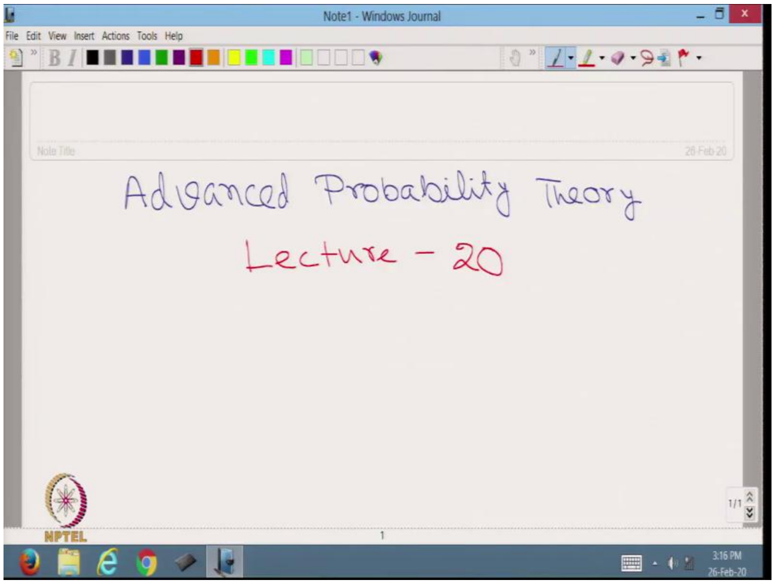
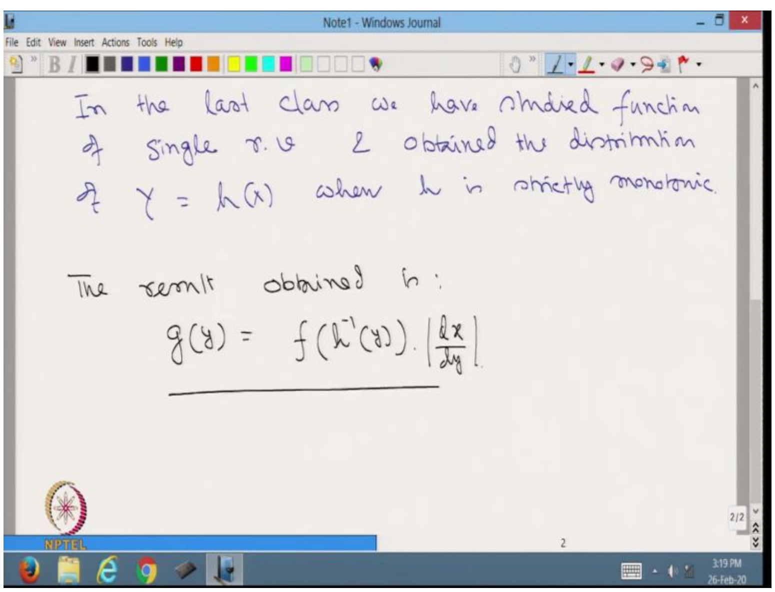
Advanced Probability Theory
Professor Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture 20

(Refer Slide Time: 0:26)



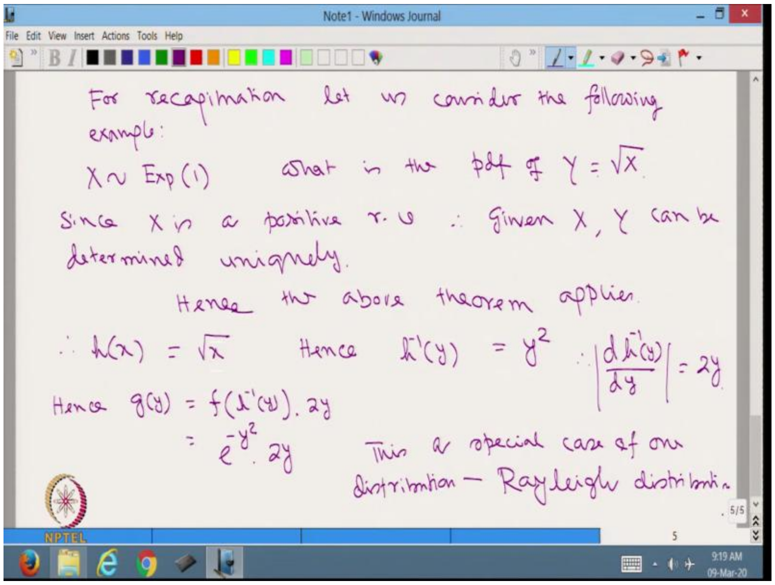
Welcome students to the MOOCS series of lectures on Advanced Probability Theory, this is lecture number 20.

(Refer Slide Time: 00:29)



In the last lecture we have studied function of single random variable and obtained the distribution of Y is equal to hx when h is strictly monotonic. The results obtained is g of y is equal f at h inverse of y multiplied by modulus of dx dy.

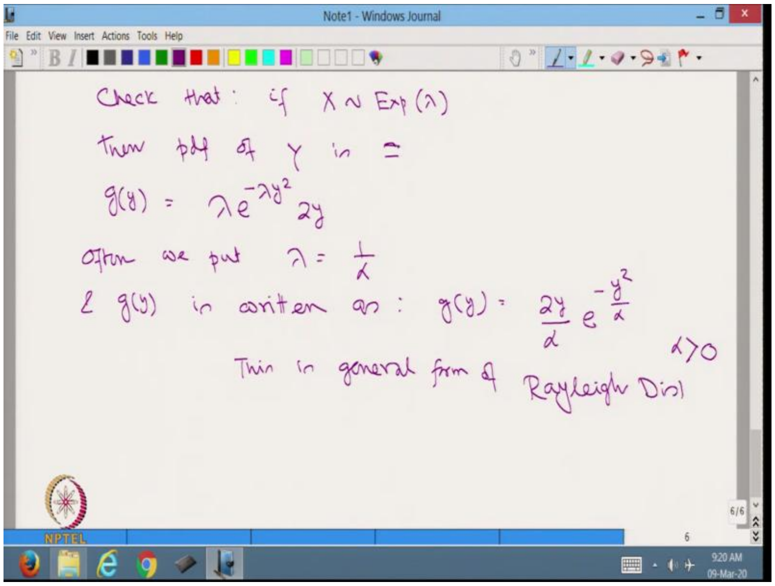
(Refer Slide Time: 01:46)



For recapitulation let us consider the following example, if X follows exponential with 1 degrees of freedom what is the pdf of Y is equal root over of X. Since, X is a positive random variable. Therefore, given X, Y can be determined uniquely. Hence, the above theorem applies straight away.

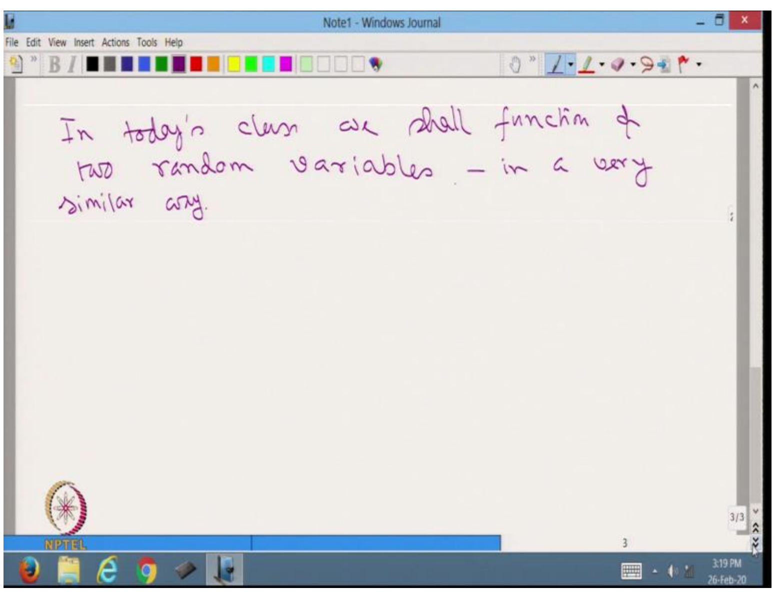
Therefore, h of x is equal to root over of x. Hence, h inverse of y is equal to y square. Therefore, d h inverse of y dy is equal to 2y. Hence, g of y is equal f at h inverse of y into 2y is equal e to the power minus y square into 2y. This is a special case of one distribution whose name is Rayleigh Distribution.

(Refer Slide Time: 04:31)



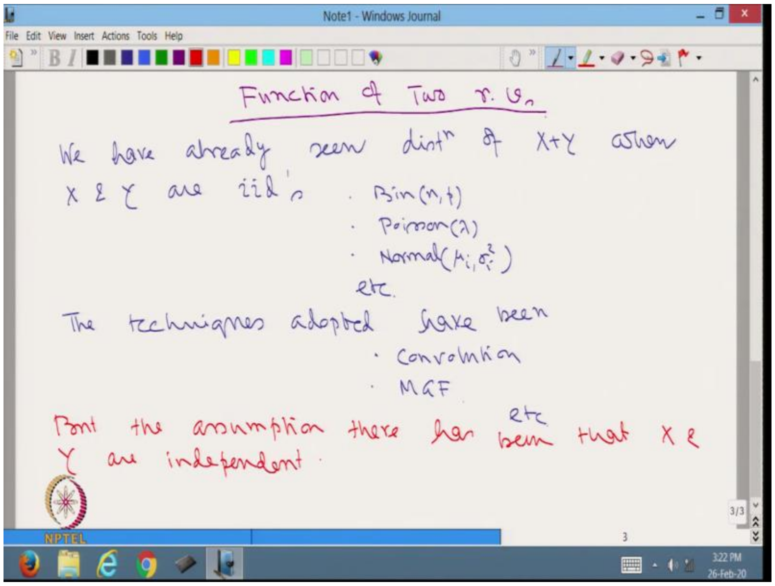
Check that if X is exponential with lambda then pdf of Y is going to be g of y is equal lambda e to the power minus lambda y square into 2y. Often we put lambda is equal 1 upon alpha and gy is written as gy is equal 2y upon alpha e to the power minus y square by alpha y greater than 0 and alpha is greater than 0. So, this a general form of Rayleigh Distribution.

(Refer Slide Time: 05:49)



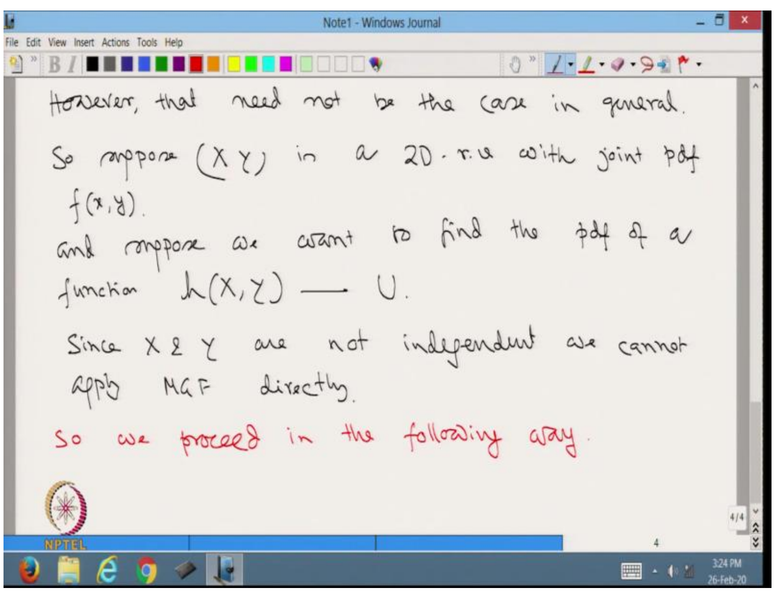
In today's class we shall see function of two random variables which as we will see in a very similar way.

(Refer Slide Time: 06:28)



So, our topic for today is function of two random variables. We have already seen distribution of X plus Y when X and Y are iid's particularly for say binomial, (())(7:08), even normal, etcetera. The techniques adapted have been through convolution or using movement generating function, etcetera. But the assumption there has been that X and Y are independent.

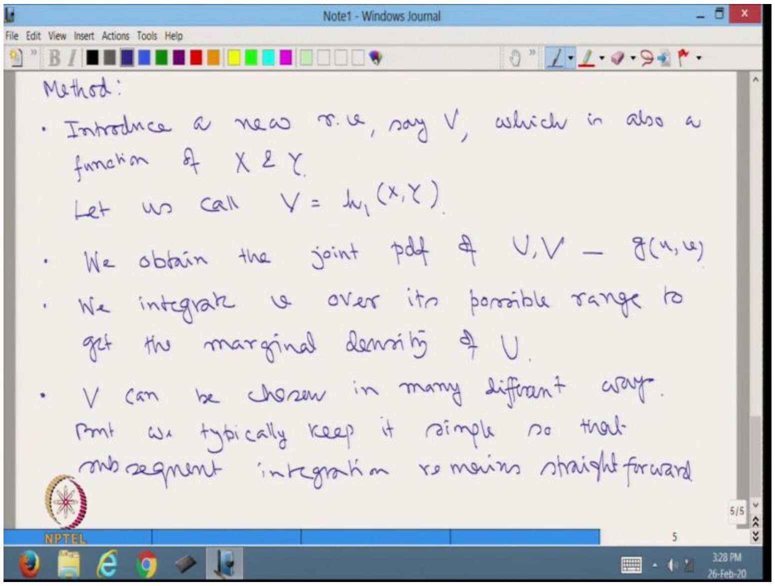
(Refer Slide Time: 08:14)



However, that need not be the case in general. So, suppose X Y is a 2D random variable with joint pdf f of x, y and suppose we want to find the pdf of a function h of X, Y. So, let us call it U.

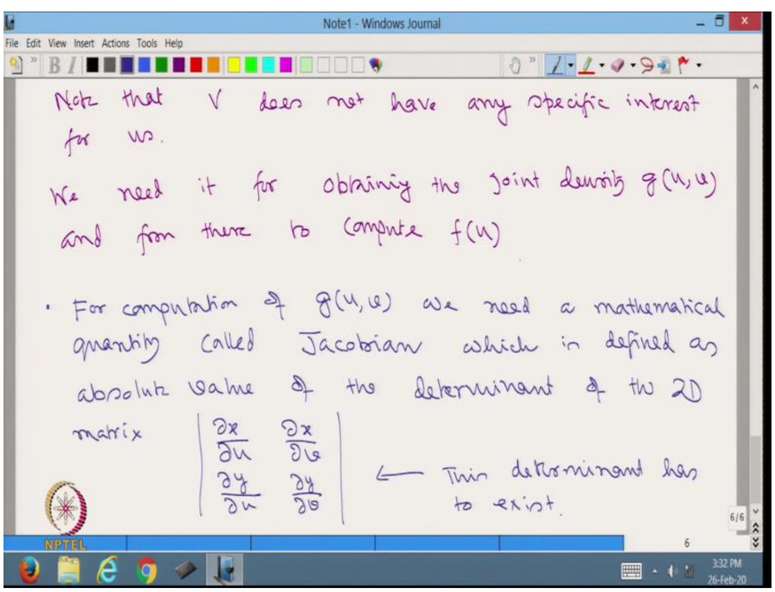
Since X and Y are not independent we cannot apply MGF directly. So, we proceed in the following way.

(Refer Slide Time: 10:01)



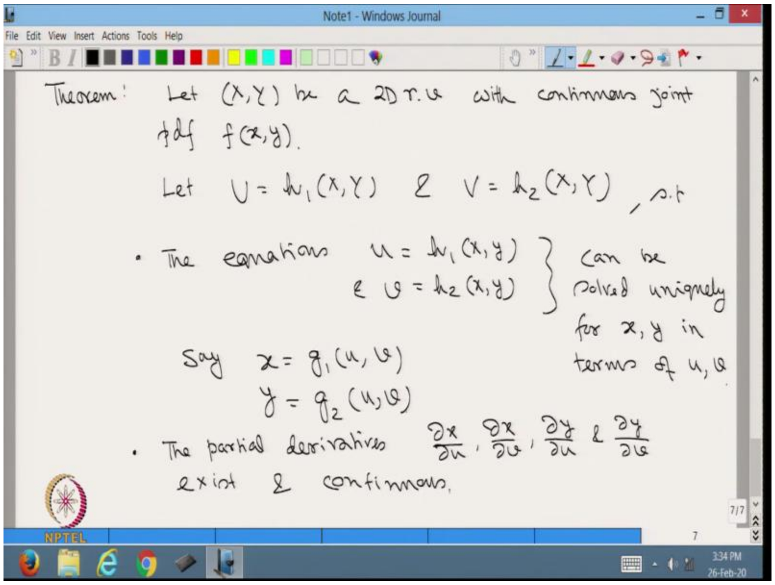
So, the method introduce a new random variable say V which is also a function of X and Y. So, let us call it V is equal h1 of X and Y. We obtain the joint pdf of U and V, so let us call it g u, v. Then, we integrate v over its possible range to get the marginal density of U. V can be chosen in many different ways but we typically keep it simple so that subsequent integration remains straight forward.

(Refer Slide Time: 12:34)



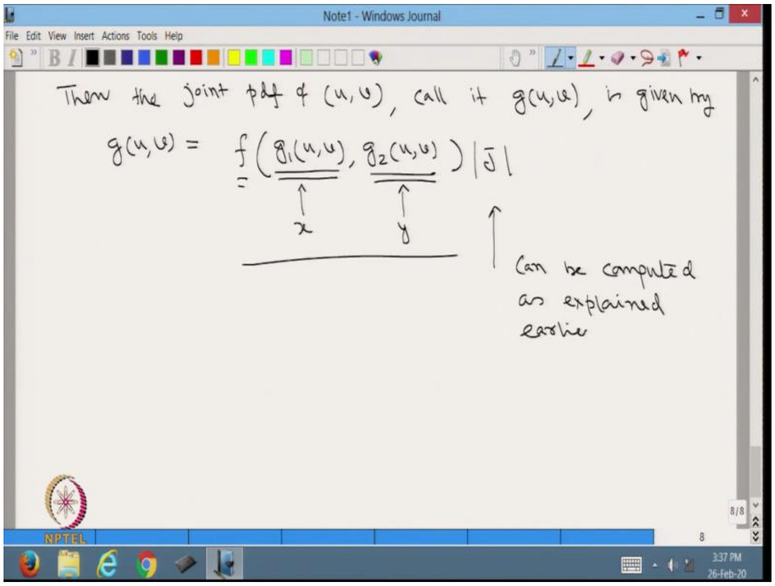
Note that V does not have any specific interest for us. We need it for obtaining the joint density g u, v and from there to compute say f at u. For computation of g u, v we need a mathematical quantity called Jacobian which is defined as the absolute value of the determinant of the 2D matrix del x del u del x del v del y del u del y del v. And therefore, we need to compute this determinant, this determinant has to exist.

(Refer Slide Time: 15:23)



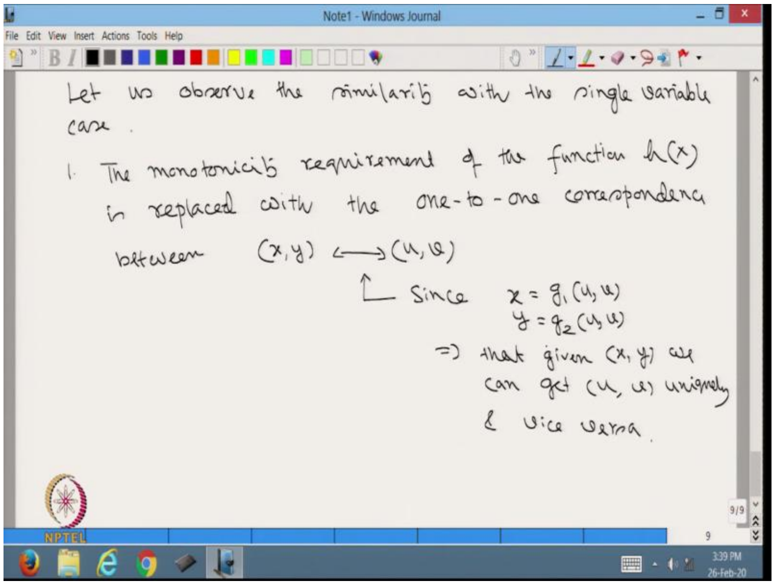
Hence, we use the following theorem. Let X, Y be a 2D random variable with continues joint pdf f x, y. Let U equal to h1 of X, Y and V is equal h2 of X, Y such that the equations U is equal h1 x, y and v is equal to h2 x, y can be solved uniquely for x, y in terms of u and v. So, say x is equal g1 of u, v and y is equal g2 of u, v. The second assumption is that the partial derivatives del x del u del x del v del y del u and del y del v exist and continuous.

(Refer Slide Time: 17:42)



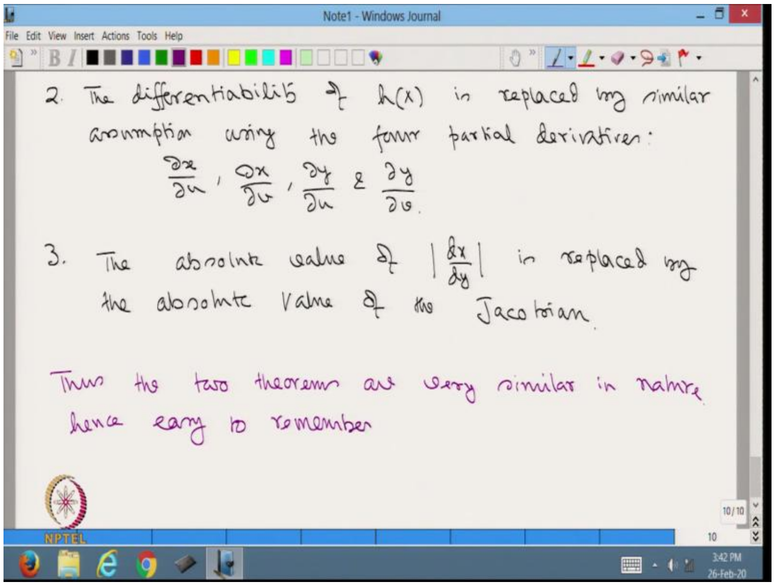
Then the joint pdf of u, v let us call it g u, v is given by g u, v is equal to f at g1 u, comma v g2 u, comma v multiplied by the Jacobian which I have already explained. So, note that this is actually corresponding to x and this is corresponding to y. Thus, f of this thing make sense and J and can be computed as explained earlier.

(Refer Slide Time: 19:05)



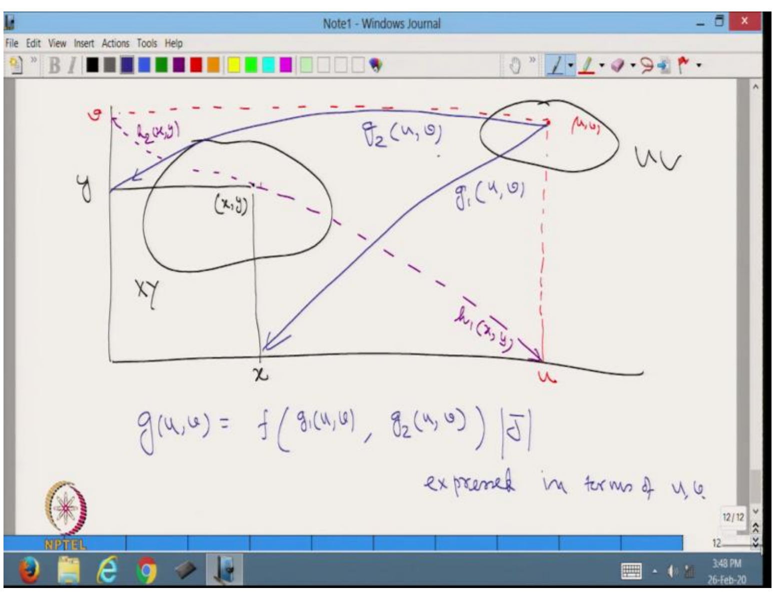
We are not proving this theorem but, let us observe this similarity with the single variable case that we studied in the last class. So, the monotonicity requirement of the function h of x is replaced with the one to one correspondence between x, y and u, v this is possible. Since, x is equal to g1 u, v and y is equal g2 u, v implies that given x, y we can get u, v uniquely and vice versa. That is given u, v we can get x, y.

(Refer Slide Time: 21:07)



Two, the differentiability of h of x the function is replaced by similar assumption using the four partial derivatives. As I have mentioned before these are del x del u del x del v del y del u and del y del v. The absolute value of dx dy is replaced by the absolute value of the Jacobian. Thus, the two theorems are very similar in nature hence easy to remember.

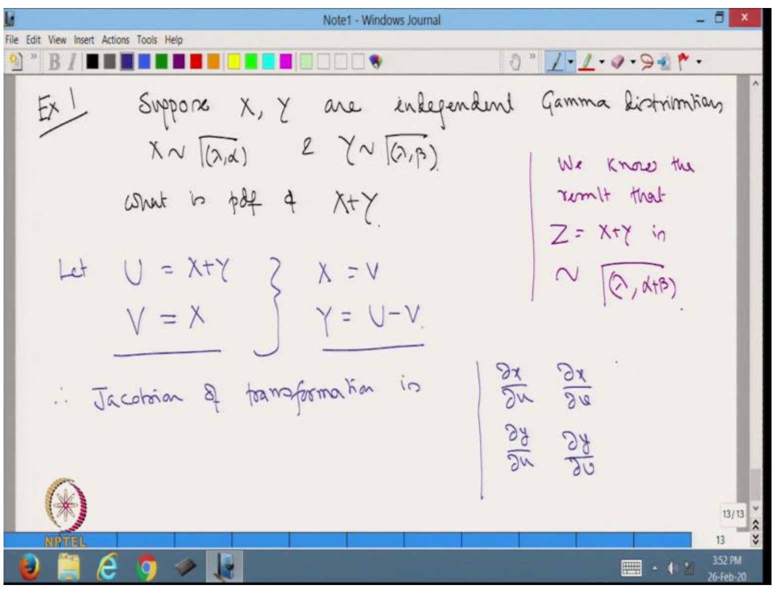
(Refer Slide Time: 23:11)



So, pictorially let us draw this. Suppose, this is the XY plane and suppose this is the uv plane we have a point x, y therefore this is the coordinate x and this is the coordinate y. We are mapping it to a point u, v therefore this is the point u and this is the point v and we have the following that we have one mapping that from x, y maps to you we are calling it h1 x, y. In a similar way, we have another mapping, which is mapping to v therefore this we are calling as h2 of x, y.

Since these are unique therefore we get inverse mapping which from u, v is mapping to x and that we have given the name g1 of u, v. And in a similar way it is mapping to y which we are calling g2 of u, v and we have got that g of u, v is equal to f at g1 at u, v g2 at u, v multiplied by the Jacobian and expressed in terms of u, v. As I said we are not going to prove this theorem but we accept this and let us solve a few problems using this theorem.

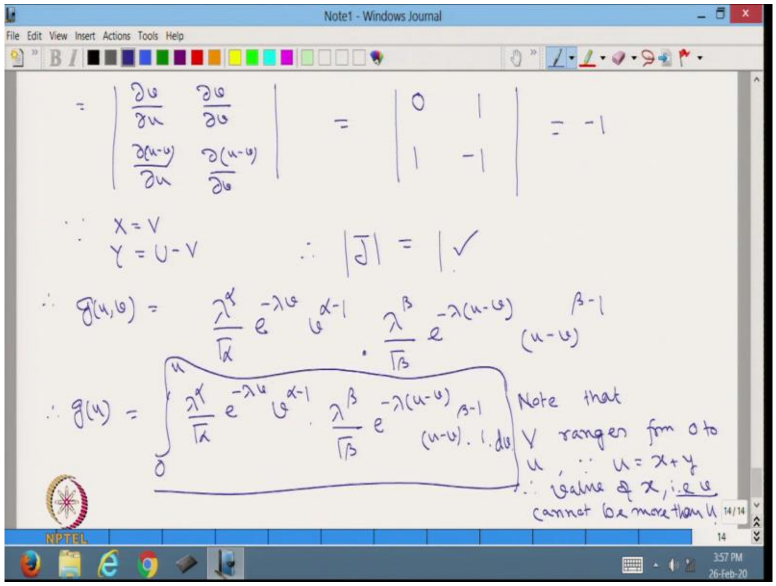
(Refer Slide Time: 25:46)



So, example 1, let us go ahead with x and y to be independent so that the joint pdf we can write as the product their individual densities. So, consider the following suppose X and Y are independent gamma distributions with X following gamma with lambda and alpha and Y following gamma with lambda and beta, what is the pdf of X plus Y.

Note we know the result that Z is equal to X plus Y is distributed as gamma with lambda, comma alpha plus with beta. We shall try to obtain the same result using the above theorem. Let U is equal to X plus Y, we have to take a V we keep it simple and suppose we take V is equal to X. Therefore, X is equal to V and Y is equal to U minus V. Thus in effect from the transformation h1 and h2 we get their inverse transformation g1 and g2 which will bring back x, y from u and v. Therefore, Jacobian of transformation is del x del u del x del v del y del u and del y del v.

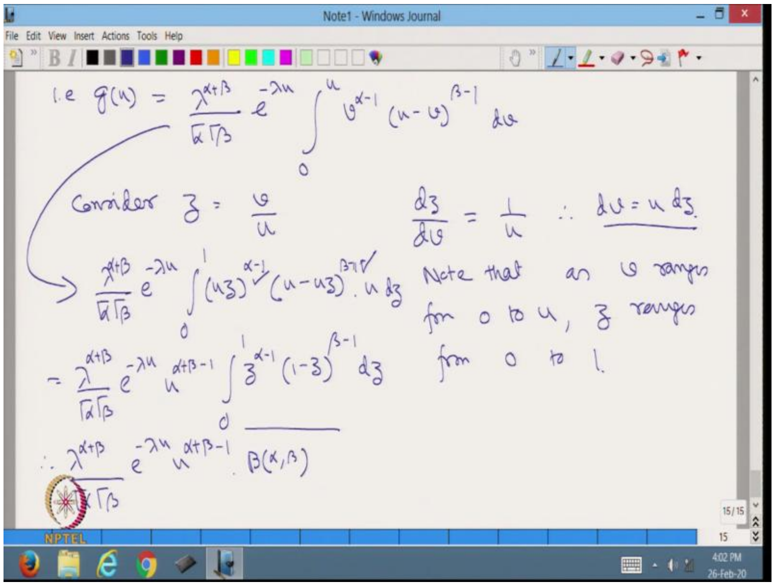
(Refer Slide Time: 28:31)



Which is equal to del v del u del v del v del u minus v del u del u minus v del v. Since, X is equal to v and Y is equal U minus V is equal 0 1 1 and minus 1. Therefore, determinant is minus 1 and therefore absolute value of J is equal to 1. Therefore, g u, comma v is equal to lambda power alpha upon gamma alpha e to the power minus lambda v v to the power alpha minus 1. This we get from the distribution of x multiplied by lambda power beta upon gamma beta e to the power minus lambda u minus v into u minus v to the power beta minus 1.

Note that, V ranges from 0 to u since u is equal to x plus y. Therefore value of x that is v cannot be more than u. So, this is the most important thing therefore we are integrating it 0 to u lambda power alpha upon gamma alpha e to the power minus lambda v v to the power alpha minus 1 multiplied by lambda power beta upon gamma beta e to the power minus lambda u minus v, u minus v to the beta minus 1 this multiplied by 1 which is the Jacobian this multiplied by dv. Therefore, this is the integration that we have to carry out.

(Refer Slide Time: 31:32)

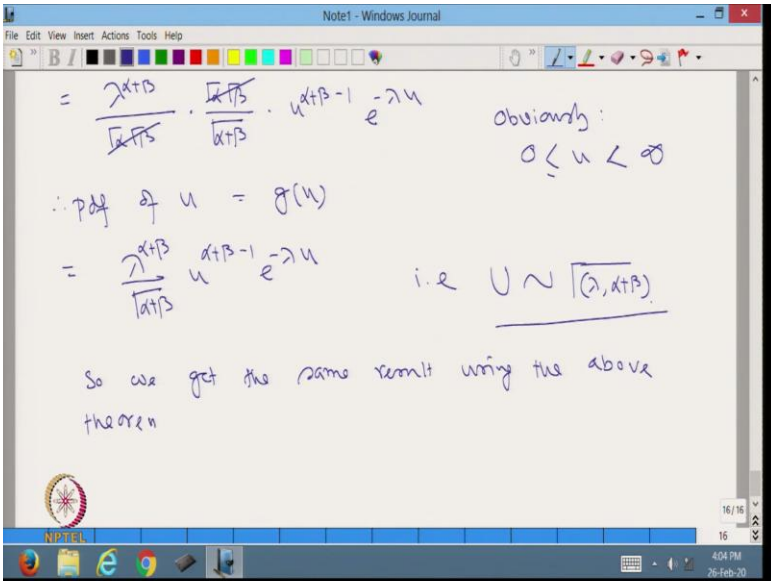


That is g of u is equal now we take out lambda power alpha plus beta we take out gamma alpha, gamma beta, we take e to the power minus lambda u as that is independent of v. Therefore, we are left with integration 0 to u v to the power alpha minus 1 u minus v to the power beta minus 1 dv. Consider z is equal v by u therefore, dz dv is equal to 1 upon u therefore dv is equal u dz. So, this is what we are going to use to compute the integral.

Note that as v ranges from 0 to u, z ranges from 0 to 1. Therefore, this buzz down to lambda power alpha plus beta upon gamma alpha gamma beta e to the power minus lambda u integration 0 to 1 u z to the power alpha minus 1 into u minus u z to the power beta minus 1 multiplied by u, comma u dz is equal to lambda power alpha plus beta gamma alpha gamma beta e to the power minus lambda u, now power of u is u to the power alpha minus 1 beta minus 1 plus u.

Therefore, we out u to the power alpha plus beta minus 1 integration 0 to 1 z to the power alpha minus 1 1 minus z to the power beta minus 1 dz this quantity as we can understand is going to be beta with alpha beta. Therefore, we have lambda to the power plus beta upon gamma alpha gamma beta e to the power minus lambda u, u to the power alpha plus beta minus 1 into beta alpha, comma beta.

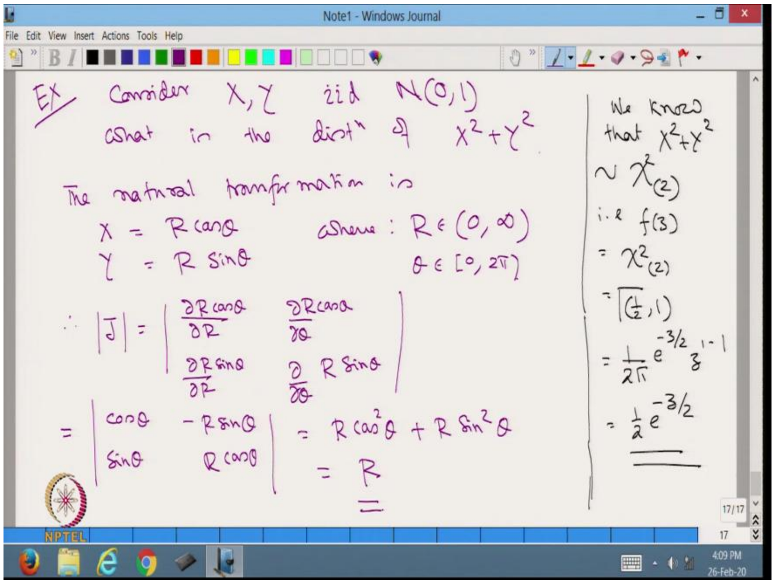
(Refer Slide Time: 34:46)



Is equal to the power alpha plus beta upon gamma alpha gamma beta now beta alpha, beta we can write it as gamma alpha gamma beta upon gamma alpha plus beta multiplied by u to the power alpha plus beta minus 1 and e to the power minus lambda u. Obviously 0 less than equal to u less than infinity.

Therefore, pdf of u is equal to gu is equal to lambda power alpha plus beta upon gamma alpha plus beta u to the power alpha plus beta minus 1 into e to the power minus lambda u that is, u is distributed as gamma with parameter lambda and alpha plus beta. So, we get the same result using the above theorem.

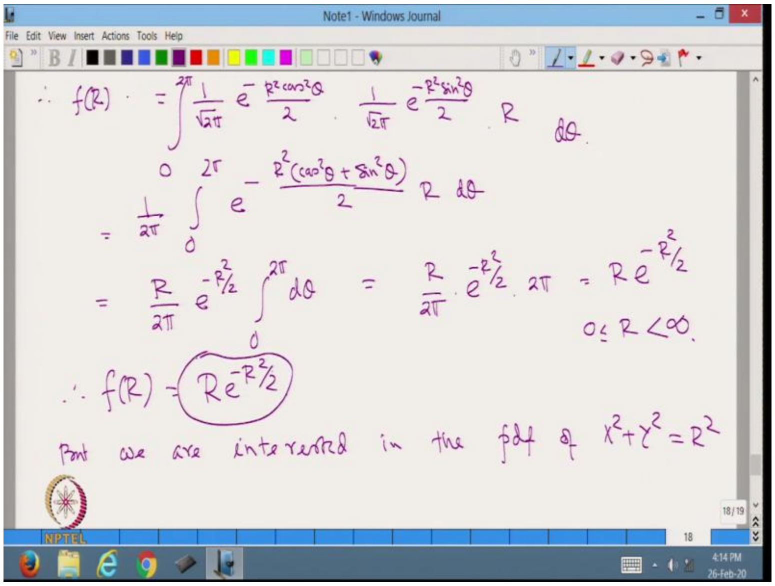
(Refer Slide Time: 36:19)



Another example, consider X and yY to be iid normal 0, 1. What is the distribution of X square plus Y square? We know that X square plus Y square is distributed as chi square with 2 degrees of freedom that is, f of z is equal to chi square with 2 is equal to gamma with half, comma 1 and gamma with half, comma 1 is equal to lambda power alpha gamma alpha e to the power minus lambda x x to the power alpha minus 1. Therefore, this is going to be half e to the power minus z by 2. So, we know the pdf anyway, let us now solve this problem using the above method.

The natural transformation is X is equal to R cos theta and Y is equal to R sin theta where, R belongs to 0 to infinity and theta belongs to 0 to 2 pi. Therefore, Jacobean is equal to del R cos theta del R del R cos theta del theta del R sin theta del R del del theta of R sin theta is equal to determinant of the matrix cos theta minus R sin theta sin theta R cos theta is equal to R cos square theta plus R sin square theta is equal to R. So, this is the Jacobean.

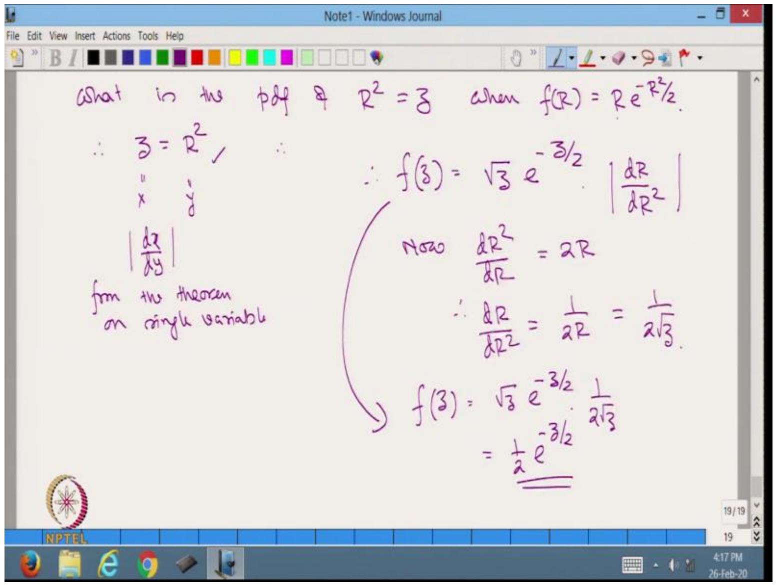
(Refer Slide Time: 39:18)

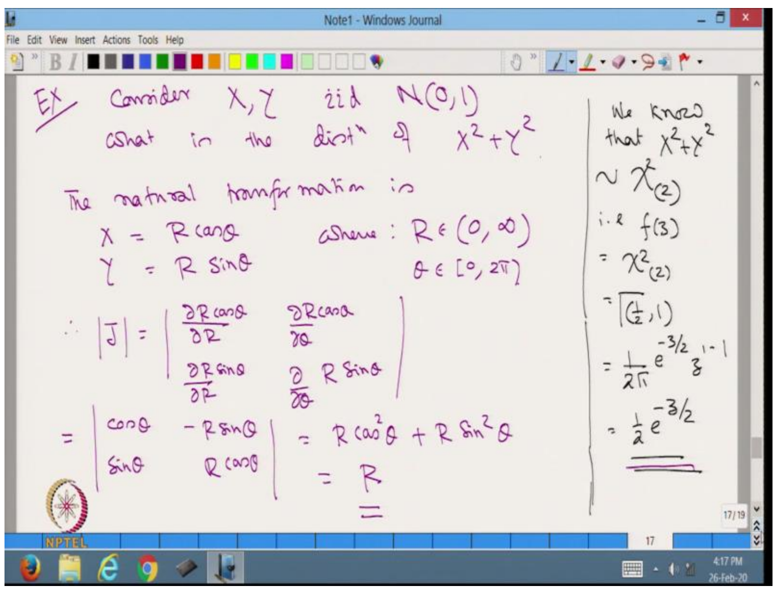


Therefore, f of R is equal to 1 over root over 2 pi e to the power minus R square cos square theta by 2 multiplied by 1 over root over 2 pi e to the power minus R square multiplied by the Jacobean which is R. And then I am integrating it for the range 0 to 2 pi with respect to theta is equal to 1 over 2 pi integration 0 to 2 pi e to the power minus R square into cos square theta plus sin square theta divided by 2 R d theta is equal to R by 2 pi e to the power minus R square by 2 integration 0 to 2 pi d theta is equal to R by 2 pi e to the power minus R square by 2 into 2 pi is equal to R e to the power minus R square by 2 when 0 less than equal to R less than infinity.

Therefore, f of R is equal to R e to the power minus R square by 2. But, we are interested in the pdf of X square plus Y square is equal to R square. And what we got is the pdf of R.

(Refer Slide Time: 41:41)

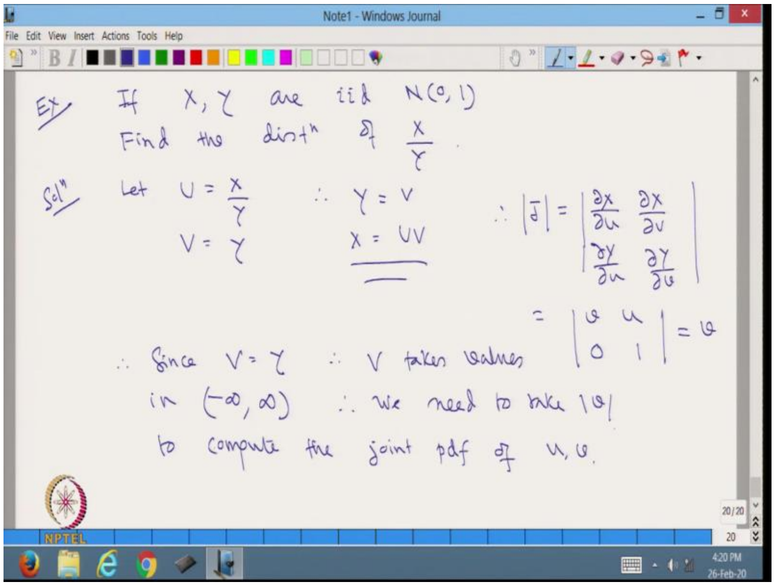




Therefore, what is the pdf of R square is equal to z when f of R is equal to R e to the power minus R square by 2. Therefore, z is equal to R square, therefore if we call it x and if we call it y then we need to compute dx dy from the theorem on single variable. Therefore, f of z is equal to root over z because R is equal to root over R square e to the power minus z by 2 into dR dR square.

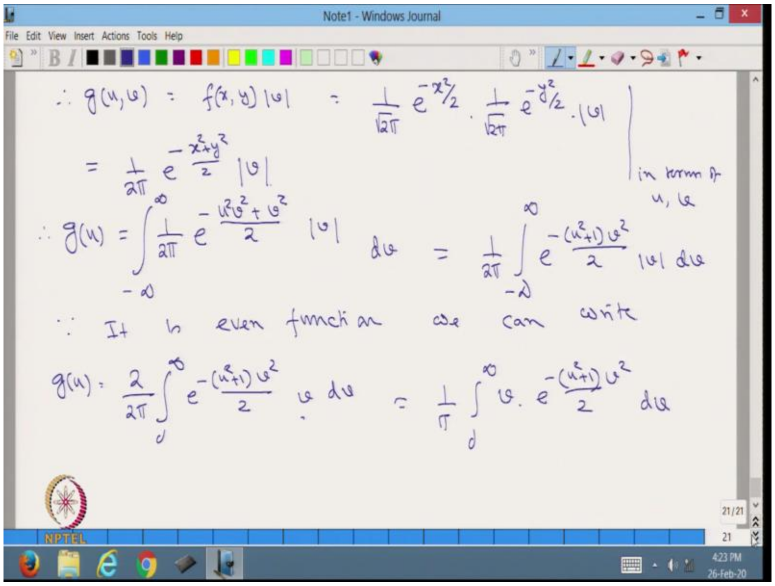
Now, dR square dR is equal to 2R, therefore dR dR square is equal to 1 upon 2R which is is equal to 1 upon 2 root z. Therefore, putting here we get f of z is equal to root over z e to the power minus z by 2 into 1 upon 2 root z is equal to half e to the power minus z by 2. Go back to that result we obtained, we see that we have got the same result here.

(Refer Slide Time: 43:50)



Another example, if X and Y are iid normal 0, 1. Find the distribution of X upon Y. So as before, let Y is equal to X by Y and let V is equal to Y, therefore Y is equal to V and X is equal to UV. So, this is the inverse transformation. Therefore, Jacobean is equal to determinant of del x del u del x del v del y del u del y del v is equal to v u 0 and 1 is equal to v. Therefore, since, V is equal to Y, therefore V takes values in minus infinity to infinity, therefore we need to take modulus of v to compute the joint pdf of u and v.

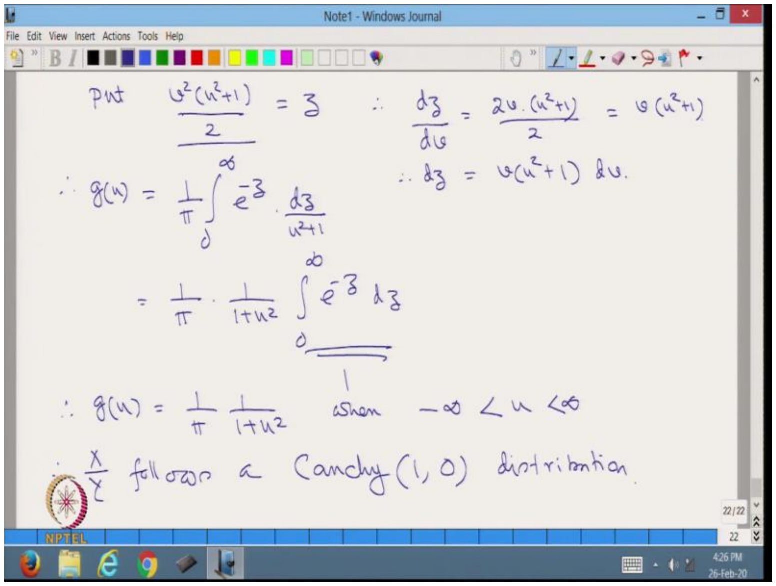
(Refer Slide Time: 45:46)



Therefore, g u, v is equal to f at x, y multiplied by modulus of v is equal to 1 over root over 2 pi e to the power minus x square by 2 multiplied by 1 over root over 2 pi e to the power minus y square by 2 multiplied by modulus of v in terms of u and v. Is equal to 1 over 2 pi e to the power minus x square plus y square by 2 into modulus of v. Therefore, gu is equal to 1 over 2 pi e to the power minus x is equal to u v, therefore u square v square plus y is equal to v therefore v square divided by 2 into modulus of v.

This we are integrating from minus infinity to infinity with respect to v, which is is equal to 1 over 2 pi minus infinity to infinity e to the power minus u square plus 1 into v square by 2 modulus of v dv. Since, it is an even function, we can write gu is equal to 2 times 0 to infinity e to the power minus u square plus 1 into v square by 2 into v dv, because v here is always positive, is equal to 1 upon pi integration 0 to infinity v into e to the power minus u square plus 1 v square by 2 dv.

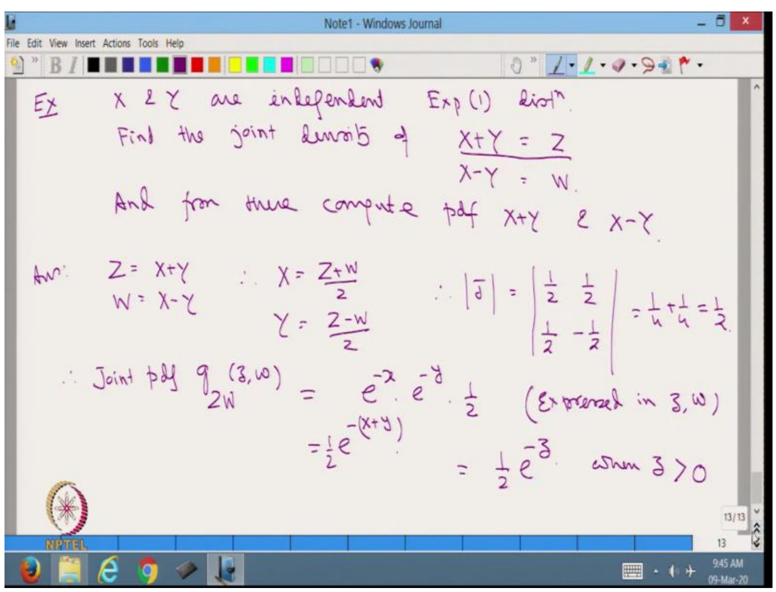
(Refer Slide Time: 48.47)



Put v square into u square plus 1 by 2 is equal to z. Therefore, dz dv is equal to 2 v into u square plus 1 upon 2 is equal to v into u square plus 1. Therefore, dz is equal to v into u square plus 1 into dv. Therefore, gu is equal to 1 over pi integration 0 to infinity e to the power minus z. I am replacing z in the equation and v dv we are writing at dz upon u square plus 1 is equal to 1 upon pi into 1 upon 1 plus u square integration 0 to infinity e to the power minus z dz.

This integrates to 1, therefore g of u is equal to 1 upon pi into 1 upon 1 plus u square when minus infinity less than u less than infinity. Therefore, X by Y follows a Cauchy distribution.

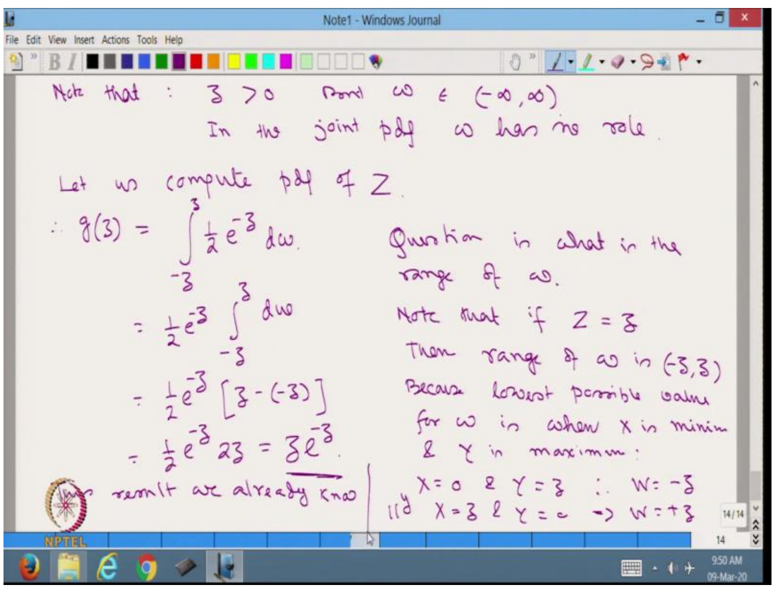
(Refer Slide Time: 50:55)



Let us now consider another example, X and Y are independent, exponential 1 distribution, find the joint density of X plus Y, so let us call it Z and X minus Y let us call it W. And from there compute pdf of X plus Y and X minus Y. So, let us move as follows. Let so Z is equal to X plus Y, W is equal to X minus Y therefore X is equal to Z plus W by 2 and Y is equal to Z minus W by 2, therefore Jacobean is equal to del x del Z del W del x del W del y del Z del y del W is equal to 1 by 4 plus 1 by 4 is equal to half.

Therefore, joint pdf g ZW of zw is equal to e to the power minus x into e to the power minus y into half expressed in zw is equal to e to the power minus x plus y into half is equal to half e to the power minus z because x plus y is equal to z when z is greater than 0. Because it is the summation of two exponential random variables.

(Refer Slide Time: 53:36)

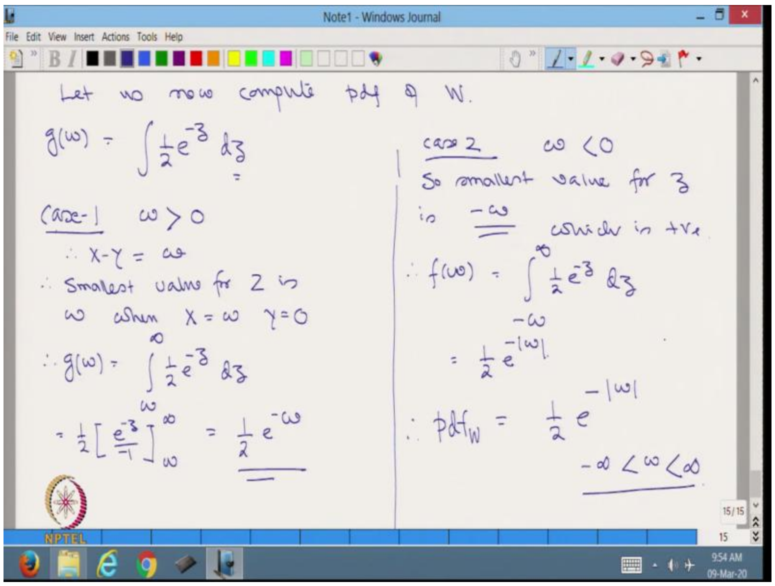


Note that, z is greater than 0 but, w belongs minus infinity to plus infinity and also note that in the joint pdf w has no role. So, let us compute pdf of Z therefore gz is equal to integration of half e to the power minus z with respect to w. Question is, what is the range of w? Note that, if Z is equal to z, then range of w is minus z to z, because lowest possible value for W is when X is minimum and Y is maximum.

Therefore, if X is equal to 0 and Y is equal to z we get the value z therefore W is equal to minus z. Similarly, X is equal to z and Y is equal to 0 implies W is equal to plus z. Hence, the limit of integration is equal to minus z to plus z of half e to the power minus z is equal to half e to the power minus z minus z to plus z dw is equal to half e to the power minus z z minus minus z is equal to half into e to the power minus z into 2z is equal to z e to the power minus z.

This in any case we knew because the sum of two exponential is going to be gamma with lambda and parameter alpha is equal to 2. So, this result we already know.

(Refer Slide Time: 57:01)



Let us now compute pdf of W, therefore gw is equal to integration of half e to the power minus z to dz. So, we are now integrating out z. Case 1, w is greater than 0, therefore X minus Y is equal to w, therefore smallest value for Z is w when X is equal to w and Y is equal to 0.

Therefore, gw is equal to integration w to infinity half e to the power minus z dz is equal to half e to the power minus z minus 1 infinity w, at infinity it is going to be 0. So, what we get is half e to the power minus w. Case 2, w is less than 0 so smallest value for z is minus w which is positive. Therefore, fw is equal to integration minus w to infinity half e to the power minus z dz. And in a similar way we get half of e to the power minus mod of w.

Therefore, pdf of W is equal to half e to the power minus mod w when minus infinity less than w less than infinity. This is called double exponential distributions. Okay friends, I stop here today. So, in today's class we have seen how to obtain the density function for an arbitrary function of two random variables. In the next class I shall continue with that.

And I shall introduce you to some important pdfs namely t and f distribution. Okay then, till then thank you.