

Considering the interpolation polynomial of degree $n + 1$ on x_0, x_1, \dots, x_n, x , we have

$$f(x) = P_{n+1}(x) = P_n(x) + f[x_0, x_1, \dots, x_n, x](x - x_0) \cdots (x - x_n).$$

21. Let i_0, i_1, \dots, i_n be a rearrangement of the integers $0, 1, \dots, n$. Show that $f[x_{i_0}, x_{i_1}, \dots, x_{i_n}] = f[x_0, x_1, \dots, x_n]$. [Hint: Consider the leading coefficient of the n th Lagrange polynomial on the data $\{x_0, x_1, \dots, x_n\} = \{x_{i_0}, x_{i_1}, \dots, x_{i_n}\}$.]

3.4 Hermite Interpolation

The Latin word *osculum*, literally a “small mouth” or “kiss”, when applied to a curve indicates that it just touches and has the same shape. Hermite interpolation has this osculating property. It matches a given curve, and its derivative forces the interpolating curve to “kiss” the given curve.

Osculating polynomials generalize both the Taylor polynomials and the Lagrange polynomials. Suppose that we are given $n + 1$ distinct numbers x_0, x_1, \dots, x_n in $[a, b]$ and nonnegative integers m_0, m_1, \dots, m_n , and $m = \max\{m_0, m_1, \dots, m_n\}$. The osculating polynomial approximating a function $f \in C^m[a, b]$ at x_i , for each $i = 0, \dots, n$, is the polynomial of least degree that has the same values as the function f and all its derivatives of order less than or equal to m_i at each x_i . The degree of this osculating polynomial is at most

$$M = \sum_{i=0}^n m_i + n$$

because the number of conditions to be satisfied is $\sum_{i=0}^n m_i + (n + 1)$, and a polynomial of degree M has $M + 1$ coefficients that can be used to satisfy these conditions.

Definition 3.8

Let x_0, x_1, \dots, x_n be $n + 1$ distinct numbers in $[a, b]$ and for $i = 0, 1, \dots, n$ let m_i be a nonnegative integer. Suppose that $f \in C^m[a, b]$, where $m = \max_{0 \leq i \leq n} m_i$.

The **osculating polynomial** approximating f is the polynomial $P(x)$ of least degree such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}, \quad \text{for each } i = 0, 1, \dots, n \quad \text{and} \quad k = 0, 1, \dots, m_i. \quad \blacksquare$$

Note that when $n = 0$, the osculating polynomial approximating f is the m_0 th Taylor polynomial for f at x_0 . When $m_i = 0$ for each i , the osculating polynomial is the n th Lagrange polynomial interpolating f on x_0, x_1, \dots, x_n .

Hermite Polynomials

The case when $m_i = 1$, for each $i = 0, 1, \dots, n$, gives the **Hermite polynomials**. For a given function f , these polynomials agree with f at x_0, x_1, \dots, x_n . In addition, since their first derivatives agree with those of f , they have the same “shape” as the function at $(x_i, f(x_i))$ in the sense that the *tangent lines* to the polynomial and the function agree. We will restrict our study of osculating polynomials to this situation and consider first a theorem that describes precisely the form of the Hermite polynomials.

Theorem 3.9

If $f \in C^1[a, b]$ and $x_0, \dots, x_n \in [a, b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at x_0, \dots, x_n is the Hermite polynomial of degree at most $2n + 1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^n f(x_j) H_{n,j}(x) + \sum_{j=0}^n f'(x_j) \hat{H}_{n,j}(x),$$

Charles Hermite (1822–1901) made significant mathematical discoveries throughout his life in areas such as complex analysis and number theory, particularly involving the theory of equations. He is perhaps best known for proving in 1873 that e is transcendental, that is, it is not the solution to any algebraic equation having integer coefficients. This led in 1882 to Lindemann’s proof that π is also transcendental, which demonstrated that it is impossible to use the standard geometry tools of Euclid to construct a square that has the same area as a unit circle.

Hermite gave a description of a general osculatory polynomial in a letter to Carl W. Borchardt in 1878, to whom he regularly sent his new results. His demonstration is an interesting application of the use of complex integration techniques to solve a real-valued problem.

where, for $L_{n,j}(x)$ denoting the j th Lagrange coefficient polynomial of degree n , we have

$$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)]L_{n,j}^2(x) \quad \text{and} \quad \hat{H}_{n,j}(x) = (x - x_j)L_{n,j}^2(x).$$

Moreover, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n + 2)!} f^{(2n+2)}(\xi(x)),$$

for some (generally unknown) $\xi(x)$ in the interval (a, b) . ■

Proof First recall that

$$L_{n,j}(x_i) = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Hence when $i \neq j$,

$$H_{n,j}(x_i) = 0 \quad \text{and} \quad \hat{H}_{n,j}(x_i) = 0,$$

whereas, for each i ,

$$H_{n,i}(x_i) = [1 - 2(x_i - x_i)L'_{n,i}(x_i)] \cdot 1 = 1 \quad \text{and} \quad \hat{H}_{n,i}(x_i) = (x_i - x_i) \cdot 1^2 = 0.$$

As a consequence

$$H_{2n+1}(x_i) = \sum_{\substack{j=0 \\ j \neq i}}^n f(x_j) \cdot 0 + f(x_i) \cdot 1 + \sum_{j=0}^n f'(x_j) \cdot 0 = f(x_i),$$

so H_{2n+1} agrees with f at x_0, x_1, \dots, x_n .

To show the agreement of H'_{2n+1} with f' at the nodes, first note that $L_{n,j}(x)$ is a factor of $H'_{n,j}(x)$, so $H'_{n,j}(x_i) = 0$ when $i \neq j$. In addition, when $i = j$ we have $L_{n,i}(x_i) = 1$, so

$$\begin{aligned} H'_{n,i}(x_i) &= -2L'_{n,i}(x_i) \cdot L_{n,i}^2(x_i) + [1 - 2(x_i - x_i)L'_{n,i}(x_i)]2L_{n,i}(x_i)L'_{n,i}(x_i) \\ &= -2L'_{n,i}(x_i) + 2L'_{n,i}(x_i) = 0. \end{aligned}$$

Hence, $H'_{n,j}(x_i) = 0$ for all i and j .

Finally,

$$\begin{aligned} \hat{H}'_{n,j}(x_i) &= L_{n,j}^2(x_i) + (x_i - x_j)2L_{n,j}(x_i)L'_{n,j}(x_i) \\ &= L_{n,j}(x_i)[L_{n,j}(x_i) + 2(x_i - x_j)L'_{n,j}(x_i)], \end{aligned}$$

so $\hat{H}'_{n,j}(x_i) = 0$ if $i \neq j$ and $\hat{H}'_{n,i}(x_i) = 1$. Combining these facts, we have

$$H'_{2n+1}(x_i) = \sum_{j=0}^n f(x_j) \cdot 0 + \sum_{\substack{j=0 \\ j \neq i}}^n f'(x_j) \cdot 0 + f'(x_i) \cdot 1 = f'(x_i).$$

Therefore, H_{2n+1} agrees with f and H'_{2n+1} with f' at x_0, x_1, \dots, x_n .

The uniqueness of this polynomial and the error formula are considered in Exercise 11. ■ ■ ■

Example 1 Use the Hermite polynomial that agrees with the data listed in Table 3.15 to find an approximation of $f(1.5)$.

Table 3.15

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959
2	1.9	0.2818186	-0.5811571

Solution We first compute the Lagrange polynomials and their derivatives. This gives

$$L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}, \quad L'_{2,0}(x) = \frac{100}{9}x - \frac{175}{9};$$

$$L_{2,1}(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}, \quad L'_{2,1}(x) = \frac{-200}{9}x + \frac{320}{9};$$

and

$$L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}, \quad L'_{2,2}(x) = \frac{100}{9}x - \frac{145}{9}.$$

The polynomials $H_{2,j}(x)$ and $\hat{H}_{2,j}(x)$ are then

$$H_{2,0}(x) = [1 - 2(x - 1.3)(-5)] \left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9} \right)^2$$

$$= (10x - 12) \left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9} \right)^2,$$

$$H_{2,1}(x) = 1 \cdot \left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9} \right)^2,$$

$$H_{2,2}(x) = 10(2 - x) \left(\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9} \right)^2,$$

$$\hat{H}_{2,0}(x) = (x - 1.3) \left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9} \right)^2,$$

$$\hat{H}_{2,1}(x) = (x - 1.6) \left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9} \right)^2,$$

and

$$\hat{H}_{2,2}(x) = (x - 1.9) \left(\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9} \right)^2.$$

Finally

$$H_5(x) = 0.6200860H_{2,0}(x) + 0.4554022H_{2,1}(x) + 0.2818186H_{2,2}(x)$$

$$- 0.5220232\hat{H}_{2,0}(x) - 0.5698959\hat{H}_{2,1}(x) - 0.5811571\hat{H}_{2,2}(x)$$

and

$$\begin{aligned} H_5(1.5) &= 0.6200860 \left(\frac{4}{27} \right) + 0.4554022 \left(\frac{64}{81} \right) + 0.2818186 \left(\frac{5}{81} \right) \\ &\quad - 0.5220232 \left(\frac{4}{405} \right) - 0.5698959 \left(\frac{-32}{405} \right) - 0.5811571 \left(\frac{-2}{405} \right) \\ &= 0.5118277, \end{aligned}$$

a result that is accurate to the places listed. ■

Although Theorem 3.9 provides a complete description of the Hermite polynomials, it is clear from Example 1 that the need to determine and evaluate the Lagrange polynomials and their derivatives makes the procedure tedious even for small values of n .

Hermite Polynomials Using Divided Differences

There is an alternative method for generating Hermite approximations that has as its basis the Newton interpolatory divided-difference formula (3.10) at x_0, x_1, \dots, x_n , that is,

$$P_n(x) = f[x_0] + \sum_{k=1}^n f[x_0, x_1, \dots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

The alternative method uses the connection between the n th divided difference and the n th derivative of f , as outlined in Theorem 3.6 in Section 3.3.

Suppose that the distinct numbers x_0, x_1, \dots, x_n are given together with the values of f and f' at these numbers. Define a new sequence $z_0, z_1, \dots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \quad \text{for each } i = 0, 1, \dots, n,$$

and construct the divided difference table in the form of Table 3.9 that uses $z_0, z_1, \dots, z_{2n+1}$.

Since $z_{2i} = z_{2i+1} = x_i$ for each i , we cannot define $f[z_{2i}, z_{2i+1}]$ by the divided difference formula. However, if we assume, based on Theorem 3.6, that the reasonable substitution in this situation is $f[z_{2i}, z_{2i+1}] = f'(z_{2i}) = f'(x_i)$, we can use the entries

$$f'(x_0), f'(x_1), \dots, f'(x_n)$$

in place of the undefined first divided differences

$$f[z_0, z_1], f[z_2, z_3], \dots, f[z_{2n}, z_{2n+1}].$$

The remaining divided differences are produced as usual, and the appropriate divided differences are employed in Newton's interpolatory divided-difference formula. Table 3.16 shows the entries that are used for the first three divided-difference columns when determining the Hermite polynomial $H_5(x)$ for x_0, x_1 , and x_2 . The remaining entries are generated in the same manner as in Table 3.9. The Hermite polynomial is then given by

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](x - z_0)(x - z_1) \cdots (x - z_{k-1}).$$

A proof of this fact can be found in [Pow], p. 56.

Table 3.16

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$	$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$	$f[z_1, z_2] = \frac{f[z_2] - f[z_1]}{z_2 - z_1}$	$f[z_0, z_1, z_2] = \frac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
$z_2 = x_1$	$f[z_2] = f(x_1)$	$f[z_2, z_3] = f'(x_1)$	$f[z_1, z_2, z_3] = \frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
$z_3 = x_1$	$f[z_3] = f(x_1)$	$f[z_3, z_4] = \frac{f[z_4] - f[z_3]}{z_4 - z_3}$	$f[z_2, z_3, z_4] = \frac{f[z_3, z_4] - f[z_2, z_3]}{z_4 - z_2}$
$z_4 = x_2$	$f[z_4] = f(x_2)$	$f[z_4, z_5] = f'(x_2)$	$f[z_3, z_4, z_5] = \frac{f[z_4, z_5] - f[z_3, z_4]}{z_5 - z_3}$
$z_5 = x_2$	$f[z_5] = f(x_2)$		

Example 2 Use the data given in Example 1 and the divided difference method to determine the Hermite polynomial approximation at $x = 1.5$.

Solution The underlined entries in the first three columns of Table 3.17 are the data given in Example 1. The remaining entries in this table are generated by the standard divided-difference formula (3.9).

For example, for the second entry in the third column we use the second 1.3 entry in the second column and the first 1.6 entry in that column to obtain

$$\frac{0.4554022 - 0.6200860}{1.6 - 1.3} = -0.5489460.$$

For the first entry in the fourth column we use the first 1.3 entry in the third column and the first 1.6 entry in that column to obtain

$$\frac{-0.5489460 - (-0.5220232)}{1.6 - 1.3} = -0.0897427.$$

The value of the Hermite polynomial at 1.5 is

$$\begin{aligned}
 H_5(1.5) &= f[1.3] + f'(1.3)(1.5 - 1.3) + f[1.3, 1.3, 1.6](1.5 - 1.3)^2 \\
 &\quad + f[1.3, 1.3, 1.6, 1.6](1.5 - 1.3)^2(1.5 - 1.6) \\
 &\quad + f[1.3, 1.3, 1.6, 1.6, 1.9](1.5 - 1.3)^2(1.5 - 1.6)^2 \\
 &\quad + f[1.3, 1.3, 1.6, 1.6, 1.9, 1.9](1.5 - 1.3)^2(1.5 - 1.6)^2(1.5 - 1.9) \\
 &= 0.6200860 + (-0.5220232)(0.2) + (-0.0897427)(0.2)^2 \\
 &\quad + 0.0663657(0.2)^2(-0.1) + 0.0026663(0.2)^2(-0.1)^2 \\
 &\quad + (-0.0027738)(0.2)^2(-0.1)^2(-0.4) \\
 &= 0.5118277.
 \end{aligned}$$



Table 3.17

1.3	0.6200860	<u>-0.5220232</u>			
1.3	0.6200860		-0.0897427		
		-0.5489460		0.0663657	
1.6	0.4554022		-0.0698330		0.0026663
		<u>-0.5698959</u>		0.0679655	-0.0027738
1.6	0.4554022		-0.0290537		0.0010020
		-0.5786120		0.0685667	
1.9	0.2818186		-0.0084837		
		<u>-0.5811571</u>			
1.9	0.2818186				

The technique used in Algorithm 3.3 can be extended for use in determining other osculating polynomials. A concise discussion of the procedures can be found in [Pow], pp. 53–57.

ALGORITHM 3.3

Hermite Interpolation

To obtain the coefficients of the Hermite interpolating polynomial $H(x)$ on the $(n + 1)$ distinct numbers x_0, \dots, x_n for the function f :

INPUT numbers x_0, x_1, \dots, x_n ; values $f(x_0), \dots, f(x_n)$ and $f'(x_0), \dots, f'(x_n)$.

OUTPUT the numbers $Q_{0,0}, Q_{1,1}, \dots, Q_{2n+1,2n+1}$ where

$$H(x) = Q_{0,0} + Q_{1,1}(x - x_0) + Q_{2,2}(x - x_0)^2 + Q_{3,3}(x - x_0)^2(x - x_1) \\ + Q_{4,4}(x - x_0)^2(x - x_1)^2 + \dots \\ + Q_{2n+1,2n+1}(x - x_0)^2(x - x_1)^2 \dots (x - x_{n-1})^2(x - x_n).$$

Step 1 For $i = 0, 1, \dots, n$ do Steps 2 and 3.

Step 2 Set $z_{2i} = x_i$;
 $z_{2i+1} = x_i$;
 $Q_{2i,0} = f(x_i)$;
 $Q_{2i+1,0} = f(x_i)$;
 $Q_{2i+1,1} = f'(x_i)$.

Step 3 If $i \neq 0$ then set

$$Q_{2i,1} = \frac{Q_{2i,0} - Q_{2i-1,0}}{z_{2i} - z_{2i-1}}.$$

Step 4 For $i = 2, 3, \dots, 2n + 1$

$$\text{for } j = 2, 3, \dots, i \text{ set } Q_{i,j} = \frac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}.$$

Step 5 OUTPUT $(Q_{0,0}, Q_{1,1}, \dots, Q_{2n+1,2n+1})$;
 STOP

The *NumericalAnalysis* package in Maple can be used to construct the Hermite coefficients. We first need to load the package and to define the data that is being used, in this case, x_i , $f(x_i)$, and $f'(x_i)$ for $i = 0, 1, \dots, n$. This is done by presenting the data in the form $[x_i, f(x_i), f'(x_i)]$. For example, the data for Example 2 is entered as

```
xy := [[1.3, 0.6200860, -0.5220232], [1.6, 0.4554022, -0.5698959],
       [1.9, 0.2818186, -0.5811571]]
```