Pelle's Equation: $x^2 - dy^2 = 1$, where d is a positive integer and it is square-free.

→ If one of the positive solution is known all solutions can be found.

→ We are interested in non-trivial solution.

Result: Let x be an arbitrary irrational number and $\frac{a}{b}$ a rational number, $b \geq 1$, $(a, b) = 1$.

If $\left| x - \frac{a}{b} \right| < \frac{1}{2b^2}$

then $\frac{a}{b}$ is one of the convergent $\frac{p_n}{q_n}$ in the continued fraction representation of x.

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<u>Theorem:</u> If p, q is a positive solution of $x^2 - dy^2 = 1$, then p/q is a convergent of the continued fraction representation of $\sqrt{d}$.

<u>Proof:</u> As p, q is a solution of $x^2 - dy^2 = 1$ then $p^2 - dq^2 = 1$

$$\therefore \left(p - q\sqrt{d}\right)\left(p + q\sqrt{d}\right) = 1$$

$\Rightarrow$ $p > q$ and

$$\frac{p}{q} - \sqrt{d} = \frac{1}{q\left(p + q\sqrt{d}\right)}$$

~~As a~~

$$0 < \frac{p}{q} - \sqrt{d} < \frac{\sqrt{d}}{q\left(q\sqrt{d} + q\sqrt{d}\right)}$$

$$= \frac{\sqrt{d}}{2q^2\sqrt{d}} = \frac{1}{2q^2}$$

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→ Converse of the above theorem is false. Note all the convergents $\frac{p_n}{q_n}$ of $\sqrt{d}$ supply solutions to $x^2 - dy^2 = 1$.

<u>Theorem:</u> Let p_k and q_k be the convergents of the continued fraction expansion of $\sqrt{d}$ and let n be the length of the expansion.

(a) If n is even, then all positive solutions of $x^2 - dy^2 = 1$ are given by $x = p_{kn-1}$, $y = q_{kn-1}$, $k = 1, 2, 3 \ldots$

(b) If n is odd, then $x = p_{2kn-1}$, $y = q_{2kn-1}$ are all positive solutions of $x^2 - dy^2 = 1$.

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Exc: Solve $x^2 - 7y^2 = 1$

$\sqrt{7} = [2, \overline{1, 1, 1, 4}]$, the

initial convergents are

$\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48},$

$\frac{590}{223}, \frac{717}{271}, \frac{1307}{494}, \frac{2024}{765}$

$n = 4$ (length of the period)

The convergents $\frac{p_{4k-1}}{q_{4k-1}}$ forms a

solution, $k = 1, 2, 3, \ldots$

$x^2 - 7y^2 = 1$

$k = 1, \quad \frac{p_3}{q_3} = \frac{8}{3}$

$k = 2, \quad \frac{p_7}{q_7} = \frac{127}{48}$

$k = 3, \quad \frac{p_{11}}{q_{11}} = \frac{2024}{765}$

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$x_1 = 8,\ y_1 = 3$

$x_2 = 127,\ y_2 = 48$

$x_3 = 2024,\ y_3 = 765$

are the first three solutions of $x^2 - 7y^2 = 1$.

Proposition: The units in $\mathbb{Z}[\sqrt{d}]$, $d > 0$ are the elements $\pm \alpha^n$ for $n \in \mathbb{Z}$, where $\alpha = p + q\sqrt{d}$ and (p, q) is the smallest positive solution of $x^2 - dy^2 = \pm 1$

Theorem: Let x_1, y_1 be the fundamental solution of $x^2 - dy^2 = 1$. Then every pair of integers x_n, y_n defined by the condition

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \ ;\ n = 1, 2, 3, \ldots$$

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Remark: $\mathbb{Z}[\sqrt{7})$

$\alpha = 8 + 3\sqrt{7}$

Units $\pm \alpha^n$

$N(\alpha) = 64 - 63 = 1$

$\alpha^2 = (8 + 3\sqrt{7})^2 = 127 + 48\sqrt{7}$

$\vdots$