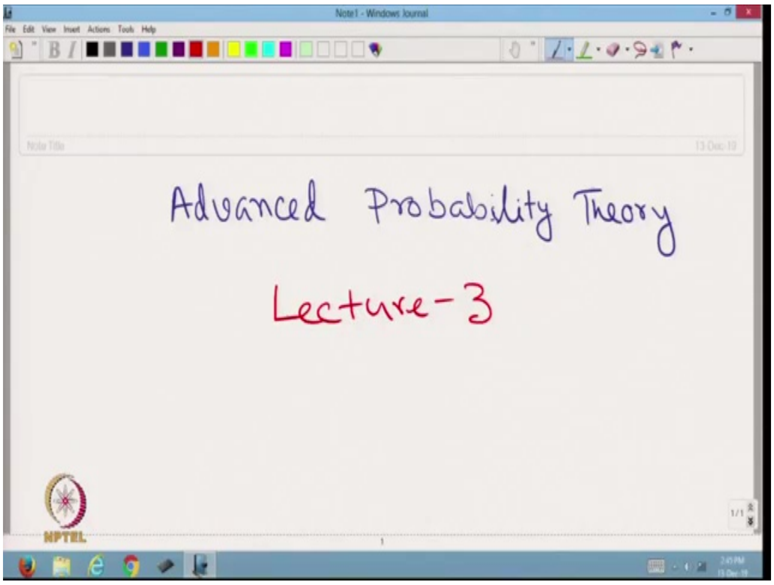

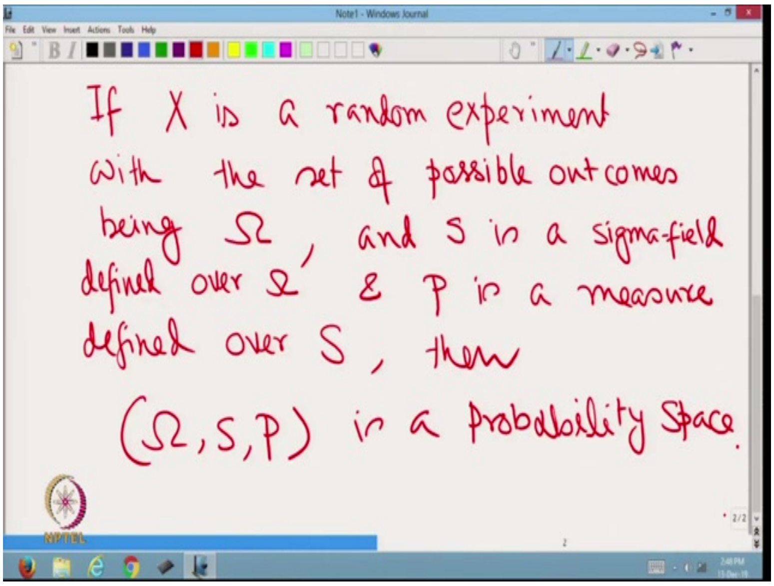

Welcome students to the MOOCs course on advanced probability theory. This is lecture number 3. In the last lecture, we have discussed the basic notion of probability. In particular, if you remember if X is a random experiment with the set of possible outcomes being Ω and S is a Σ field defined over Ω and P is a measure defined over S, then (Ω, S, P) is a probability space.

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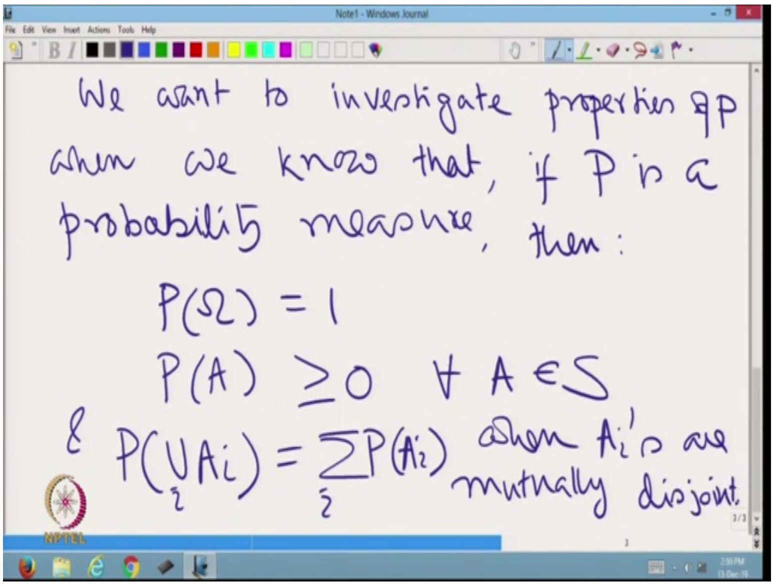

So, this we have discussed in the last class and today we want to investigate properties of P when we know that if P is a probability measure then

$P(\Omega) = 1$

$P(\mathrm{A}) \geq 0 \ \forall \ A \ \in S$

and

$P(\cup_i A_i)\ ; where\ each\ A_i\ is\ a\ member\ of\ S$

$P(\cup_i A_i) \ = \ \sum_i P\ (A_i)\ \ When\ A_i's$ are mutually disjoint.

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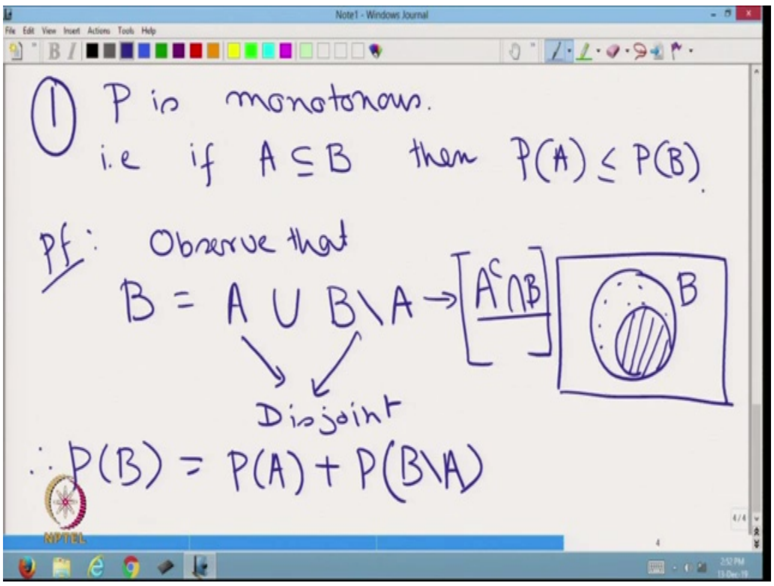

This much we know now let us investigate certain properties of P.

(1) P is monotonous. What does it mean?

It means

$$i.e\ if\ A \subseteq B\ \ then\ \ P(A) \leq P(B)$$

it is very simple. So, let me give you a proof. Suppose, this is my Ω and suppose this is my B and suppose,

$A \subseteq B$. So, suppose this is my A,

so observed that $B = A \cup B\backslash A$ that means $B\backslash A \in B$, but not in A.

Therefore, A, B are disjoint that is very clear from the Venn diagram.

Therefore,

$$P(B) = P(A) + P(B\backslash A)$$

this is that set difference. Now,

$$B\backslash A \rightarrow [A^c \cap B]$$

you may find this notation in some other books, but basically these are same.

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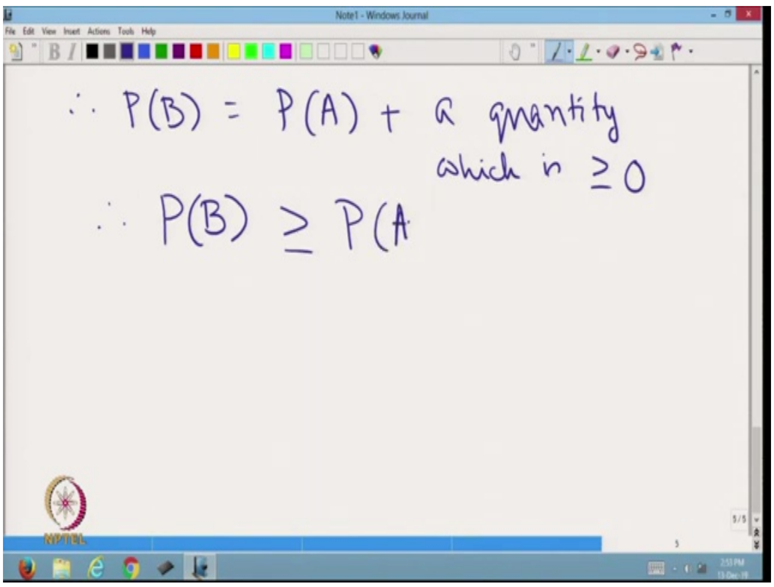

Therefore, what we find that

$P(B) = P(A)$ +a quantity which is ≥ 0

Therefore,

$$P(B) \geq P(A)$$

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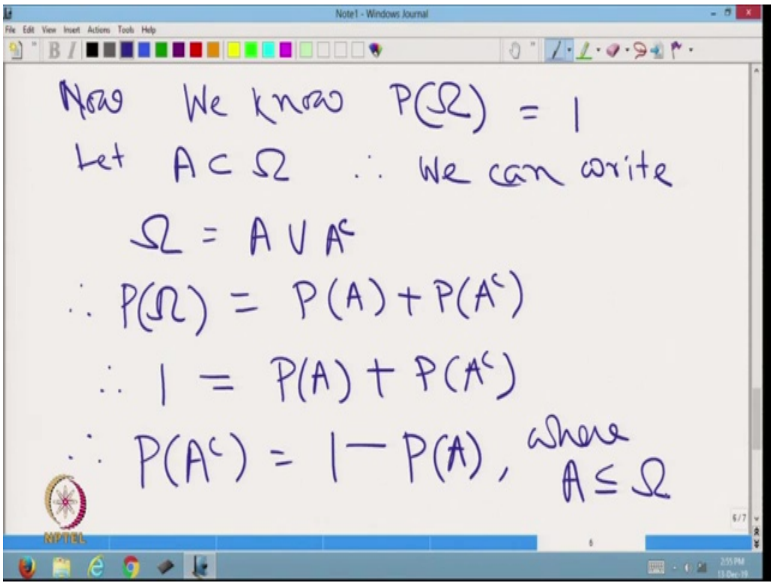

Now, we know $P(\Omega) = 1$,

let $A \subset \Omega$.

Therefore, we can write $\Omega = A \cup A^c$.

Therefore, $P(\Omega) = P(A) + P(A^c)$

Therefore,

$$1 = P(A) + P(A^c)$$

Therefore $P(A^c) = 1 - P(A),\ where\ A \subseteq \Omega$

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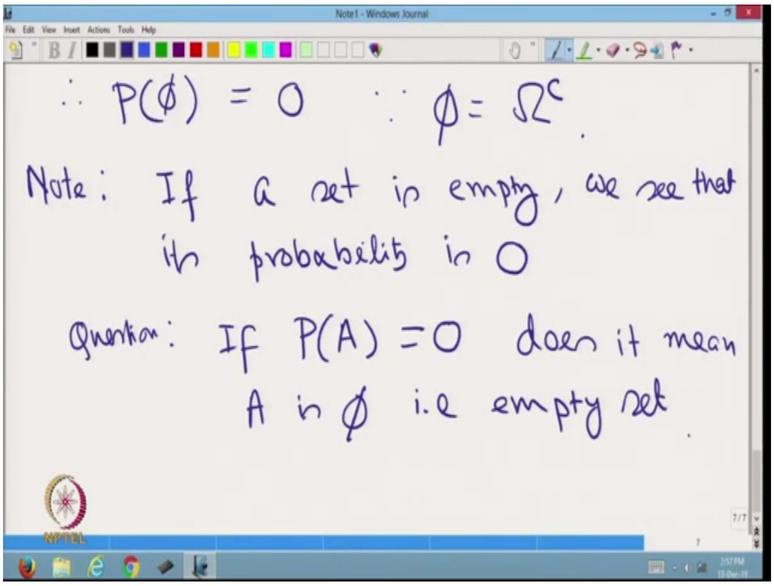

So, this allows us to say

$$P(\phi) = 0 \quad \because \ \phi = \Omega^c$$

Now, note if a set is empty we see that its probability is 0.

Question is, is the converse true? If $P(A) = 0$, does it mean $A \text{ is } \phi$? That is empty set. That is the question.

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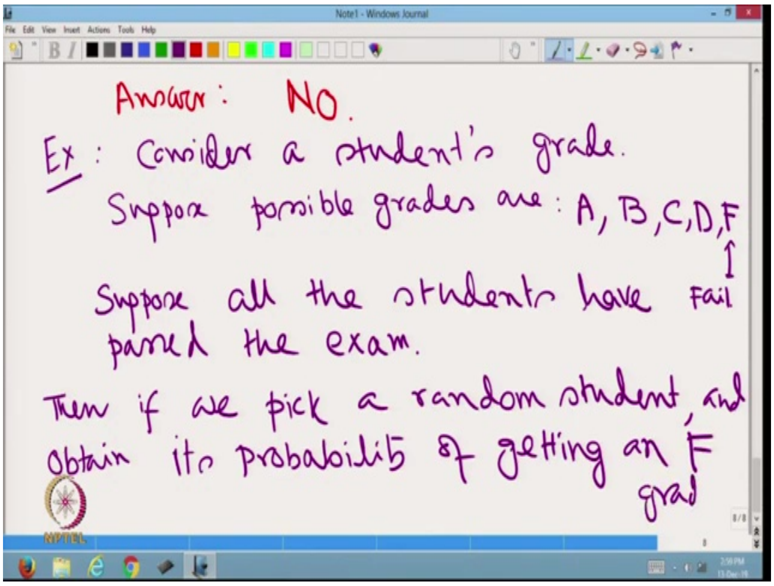

The answer is that, no.

Example, consider a student's grade. Suppose possible grades are A, B, C, D and F suppose this means fail. Now, suppose all the students have passed the exam then if we pick a random student and obtain its probability of getting an F grade.

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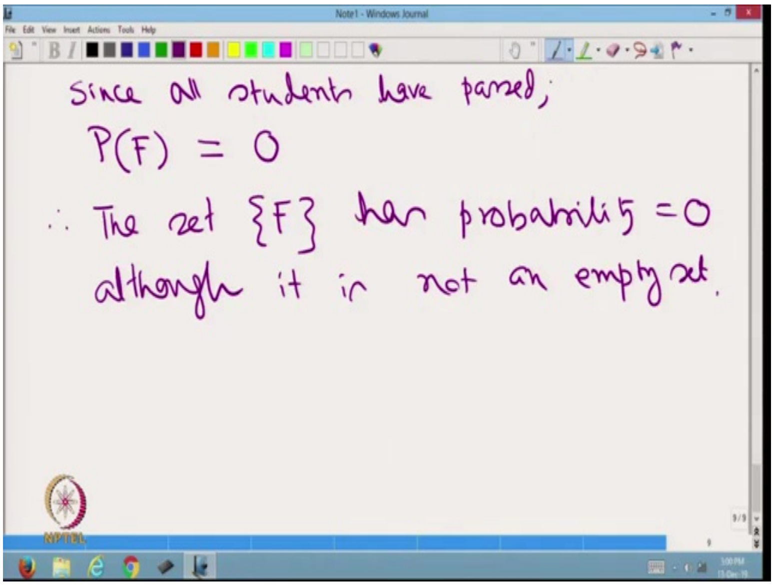

Since all students have passed $P(F) = 0$. Therefore $\{F\}$ has probability 0 although it is not an empty set. Okay students, so I hope the distinction is clear. We are sure that if the set is empty, then its probability is going to be 0. But we cannot conclude that given a set has probability 0 or an event has probability 0, the probability of that event is going to be then the event is going to be null that is not correct.

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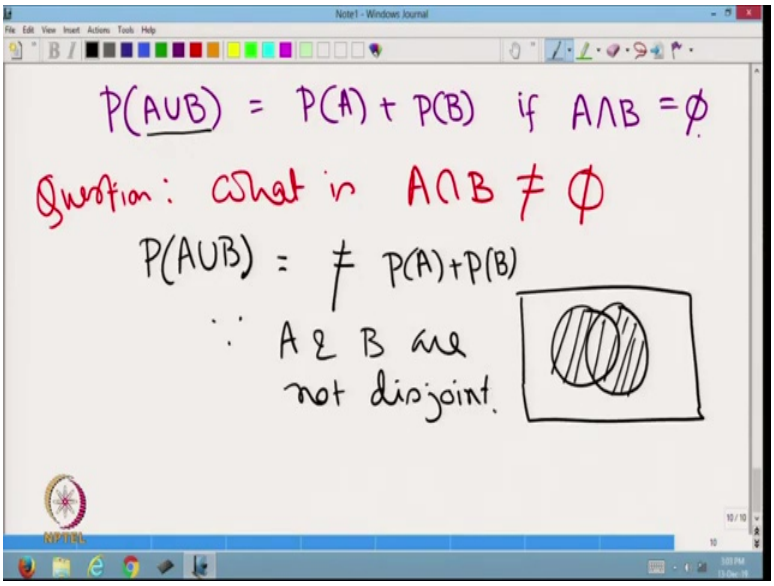

Now, in our last class, we have seen and we have recapitulated today that

$P(A \cup B) = P(A) + P(B)\,, if\ \ A \cap B = \phi$

Question, what is if $A \cap B \neq \emptyset$? Example, suppose this is my Ω this is my A and this is my B and we are looking at $P(A \cup B)$ this is the probability that the entire portion is covered by the event $A \cup B$. What is going to be its probability?

$$P(A \cup B) \neq P(A) + P(B) \quad \because A \,\&\, B \text{ are not disjoint.}$$

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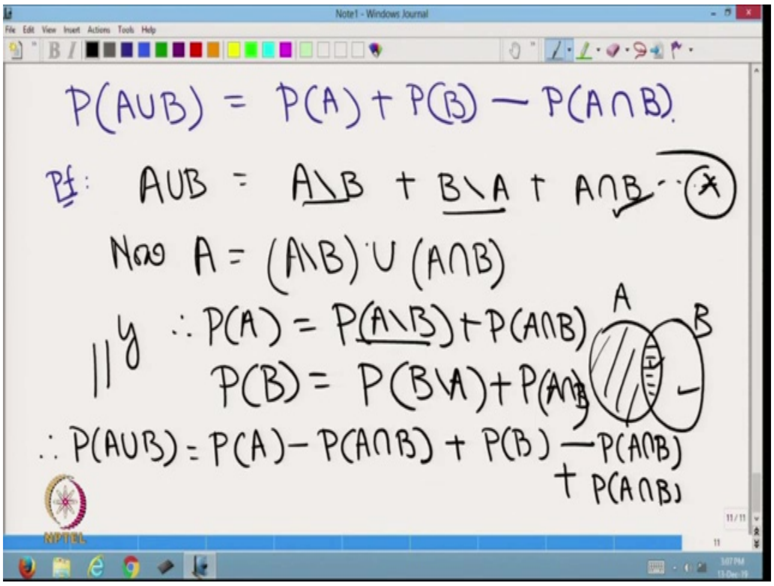

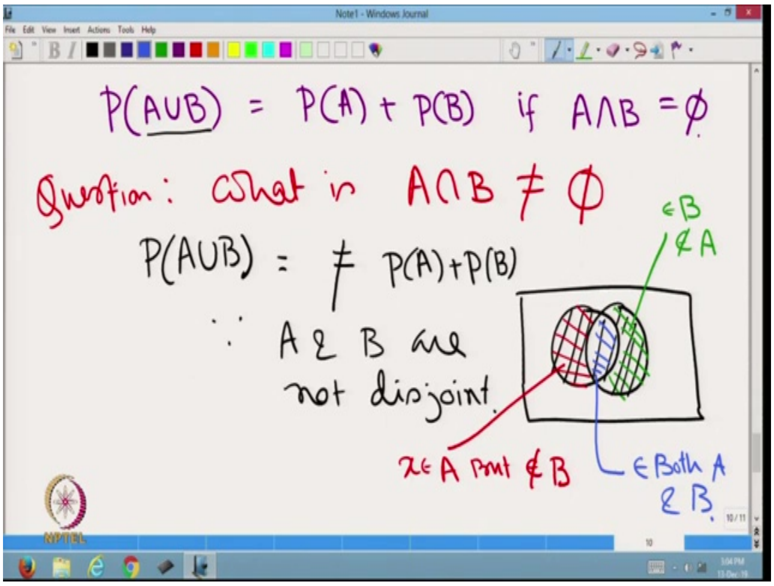

The result is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof : if we go back to that diagram, we can see that

$A \cup B$ can be decomposed into 3 parts. 1 is this, these are the elements which belong to A, but does not belong to B i.e $(A\backslash B)$. These are the elements which belongs to B, but does not belong to A i.e $(B\backslash A)$and these are the elements which belongs to both A and B i.e $(A \cap B)$ thus we can see that we can split the A union B event into union of 3 disjoint events, which we can write as follows,

$$A \cup B = A\backslash B + B\backslash A + A \cap B \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \ *$$

This is very clear, if this is my A and if this is my B, then we can say that

$$A = (A\backslash B) \cup (A \cap B) \quad A \,\&\, B \text{ are disjoint}$$

Therefore,

$$P(A) = P(A\backslash B) + P(A \cap B) \,; \text{since } A \,\&\, B \text{ are disjoint}$$

Similarly,

$$P(B) = P(B\backslash A) + P(A \cap B)$$

this region union this region. Therefore, from star, we can write

$$P(A \cup B) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B)$$

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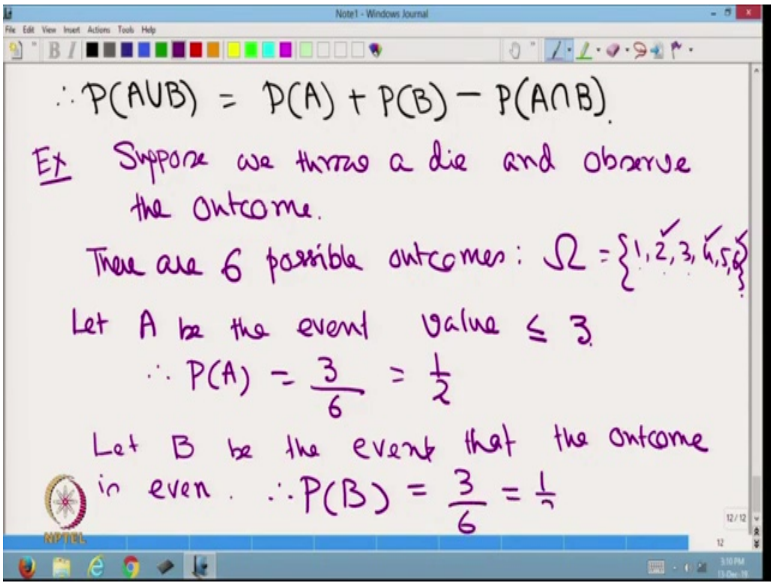

So after cancellations we get

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example suppose, we throw a die and observe the outcome. We know that there are 6 outcomes , $\Omega = \{1,2,3,4,5,6\}$

let A be the event that the $value \leq 3$.

Therefore, the $P(A)$ is equal to this event will be satisfied when the outcome is 1 of 1, 2 or 3 therefore $P(A)$ is going to be 3.

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

Let B be the event that the outcome is even. Therefore, we know the $P(B)$ is the event B occurs when the outputs are 2, 4 and 6. Therefore, $P(B) = \frac{3}{6} = \frac{1}{2}$

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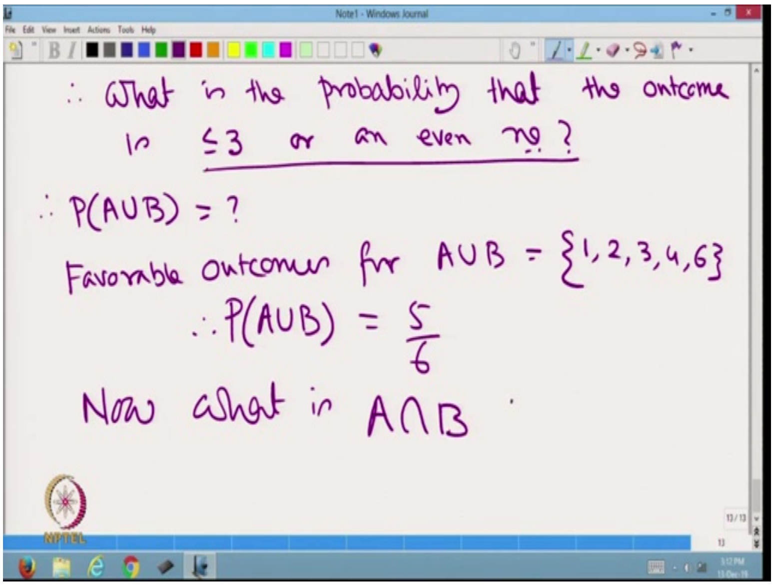

Therefore, what is the probability that the outcome is ≤ 3 or and even number that is $A \cup B$ that is $P(A \cup B)$ is equal to… Now, favorable outcomes for $A \cup B = \{1,2,3,4,6\}$. Therefore, $P(A \cup B) = \frac{5}{6}$. Now, what is $A \cap B$?

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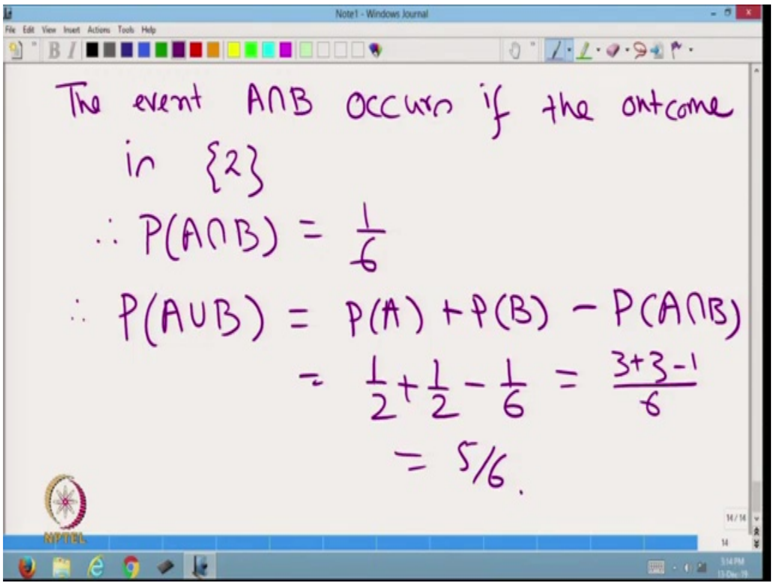

The event $A \cap B$ occurs if the outcome is only 2, because $2 \leq 3$, and 2 is an even number. Therefore, probability of A intersection B is equal to 1 by 6 $P\ (A \cap B) = \frac{1}{6}$.

Therefore, by our formula, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

is equal to $\frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \frac{3+3-1}{6} = \frac{5}{6}$

So, we verify the formula with respect to ONE example, as well.

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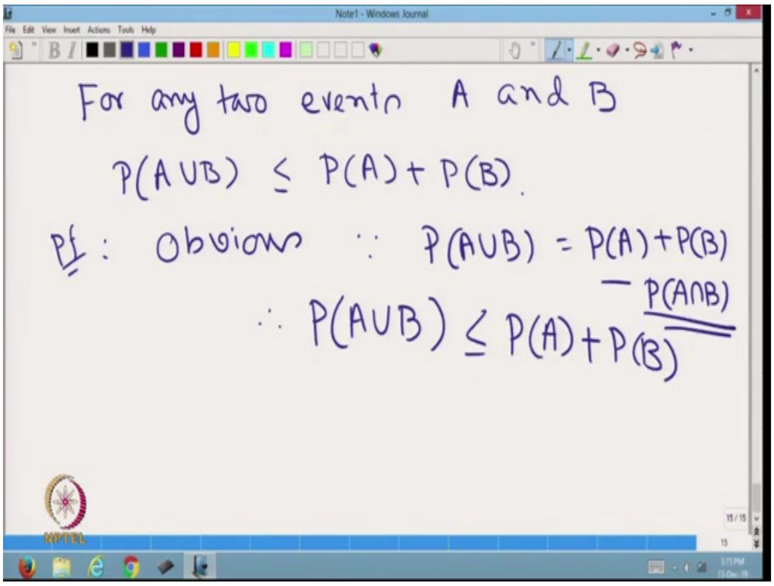

For any two events A and B

$P(A \cup B) \ \leq P(A) + P(B)$

This is obvious, why?

Since

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Therefore, this is a positive quantity, which we are subtracting

therefore,

$$P(A \cup B) \ \leq P(A) + P(B) \, ; if \ and \ only \ if \ P(A \cap B) = 0$$

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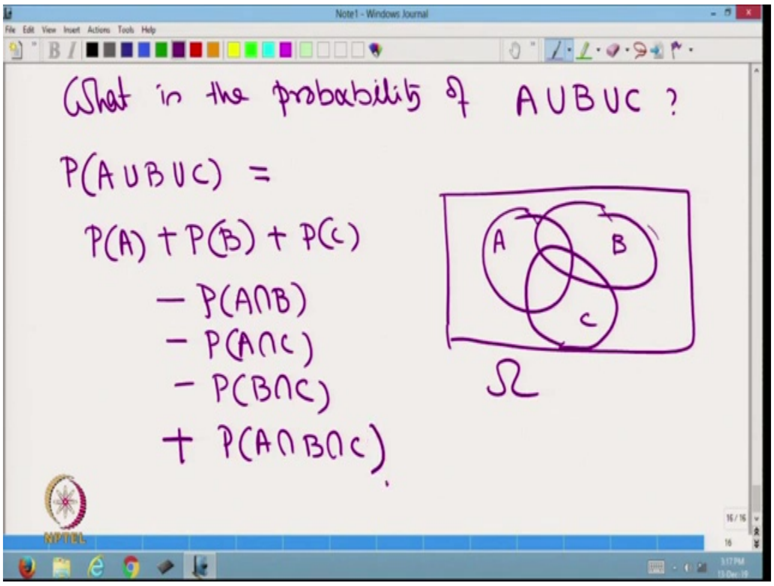

Now, suppose the question is what is the $P\ (A \cup B \cup C)$? So, as before let me explain with a diagram. Suppose this is my Ω , suppose this is one event A this is another event B and this is another event C. So, we are looking at the $P(A \cup B \cup C)$ that means, this entire thing should happen. What is the probability?

Formula is $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

So, that is a formula.

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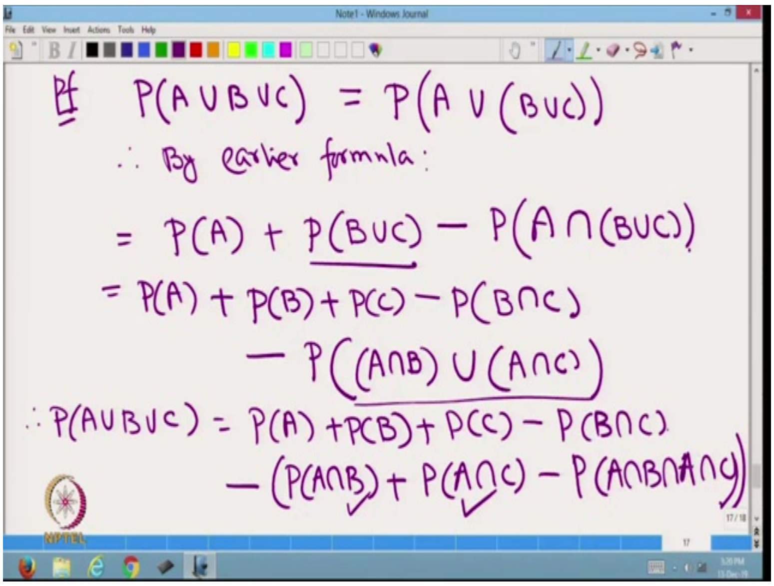

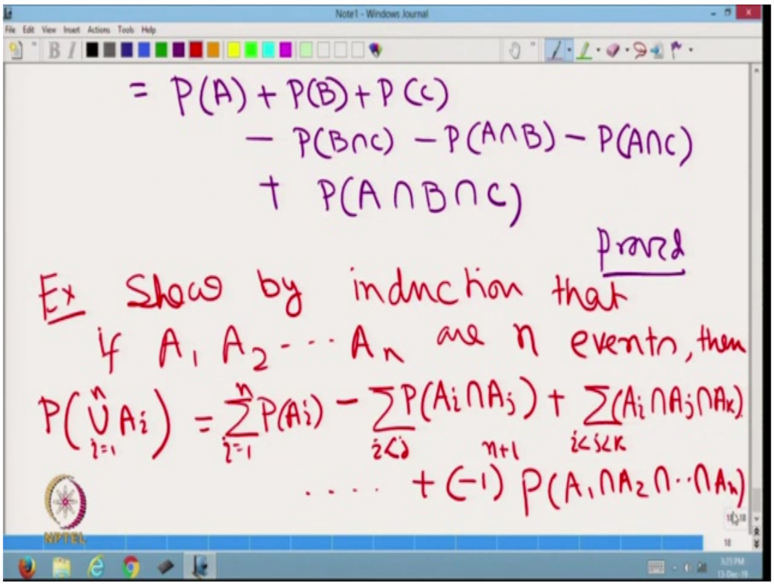

Let me prove it.

Proof,

$P(A \cup B \cup C) = P\big(A \cup \ (B \cup C)\big)$

Now, since it is union of 2 events, therefore by earlier formula this is equal to

$P(A) + P(B \cup C) - P\big(A \cap \ (B \cap C)\big)$

this is equal to

$$P(A) + P(B) + P(C) - P(B \cap C) - P\big((A \cap B) \cup (A \cap C)\big)$$

now note that it is also union of 2 events.

Therefore, we can apply the same formula therefore, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - \big(P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C)\big)$

which is equal to $P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

So, that proves the formula,

I give you a small thing to practice show by induction that if -A_1, A_2, A_n are n events

then $P(\bigcup_{i=1}^{n} A_i)$

That means

$$P(A_1) \cup \ P(A_2) \dots \ P(A_n)$$

is equal to

$$\sum_{i=1}^{n} P(A_i)$$

That means, I am summing up the individual probabilities

$$- \ \textstyle\sum_{i<j} P\left(A_i \cap A_j\right) + \ \textstyle\sum_{i<j<k}\left(A_i \cap A_j \cap A_k\right)$$

So, here you are looking at all possible pairs of sets. Here we are looking at all possible triplets of sets.

And the last term is going to be $(-1)^{n+1} \, P(A_1 \cap A_2 \cap \dots \cap A_n)$

The proof is simple,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P\left(A_i \cap A_j\right) + \sum_{i<j<k} \left(A_i \cap A_j \cap A_k\right) + \cdots$$
$$+ \ (-1)^{n+1} \, P(A_1 \cap A_2 \cap \dots \cap A_n)$$

basically the way we have introduced C and then used the distributivity of intersection of our union in a very similar approach, you will be able to prove this for n or in other words, you assume it for n and you show that it is true for $n + 1$.

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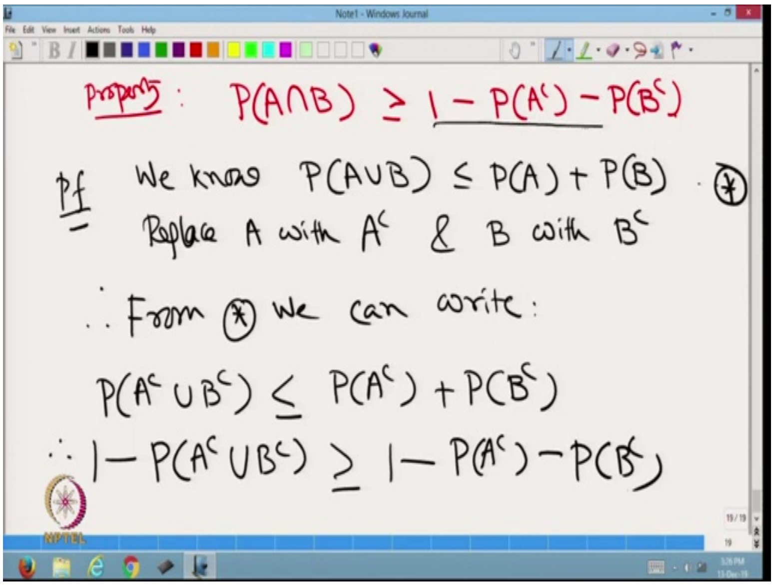

Let me give you another property

$P(A \cap B) \ \geq 1 - P(A^c) - P(B^c)$

this is very simple.

Proof, we know

$P(A \cup B) \ \leq P(A) + P(B) \qquad \ldots \ldots *$

Replace A with (A^c) and B with (B^c).

Therefore, from star we can write

$P(A^c \ \cup \ B^c) \ \leq \ P(A^c) + P(B^c)$

Therefore,

$$1 - \ P(A^c \ \cup \ B^c) \ \geq \ 1 - P(A^c) - P(B^c)$$

So, you can see that we have already got the right hand side, we have got the inequality.

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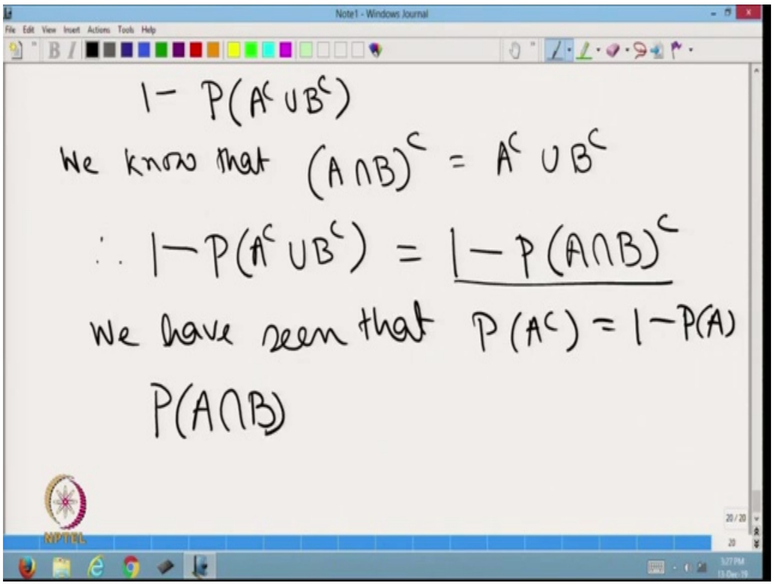

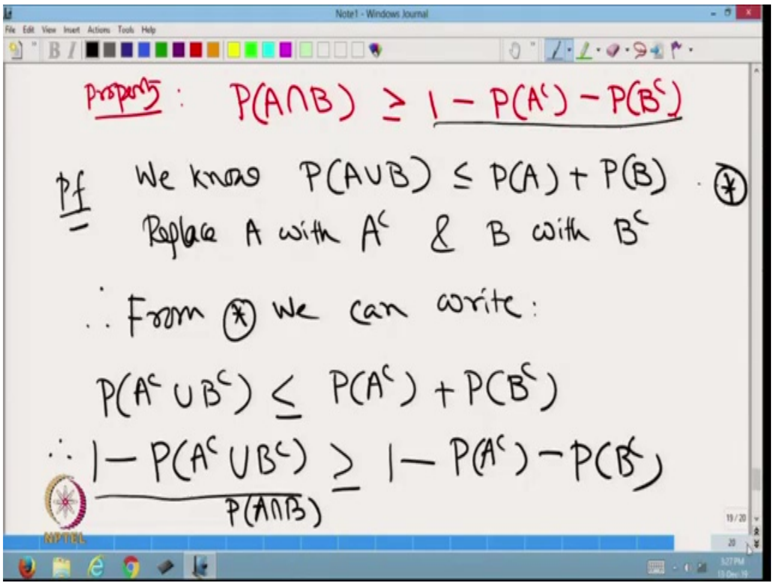

Now, what is

$1 - P(A^c \cup B^c)$

we know that

$$(A \cap B)^c = A^c \cup B^c$$

This is from our elementary knowledge of set theory.

Therefore,

$$1 - P(A^c \cup B^c) = 1 - P(A \cap B)^c$$

And we have seen that

$P(A^c) = 1 - P(A)$

Therefore, this whole thing is going to be

$P(A \cap B)$.

Hence, this is

$$P(A \cap B) \geq 1 - P(A^c) - P(B^c)$$

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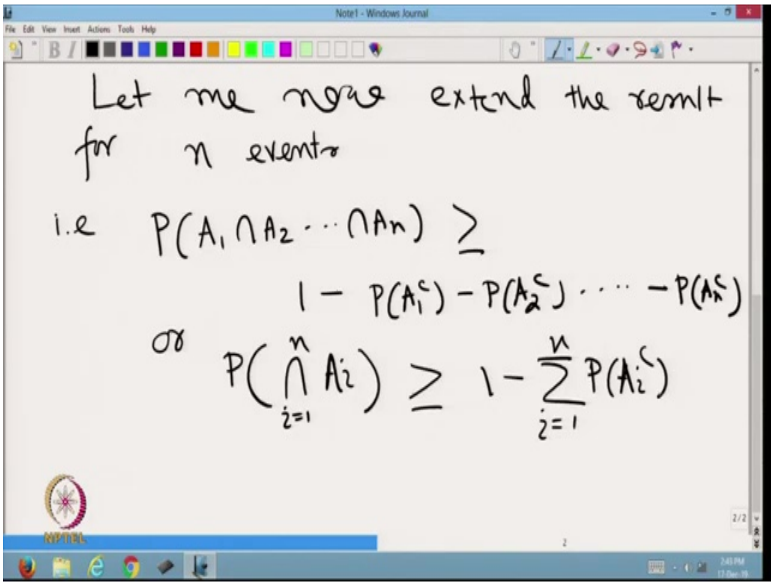

Okay, let me know extend the result for n events that is

$i.e\ P(A_1 \cap A_2 \ldots \cap A_n) \geq 1 - P(A_1^c) - P(A_2^c) \ldots - P(A_n^c)$

or we can write it as

$P(\bigcap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$

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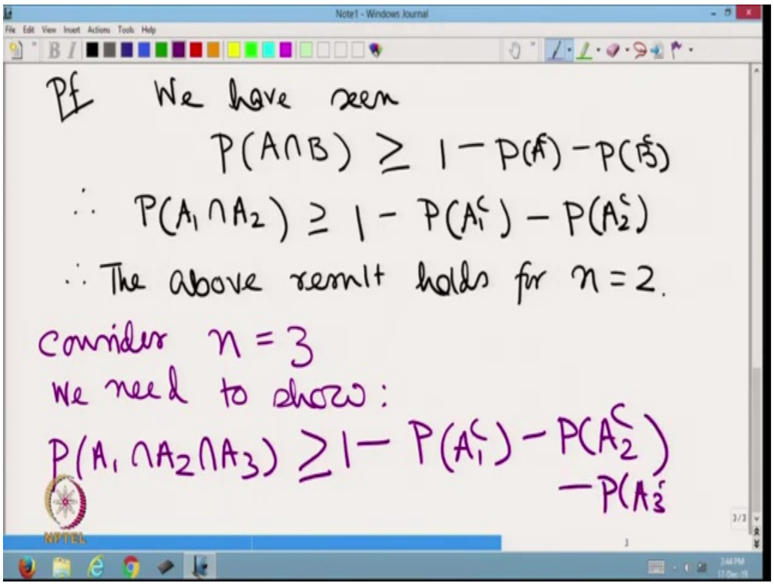

How to prove it? We have seen

$P(A \cap B) \geq 1 - P(A) - P(B)$

Therefore, $P(A_1 \cap A_2) \geq 1 - P(A_1^c) - P(A_2^c)$

Therefore, the above result holds for $n = 2$. Consider 3 $n = 3$ therefore we need to show

$P(A_1 \cap A_2 \cap A_3) \geq 1 - P(A_1^c) - P(A_2^c)$

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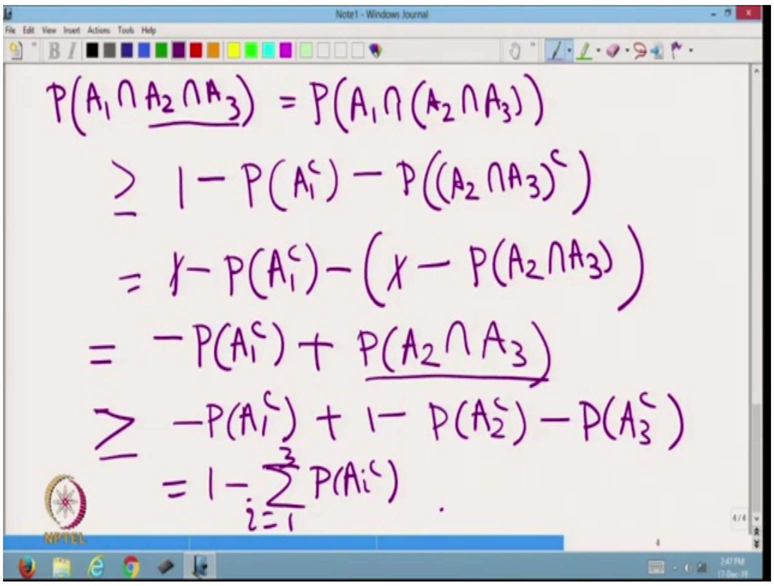

So,

$P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap (A_2 \cap A_3))$

Since now there are 2 events, we can write for $n = 2$, from there we can write, this is $\geq 1 - P(A_1^c) - P((A_2 \cap A_3)^c)$

Which is equal to $= 1 - P(A_1^c) - (1 - P(A_2 \cap A_3))$

because some of an event

$$P(A) + P(A^c) = 1$$

that we have already seen is equal to, now this 1 cancels with this,

we are left with $-P(A_1^c) + P(A_2 \cap A_3)$

Now, because of the hypothesis that the inequality is true for 2 events, therefore, we can write it as

$$\geq -P(A_1^c) + 1 - P(A_2^c) - P(A_3^c)$$

which is nothing but

$$= 1 - \sum_{i=1}^{3} P(A_i^c)$$

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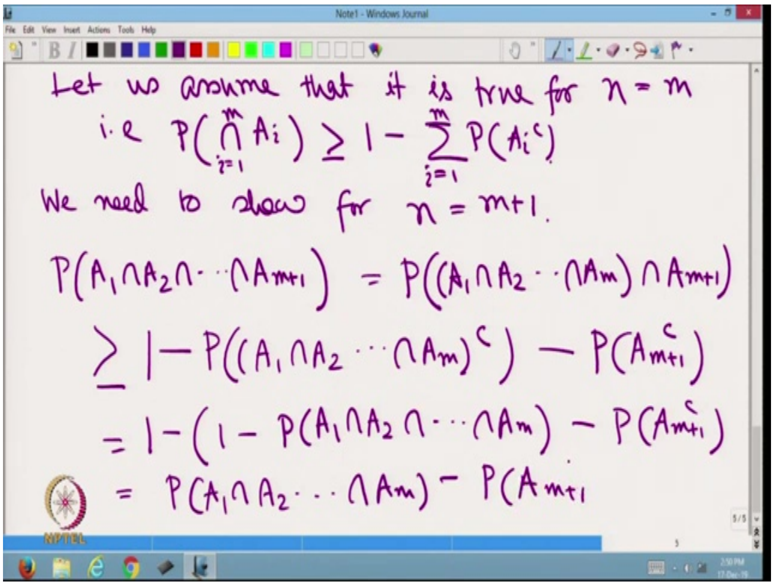

Let us assume that it is true for $n = m$ that is

$P(\cap_{i=1}^{m} A_i) \geq 1 - \sum_{i=1}^{m} P(A_i^c)$

we need to show for $n = m + 1$.

So, $P(A_1 \cap A_2 \cap \dots A_{m+1}) = P\big((A_1 \cap A_2 \dots \cap A_m) \cap \ A_{m+1}\big)$

again we have split it into intersection of 2 events.

Therefore, this is

$\geq 1 - P((A_1 \cap A_2 \dots \cap A_m)^c) \ - P(A_{m+1}^c)$

which is equal to

$= 1 - \ \big(1 - P(A_1 \cap A_2 \cap \dots \cap A_m)\big) - P(A_{m+!}^c)$

which is equal to $P(A_1 \cap A_2 \cap \dots \cap A_m) - P(A_{m+1})$

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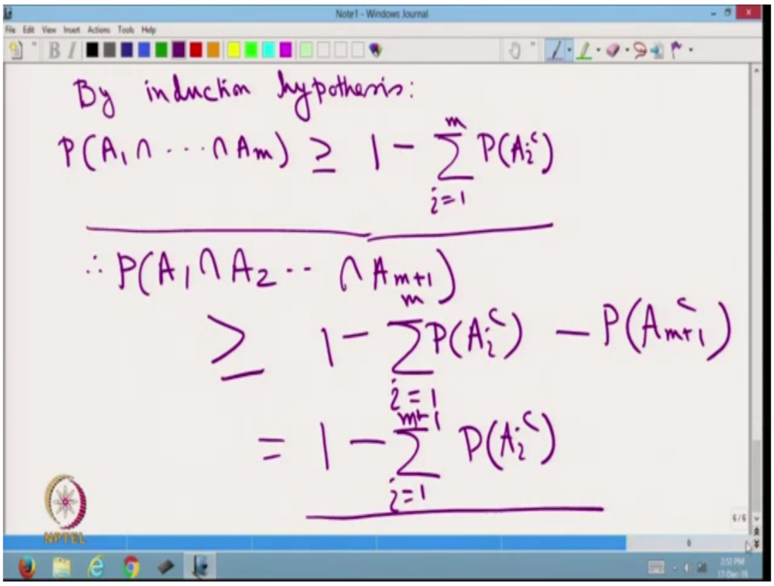

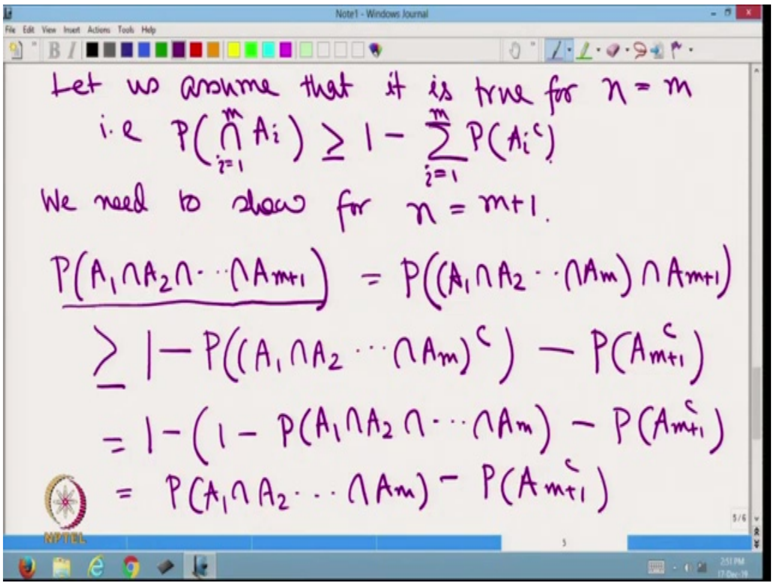

By induction hypothesis.

$$P(A_1 \cap \ldots \cap A_m) \ \geq 1 - \sum_{i=1}^{m} P(A_i^c)$$

Therefore, this result, when you plug in here, we get that

$$P(A_1 \cap A_2 \ldots \cap A_{m+1}) \ \geq \ 1 - \sum_{i=1}^{m} P(A_i^c) - P(A_{m+1}^c)$$

which is nothing but $= 1 - \sum_{i=1}^{m+1} P(A_i^c)$

So, that proves the result. Therefore, by induction hypothesis, this is true for all n for any positive integer.

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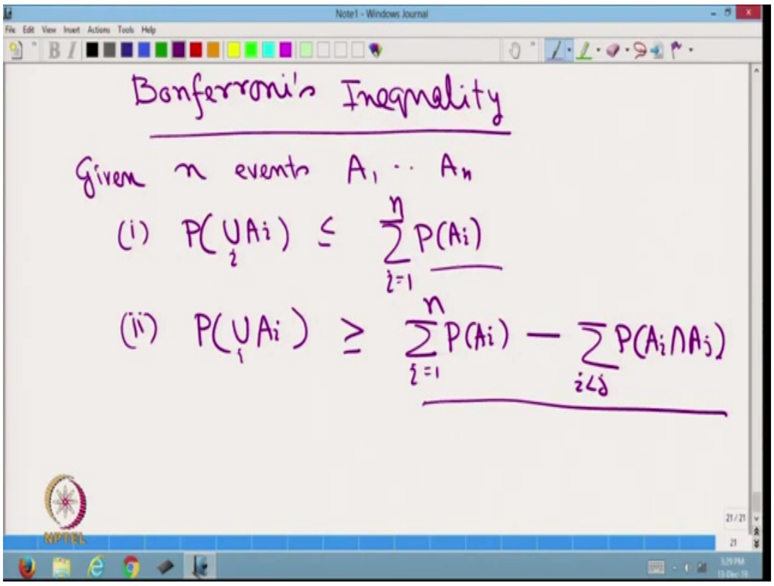

Let me now, state another property, which is called Bonferroni's property or Bonferroni's inequality, it states that given n events - A_1, A_2, A_n

$P(\bigcup_i A_i) \ \leq \ \sum_{i=1}^{n} P(A_i)$

And secondly,

$$P\left(\bigcup_i A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P\left(A_i \cap A_j\right)$$

or in other words. We have already seen the formula for $P(\bigcup_i A_i)$ That is a pretty long formula. If the value of n is pretty large. But these 2 inequality suggests that the value of this probability will lie between these 2 quantities from the lower side, this is the bound and from the upper side that is the bound.

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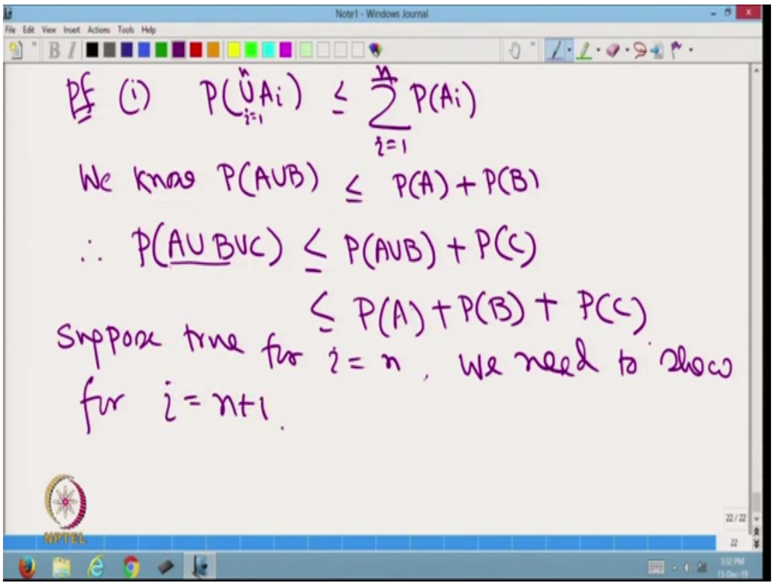

So, proof 1

$P(\bigcup_{i=1}^{n} A_i) \ \leq \ \sum_{i=1}^{n} P(A_i)$

We know

$$P(A \cup B) \ \leq P(A) + P(B)$$

Therefore,

$$P(A \cup B \cup C) \ \leq P(A) + P(B) + P(C)$$

That is, we are writing this event as

$$P(A \cup B \cup C) \le P(A \cup B) + P(C)$$

Since there are 2 events we can apply this formula and thereby, we get this and then we extend it 1 step further, because $P(A \cup B) \le P(A) + P(B)$. Therefore, we get this inequality. Suppose, true for $i = n$ we need to show for $i = n + 1$.

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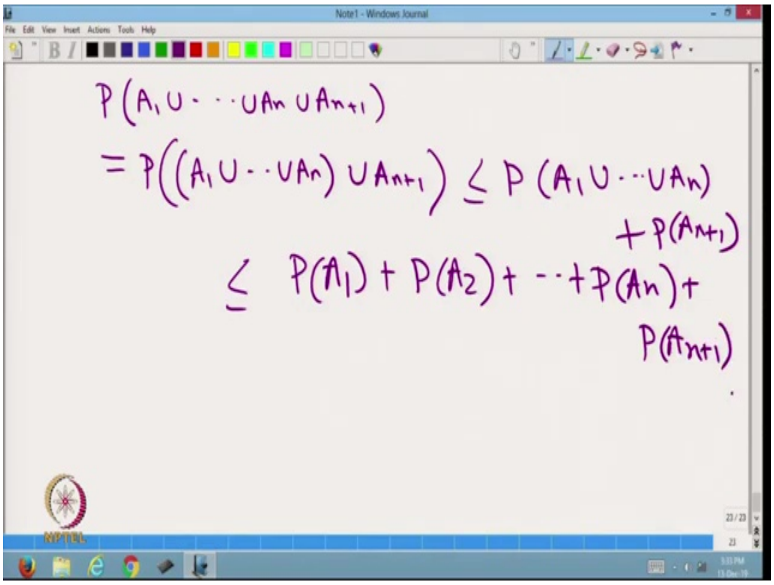

It is very simple

$$P(A_1 \cup \ldots \cup A_n \cup A_{n+1}) = P\big((A_1 \cup \ldots \cup A_n) \cup A_{n+1}\big)$$

Thus we have divided it into union of 2 events.

Therefore, this is $\le P(A_1 \cup \ldots \cup A_n) + P(A_{n+1})$

And by induction hypothesis, this is $\le P(A_1) + P(A_2) + \cdots + P(A_n)$

therefore whole thing is

$$\le P(A_1) + P(A_2) + \cdots + P(A_n) + P(A_{n+1})$$

So, this is pretty straight forward.

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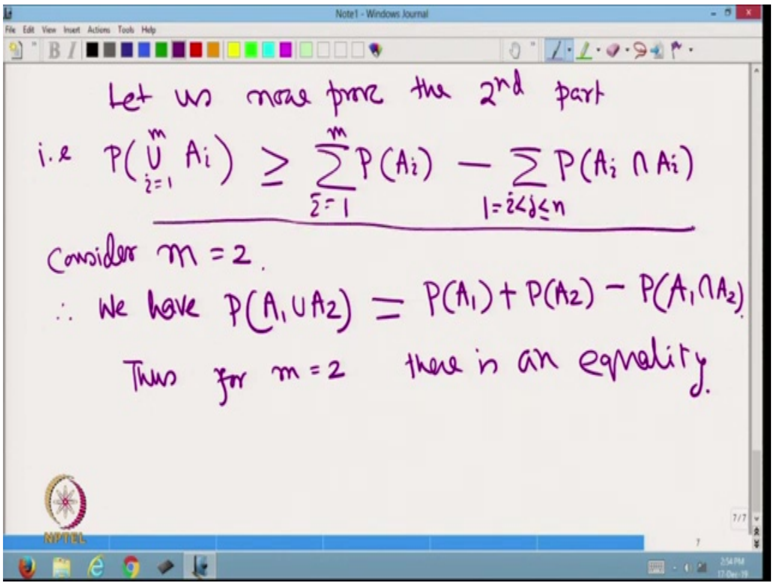

So, let me now second part

that is $i.e\ P(\bigcup_{i=1}^{m} A_i) \geq \sum_{i=1}^{m} P(A_i) - \sum_{1=i<j\leq n} P(A_i \cap A_j)$

let us write it like this considered $n = 2$

therefore, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Thus for $m = 2$, there is the equality this we have already seen, but it satisfies this equation, or this in equation.

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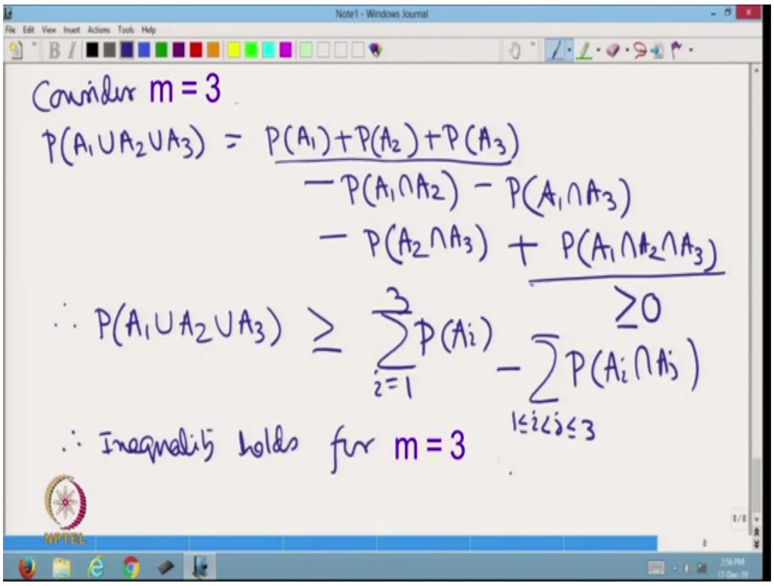

Considers m is equal to 3 $m = 3$, now

$$P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

Now, . $P(A_1 \cap A_2 \cap A_3) \geq 0$

Therefore,

$$P(A_1 \cup A_2 \cup A_3) \geq \sum_{i=1}^{3} P(A_i) - \sum_{1 \leq i < j \leq 3} P(A_i \cap A_j)$$

because $P(A_1 \cap A_2 \cap A_3) \geq 0$. Therefore, inequality holds for $m = 3$.

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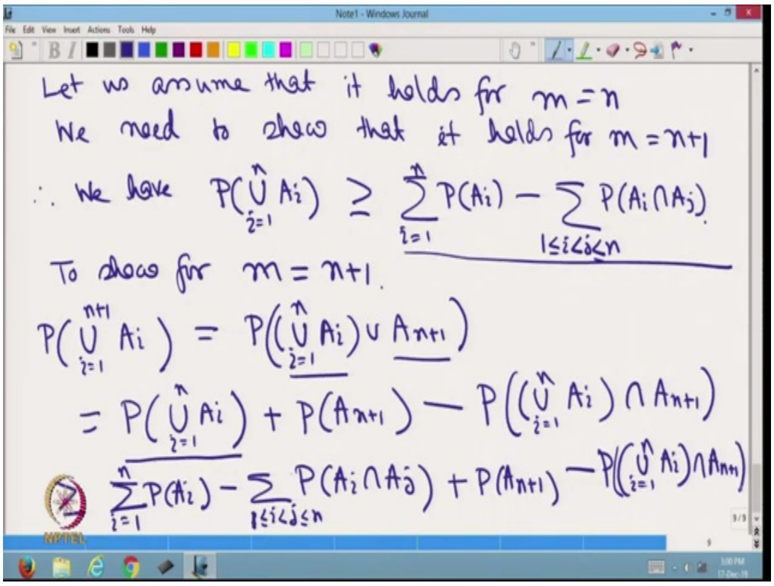

Let us assume that it holds for, $m = n$, we need to show that it holds for $m = n + 1$. Therefore, by induction hypothesis we have

$P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1=i<j\leq n} P(A_i \cap A_j)$

To show for $m = n + 1$. Now, $P(\bigcup_{i=1}^{n+1} A_i) = P\left((\bigcup_{i=1}^{n} A_i) \cup A_{n+1}\right)$

Therefore, this is now we have written as union of 2 events therefore, we can use our earlier result. Therefore, this is equal to

$P(\bigcup_{i=1}^{n+1} A_i) + P(A_{n+1}) - P\left((\bigcup_{i+1}^{n} A_i) \cap A_{n+1}\right)$

Because this is one event this is another event therefore, this equality comes. Now, let us considered $\geq \sum_{i=1}^{n} P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \cap A_j) + P(A_{n+1}) - P(\bigcup_{i=1}^{n} A_i) \cap A_{n+1}$

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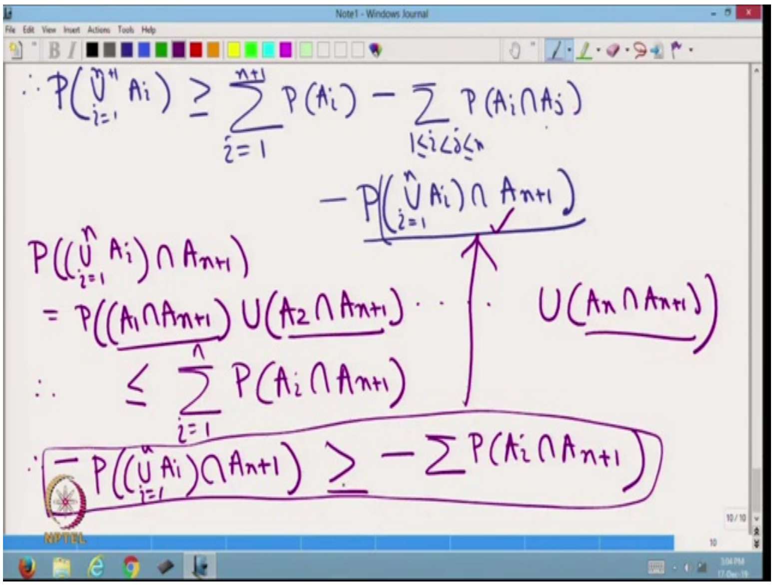

Therefore, this we can combined with this and we can write

$$P(\cup_{i=1}^{n+1} A_i) \geq \sum_{i=1}^{n+1} P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \cap A_j) - P\left((\cup_{i=1}^{n} A_i) \cap A_{n+1}\right)$$

So, let us focus on this part $P\left((\cup_{i=1}^{n} A_i) \cap A_{n+1}\right)$ is equal to probability of because of the distributivity of intersection over union.

We can write it as let me expand it for your understanding.

$$P\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right) = P\big((A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \ldots\ldots\ldots \cup (A_n \cap A_{n+1})\big)$$

Now, this is now union over n events and therefore, by the first result, we can write that this is

$$\leq \sum_{i=1}^{n} P(A_i \cap A_{n+1})$$

because these are union of n events we have seen in the first result that this union has to be less than equal to the summation of probabilities of the individual event.

Therefore, $-P\left((\cup_{i=1}^{n} A_i) \cap A_{n+1}\right)$ that is this quantity is because this is less than equal to this, when we are taking the negative sign the inequality changes. This is $-\sum_{i=1}^{n} P(A_i \cap A_{n+1})$. Now, therefore, this whole thing if I now consider this and I plug it in here we get that inequality is still satisfied because this part is greater than equal to this.

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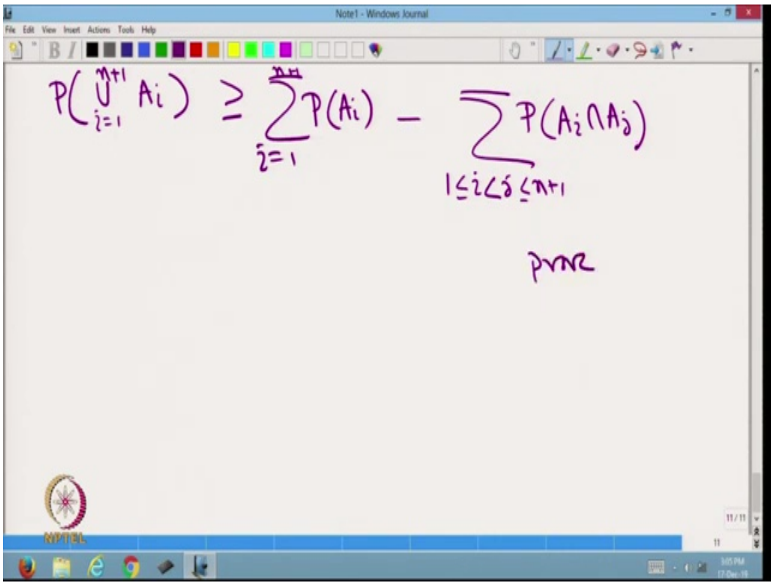

Therefore, finally we get the result that

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) \geq \sum_{i=1}^{n+1} P(A_i) - \sum_{1 \leq i < j \leq n+1} P\left(A_i \cap A_j\right)$$

proved.

Okay friends, I stop here today in the next class I shall solve 1 or 2 interesting problems with probability and then I shall go to conditional probability, which is a very practical, an interesting aspect of the theory of probability. Okay friends. Thank you so much.