Stimulation And Modeling Assignment 10

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Abstract

Assignment 10.1 (19-10-2020)
Triangle distribution
Associate inverse transformation
Assignment 10.2
Given grouped frequency table of a population
Visualize the Distribution function as Pricewise Linear
Provide an algorithm to generate Random deviates population
Assignment 10.3
Tabulate the univariate distributions (continuous as well as discrete)
List the distributions for which you can generate Random deviates by inverse transformation method then the method.
For the rest why you could not provide such method.

1 Associate inverse transformation for triangle distribution

1.1 PDF of Triangle distribution:

$$f(x) = \begin{cases} \frac{k(x-a)}{b-a} & a \leq x \leq b \\ \\ \frac{k(c-x)}{c-b} & b \leq x \leq c \\ \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Where $k = \frac{2}{c-a}$

1.2 Calculating CDF

We will calculate CDF $F(x)$for different intervals

1.2.1 $x \leq a$

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0dx = 0 \tag{2}$$

1.2.2 $a \leq x \leq b$

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x)dx \\
&= \int_{-\infty}^{a} f(x)dx + \int_{a}^{x} f(x)dx \text{ \{Integration by parts\}} \\
&= 0 + \int_{a}^{x} f(x)dx \text{ \{ From equation 2 \}} \\
&= \int_{a}^{x} \frac{k(x-a)}{b-a}dx \text{ \{ From equation 1\}} \\
&= \left.\frac{k(x-a)^2}{2(b-a)}\right|_{a}^{x} \\
&= \frac{k(x-a)^2}{2(b-a)}
\end{aligned} \tag{3}$$

1.2.3 $b \leq x \leq c$

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x)dx \\
&= \int_{-\infty}^{b} f(x)dx + \int_{b}^{x} f(x)dx \text{ \{Integration by parts\}} \\
&= \frac{k(b-a)}{2} + \int_{b}^{x} f(x)dx \text{ \{ From equation 3 \}} \\
&= \frac{k(b-a)}{2} + \int_{a}^{x} \frac{k(c-x)}{c-b}dx \text{ \{ From equation 1\}} \\
&= \frac{k(b-a)}{2} - \frac{k(c-x)^2}{2(c-b)}\bigg|_{b}^{x} \\
&= \frac{k(b-a)}{2} + \frac{k(c-b)}{2} - \frac{k(c-x)^2}{2(c-b)} \\
&= \frac{k(c-a)}{2} - \frac{k(c-x)^2}{2(c-b)}
\end{aligned} \tag{4}
$$

From equation

$$F(c) = 1 = \frac{k(c-a)}{2}$$

implies

$$k = \frac{2}{c-a}$$

1.2.4 $x \geq c$

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x)dx \\
&= \int_{-\infty}^{c} f(x)dx + \int_{c}^{x} f(x)dx \text{ \{Integration by parts\}} \\
&= 1 + \int_{b}^{x} f(x)dx \text{ \{ From equation 5 \}} \\
&= 1 + 0 \text{ \{ From equation 1\}} \\
&= 1
\end{aligned} \tag{5}$$

1.2.5 The CDF for Triangle distribution

$$F(x) = \begin{cases} 0 & x \leq a \\ \frac{k(x-a)^2}{2(b-a)} & a \leq x \leq b \\ \frac{k(c-a)}{2} - \frac{k(c-x)^2}{2(c-b)} & b \leq x \leq c \\ 1 & x \geq c \end{cases} \tag{6}$$

Where $k = \frac{2}{c-a}$

1.3 Calculating the inverse transform

Let u be a random number such that u is generated from a uniform random number generator and $u \in (0, 1)$. We have to find x such that $F(x) = u$, i.e. $x = F^{-1}(u)$ We divide F into different intervals to calculate the inverse.

1.3.1 $x \leq a$

$u = 0$ Since $u \in (0, 1)$, we can ignore this region.

1.3.2 $x \geq c$

$u = 1$ Since $u \in (0, 1)$, we can ignore this region.

1.3.3 $a \leq x \leq b$

$$
\begin{aligned}
u =& F(x) \\
u =& \frac{k(x-a)^2}{2(b-a)} \text{ \{From Equation 6\}} \\
(x-a)^2 =& \frac{2(b-a)(u)}{k} \\
x =& a \pm \sqrt{\frac{2(b-a)(u)}{k}}
\end{aligned}
\tag{7}
$$

Since $a \leq x$

$$
x = a + \sqrt{\frac{2(b-a)(u)}{k}}
$$

Substituting the above equation for x in $a \leq x \leq b$ we get the interval for u where the equation is valid as

$$
0 < u \leq \frac{k(b-a)}{2}
$$

1.3.4 $b \leq x \leq c$

$$
\begin{aligned}
u =& F(x) \\
u =& \frac{k(c-a)}{2} - \frac{k(c-x)^2}{2(c-b)} \text{ \{From Equation 6\}} \\
(x-c)^2 =& \frac{(c-b)(k(c-a)-2u)}{k} \\
x =& c \pm \sqrt{\frac{(c-b)(k(c-a)-2u)}{k}}
\end{aligned}
\tag{8}
$$

Since $x < c$

$$
x = c - \sqrt{\frac{(c-b)(k(c-a)-2u)}{k}}
$$

Substituting the above equation for x in $a \leq x \leq b$ we get the interval for u where the equation is valid as

$$\frac{k(b-a)}{2} \leq u < \frac{k(c-a)}{2}$$

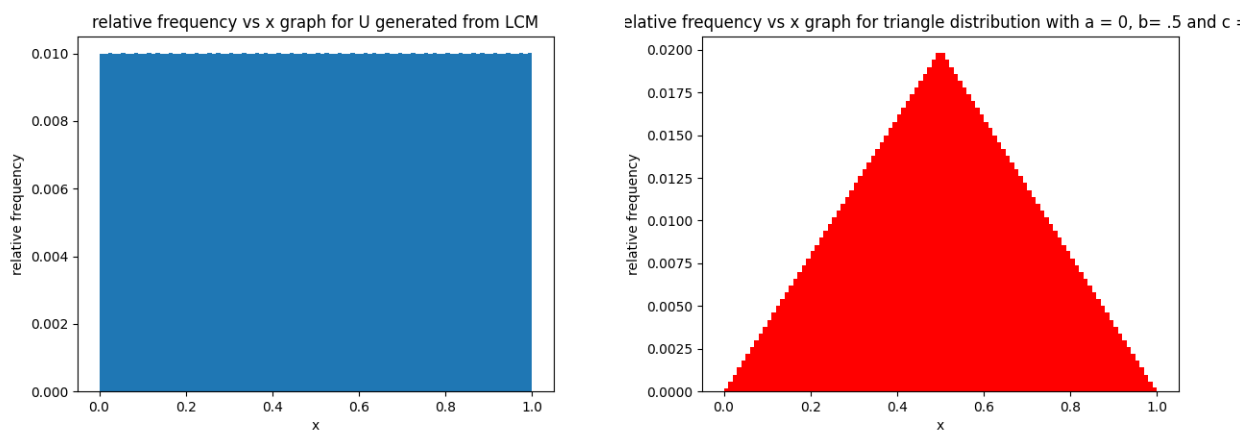
1.3.5 The inverse transform for Triangle distribution

$$x = \begin{cases} a + \sqrt{\frac{2(b-a)(u)}{k}} & 0 < u \leq \frac{k(b-a)}{2} \\ c - \sqrt{\frac{(c-b)(k(c-a)-2u)}{k}} & \frac{k(b-a)}{2} \leq u < \frac{k(c-a)}{2} \end{cases} \tag{9}$$

Where $k = \frac{2}{c-a}$ and $0 < u < 1$

1.4 An Example

We try to convert a uniform random to a triangle distribution.



Triangle distribution with a = 0, c = .5, c = 1.

2 Given grouped frequency table of a population. Visualize the Distribution function as Picewise Linear. Provide an algorithm to generate Random deviates population

2.1 Assumption of Input

We are assuming the input will be a table with two rows, one with the maximum limit under which the cumulative frequency is valid and another one with its associated cumulative frequency, in increasing order, with cf of 0 in the beginning Given grouped frequency table of a population, it can easily be converted to one such table.

Max Interval	Cumulative frequency
.5	.31
1	.41
1.5	.66
2.0	1.00

2.2 Algorithm

Let the table be an array called data, data[i].mx represents the max interval for the cumulative frequency and data[i].cf the corresponding cumulative frequency of i^{th} entry. data[0].mx = 0 and data[0].cf = 0 by default

1. Take a $u \in (0,1)$ as input.
2. Compare u to data[i].cf $\forall i > 0$ i.e. $i = 1, 2, ..$ till $data[i].cf \geq u$ is found .
3. If $data[i].cf \geq u$ is found do the steps given below.
4. Calculate $x = F^{-1}(u)$ with the formula

$$x = data[i].mx + \frac{data[i].cf - data[i-1].cf}{data[i].mx - data[i-1].mx}$$

5. return x
6. Return none if $u \notin (0,1)$

2.3 An example

Interval (Hours)	Frequency	Relative Frequency	Cumulative Frequency
$0 \leq x \leq 0.5$	31	0.31	0.31
$0.5 \leq x \leq 1.0$	10	0.10	0.41
$1.0 \leq x \leq 1.5$	25	0.25	0.66
$1.5 \leq x \leq 2.0$	34	0.34	1.00

Figure 1: The table with data to model

Max Interval	Cumulative frequency
.5	.31
1	.41
1.5	.66
2.0	1.00

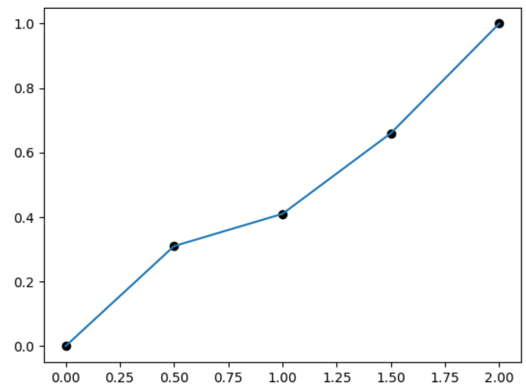


Figure 2: The piece wise visualisation for graph in above table

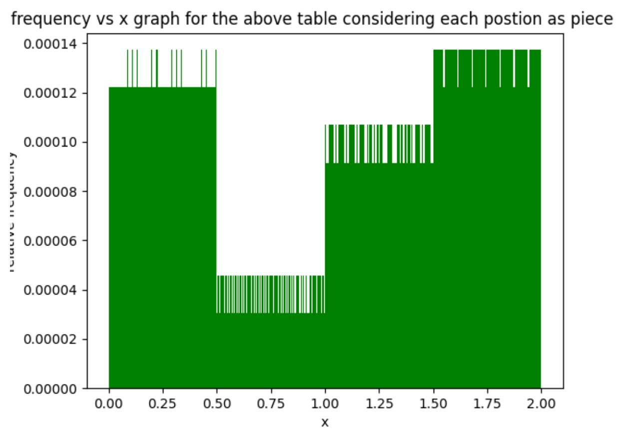


Figure 3: The relative frequency graph generated by generating Random deviates population, the random numbers are generated from an LCM

3 Tabulation of Continuous Uni-variate Distribution with methods to compute distribution using inverse transform

The steps are as follows:

1. Generate a uniform random number $u \in (0, 1)$
2. Use u to generate $x = F^{-1}(u)$ where F is the cdf associated with the random number x.

3.1 Exponential distribution

3.1.1 Probability Density Function

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ \\ 0 & x < 0 \\ \end{cases}$$

3.1.2 Cumulative Density Function

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ \\ 0 & x < 0 \\ \end{cases}$$

3.1.3 Inverse Transform

$$x = \frac{\ln \frac{1}{1-u}}{\lambda}$$

where $u \in (0,1)$ and is from a uniform distribution For The sake of generating a random distribution, we can take $1-u$ as u, since both will be uniform distributions. hence the equation becomes:

$$x = \frac{\ln \frac{1}{u}}{\lambda}$$

3.2 Uniform distribution

3.2.1 Probability Density Function

$$f(x) = \begin{cases} \frac{1}{b-a} & a \geq x \geq b \\ \\ 0 & \text{otherwise} \\ \end{cases}$$

3.2.2 Cumulative Density Function

$$F(x) = \begin{cases} 1 & x < a \\ \\ \frac{a-x}{b-a} & a \geq x \geq b \\ \\ 0 & x > b \\ \end{cases}$$

3.2.3 Inverse Transform

$$x = a + u(b - a)$$

where $u \in (0, 1)$ and is from a uniform distribution

3.3 Weibull distribution

3.3.1 Probability Density Function

$$f(x) = \begin{cases} \{\frac{\beta}{\alpha^{\beta}}\}x^{\beta-1}e^{-(\frac{x}{\alpha})^{\beta}} & x \geq 0 \\ \\ 0 & \text{otherwise} \\ \end{cases}$$

3.3.2 Cumulative Density Function

$$F(x) = \begin{cases} 1 - e^{-(\frac{x}{\alpha})^{\beta}} & x \geq 0 \\ \\ 0 & x < 0 \\ \end{cases}$$

3.3.3 Inverse Transform

$$x = \left(\alpha \ln \frac{1}{1 - u}\right)^{1/\beta}$$

where $u \in (0, 1)$ and is from a uniform distribution For The sake of generating a random distribution, we can take $1 - u$ as u, since both will be uniform distributions. hence the equation becomes:

$$x = \left(\alpha \ln \frac{1}{u}\right)^{1/\beta}$$

3.4 Normal distribution

3.4.1 Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

3.4.2 Cumulative Density Function

$$F(x) = \int_{\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{t-\mu}{\sigma})^2} dt$$

The above integral doesn't exist as a nice function, hence we can't find an inverse, i.e inveerse is not tractable

4 Tabulation of Discrete Uni-variate Distribution with methods to compute distribution using inverse transform

Assume that X is discrete random variable such that $P(X = x_i) = p_i$

The steps are as follows:

1. Generate a uniform random number $u \in (0, 1)$
2. Determine index k such that $\sum_{i=1}^{k-1} p_i \leq u < \sum_{i=1}^{k} p_i$

4.1 Bernoulli distribution

4.1.1 Probability Mass Function

$$Pr(X = k) = \begin{cases} q = 1 - p & k = 0 \\ p & k = 1 \end{cases}$$

Where $k \in \{0, 1\}$

4.1.2 Inverse transform

$$x = \begin{cases} 0 & u \leq 1 - p \\ 1 & u > 1 - p \end{cases}$$

where $u \in (0, 1)$ and is from a uniform distribution

4.2 Binomial distribution

4.2.1 Probability Mass Function

$$\binom{n}{k} p^k q^{n-k}$$

Where $k \in \{0, 1, ..., n\}$ and $q = 1 - p$

4.2.2 Inverse transform

Check for k such that $\sum_{i=1}^{k-1} \binom{n}{i} p^i q^{n-i} \leq u < \sum_{i=1}^{k} \binom{n}{i} p^i q^{n-i}$ where $u \in (0,1)$ and is from a uniform distribu

4.3 Uniform distribution

4.3.1 Probability Mass Function

$$Pr(X = k) = \frac{1}{b - a + 1}$$

Where $k \in \{a, a+1, a+2, ..., b-1, b\}$

4.3.2 Inverse transform

- Check for k such that $\sum_{i=a}^{k-1} \frac{1}{b-a+1} \leq u < \sum_{i=a}^{k} \frac{1}{b-a+1}$ where $u \in (0,1)$ and is from a uniform distribut
- We can rewrite as $\frac{k-1-a+1}{b-a+1} \leq u < \frac{k-a+1}{b-a+1}$
- $k - 1 - a + 1 \leq u(b - a + 1) < k - a + 1$
- $k - 1 \leq u(b - a + 1) + a - 1 < k$
- Hence we have find k such that $k - 1 \leq u(b - a + 1) + a - 1 < k$ which is equivalent to finding $\lfloor u(b - a + 1) + a - 1 \rfloor$

4.4 Poisson distribution

4.4.1 Probability Mass Function

$$Pr(X = k) = \frac{\lambda^k e^{-k}}{k!}$$

Where $k \in \{0, 1, 2, 3, 4...,\}$ and $q = 1 - p$

4.4.2 Inverse transform

Check for k such that $\sum_{i=1}^{k-1} \frac{\lambda^i e^{-i}}{i!} \leq u < \sum_{i=1}^{k} \frac{\lambda^i e^{-i}}{i!}$ where $u \in (0,1)$ and is from a uniform distribution

4.5 Geometric distribution

4.5.1 Probability Mass Function

$$Pr(X = k) = (1 - p)^{k-1} p$$

Where $k \in \{1, 2, 3, 4...,\}$ and $q = 1 - p$

4.5.2 Inverse transform

Check for k such that $\sum_{i=1}^{k-1}(1-p)^{i-1}p \leq u < \sum_{i=1}^{k}(1-p)^{i-1}p$
where $u \in (0,1)$ and is from a uniform distribution

This will give $1-(1-p)^{k-1} \leq u < 1-(1-p)^{k}$

$(1-p)^{k-1} \geq 1-u > (1-p)^{k}$

$(k-1)\ln(1-p) \geq \ln(1-u) > k\ln(1-p)$

$(k-1) \leq \frac{\ln(1-u)}{\ln 1-p} < k$ since ln(1 - u) is negative we have to change signs

hence $X = \left\lfloor \frac{\ln(1-u)}{\ln 1-p} \right\rfloor$

Since 1 - u is uniform if u is uniform we can replace one with other to generate a distribution

hence $X = \left\lfloor \frac{\ln(u)}{\ln 1-p} \right\rfloor$