수리통계 I HW3

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1.8. $f_{X,Y}(x,y) = 10xy^2 I_{(0<x<y<1)}$

(a) $\text{pdf}\ f_Y(y) = \int_0^y 10xy^2 dx\, I_{(0,1)}(y)$

$= 5y^4 I_{(0,1)}(y)$

(b) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{2x}{y^2} I_{(0<x<y<1)}$

(c) $E(X|Y) \Rightarrow \int_0^y \frac{2x^2}{y^2} dx = \frac{1}{y^2}\left[\frac{2}{3}x^3\right]_0^y = \frac{2}{3}y.$

$\therefore E(X|Y) = \frac{2}{3}Y$

$Var(X|Y) = E(X^2|Y) - \{E(X|Y)\}^2$

$\Rightarrow \int_0^y \frac{2x^3}{y^2} dx = \frac{Y^2}{2},\ Var(X|Y) = \frac{Y^2}{18}$

(d) $E(Y) = \int_0^1 5y^5 dy = \frac{5}{6},\ Var(Y) = \int_0^1 5y^6 dy - \frac{25}{36} = \frac{5}{252}$

$E(Var(X|Y)) = \frac{1}{18}E(Y^2) = \frac{1}{18} \times \frac{5}{7} = \frac{5}{126}$

$Var(E(X|Y)) = Var(\frac{2}{3}Y) = \frac{4}{9} \times \frac{5}{252}$

(e) $\text{pdf}\ f_X(x) = \int_x^1 10xy^2 dy\, I_{(0<x<1)} = \frac{10}{3}x(1-x^3) I_{(0,1)}(x)$

$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$

$= \frac{5}{126} + \frac{5}{252} \times \frac{4}{9}$

$= \frac{5}{126} \times \left(1 + \frac{2}{9}\right) = \frac{55}{9 \times 126}$

1.10.

(a) $f_2(y) = \int_0^y 2dx\, I_{(0,1)}(y) = 2y I_{(0,1)}(y)$

$f_{1|2}(x|y) = \dfrac{f_{1,2}(x,y)}{f_2(y)} = \dfrac{2 I_{(0<x<y<1)}}{2y I_{(0,1)}(y)}$

$= \frac{1}{y} I_{(0<x<y)}$

$E(X|Y) = \int_0^y \frac{x}{y} dx = \frac{1}{2}Y$

$Var(X|Y) = E(X^2|Y) - \{E(X|Y)\}^2$

$= \frac{1}{3}Y^2 - \frac{1}{4}Y^2 = \frac{1}{12}Y^2$

$Var(E(X|Y)) = Var(\frac{1}{2}Y) = \frac{1}{4}Var(Y)$

$E(Var(X|Y)) = E(\frac{1}{12}Y^2) = \frac{1}{12}E(Y^2)$

$E(Y) = \int_0^1 2y^2 dy = \frac{2}{3},\ E(Y^2) = \int_0^1 2y^3 dy = \frac{1}{2}$

$\therefore Var(E(X|Y)) = \frac{1}{72},\ E(Var(X|Y)) = \frac{1}{24}$

(b) $f_2(y) = \int_0^y e^{-y} x dx\, I_{(0,\infty)}(y) = \frac{y^2}{2}e^{-y} I_{(0,\infty)}(y)$

$E(Y) = \int_0^\infty \frac{y^3}{2}e^{-y} dy = \frac{3!}{2} = 3,\ E(Y^2) = \int_0^\infty \frac{y^4}{2}e^{-y} dy = \frac{4!}{2} = 12$

$f_{1|2}(x|y) = \frac{2x}{y^2} I_{(0<x<y)}$

$E(X|Y) = \int_0^y \frac{2x^2}{y^2} dx = \frac{2}{3}Y,\ E(X^2|Y) = \frac{1}{2}Y^2$

$Var(Y) = 3,\ Var(X|Y) = \frac{1}{18}Y^2$

$\therefore Var(E(X|Y)) = \frac{4}{9}Var(Y) = \frac{4}{3}$

$E(Var(X|Y)) = \frac{1}{18}E(Y^2) = \frac{2}{3}$

2.12. $U(X) := C^tX+d$, $V(X) := \mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)$

(a) $cov(Y-V(X), U(X))=0$임을 보이자.

$cov(Y-\mu_2-\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1), C^tX+d)$

$=cov(Y-\sigma_{21}\Sigma_{11}^{-1}X, C^tX)$

$=\{cov(Y,X)-cov(\sigma_{21}\Sigma_{11}^{-1}X, X)\}C$

$=\{\sigma_{21}-\sigma_{21}\Sigma_{11}^{-1}cov(X,X)\}C$

$=0C=0$.

(b) $Y-U(X)=Y-V(X)+V(X)-U(X)$로 두면

$$E(Y-U(X))^2=E(Y-V(X))^2+E(V(X)-U(X))^2+2E(Y-V(X))(V(X)-U(X))$$

이때 $V(X)-U(X)$는 일차함수이므로 $cov(Y-V(X), V(X)-U(X))=0$

즉 $E(Y-U(X))^2=E(Y-V(X))^2+E(V(X)-U(X))^2$

$\geq E(Y-V(X))^2$. 즉 임의의 U에 대해 이가 성립한다

(c) $corr(Y, a^tX+b)=\dfrac{\sigma_{21}a}{\sqrt{\sigma_{22}}\sqrt{a^t\Sigma_{11}a}}$

$b=\Sigma_{11}^{\frac{1}{2}}a$로 두면 $corr^2=\dfrac{(\sigma_{21}\Sigma_{11}^{-\frac{1}{2}}b)^2}{\sigma_{22}b^tb}$

$$=\frac{(\|b^t\Sigma_{11}^{-\frac{1}{2}}\sigma_{12}\|)^2}{\sigma_{22}\|b\|^2}\leq\frac{\|b\|^2\|\Sigma_{11}^{-\frac{1}{2}}\sigma_{12}\|^2}{\sigma_{22}\|b\|^2}$$

$$=\frac{\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}}{\sigma_{22}}$$

(d) $Var(Y-\mu_2-\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1))=$

$Var(Y)+Var(\mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1))$

$-2cov(Y, \mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1))$

$=\sigma_{22}+\sigma_{21}\Sigma_{11}^{-1}Var(X)\Sigma_{11}^{-1}\sigma_{12}-2\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}$

$=\sigma_{22}-\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}=\sigma_{22}\left(1-\dfrac{\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}}{\sigma_{22}}\right)=\sigma_{22}(1-\rho^2)$

2.14.

(a) $mgf_{1,2}(t_1,t_2)=E(e^{t_1X+t_2Y})$

$$=\int_0^\infty 6e^{(t_2-2)y}\int_0^y e^{(t_1-1)x}dxdy$$

$$=\frac{6}{t_1-1}\int_0^\infty e^{(t_2-2)y}(e^{(t_1-1)y}-1)dy$$

$$=\frac{6}{t_1-1}\left(\frac{1}{t_2-2}-\frac{1}{t_1+t_2-3}\right)\quad(t_2<2,\ t_1+t_2<3)$$

$$=\frac{6}{(t_1+t_2-3)(t_2-2)}$$

(b) $Var(X+Y)=Var(X)+Var(Y)+2cov(X,Y)$

mgf를 거듭제곱 꼴로 전개하면

$$mgf_{1,2}(t_1,t_2)=\sum_{r=0}^{\infty}\left(\frac{t_2}{2}\right)^r\times\sum_{r=0}^{\infty}\left(\frac{t_1+t_2}{3}\right)^r$$

$$=1+\frac{1}{3}t_1+\frac{5}{6}t_2+\frac{1}{9}t_1^2+\frac{7}{18}t_1t_2+\frac{19}{36}t_2^2+\cdots$$

$EX=\frac{1}{3}$ $EY=\frac{5}{6}$ $EX^2=\frac{2}{9}$ $EXY=\frac{7}{18}$ $EY^2=\frac{19}{18}$

$Var(X)=\frac{1}{9}$, $Var(Y)=\frac{13}{36}$, $cov(X,Y)=\frac{1}{9}$

$\therefore Var(X+Y)=\frac{25}{36}$

(c) $6e^{-x-2y}I_{(0<x<y<\infty)}=3e^{-3x}I_{(x>0)}2e^{-2(y-x)}I_{(y-x>0)}$

즉 X와 $Z=Y-X$는 서로 독립이다.

$M_X(t)=\dfrac{3}{3-t}$, $M_Z(s)=\dfrac{2}{2-s}$

$M_{X,Z}(t,s)=\dfrac{6}{(3-t)(2-s)}$ $(\because X\perp\!\!\!\perp Z)$

2.

(a) $f_{1,3}(x,z)=\int_x^z 6e^{-x-z}e^{-y}\,dy\,I_{(0\le x<z<\infty)}$

$=6e^{-x-z}(e^{-x}-e^{-z})I_{(0\le x<z<\infty)}$

$f_1(x)=6\int_x^\infty e^{-2x-z}-e^{-x-2z}\,dz\,I_{(0\le x<\infty)}$

$=3e^{-3x}I_{(0,\infty)}(x)$

$f_{3|1}(z|x)=\dfrac{6e^{-x-z}(e^{-x}-e^{-z})}{3e^{-3x}}I_{(0\le x<z)}$

$=2e^{2x-z}(e^{-x}-e^{-z})I_{(0\le x<z)}$

(b) $Var(Z-E(Z|X)-X)$

$=Var(Z-E(Z|X))+Var(X)+2\cancel{cov(Z-E(Z|X),X)}$ →영

$=Var(Z-X-\frac{3}{2})+Var(X)$

$=2Var(X)+Var(Z)+2cov(X,Y)$

$E(e^{tX+sZ})=\int_0^\infty\int_0^z 6e^{(t-2)x}e^{(s-1)z}-6e^{(t-1)x}e^{(s-2)z}\,dx\,dz$

$=\dfrac{1}{1-\frac{s}{2}}\times\dfrac{1}{1-s}\times\dfrac{1}{1-\frac{s+t}{3}}=\Sigma(\frac{s}{2})^r\Sigma s^r\Sigma(\frac{s+t}{3})^r$

$EX=\frac{1}{3},\ EZ=\frac{11}{6},\ EX^2=\frac{2}{9},\ EZ^2=\frac{85}{18},$

$EXZ=\frac{13}{18},\ Var(X)=\frac{1}{9},\ Var(Z)=\frac{49}{36},$

$cov(X,Z)=\frac{1}{9}$

$\therefore \frac{2}{9}+\frac{2}{9}+\frac{49}{36}=\frac{65}{36}$

(d) $f_{1,2,3}(x,y,z)=3e^{-3x}I_{(x>0)}\,2e^{-2(y-x)}I_{(y>x)}\,e^{-(z-y)}I_{(z>y)}$

즉 X, $Y-X$, $Z-Y$는 서로 독립이다.

따라서 X와 $Y-X$도 독립이다.

(c) 지수분포의 $mgf=\dfrac{\lambda}{\lambda-t}$ 꼴이므로

$mgf_X(t)=\dfrac{3}{3-t},\ mgf_{Y-X}(s)=\dfrac{2}{2-s}$

$\therefore mgf_{X,Y-X}(t,s)=\dfrac{6}{(3-t)(2-s)}$

3.

(a) X가 μ와 다른 값을 가지는 사건

$|X-\mu|>0=\bigcup_{n=1}^{\infty}(|X-\mu|\ge\frac{1}{n})$이다.

$P(|X-\mu|>0)\le\sum_{i=1}^{\infty}P(|X-\mu|\ge\frac{1}{i})$이고

$P(|X-\mu|\ge\frac{1}{i})\le i^2Var(X)=0$이다.

$P(|X-\mu|>0)\ge0$이고 $P(|X-\mu|>0)\le0$이므로

$P(|X-\mu|>0)=0,\ P(|X-\mu|=0)=1.$

(b) $p>1$에서 $|X_n-W|<|X_n-W|^p$이므로

$0\le\lim_{n\to\infty}E|X_n-W|\le\lim_{n\to\infty}E|X_n-W|^p=0$이고

$\lim_{n\to\infty}E|X_n-W|=0$이다.

마르코프 부등식에 의해 $P(|X_n-W|\ge\varepsilon)\le\dfrac{E|X_n-W|}{\varepsilon}$

$0\le\lim_{n\to\infty}P(|X_n-W|\ge\varepsilon)\le\dfrac{\lim E|X_n-W|}{\varepsilon}=0.$

$\therefore\lim_{n\to\infty}P(|X_n-W|\ge\varepsilon)=0.$

(c) $Var\left(\dfrac{Y-\mu_2}{\sigma_2}-\rho\dfrac{X-\mu_1}{\sigma_1}\right)=1-\rho^2$이다.

$(\rho=1\Rightarrow P(\frac{Y-\mu_2}{\sigma_2}-\frac{X-\mu_1}{\sigma_1})=1)$

$\rho=1$이므로 $Var\left(\dfrac{Y-\mu_2}{\sigma_2}-\dfrac{X-\mu_1}{\sigma_1}\right)=0$이고

$E\left(\dfrac{Y-\mu_2}{\sigma_2}-\dfrac{X-\mu_1}{\sigma_1}\right)=0$이므로 (a)에 의해

$P\left(\dfrac{Y-\mu_2}{\sigma_2}-\dfrac{X-\mu_1}{\sigma_1}=0\right)=1$이다.

$(\Leftarrow)\ P\left(\dfrac{Y-\mu_2}{\sigma_2}=\dfrac{X-\mu_1}{\sigma_1}\right)=1$이므로

$\rho=\dfrac{E(X-\mu_1)(Y-\mu_2)}{\sigma_1\sigma_2}=E\left(\dfrac{X-\mu_1}{\sigma_1}\right)\left(\dfrac{Y-\mu_2}{\sigma_2}\right)$

$=E\left(\dfrac{X-\mu_1}{\sigma_1}\right)^2=\dfrac{1}{\sigma_1^2}E(X-\mu_1)^2=\dfrac{\sigma_1^2}{\sigma_1^2}=1.$