Advanced Probability Theory.
Professor Niladri Chatterjee.
Department of Mathematics.
Indian Institute of Technology, Delhi.
Lecture 10.

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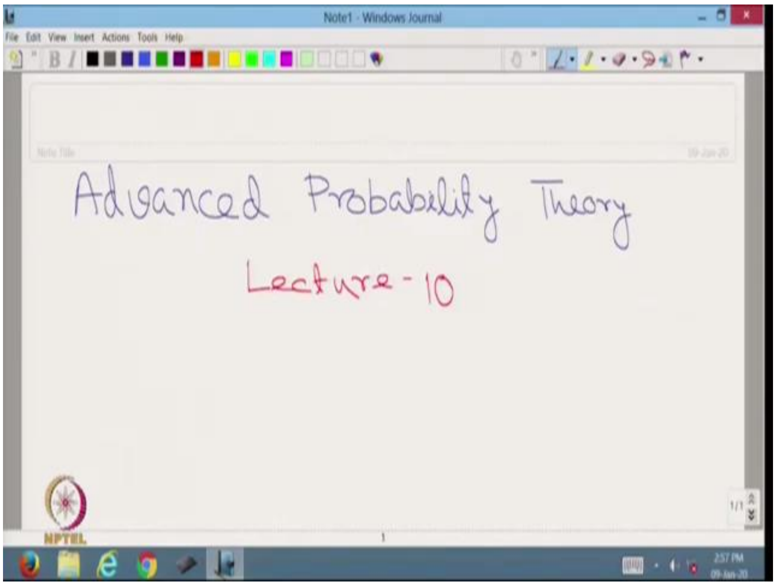

Welcome students to the MOOC's series of lectures on Advanced Probability Theory. This is lecture number 10.

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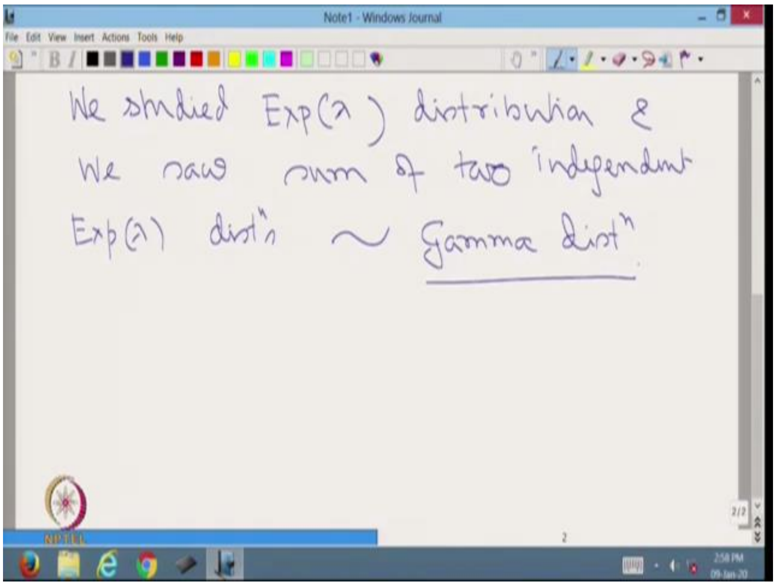

In the last class towards the end, we studied the exponential distribution. And I mentioned that sum of two independent exponential lambda distributions follow the gamma distribution, question is what is a gamma distribution? In today's lecture, we shall start with that.

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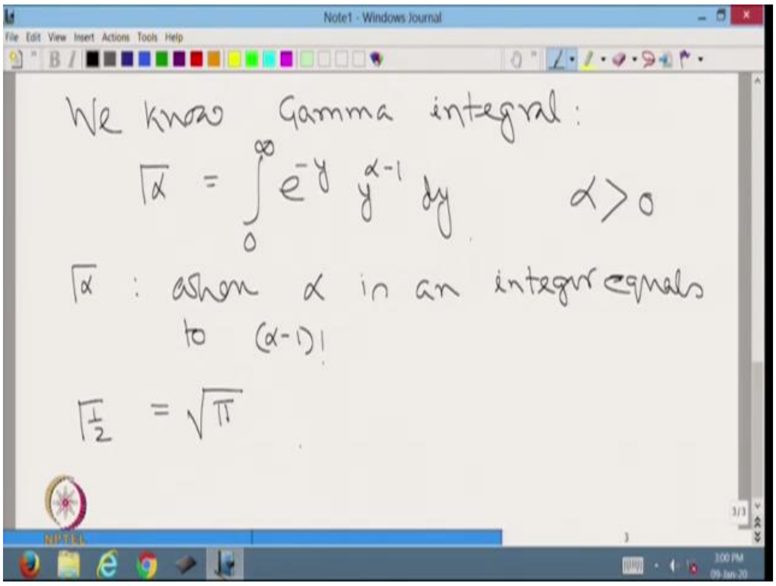

All of you definitely know gamma integral. So, for recollection gamma alpha is equal to integration 0 to infinity e to the power minus y, y to the power alpha minus 1 dy, this we should know from our background of mathematics. Now, what is gamma alpha? Gamma alpha, when alpha is an integer equals to factorial alpha minus 1, also gamma half is equal to root over pi. So, we need these two knowledge to proceed further.

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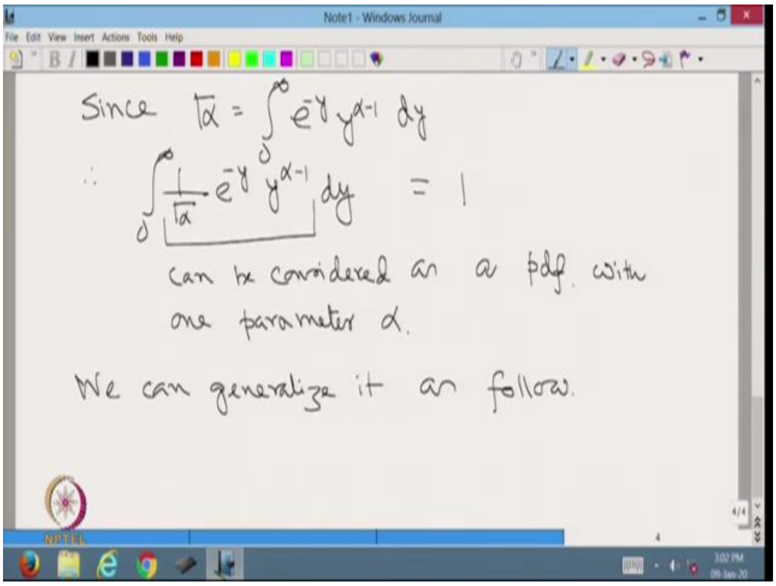

Since, gamma alpha is equal to 0 to infinity, e to the power minus y, y to the power alpha minus 1 dy. Therefore, 0 to infinity 1 upon gamma alpha e to the power minus y, y to the power alpha minus 1 dy is equal to 1, that is obvious. And therefore, this quantity can be considered as a pdf with 1 parameter alpha. We can generalize it as follows.

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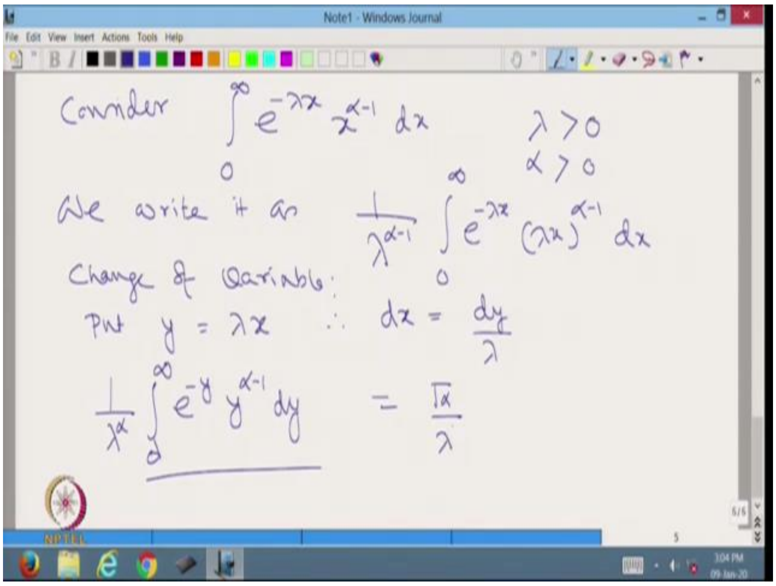

Consider 0 to infinity e to the power minus lambda x, x to the power alpha minus 1 dx when lambda is greater than 0 and obviously, alpha is greater than 0. To bring it to the standard form, we transform it as follows, 1 upon lambda power alpha minus 1, integration 0 to infinity e to the power minus lambda x lambda x to the power alpha minus 1 dx. Now make a change of variable, put y is equal to lambda x, therefore, dx is equal to dy upon lambda.

Therefore, this we can write it as 1 upon lambda power alpha 0 to infinity e to the power minus y, y to the power alpha minus 1 dy. And this is something that we have just seen is equal to gamma alpha.

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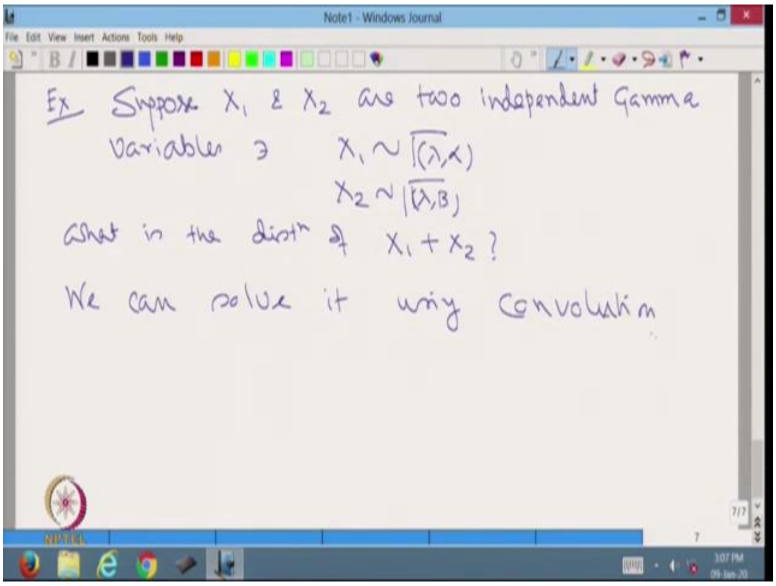

Hence, lambda power alpha upon gamma alpha e to the power minus lambda x, x to the power alpha minus 1, where 0 less than x less than infinity and lambda is greater than 0 can be considered as a probability density function. Any random variable X having the above pdf is said to be gamma random variable, X follows gamma distribution with two parameters lambda and alpha.

Suppose, X 1 and X 2 are two independent gamma variables, such that the X 1 follows gamma lambda alpha and X 2 follows gamma lambda beta, what is the distribution of X1 plus X2? We can do it using convolution.

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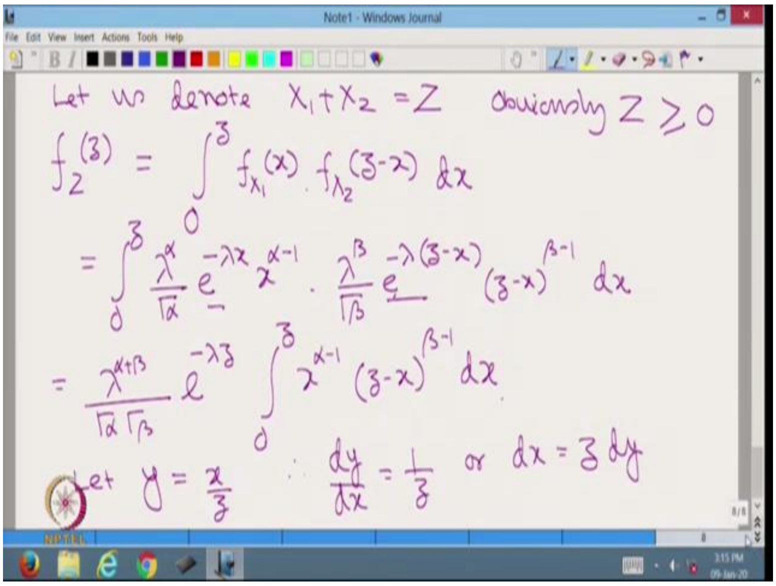

So, let us denote X 1 plus X 2 is equal to the random variable Z, obviously Z is greater than equal to 0. Now, f Z at a point z, we can write using convolution as 0 to z f X 1 at the point x multiplied by f X2 at the point z minus x dx, this we have already seen in our earlier lectures. So, we expand it as follows, 0 to z lambda power alpha upon gamma alpha e to the power minus lambda x, x to the alpha minus 1 multiplied by lambda power beta upon gamma beta e to the power minus lambda z minus x, z minus x to the power beta minus 1 dx, is equal to now we take out the constants.

So, we get lambda to the power of alpha plus beta upon gamma alpha gamma beta multiplied by, if we consider this term and this term, we see that it is going to be e to the power minus lambda z, then we write integration 0 to z, x to the power alpha minus 1, z minus x to the power beta minus 1 dx. Now, we make a change of variable, y is equal to x by z. Therefore, dy dx is equal to 1 upon z or dx is equal to z dy. We make the substitution here, and therefore you note that y, since x ranges from 0 to z, y ranges from 0 to 1.

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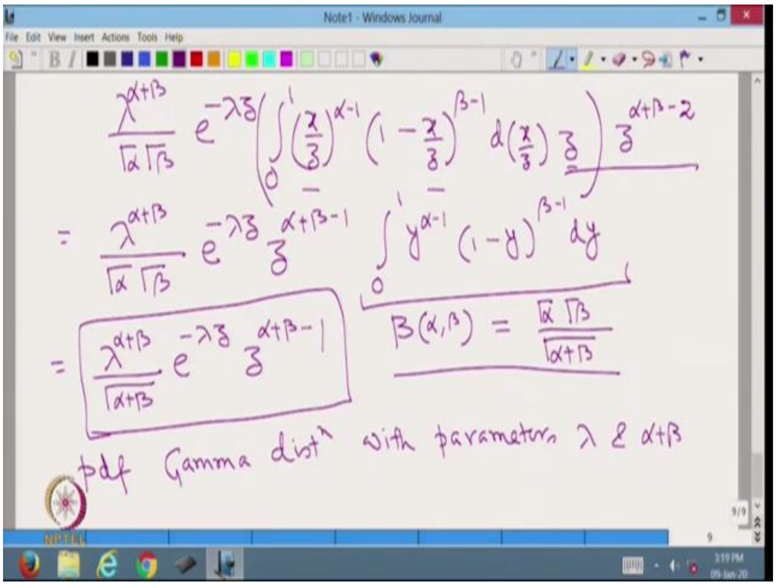

Therefore, this we can write it as lambda to the power alpha plus beta upon gamma alpha gamma beta e to the power minus lambda z integration 0 to 1 x by z to the power alpha minus 1, 1 minus x by z to the power beta minus 1 into dx by z into z. The whole thing now we multiply by z to the power alpha plus beta minus 2, because we have to compensate for this z and compensate for this z, so these two together will get alpha plus beta minus 2.

But note that we have an extra z here. Therefore, we can write it is lambda to the power alpha plus beta upon gamma alpha gamma beta e to the power minus lambda z, z to the power together we get alpha plus beta minus 1 integration 0 to 1. Now, let us put x by z is equal to y, y to the power alpha minus 1, 1 minus y to the power beta minus 1 dy. Now, this quantity is called beta alpha comma beta which is equal to gamma alpha, gamma beta upon gamma alpha plus beta.

This also you might have read in your high level mathematics, therefore, we can write lambda power alpha plus beta upon gamma alpha plus beta e to the power minus lambda z, z to the power alpha plus beta minus 1. Now, you should be able to recognize this as the pdf of gamma distribution with parameters lambda and alpha plus beta.

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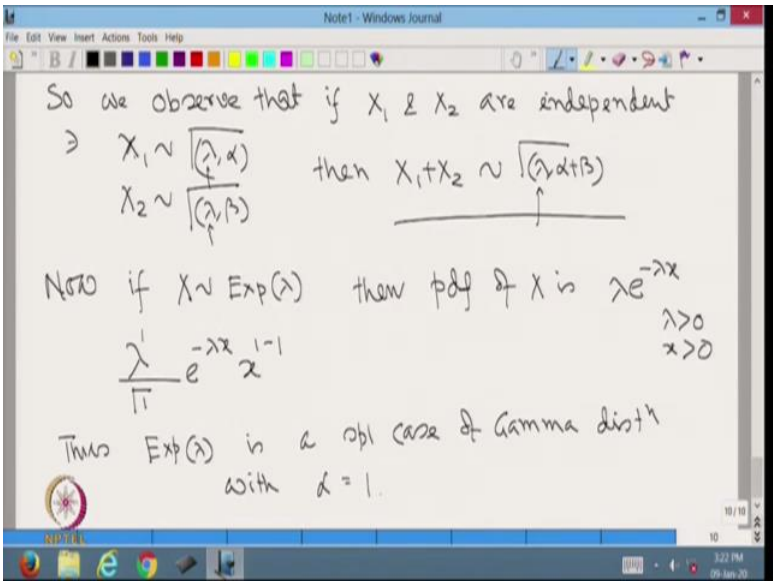

So, what we observe that, if X1 and X2 are independent, such that X1 follows gamma with lambda alpha and X2 follows gamma with lambda beta, then X1 plus X2 follows gamma with lambda and alpha plus beta. So, note that this parameter has to be the same. It is the same lambda for both X1 and X2 and if that happens, then we can add the other parameter values to get the distribution of sum of two different independent gamma random variables.

Now, if X follows exponential lambda then pdf of X is lambda e to the power minus lambda x, lambda greater than 0, x greater than 0. Therefore, we can write it as lambda power 1 upon gamma 1 e to the power minus lambda x, x to the power 1 minus 1. Thus, exponential lambda is a special case of gamma distribution with all alpha is equal to 1.

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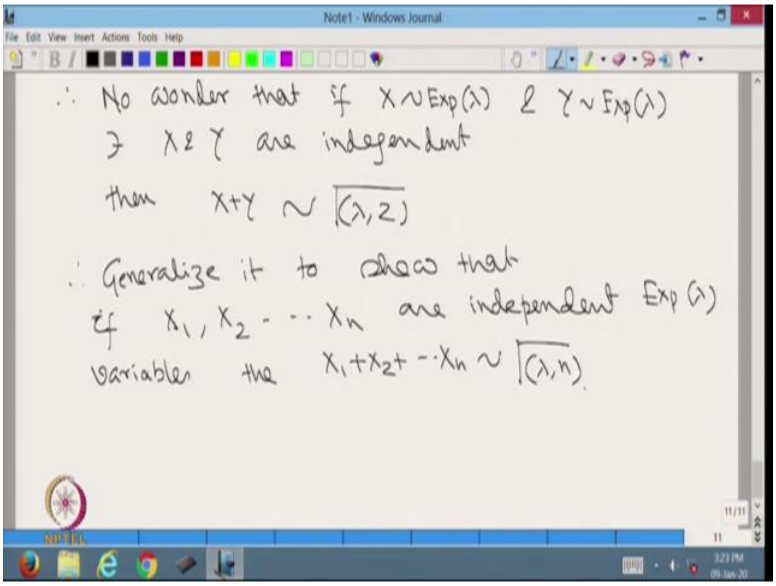

No wonder that if X is exponential lambda, and y is exponential lambda such that x and y are independent, then x plus y will follow gamma with lambda and 2. Generalize it to show that if X1, X2, Xn are independent exponential lambda variables, then X1 plus X2 plus Xn will follow gamma with lambda in beta distribution.

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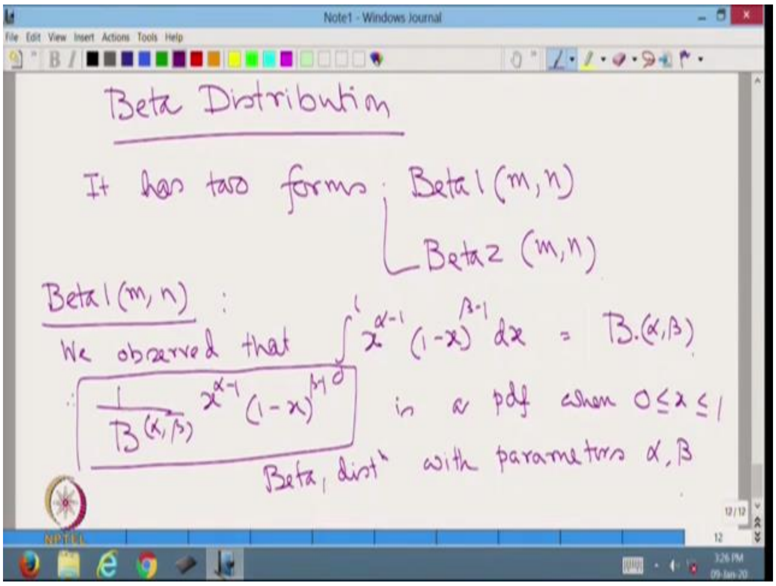

This is another important continuous random distribution and each has two forms, beta 1 with two parameters m and n and another one is beta 2 with parameters m, n. So, let us consider the first one. We observed that integration 0 to 1 x to the power alpha minus 1, 1 minus x to the power beta minus 1 dx. This, we have denoted as beta alpha, beta. Therefore, 1 upon beta alpha, beta, x to the power alpha minus 1, 1 minus x to the power beta minus 1 is a pdf, when

0 less than equal to x less than equal to 1. This particular pdf is called beta 1 distribution with parameters alpha and beta.

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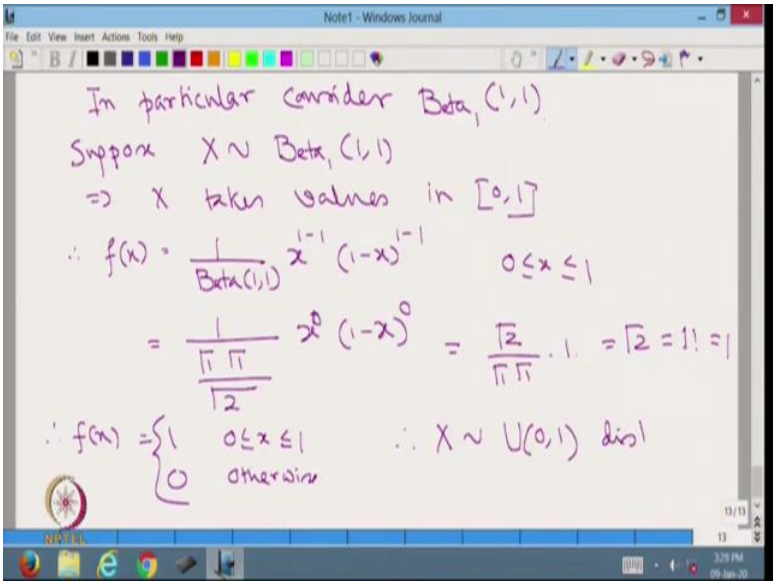

In particular, consider beta 1 with parameters 1, 1. So, suppose x follows beta 1 1, 1 implies X takes values in 0, 1 and therefore, fx is equal to 1 upon beta 1, 1 x to the power 1 minus 1, 1 minus x to the power 1 minus 1, when 0 less than equal to x less than equal to 1, is equal to 1 upon gamma 1 gamma 1 upon gamma 2 into x to the power 0, 1 minus x to the power 0 is equal to gamma 2 upon gamma 1, gamma 1 multiplied by 1 is equal to gamma 2, which we have already seen is equal to 1 factorial is equal to 1.

Therefore, fx is equal 1, if 0 less than equal to x less than equal to 1, which is 0 otherwise. Therefore, X follow uniform 0, 1 distribution.

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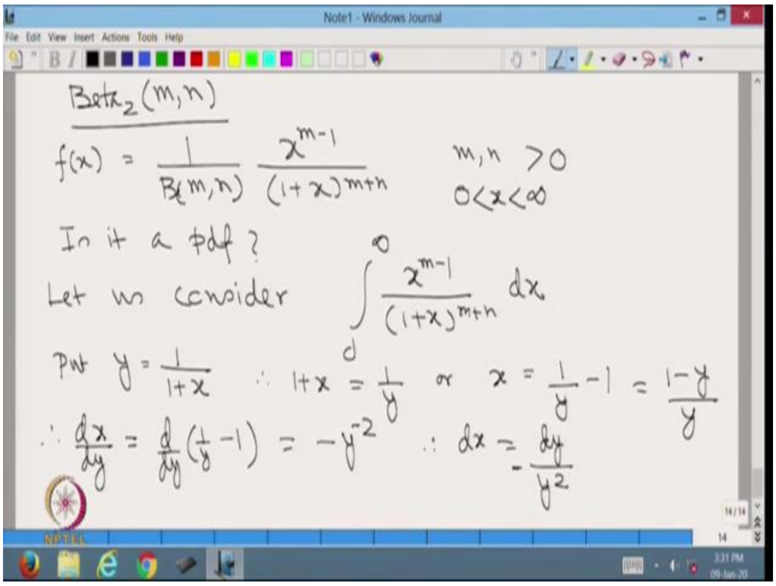

Beta 2 with parameters m and n. In this case, the f of x is equal to 1 upon beta m, n x to the power m minus 1, 1 plus x to the power m plus n, where m, n greater than 0 and x ranges from 0 to infinity. Is it a pdf? So, let us consider integration 0 to infinity x to the power m minus 1, 1 plus x to the power m plus n dx. Put y is equal to 1 upon 1 plus x. Therefore, 1 plus x is equal to 1 upon y or x is equal to 1 upon y minus 1, is equal to 1 minus y upon y. Therefore dx dy is equal to d dy of 1 upon y minus 1 is equal to minus y to the power minus 2. Therefore, dx is equal to dy upon minus y square.

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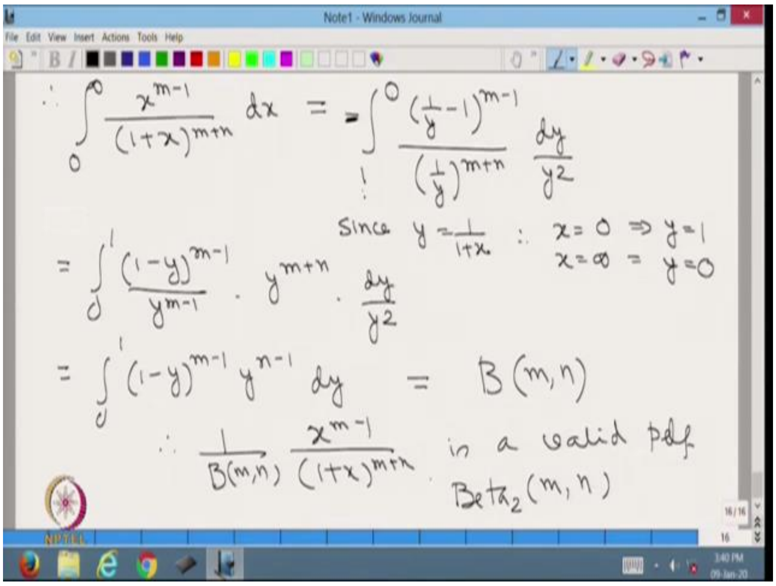

Therefore, integration 0 to infinity, x to the power m minus 1 upon 1 plus x whole to the power m plus n dx is equal to minus 1 to 0, 1 upon y minus 1 whole to the power m minus 1 upon 1 upon y whole to the power m plus n dy upon y square. How do we get the values of the limits? Since y is equal to 1 upon 1 plus x, therefore, x is equal to 0 implies y is equal to 1 and x is equal to infinity implies y is equal to 0. So, that is why we have the change in the limit of integration.

And this minus sign that we have there, that will allow us to change the limit to integration 0 to 1, 1 minus y to the power m minus 1 upon y to the power m minus 1 and multiplied by y to the power m plus n dy upon y square. This is equal to, if we cancel out the powers, we get 1 minus y to the power m minus 1, y to the power n minus 1 dy, is equal to beta m, n. Therefore, 1 upon beta m, n multiplied by x to the power m minus 1 upon 1 plus x whole to the power m plus n is a valid pdf. And this in particular is called beta 2 with m, n.

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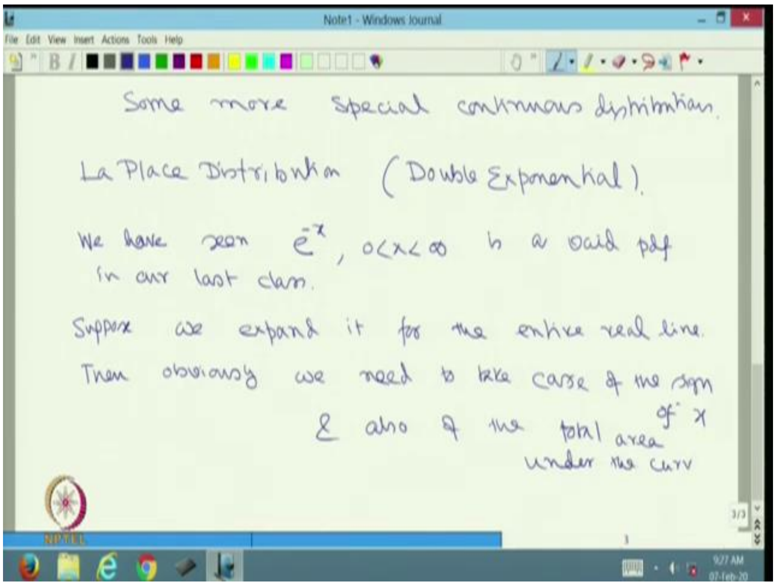

Let us now study some more special continuous distributions. For example, first one is Laplace distribution, which is also known as double exponential. We have seen e to the power minus x, 0 less the x less than infinity is a valid pdf in our last class. Suppose we expand it for the entire real line, then obviously we need to take care of the sign of x and also of the total area under the curve.

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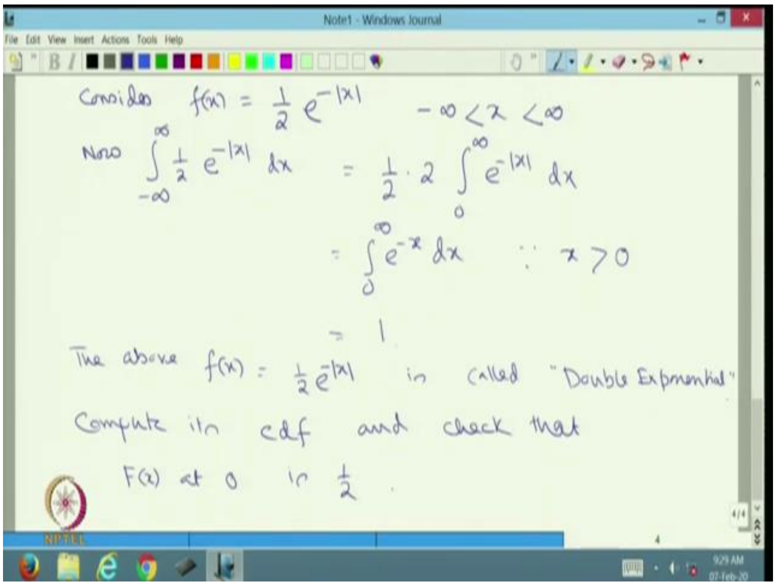

So, consider fx is equal to half e to the power minus mod x minus infinity less than x less than infinity. Now, integration minus infinity to infinity half e to the power minus mod x dx is equal to half into 2 times 0 to infinity into e to the power minus mod x dx, is equal to integration 0 to infinity, e to the power minus x dx, since x is greater than 0, is equal to 1. The above fx is equal to half e to the power minus mod x is called double exponential, as I have said earlier, compute its cdf and check that fx at 0 is half.

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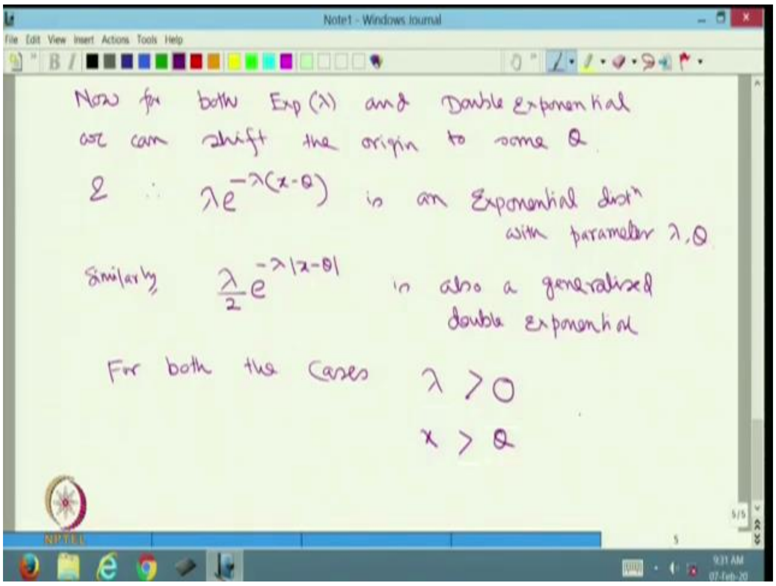

Now, for both exponential lambda and double exponential we can shift the origin to some theta and therefore, lambda e do the power minus lambda x minus theta is an exponential distribution, with parameters lambda and theta. Similarly, lambda by 2 into e to the power minus lambda mod x minus theta is also a generalized double exponential. For both the cases lambda is greater than 0 and defined for x greater than theta.

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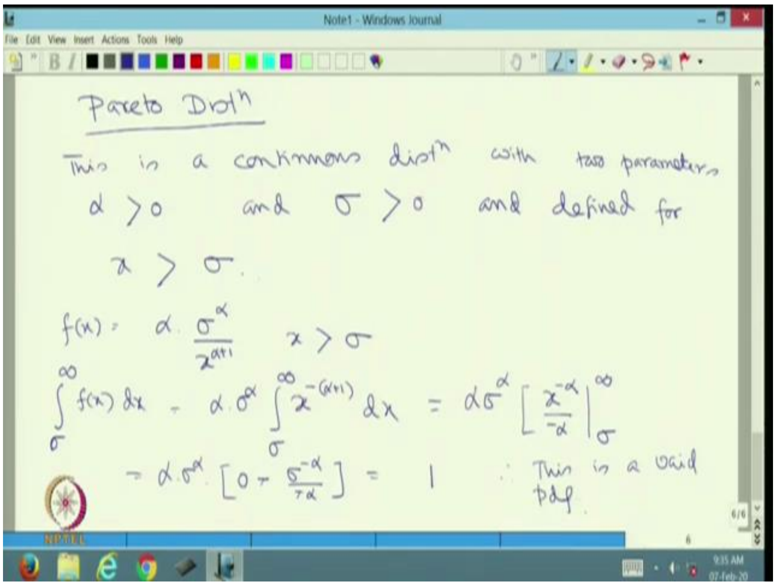

Pareto distribution. This is a continuous distribution with two parameters alpha, which is greater than 0 and sigma which is also greater than 0 and defined for x greater than sigma. So f x is equal to alpha sigma times alpha upon x to the power alpha plus 1, where x is greater than sigma, so let us integrate. Integration of sigma to infinity fx dx is equal to alpha sigma power alpha integration sigma to infinity x to the power minus alpha plus 1 dx, is equal to alpha sigma powered alpha x to the power minus alpha upon minus alpha in the range sigma to infinity, is equal to alpha sigma to the power alpha 0 minus sigma to the power minus alpha upon minus alpha is equal to 1, therefore, this is a valid pdf.

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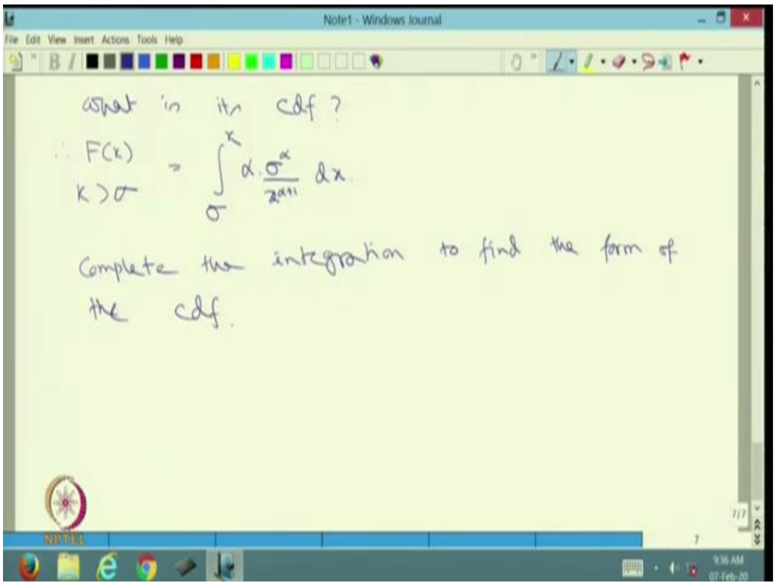

What is going to be cumulative distribution function? Therefore, F of k, k greater than sigma is equal to integration sigma to k alpha sigma to the power alpha, upon x to the power alpha plus 1 dx, complete the integration to find the form of the cdf.

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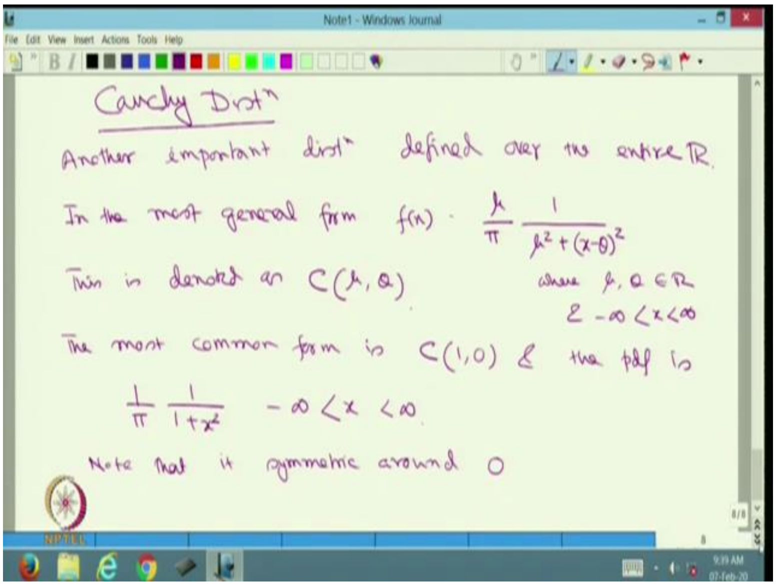

Cauchy distribution. This is another important distribution defined over the entire real line. In the most general form, fx is equal to mu upon pi into 1 upon mu square plus x minus theta whole square, where mu theta belong to R and minus infinity less than x less than infinity. This is denoted as C with mu, theta, the most common form is C 1, 0 and the pdf is 1 over pi 1 upon 1 plus x square minus infinity less than x less than infinity. And note that it is symmetric around 0.

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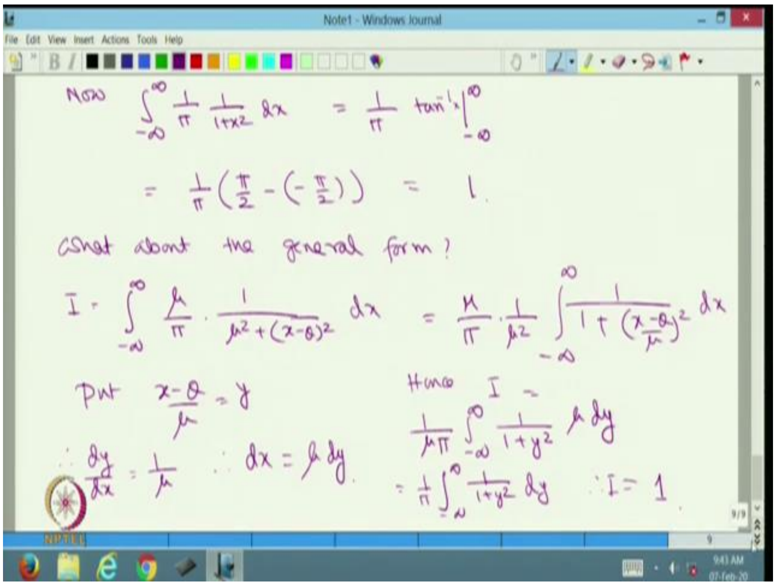

Now, integration minus infinity to infinity, 1 upon pi 1 upon 1 plus x square dx is equal to 1 upon pi into tan inverse x from minus infinity to infinity, is equal to 1 over pi into pi by 2 minus minus pi by 2 is equal to 1. What about the general form? Now, let I be integration minus infinity to infinity, mu upon pi into 1 upon mu squared plus x minus theta whole square dx, is equal to mu upon pi into 1 over mu square integration of minus infinity to infinity 1 upon 1 plus x minus theta over mu whole square dx.

Put x minus theta over mu is equal to y. Therefore, dy dx is equal to 1 over mu. Therefore, dx is equal to mu dy. Hence, I is equal to 1 over mu pi integration minus infinity to infinity 1 upon 1 plus y squared into mu dy, is equal to 1 over pi minus infinity to infinity 1 upon 1 plus y square dy and therefore I is equal to 1.

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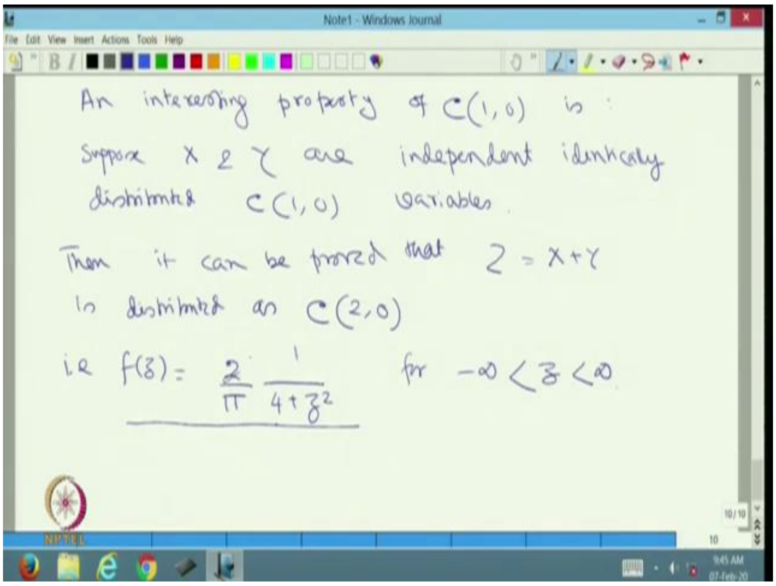

And interesting property of C 1, 0 is suppose X and Y are independent identically distributed C 1, 0 variables, then it can be proved that Z is equal to X plus Y is distributed as C 2, 0. That is, f of z is equal to 2 by pi 1 upon 4 plus z square for minus infinity less z less than infinity. This is a very difficult result to prove mathematically, so I am not doing that in this class, but we should remember the result that sum of two independent Cauchy distributions is also Cauchy distribution with the first parameter mu changing from 1 to 2.

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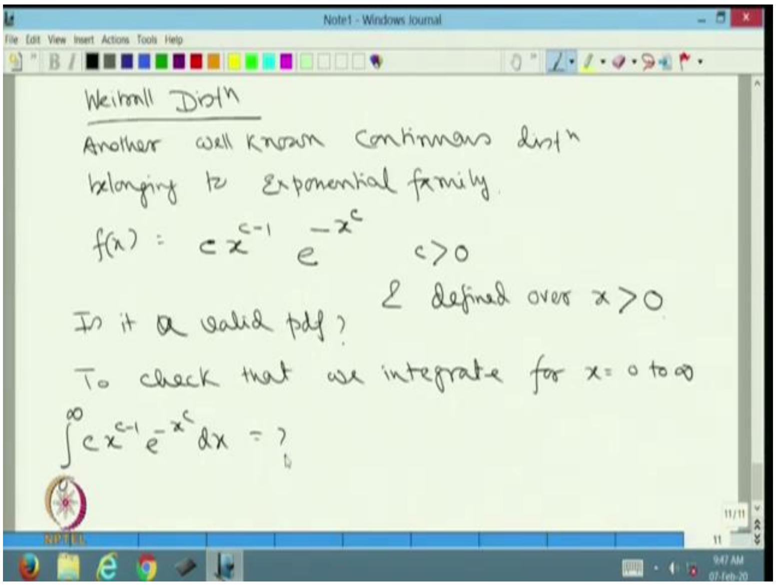

Weibull distribution, this is another well-known continuous distribution belonging to exponential family, where f of x is equal to c into x to the power c minus 1, e to the power minus x to the power c, where c is greater than 0 and defined over x greater than 0, is it a valid pdf? To check that, we integrate for x is equal to 0 to infinity. Now, integration 0 to infinity, c x to the power c minus 1, e to the power minus x to the power c dx is equal to what?

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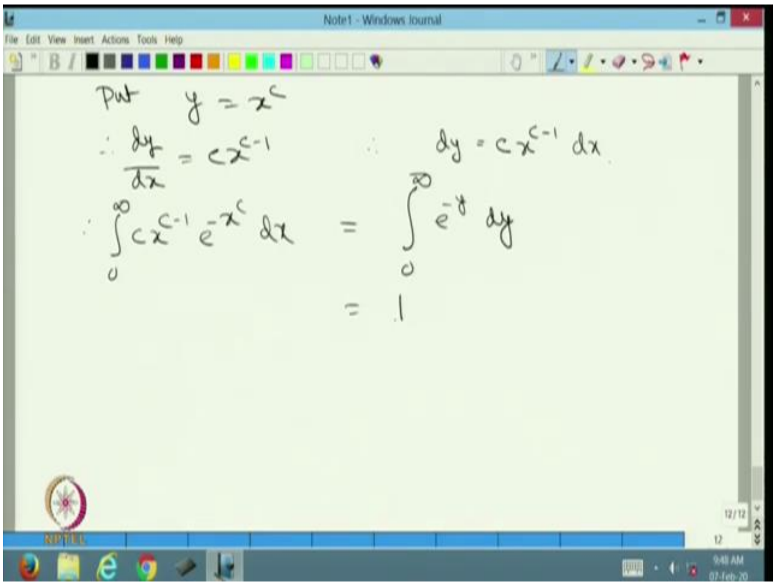

So, put y is equal to x to the power c, therefore, dy dx is equal to c x to the power c minus 1. Therefore, dy is equal to c x to the power c minus 1 dx. Therefore, integration 0 to infinity c x to the power c minus 1, e to the power minus x to the power c dx is equal to integration 0 to infinity e to the power minus y dy, which we know is equal to 1.

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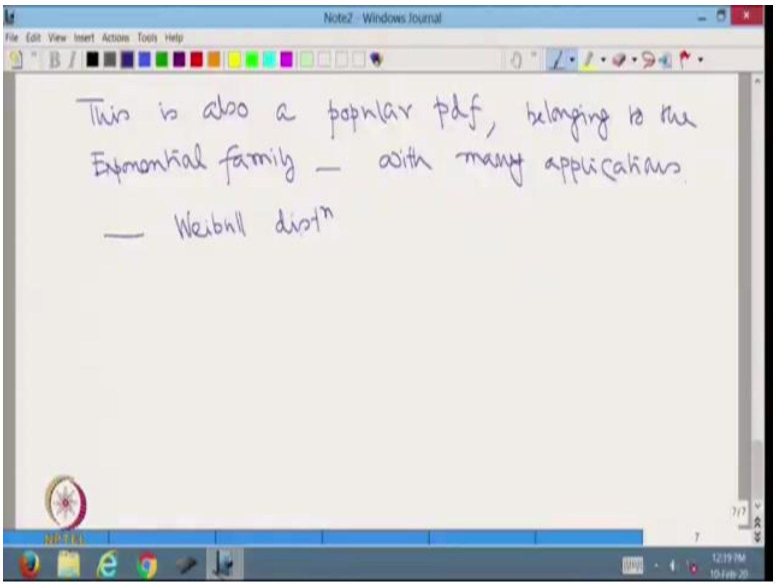

This is also a very popular pdf, belonging to the exponential family with many applications and as I said, the name is Weibull distribution.

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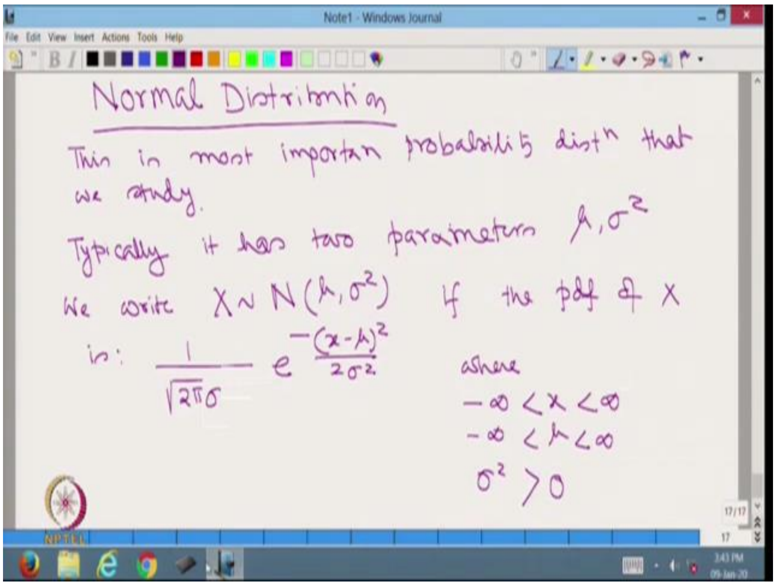

We shall conclude our discussion on continuous random variables by discussing perhaps the most important distribution, namely Normal Distribution. Undoubtedly, this is most important probability distribution that we study. So, typically it has two parameters mu and sigma square or we write X follows normal mu, sigma square, if the pdf of X is 1 over root over 2 pi sigma e to the power minus x minus mu whole square upon two sigma square, where minus infinity less than x less than infinity, minus infinity less than mu less than infinity, mu can be any particular value and obviously sigma square has to be positive that is, it is greater than 0.

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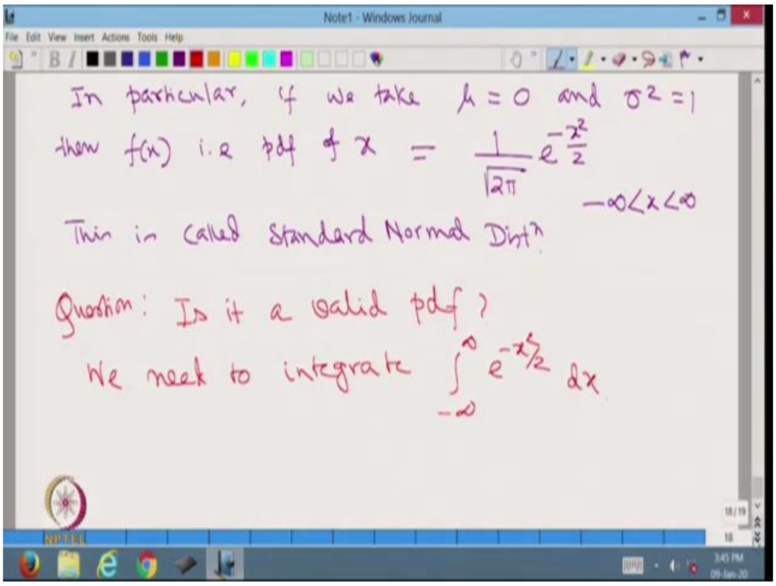

In particular, if we take mu is equal to 0 and sigma square is equal to 1, then f of x, that is the pdf becomes 1 over root over 2 pi e to the power minus x square by 2. This is called standard normal distribution. Question is, is it a valid pdf. So we need to integrate minus infinity to infinity e to the power minus x square by 2 dx.

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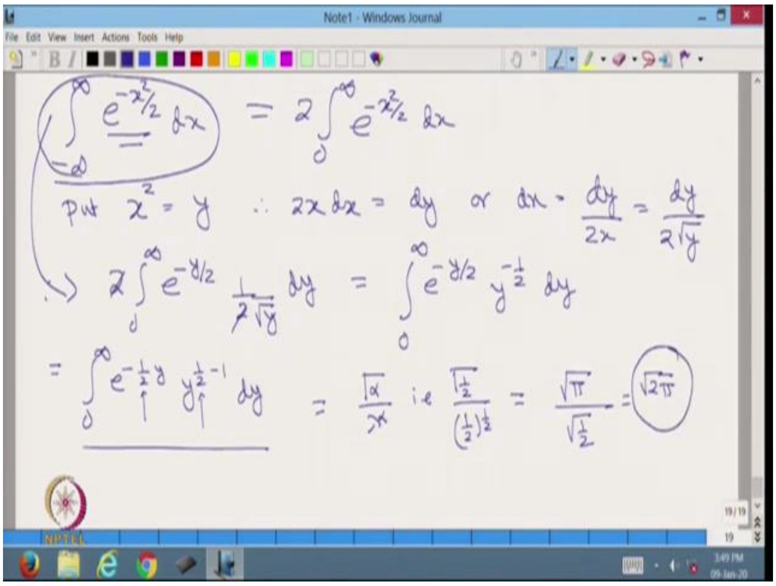

So, let us do that minus infinity to infinity e to the power minus x square by 2 dx. Since, this is an even function, we can write it as two times 0 to infinity e to the power minus x square by 2 dx. Put x square is equal to y, therefore 2x dx is equal to dy or dx is equal to dy upon 2x is equal to dy upon 2 root y. Therefore, this above integral now becomes 2 into integration 0 to infinity, e to the power minus y by 2 into 2 root y dy.

Now, 2 cancels, therefore what we are having is 0 to infinity e to the power minus y by 2, y to the power minus of dy, is equal to 0 to infinity e to the power minus half y and y to the power half minus 1 dy. This looks like a gamma integral with the parameter lambda is equal to half and alpha is equal to half. Therefore, it will integrate to gamma alpha upon lambda power alpha, that we know, that is gamma half upon half to the power half, is equal to, we have already mentioned that gamma half is equal to root over pi and here we get square root of half, which is equal to root 2 pi. Therefore, by integrating this quantity, we get root over 2 pi.

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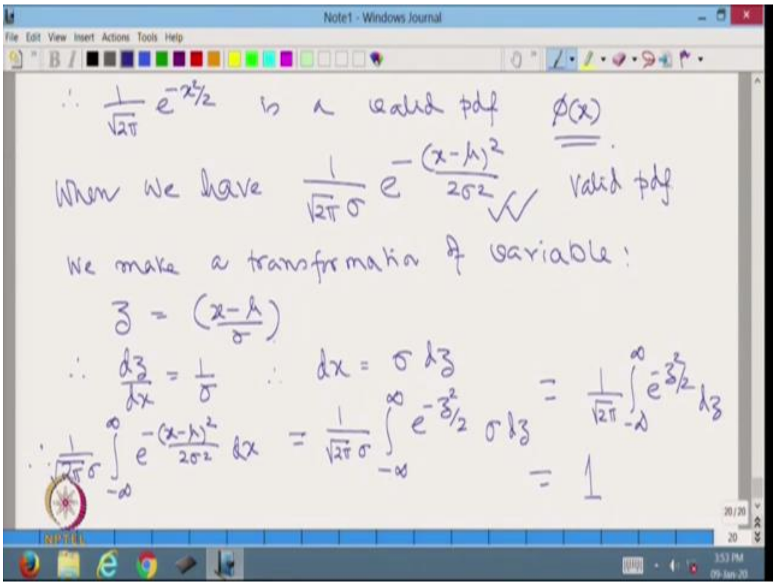

Therefore, 1 over root over 2 pi into e to the power minus x square by 2 is a valid pdf, which we call phi of x in general. When we have 1 over root over 2 pi sigma e to the power minus x minus mu whole square by 2 sigma square, we make a transformation of variable z is equal to x minus mu upon sigma, therefore dz dx is equal to 1 upon sigma. Therefore, dx is equal to sigma dz. Therefore, 1 over root over 2 pi sigma integration minus infinity to infinity e to the power minus x minus mu whole square upon 2 sigma square dx.

Is equal to 1 over root over 2 pi sigma integration minus infinity to infinity e to the power minus z square by 2 sigma dz, which is equal to 1 over root over 2 pi integration minus infinity to infinity e to the power minus z square by 2 dz and therefore, it is equal to 1. Therefore, this is also a valid pdf.

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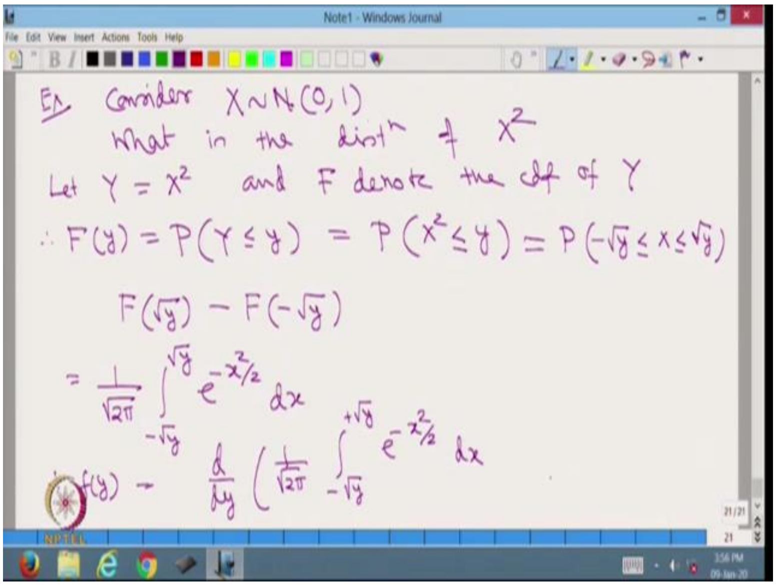

Example, consider X to be standard normal that is distributed as normal 0, 1. What is the distribution of X square? Let Y equal to X square and F denote the cdf of Y. Therefore, F y is equal probability, Y less than equal to y is equal to probability X square less than equal to y is equal to probability minus root y less than equal to x less than equal to root y.

And we know from the concept of cumulative distribution function, that this is going to be f at root y minus f at minus root y is equal to 1 over root over 2 pi integration minus root y to plus root y, e to the power minus x squared by 2 dx. Therefore, f at y is equal to d dy of 1 over root over 2 pi minus root y to plus root y,y e to the power minus x square by 2 dx.

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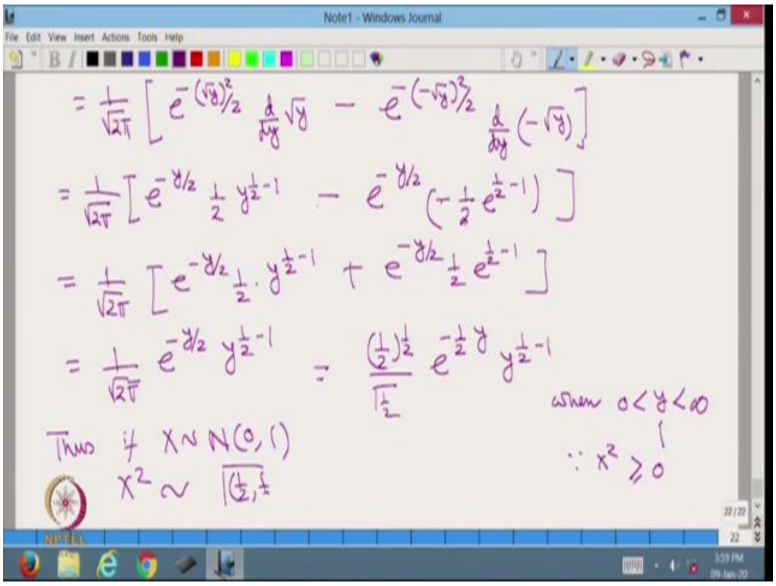

Is equal to 1 over root over 2 pi, e to the power minus root y square by 2 into d dy of root y minus e to the power minus minus root y square by 2 d dy of minus root y, is equal to 1 over root over 2 pi, e to the power minus y by 2, half y to the power half minus 1 minus e to the power minus y by 2 minus half e to the power half minus 1 minus and minus becomes plus,, therefore half plus half, it becomes 1 is equal to 1 over root over 2 pi, e to the power minus y by 2 y to the power half minus 1, is equal to half to the power half upon gamma half.

We know that root pi is equal to gamma half, e to the power minus half y, y to the power half minus 1, when 0 less than y less than infinity. And this has to be positive, because we are looking at x square which is greater than equal to 0. Thus, if X follows normal 0,1, X square follows gamma with half, half.

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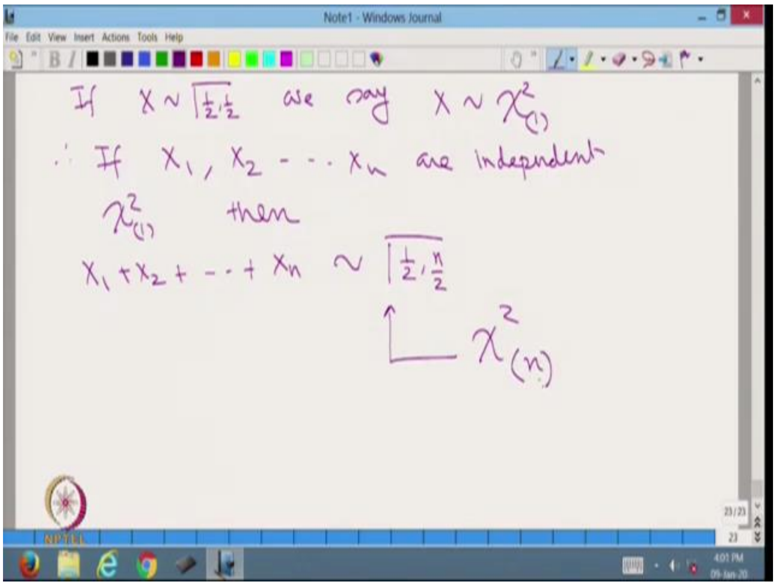

If X follows gamma half, half, we say x is distributed as chi square with 1 degree of freedom. Therefore, if X1, X2, Xn are independent chi square with 1 degrees of freedom, then X1 plus X2 plus Xn is distributed as gamma half, n by 2, which we say is the pdf of chi square with n degrees of freedom. So, this way we get another continuous random variable or continuous distribution, which is called chi square and that is going to be a positive random variable.

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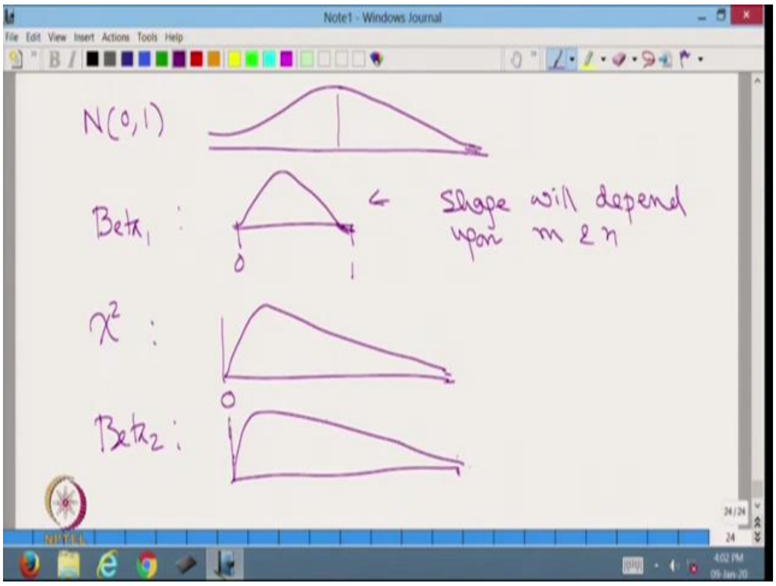

In general, a normal 0, 1 will be having a shape of the pdf like this beta 1 will have a pdf like this and this shape will depend upon m and n, chi square will have a shape like this. Similarly, beta 2 as we can understand that it is a positive random variable, therefore, as the value of x goes to infinity, it will also be asymptotically going towards 0, but before that it will have a similar type of shape. Okay friends, I stop today. In the next class, we shall study some important properties of a distribution, namely expectation and variance. Okay friends, thank you.