Advanced Probability Theory
Professor Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology Delhi
Lecture 29

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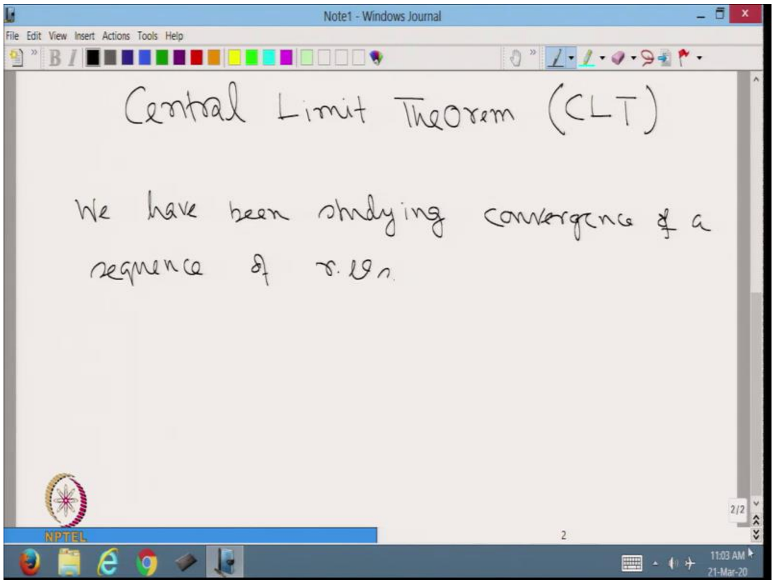

Welcome students to the mock series of lectures on Advanced Probability Theory, this is lecture number 29. As I said at the end of the last class that today we shall be studying Central Limit Theorem, a very important concept of probability which in short we call CLT. Over the last few lectures we have been studying convergence of a sequence of random variables.

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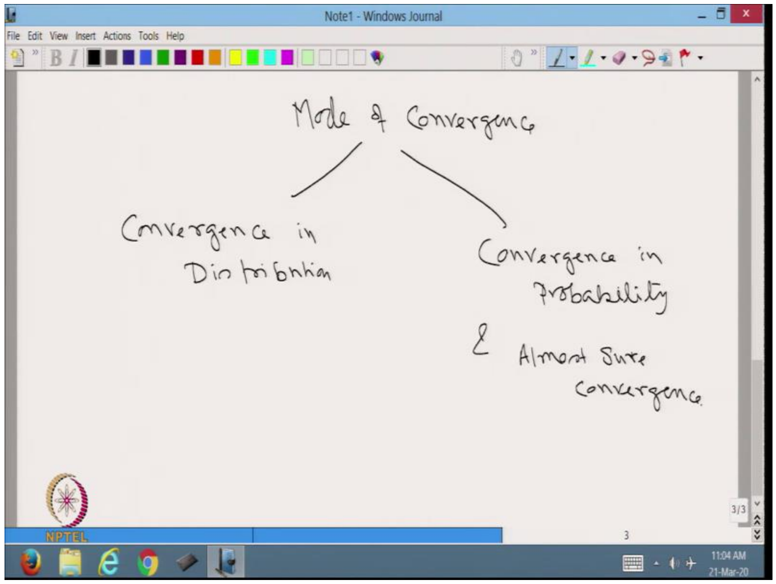

So, first we studied mode of convergence, here we looked at convergence in distribution and also we have looked at convergence in probability and almost sure convergence.

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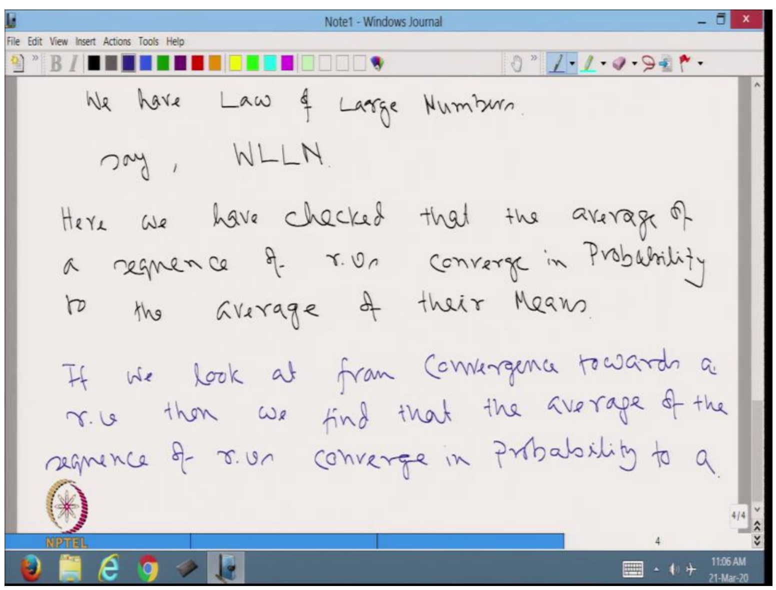

If we look at your application, we have seen laws of large numbers in particular say weak law of large numbers, here we have checked that the average of a sequence of random variables converge in probability to the average of their means. If we look at it from convergence towards a random variables, random variable then we find that the average of the sequence of the random variables converge in probability to a degenerate random variable.

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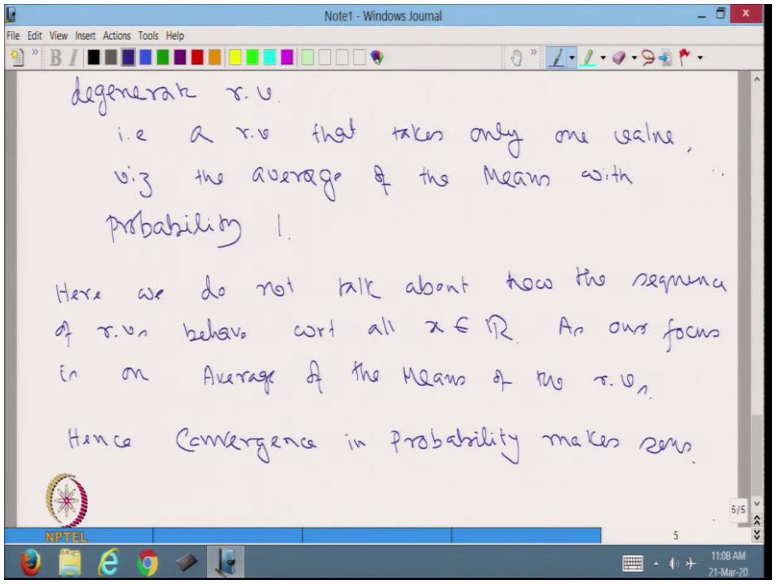

That is, a random variable that takes only one value namely the average of the means with probability 1. Here we do not talk about how the sequence of random variables behave with respect to all x belonging to R as our focus is on average of the means of the random variables. Hence convergence in probability makes sense.

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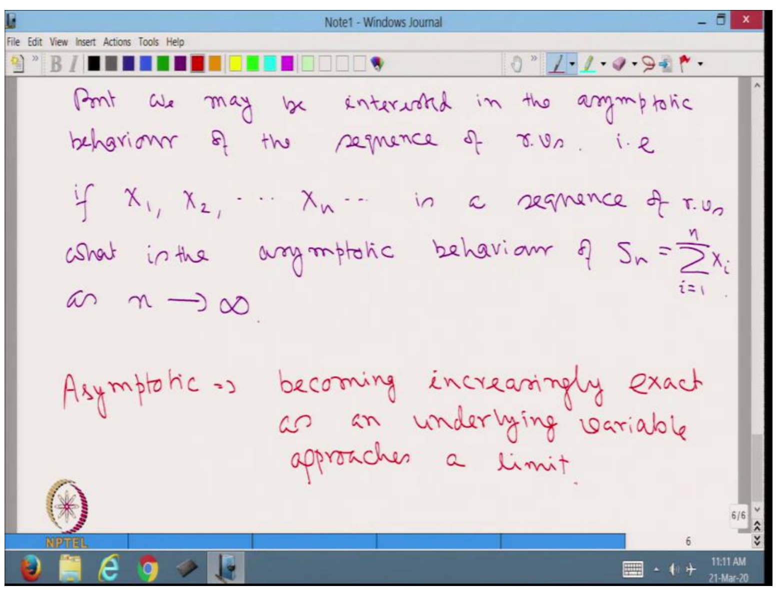

But we may be interested in the asymptotic behaviour of the sequence of random variables that is, if X1, X2, Xn is a sequence of random variable, what is the asymptotic behaviour of Sn is equal to sigma Xi i is equal to 1 to n as n goes to infinity. Now, some of you may not know the meaning of the word asymptotic, so here I define for you, asymptotic means becoming increasingly close, increasingly exact as an underlying variable approaches a limit.

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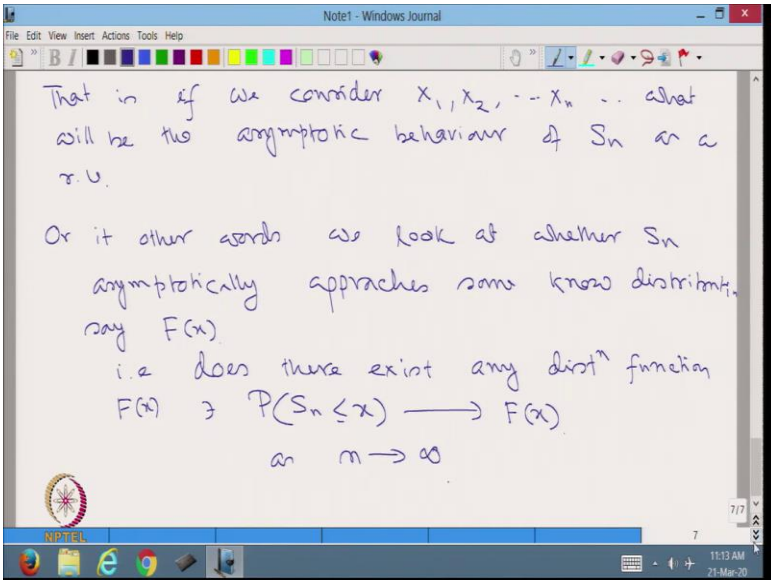

That is if we consider X1, X2, Xn, what will be the asymptotic behaviour of Sn as a random variable? Or in other words we look at whether Sn asymptotically approaches some known distribution say Fx that is does there exist any distribution function Fx such that probability Sn less than equal to X converges to Fx as n goes to infinity.

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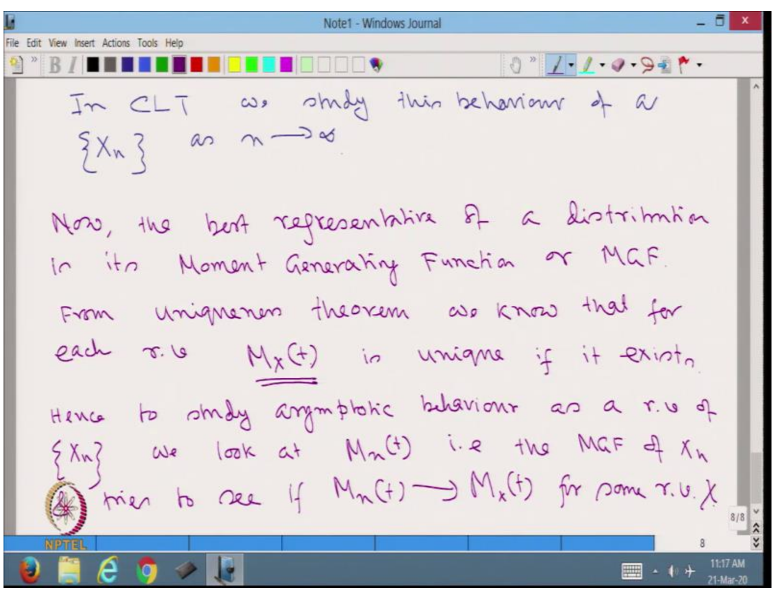

In central limit theorem we study this behaviour of a sequence of random variable Xn as n goes to infinity. Now, the best representative of a distribution is its moment generating function or MGF, from uniqueness theorem we know that for each random variable X MX t that is moment generating function of X at t is unique if it exists. Hence to study asymptotic behaviour as a random variable of Xn is sequence of random variables we look at Mn t that is the MGF of Xn and tries to see if Mn t converges to MX t for some random variable X.

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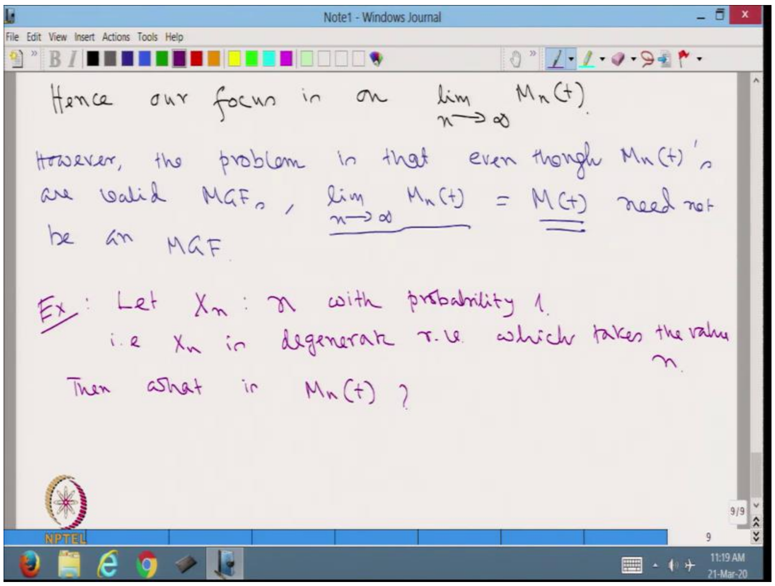

Hence, our focus is on limit n going to infinity Mn t. However, the problem is that even though Mn t’s are valid moment generating functions limit n going to infinity Mn t is equal to Mt need not be an MGF. That is the sequence of moment generating functions converging to something which is not a moment generating function itself. Example, let Xn with distributed as follows it takes the value n with probability 1, that is Xn is a degenerate random variable, which takes the value n. Then what is Mn t?

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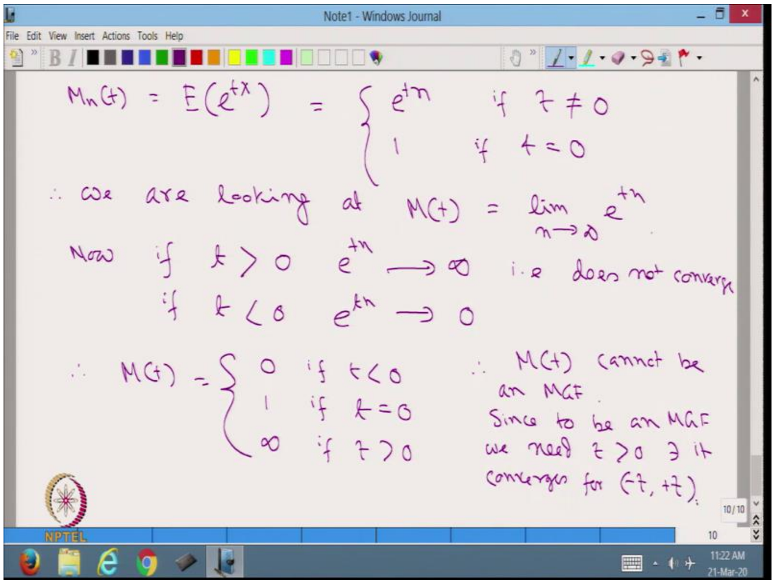

Mn t is equal to expected value of e to the power tX equal to e to the power tn if t not equal to 0 and is equal to 1 if t is equal to 0, therefore we are looking at Mt is equal to limit n going to infinity e to the power tn. Now, if t is greater than 0, e to the power tn goes to infinity that is does not converge and if t is less than 0, e to the power tn goes to 0. Therefore, Mt is equal to 0 if t is less than 0, 1if t is equal to 0 and infinity if t is greater than 0, therefore Mt cannot be an MGF, since to be an MGF we need t greater than 0 such that it converges for minus t to plus t.

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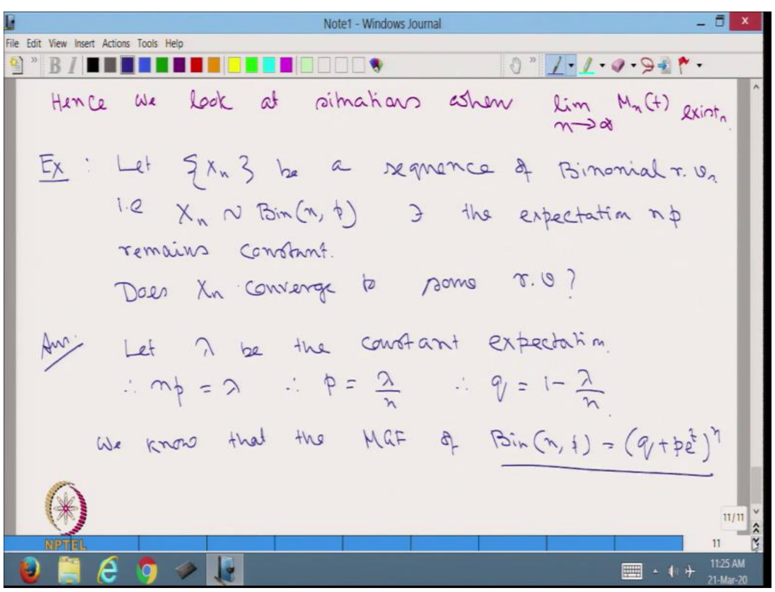

Hence, we look at situations when limit n going to infinity Mn t exist. Example, let Xn be a sequence of binomial random variables that is Xn is distributed as binomial n comma p such that the expectation np remains constant. Does Xn converge to some random variable? That is the question. So, let lambda be the constant expectation, therefore np is equal to lambda, therefore p is equal to lambda by n, therefore q is equal to 1 minus lambda n. We know that the MGF of binomial n, p is equal to q plus p e to the power t whole to the power n.

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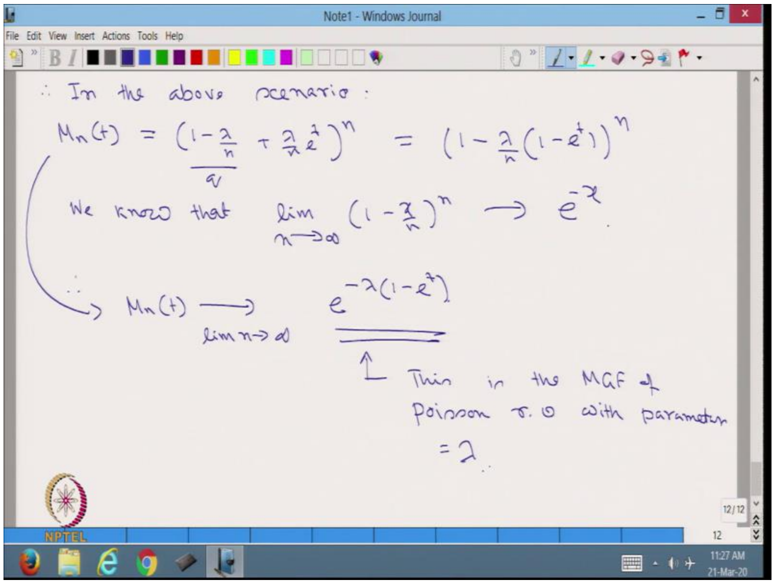

Therefore, in the above scenario Mn t is equal to 1 minus lambda by n that is q plus lambda by n e to the power t whole to the power n is equal to 1 minus lambda by n into 1 minus e to the power t whole to the power n, we know that limit n going to infinity 1 minus x by n whole to the power n converges to e to the power minus x, therefore this Mn t converges to limit n going to infinity e to the power minus lambda into 1 minus e to the power t, this is the MGF of Poisson random variable with parameter is equal to lambda.

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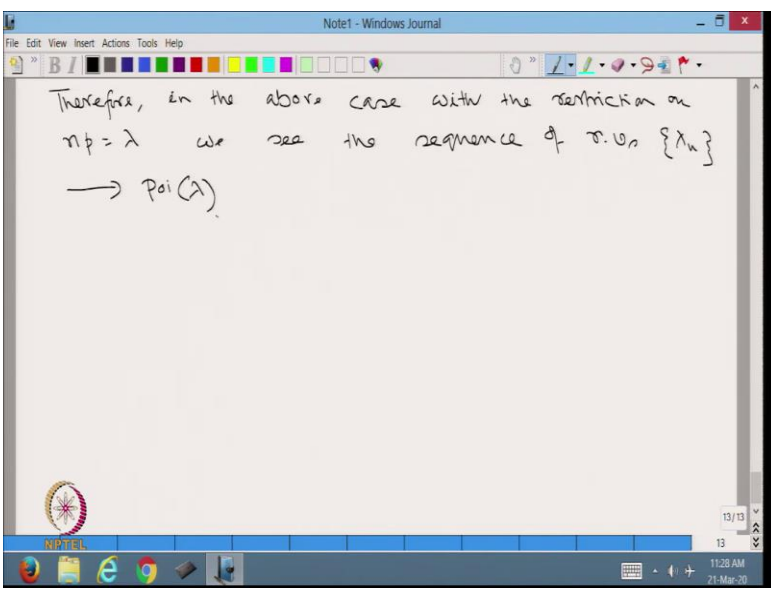

Therefore, in the above case with the restriction on np is equal to lambda we see the sequence of random variables Xn converges to Poisson with lambda.

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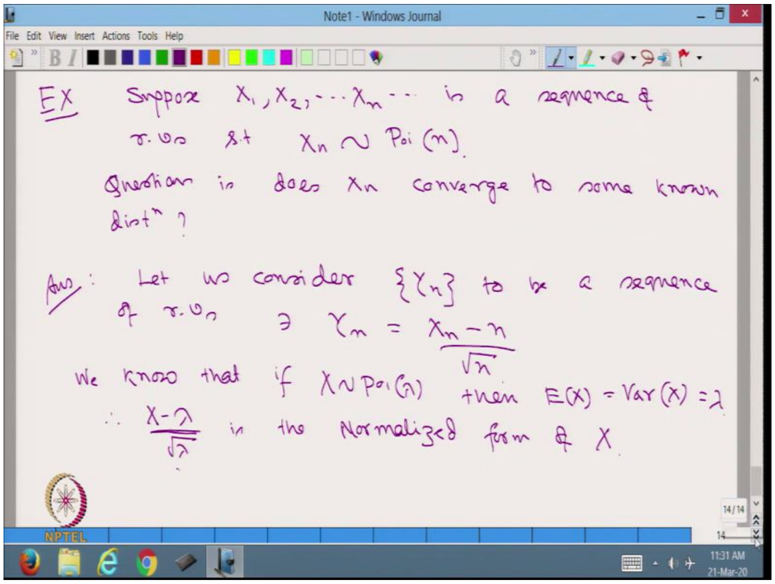

Let us, now consider another example, suppose X1, X2, Xn is a sequence of random variables such that Xn is distributed as Poisson with parameter n, question is, does Xn converge to some known distribution? Answer, let us consider Yn to be a sequence of random variables such that Yn is equal to Xn minus n over root n. We know that if X is distributed as Poisson with lambda, then expectation of X is equal to variance of X is equal to lambda. Therefore, x minus lambda over upon root over lambda is the normalized form of X. Because it is variable minus mean divided by standard deviation.

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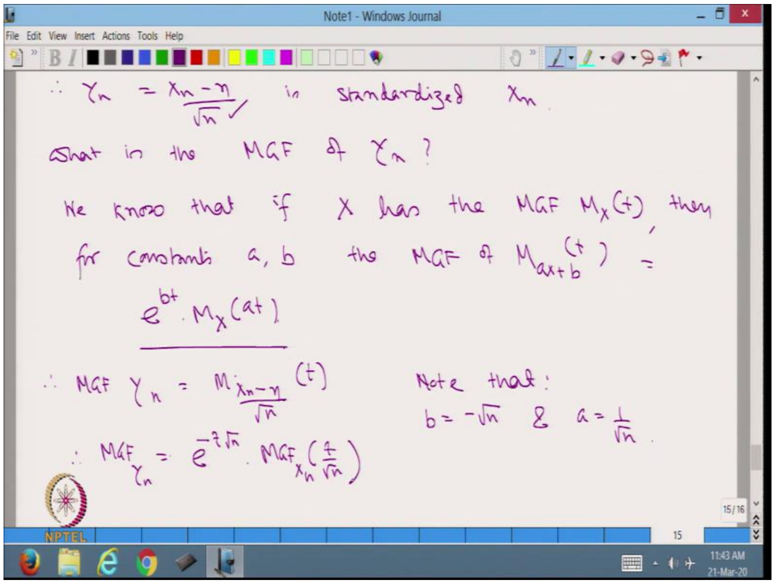

Therefore, Yn which is equal to Xn minus n over root over n is standardized or normalized Xn. What is the MGF of Yn? We know that, if X has the moment generating function Mx t, then for constants a and b the MGF of ax plus b at t is equal to e to the power bt into MG of X at the point a t, therefore MGF of Yn is equal to moment generating function of Xn minus n upon root n at t, note that b is equal to minus root n and a is equal to 1 upon root n, therefore MGF of Yn is equal to e to the power minus t root n into MGF of Xn at t upon root n.

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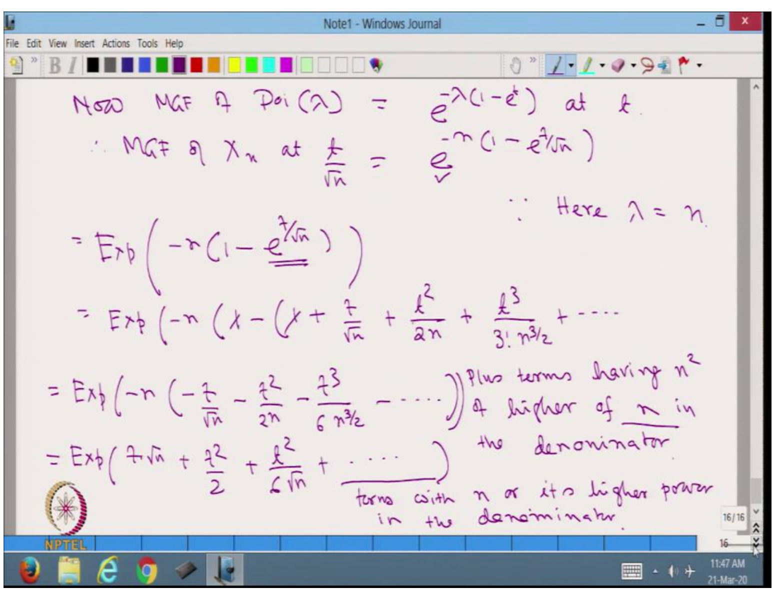

Now, MGF of Poisson random variable with parameter lambda is equal to e to the power minus lambda into 1 minus e to the power t at t. Therefore, MGF of Xn at t over root n is equal to e to the power minus n into 1 minus e to the power t by root n, since here lambda is equal to n, then is equal to exponentiation of, I am writing this as exponentian function minus n into 1 minus e to the power t by root n is equal to exponentiation of minus n into 1 minus, now let us expand e to the power t upon root n, this is equal to 1 plus t upon root n plus t square upon 2n plus t cube upon factorial 3 n to the power 3 by 2 plus term having n square or higher power of n in the denominator is equal to exponentiation of minus n into this one cancels with this, therefore minus t by root n minus t square upon 2n minus t cube upon 6 n to the power 3 by 2 minus terms with higher power of n in the denominator is equal to exponentiation of t root n plus t square upon 2 plus t cube upon 6 square root of n plus terms with n or its higher power in the denominator.

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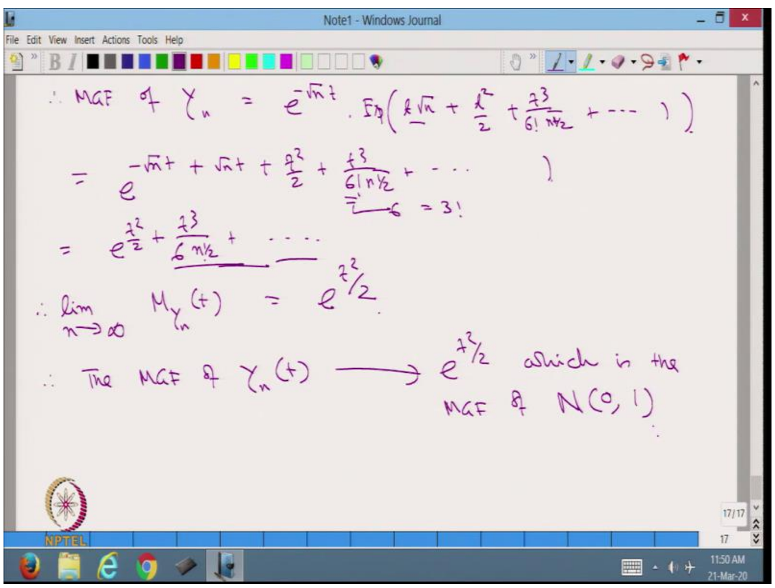

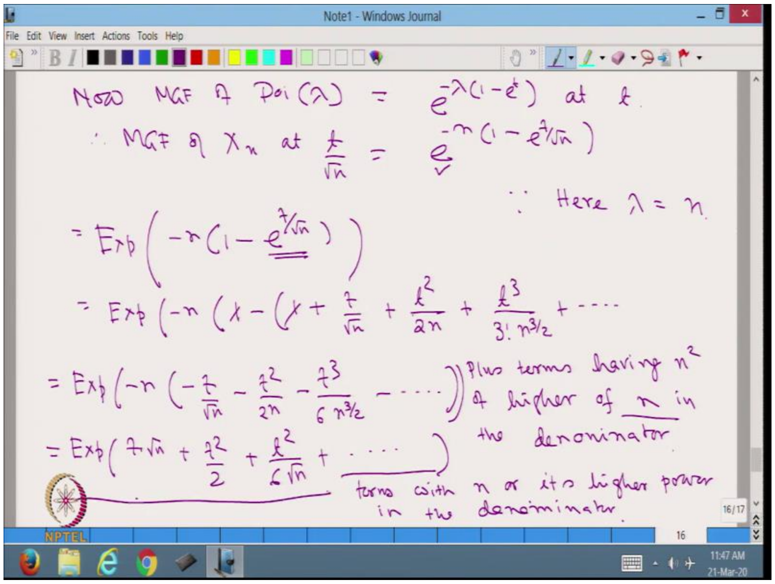

Therefore, MGF of Yn is equal to e to the power minus root n into t multiplied by exponentiation of this term that is t root n plus t square upon 2 plus t cube upon 6 into n to the power half plus terms with higher power of n in the denominator, is equal to e to the power minus root n t plus root n t plus t square by 2 plus t cube upon 6 n to the power half plus other terms is equal to e to the power t square by 2 plus t cube into 6 this is 6 equal to factorial 3, n to the power half plus other terms. Therefore, limit n going to infinity M Yn at t is equal to e to the power t square by 2, because all these terms are going to 0 as n goes to infinity. Therefore, the MGF of Yn t converges to e to the power t square by 2, which is the MGF of standard normal with parameter 0, 1.

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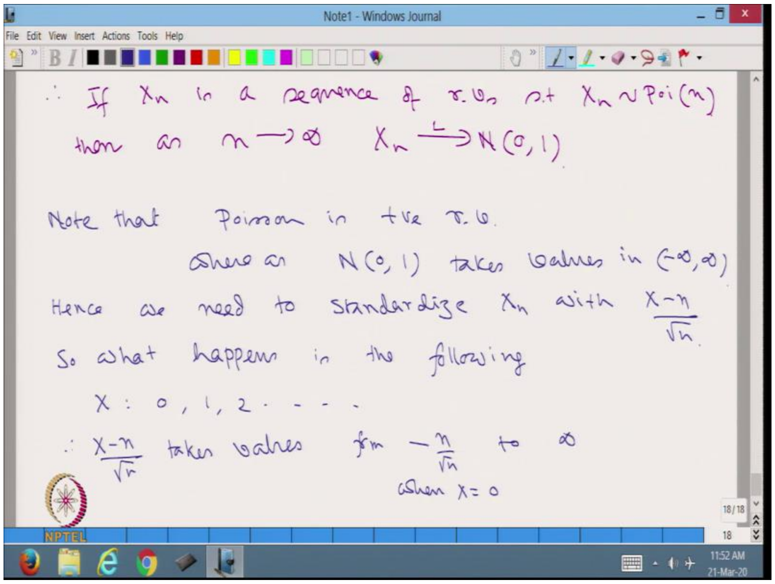

Therefore, if Xn is a sequence of random variables, such that Xn is distributed as Poisson with parameter n than as n goes to infinity Xn converges in distribution to normal 0, 1. Note that, Poisson is a positive random variable, whereas normal 0, 1 takes values in minus infinity to plus infinity. Hence, we need to standardize or normalize Xn with Xn minus n upon root over n, So, what happens is the following, X takes values 0, 1, 2, etcetera, therefore X minus n upon root n takes values from minus n upon root n when X is equal to 0 to infinite.

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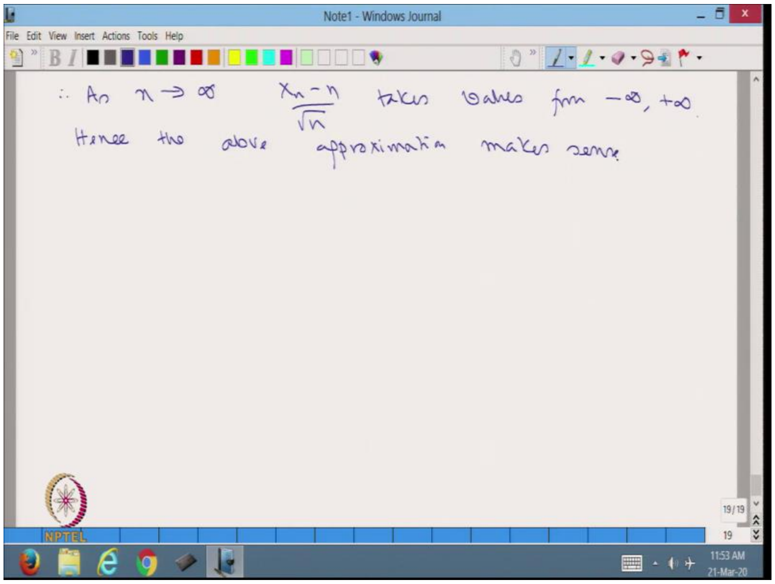

Therefore, as n goes to infinity Xn minus n upon root n takes values from minus infinity to infinity, hence the above approximation makes sense.

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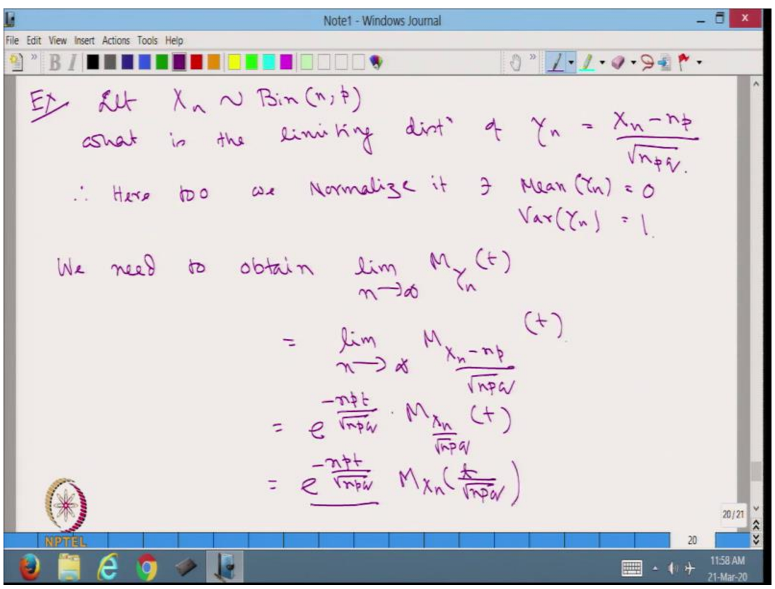

Another example, let Xn is distributed as binomial n, p, what is the limiting distribution of Yn is equal to Xn minus np upon root over npq. Therefore, here too we normalize it such that mean of Yn is equal to 0 and variance of Yn is equal 1. Therefore, we need to obtain limit n going to infinity M Yn at a point t is equal to limit n going to infinity M Xn minus np upon root of over npq at the point t is equal to e to the power minus npt upon root over npq into M Xn upon root over npq at the point t is equal to e to the power minus npt upon root over npq into M Xn at t upon root over npq.

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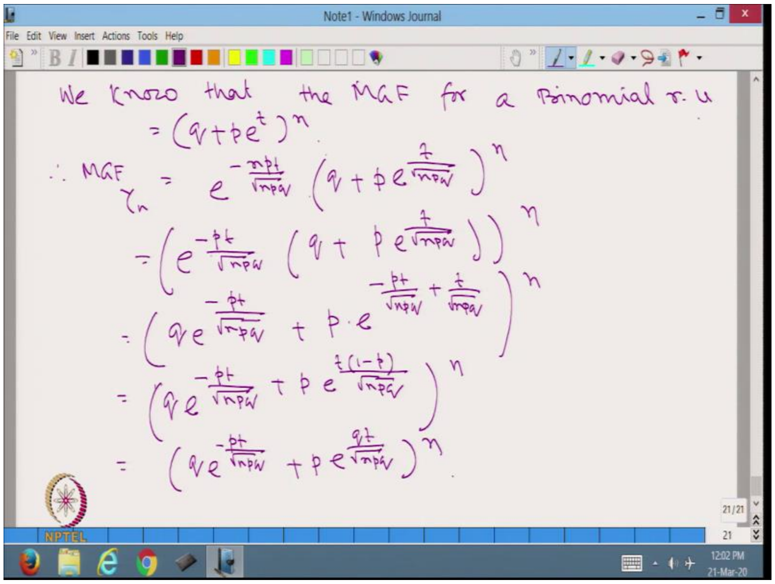

We know that, the moment generating function for a binomial random variable is equal to q plus p e to the power t whole to the power n. Therefore, MGF of Yn is equal to e to the power minus npt root over npq multiplied by q plus p e to the power t upon root over npq whole to the power n, is equal to e to the power minus pt root over npq multiplied by q plus p e to the power t upon root over npq whole to the power n is equal to q e the power minus pt root over npq plus p times e to the power minus pt root over npq plus t root over npq whole to the power n, is equal to q times e to the power minus pt upon root over npq plus p times e to the power t into 1 minus p upon root over npq whole to the power n, is equal to q times e to the power minus pt root over npq plus p times e to the power qt root over npq whole to the power n.

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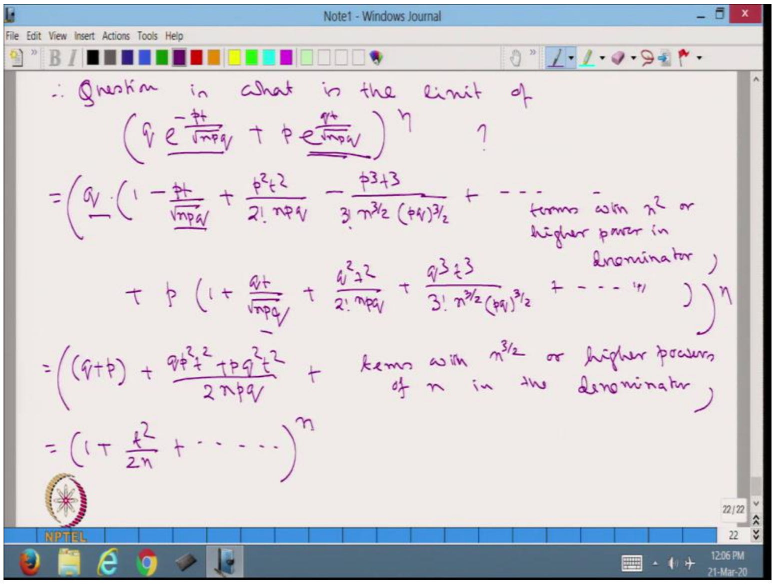

Therefore, question is, what is the limit of q times e to the power minus pt root over npq plus p times e to the power qt root over npq whole to the power n. So we do this by expending this exponential terms, therefore we are writing it as q times 1 minus pt upon root over npq plus p square t square upon 2 into npq minus p cube t cube upon n to the power 3 by 2 pq to the power 3 by 2 into factorial 3 plus terms with n square or higher power in denominator plus p times now we are expanding this, 1 plus qt upon root over npq plus q square t square upon factorial 2 into npq plus q cube t cube upon factorial 3 n to the power 3 by 2 pq to the power 3 by 2 plus like as before higher powers of n square in the denominator whole to the power n is equal to, now we are adding so we get q plus p minus qpt plus qpt, therefore they get cancelled plus q p square t square plus p q square t square upon 2 npq plus terms with n to the power 3 by 2 or higher powers of n in the denominator, is equal to 1 plus if we take pq common it is t square upon 2 and that pq cancels with this plus n plus other terms whole to the power n.

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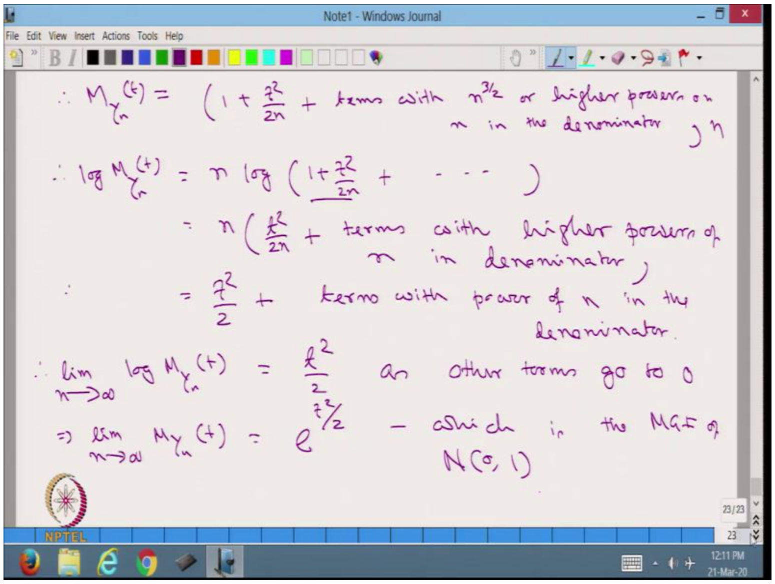

Therefore, moment generating function of Yn is equal to what we get from here is 1 plus t square upon 2n plus terms with n to the power 3 by 2 or higher powers of n in the denominator whole to the power n. Therefore, M Yn at t, here also there should have been a t is equal to n times log of 1 plus t square upon 2n plus such terms is equal to n times t square upon 2 by n plus terms with higher powers of n in denominator is equal to t square 2 plus terms with power of n in the denominator. Therefore, limit n going to infinity log of M Yn at t is equal to t square upon 2 as other terms go to 0, implies limit n going to infinity M yn to the power of t is equal to e to the power t square by 2 which is the MGF of standard normal 0, 1.

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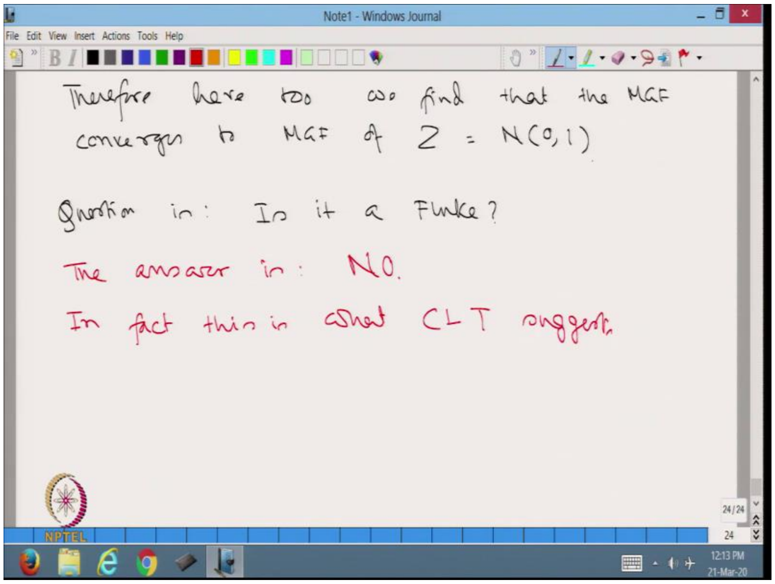

Therefore, here too we find that the MGF converges to MGF of Z which is is equal to normal 0, 1. Question is, is it a fluke? The answer is, no, in fact this is what central limit theorem suggest.

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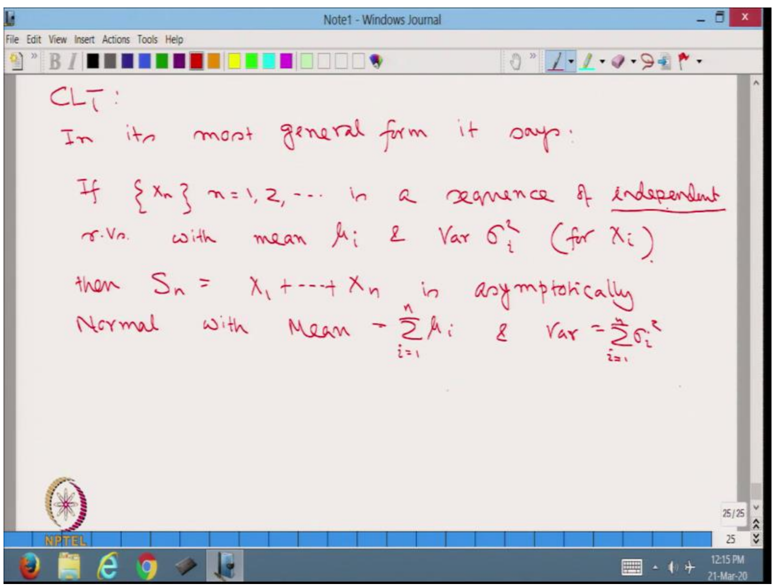

So, what is central limit theorem? In its most general form, it says if Xn n is equal to 1, 2 etcetera is a sequence of independent random variables with mean mu i and variance sigma square i for Xi then Sn is equal to X1 plus X2 up to Xn is asymptotically normal with mean is equal to sigma mu i and variance is equal to summation sigma i square i is equal to 1 to n, here also i is equal to .1 to n. So, this is the most general form when the only assumption is that X1, X2, Xn are independent. In this class we shall not prove this, what we should do that if they are not only independent, but identically distributed then same result holds. Okay students, I stop here today, in the next class I shall start with this statement. Thank you.