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Tabular Form

$$\frac{95}{42} = [2, 3, 1, 4, 2]$$

$$= 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

k	0	1	2	3	4
a_k	2	3	1	4	2
p_k	$a_0 = 2$	$\overset{a_1 a_0 + 1}{7}$	9	43	95
q_k	$q_0 = 1$	$\overset{a_1}{3}$	4	19	42
c_k	$\frac{2}{1}$	$\frac{7}{3}$	$\frac{9}{4}$	$\frac{43}{19}$	$\frac{95}{42}$

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$$p_2 = a_2 a_1 + p_0 = 1.7 + 2 = 9$$

$$q_2 = a_2 a_1 + q_0 = 1.3 + 1 = 4$$

$$p_3 = a_3 p_2 + p_1 = 4.9 + 7 = 43$$

$$q_3 = a_3 q_2 + q_1 = 4.4 + 3 = 19$$

$$p_4 = a_4 p_3 + p_2 = 2.43 + 9 = 95$$

$$q_4 = a_4 q_3 + q_2 = 2.19 + 4 = 42$$

Theorem: If $C_k = \dfrac{p_k}{q_k}$ is the qth convergent of the finite simple continued fraction $[a_0, a_1, \ldots a_n]$, then

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, \quad 1 \le k \le n.$$

(Hint: Use Induction)

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Corollary: For $1 \le k \le n$, p_k and q_k are relatively prime.

Proof: Let $d = \gcd(p_k, q_k)$

From the result

$$p_k(q_{k-1}) + q_k(-p_{k-1}) = (-1)^{k-1}$$

$$\Rightarrow \quad d \mid (-1)^{k-1}$$

$$\Rightarrow \quad d = 1 \quad \text{as} \quad d > 0$$

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How to Solve Linear Diophantine Equation Using Continued Fraction:

Linear Diophantine Equation

$$ax + by = c \quad (*)$$

is Solvable iff

$$\gcd(a, b) = d \mid c.$$

Equation $(*)$ becomes

$$\frac{a}{d}x + \frac{b}{d}x = \frac{c}{d}$$

$$\left(\frac{a}{d}, \frac{b}{d}\right) = 1$$

Expand $\frac{a}{b}$ as a simple Continued fraction.

$$\frac{a}{b} = [a_0, a_1, \ldots, a_n]$$

$$C_{n-1} = \frac{p_{n-1}}{q_{n-1}}, \quad C_n = \frac{p_n}{q_n} = \frac{a}{b}$$

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$$\gcd(p_n, q_n) = 1 = \gcd(a, b)$$

$$\Rightarrow p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$$

$$\Rightarrow a q_{n-1} - b p_{n-1} = (-1)^{n-1}$$

$\therefore$ for $x = q_{n-1}$ & $y = -p_{n-1}$,

We have

$$ax + by = (-1)^{n-1}$$

If n is odd, $ax + by = 1$

has solution $x_0 = q_{n-1}$, $y_0 = -p_{n-1}$

If n is even, $x_0 = -q_{n-1}$, $y_0 = p_{n-1}$

is the solution.

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Solve

$$172x + 20y = 1000 \quad - \circledast$$

$$\gcd(172, 20) = 4 \text{ \& } 4 \mid 1000$$

$\Rightarrow$ Equation is solvable.

Equation $\circledast$ becomes

$$43x + 5y = 250$$

$$\frac{43}{5} = [8, 1, 1, 2]$$

n	0	1	2	3
a_n	8	1	1	2
p_n	8	9	17	43
q_n	1	1	2	5
C_n	8	9	17/2	43/5

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$$p_3 q_2 - q_3 p_2 = (-1)^{3-1}$$

$$43.2 - 5.17 = 1$$

$$\Rightarrow \quad 43.500 + 5(-4250) = 250$$

$$\Rightarrow \quad x_0 = 500$$

$$y_0 = -4250$$

General solution is

$$x = 500 + 5t$$

$$y = -4250 - 43t, \quad t = 0, 1, 2, 3, \ldots$$

(99)

Infinite Continued Fraction:

Infinite continued fraction is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}$$

$a_0, a_1, a_2, \ldots, b_1, b_2, b_3, \ldots$ are all real numbers.

Infinite Simple Continued Fraction:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots}}}} = [a_0; a_1, a_2, \ldots]$$

$a_1, a_2, a_3, a_4, \ldots$ are all positive except a_0.

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Def: If $a_0, a_1, a_2 \ldots$ is an infinite sequence of integers all positive except possibly a_0, then the infinite simple continued fraction $[a_0; a_1, a_2, \ldots]$ has the value $\lim_{n\to\infty} [a_0, a_1, \ldots a_n] = \lim_{n\to\infty} C_n$

Periodic Continued Fraction: If an infinite continued fraction $[a_0, a_1, a_2, \ldots]$ contains a block of partial denominators $b_1, b_2, \ldots b_n$ that repeats indefinately, the fraction is called periodic.

e.g $[a_0, a_1, \ldots a_n, \overline{b_1, b_2, \ldots b_n}]$

(101)

Determine the Unique irrational number represented by the infinite continued fraction

$$x = [3, 6, \overline{1, 4}]$$

$$\Rightarrow \quad x = [3, 6, y]$$

$$y = [\overline{1, 4}] = [1, 4, y]$$

$$= 1 + \cfrac{1}{4 + \cfrac{1}{y}}$$

$$= \frac{5y+1}{4y+1}$$

$$\Rightarrow \quad y = \frac{1+\sqrt{2}}{2} \quad \text{as} \quad y > 0$$

$$\Rightarrow \quad x = [3, 6, y] = 3 + \cfrac{1}{6 + \cfrac{2}{1+\sqrt{2}}}.$$

$$= \frac{25 + 19\sqrt{2}}{8 + 6\sqrt{2}}$$

$$= \frac{14 - \sqrt{2}}{4}$$

(102)

Theorem: Every irrational number has a unique representation as an infinite continued fraction.

→ Represent $\sqrt{23}$ as an infinite continued fraction.

$$x_0 = \sqrt{23} = 4 + (\sqrt{23} - 4)$$

$$a_0 = 4$$

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{7}$$

$$= 1 + \frac{\sqrt{23}-3}{7}$$

$$[x_1] = 1 = a_1$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{7}{\sqrt{23}-3} = \frac{\sqrt{23}+3}{2}$$

$$= 3 + \frac{\sqrt{23}-3}{2}$$

$$a_2 = [x_2] = 3$$

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$$x_3 = \frac{1}{x_2 - [x_2]}$$

$$= \frac{2}{\sqrt{23} - 3} \times \frac{\sqrt{23} + 3}{7}$$

$$= 1 + \frac{\sqrt{23} - 4}{7}$$

$$a_3 = [x_3] = 1$$

$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{7}{\sqrt{23} - 4} = \sqrt{23} + 4$$

$$= 8 + (\sqrt{23} - 4)$$

$$a_4 = [x_4] = 8$$

$$x_5 = \frac{1}{\sqrt{23} - 4} = x_1, \quad \text{also } x_6 = x_2, \; x_7 = x_3 \; \vdots$$

1, 3, 1, 8. repeats indefinitely.

$$\sqrt{23} = [4, 1, 3, 1, 8, 1, 3, 1, 8, \ldots]$$

$$= [4, \overline{1, 3, 1, 8}]$$