수리통계 I HW6 2020-15709 나승찬

1.4.3.

$u : \begin{cases} Y_1 = \frac{X_1}{X_2} \\ Y_2 = X_2 \end{cases} \Longleftrightarrow u^{-1} : \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_2 \end{cases}$ 이면

u는 일대일 대응이고 $|J_{u^{-1}}| = |y_2| = y_2$이므로

$pdf_{Y_1,Y_2}(y_1, y_2) = y_2 f(y_1 y_2, y_2) = 2y_1 I_{(0,1)}(y_1) 4y_2^3 I_{(0,1)}(y_2)$

이므로 $Y_1 \perp Y_2$이다.

1.4.4. 확률변수의 변환을 사용하지 않더라도

$f(x_1, x_2, x_3) = 3e^{-3x_1} I(x_1 \geq 0) 2e^{-2(x_2-x_1)} I(x_2 - x_1 \geq 0) e^{-(x_3-x_2)} I(x_3 - x_2 \geq 0)$

$f(y_1, y_2, y_3) = 3e^{-3y_1} I_{(0,\infty)}(y_1) \cdot 2e^{-2y_2} I_{(0,\infty)}(y_2) \cdot e^{-y_3} I_{(0,\infty)}(y_3)$

이므로 Y_1, Y_2, Y_3은 서로 독립

1.4.6.

(a) $u : \begin{cases} Y_1 = \frac{X_1}{X_2} \\ Y_2 = X_2 \end{cases} \Longleftrightarrow u^{-1} \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_2 \end{cases}$ 로 두면 u는

$\mathcal{X}$에서 $\mathcal{Y}$로의 one-to-one 함수이므로 $|J_{u^{-1}}| = y_2$에서

$pdf_{Y_1,Y_2}(y_1, y_2) = 2y_2 e^{-(y_1+1)y_2} I_{(0 < y_1 y_2 < y_2 < \infty)}$

$pdf_{Y_1}(y_1) = \int_0^\infty 2y_2 e^{-(y_1+1)y_2} dy_2 \, I_{(0,1)}(y_1) = \frac{2 I_{(0,1)}(y_1)}{(y_1+1)^2}$

(b) $pdf_{Y_2|Y_1=y_1}(y_2 | y_1) = (y_1+1)^2 y_2 e^{-(y_1+1)y_2}$

$E(Y_2 | Y_1 = y) = \int_0^\infty (y_1+1)^2 y_2^2 e^{-(y_1+1)y_2} dy_2$

$= \frac{2}{y_1+1}$

$E(Y_2^2 | Y_1 = y) = \int_0^\infty (y_1+1)^2 y_2^3 e^{-(y_1+1)y_2} dy_2$

$= \frac{6}{(y_1+1)^2}$

$Var(Y_2^2 | Y_1 = y) = \frac{2}{(y_1+1)^2}$

1.4.19. $pdf_U(u) = I_{(0,1)}(u)$, $X = \log\left(\frac{U}{1-U}\right)$에서

$\log\left(\frac{U}{1-U}\right)$는 $(0,1)$에서 $\mathbb{R}$로의 one-to-one 함수이고

$U = \frac{e^x}{1+e^x}$이므로 $\frac{du}{dx} = \frac{e^x}{(1+e^x)^2}$

$pdf_X(x) = pdf_U\left(\frac{e^x}{1+e^x}\right) \frac{e^x}{(1+e^x)^2}$

$= \frac{e^x}{(1+e^x)^2} I_{(-\infty,\infty)}(x)$이고 이는

$\mu = 0$, $s = 1$인 로지스틱 분포이다.

1.4.20. $pdf_U(u) = I_{(0,1)}(u)$, $X = \tan\left(\pi\left(U - \frac{1}{2}\right)\right)$에서

$\tan\left(\pi\left(U - \frac{1}{2}\right)\right)$은 $(0,1)$에서 $\mathbb{R}$로의 one-to-one 함수이고

$U = \frac{\arctan x}{\pi} + \frac{1}{2}$, $\frac{du}{dx} = \frac{1}{\pi(1+x^2)}$이다.

$pdf_X(x) = \frac{1}{\pi(1+x^2)} I_{(-\infty,\infty)}(x)$이고 이는 $\mu = 0$, $s = 1$인 코시 분포이다.

2.4.8.

(a) $(-2,2)=(\overset{\mathcal{X}_1}{-2,0})\sqcup(\overset{\mathcal{X}_2}{0,2})$로 두면 여기서부터 $(0,4)$로의 two-to-one 함수로부터 $\mathcal{X}_1$에서 $u_1^{-1}: X=-\sqrt{Y}$, $\mathcal{X}_2$에서 $u_2^{-1}: X=\sqrt{Y}$이다.

그러면 $\mathcal{X}_1$에서 $pdf_Y(y) \Rightarrow \frac{1}{8}I_{(-2,0)}(-\sqrt{y})=\frac{1}{4}I_{(0,4)}(y)\times\frac{1}{2\sqrt{y}}$

$\mathcal{X}_2$에서 $pdf_Y(y) \Rightarrow \frac{1}{8}I_{(0,2)}(\sqrt{y})=\frac{1}{4}I_{(0,4)}(y)\times\frac{1}{2\sqrt{y}}$

$\therefore pdf_Y(y)=\frac{1}{4\sqrt{y}}I_{(0,4)}(y)$

(b) $\mathcal{X}_1:=(-1,0)$, $\mathcal{X}_2:=(0,3)$으로 두면 $\mathcal{X}_1\sqcup\mathcal{X}_2$ 상인 two-to-one 함수로부터

$\mathcal{X}_1$에서 $u_1^{-1}: X=-\sqrt{Y}$, $u_2^{-1}: X=\sqrt{y}$이고

$\mathcal{X}_1$에서 $\frac{1}{4}\times\frac{1}{2\sqrt{y}}\times I_{(-1,0)}(-\sqrt{y})=\frac{I_{(0,1)}(y)}{8\sqrt{y}}$

$\mathcal{X}_2$에서 $\frac{1}{4}\times\frac{1}{2\sqrt{y}}\times I_{(0,3)}(\sqrt{y})=\frac{I_{(0,9)}(y)}{8\sqrt{y}}$

$\therefore pdf_Y(y)=\frac{1}{4\sqrt{y}}I_{(0,1)}(y)+\frac{1}{8\sqrt{y}}I_{(1,9)}(y)$

2.4.9.

(a) $u:\begin{pmatrix}R=\sqrt{X_1^2+X_2^2}\\ \theta=\arctan\left(\frac{X_2}{X_1}\right)\end{pmatrix} \Leftrightarrow u^{-1}:\begin{pmatrix}X_1=R\cos\theta\\ X_2=R\sin\theta\end{pmatrix}$로 두면

U는 $\{0<x_1^2+x_2^2<1\}\to\{0<r<1, 0<\theta<2\pi\}$상인 one-to-one 함수.

$pdf_{R,\theta}(r,\theta)=\frac{1}{\pi}\left|\det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\right|I_{(0,1)}(r)I_{(0,2\pi)}(\theta)$

$=\frac{r}{\pi}I_{(0,1)}(r)I_{(0,2\pi)}(\theta)$

(b) $u_1:\begin{pmatrix}Y_1=\sqrt{X_1^2+X_2^2}\\ Y_2=\frac{X_1}{\sqrt{Y_1^2+Y_2^2}}\end{pmatrix}$ $u_1^{-1}:\begin{pmatrix}X_1=Y_1Y_2\\ X_2=Y_1\sqrt{1-Y_2^2}\end{pmatrix}$

$u_2:\begin{pmatrix}Y_1=\sqrt{X_1^2+X_2^2}\\ Y_2=X_1/\sqrt{X_1^2+X_2^2}\end{pmatrix}$ $u_2^{-1}:\begin{pmatrix}X_1=Y_1Y_2\\ X_2=-Y_1\sqrt{1-Y_2^2}\end{pmatrix}$

이때 $||J_{u_1^{-1}}||=||J_{u_2^{-1}}||=\frac{y_1}{\sqrt{1-y_2^2}}$이므로

$pdf_{Y_1,Y_2}(y_1,y_2)=\frac{y_1}{\pi\sqrt{1-y_2^2}}[I\{0<y_1<1,-1<y_2<0\}+I\{0<y_1<1,0<y_2<1\}]$

$=\frac{2y_1}{\pi\sqrt{1-y_2^2}}I\{0<y_1<1,-1<y_2<0\}$

2.4.10. $W_1, W_2-W_1, \cdots, W_k-W_{k-1} \overset{iid}{\sim} Exp\left(\frac{1}{\lambda}\right)$

~~(a)~~ $Z_i=\begin{cases}W_1 & (i=1)\\ W_i-W_{i-1} & (i=2,3,\cdots,k+1)\end{cases} \Rightarrow Z_i\sim Gamma\left(1,\frac{1}{\lambda}\right)$

(a) $Y_i=\frac{W_i-W_{i-1}}{W_1\oplus(W_2-W_1)\oplus\cdots\oplus(W_{k+1}-W_k)}$

$=\frac{Z_i}{Z_1\oplus\cdots\oplus Z_{k+1}}$이므로

$Y=(Y_1,\cdots,Y_k)^t\sim Dirichlet(1,1,\cdots,1)$

(b) $U:=X$, $V:=Y-X$로 두면

$U,V \overset{iid}{\sim} Gamma\left(2,\frac{1}{\lambda}\right)$이므로 $(U,V)\sim Dirichlet(2,2,1)$

$pdf_{U,V}(u,v)=\frac{\Gamma(5)uv}{\Gamma(2)\Gamma(2)\Gamma(1)}I\{u>0,v>0,u+v<1\}$

$pdf_{X,Y}(x,y)=\frac{1\times\Gamma(5)x(y-x)}{\Gamma(2)\Gamma(2)\Gamma(1)}I\{0<x<y<1\}$

$=24x(y-x)I\{0<x<y<1\}$

2.4.18.

(a) $u: X=\beta(-\log(1-U))^{\frac{1}{\alpha}} \Leftrightarrow u^{-1}: U=1-e^{-\left(\frac{x}{\beta}\right)^\alpha}$에서

u는 $(0,1)$에서 $(0,\infty)$로의 one-to-one 함수이고

$\frac{du}{dx}=\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}>0$이므로 확률변수 변환에 의해

$pdf_X(x)=\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}I_{(0,\infty)}(x)$

(b) $cdf_X(x)=\int_0^x\frac{\alpha}{\beta^\alpha}t^{\alpha-1}e^{-\left(\frac{t}{\beta}\right)^\alpha}dt=\left[-e^{-\left(\frac{t}{\beta}\right)^\alpha}\right]_0^x$

$=1-e^{-\left(\frac{x}{\beta}\right)^\alpha}$이므로 $1-F(x)=e^{-\left(\frac{x}{\beta}\right)^\alpha}$

$\frac{f(x)}{1-F(x)}=\frac{\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}}{e^{-\left(\frac{x}{\beta}\right)^\alpha}}=\frac{\alpha}{\beta^\alpha}x^{\alpha-1}$

2.4.23.

$$u:\begin{cases} R=\sqrt{X_1^2+X_2^2+X_3^2} \\ \theta_1=\arctan\frac{X_2}{X_1} \\ \theta_2=\arccos\frac{X_3}{\sqrt{X_1^2+X_2^2+X_3^2}} \end{cases} \Leftrightarrow u^{-1}:\begin{cases} X_1=R\cos\theta_1\sin\theta_2 \\ X_2=R\sin\theta_1\sin\theta_2 \\ X_3=R\cos\theta_2 \end{cases}$$

로 두면 u는 $\mathbb{R}^3\to\mathbb{R}^3$의 one-to-one 함수이고

$|J_{u^{-1}}| = r^2\sin\theta_2$이므로 $pdf_{X_1,X_2,X_3}(x_1,x_2,x_3)=\frac{1}{(\sqrt{2\pi})^3}e^{-\frac{x_1^2+x_2^2+x_3^2}{2}}I_{(0<x_1,x_2,x_3<1)}$

에서 $pdf_{R,\theta_1,\theta_2}(r,\theta_1,\theta_2)=\left(\frac{1}{\sqrt{2\pi}}\right)^3e^{-\frac{r^2}{2}}r^2\sin\theta_2 I\left\{\begin{matrix}0\le r<\infty,\\0\le\theta_1<2\pi\\0\le\theta_2<\pi\end{matrix}\right\}$

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3. $pdf_{X_1,X_2}(x_1,x_2)=\frac{1}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}}I_{(-\infty<x_1,x_2<\infty)}$

(a) $u:\begin{cases} Y_1=\frac{\sigma X_1+\mu X_2}{X_2} \\ Y_2=X_2\end{cases}$ 로 두면 $u^{-1}:\begin{cases} X_1=\frac{Y_1-\mu}{\sigma} \\ X_2=Y_2\end{cases}$

이고 u가 $\mathbb{R}^2\to\mathbb{R}^2$의 one-to one function이므로

$\|J_{u^{-1}}\| = \left\|\begin{vmatrix}\frac{1}{\sigma} & 0\\ 0 & 1\end{vmatrix}\right\| = \frac{1}{\sigma}$, $pdf_{Y_1,Y_2}(y_1,y_2)=\frac{e^{-\frac{(y_1-\mu)^2}{2\sigma^2}-\frac{y_2^2}{2}}}{2\pi\sigma}I_{(-\infty<y_1,y_2<\infty)}$

$=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y_1-\mu)^2}{2\sigma^2}}I_{(-\infty,\infty)}(y_1)\frac{1}{\sqrt{2\pi}}e^{-\frac{y_2^2}{2}}I_{(-\infty,\infty)}(y_2)$ 이다.

$\therefore Y_1\sim N(\mu,\sigma^2)$, $Y\sim N(\mu,\sigma^2)$

(b) 2.4.9의 u_1, u_2를 활용하되, 범위는 $\mathcal{X}_1:=\mathbb{R}\times\mathbb{R}^+$, $\mathcal{X}_2:=\mathbb{R}\times\mathbb{R}^-$ 로 두면 $\mathcal{X}=\mathcal{X}_1\sqcup\mathcal{X}_2$에서 $\mathcal{Y}=(-1,1)\times\mathbb{R}^+$로의 2-to-one 대응이다.

$u_1:\begin{cases} Z_1=\frac{X_1}{\sqrt{X_1^2+X_2^2}} \\ Z_2=\sqrt{X_1^2+X_2^2}\end{cases}$ $u_1^{-1}:\begin{cases} X_1=Z_1Z_2 \\ X_2=Z_2\sqrt{1-Z_1^2}\end{cases}$

$u_2:\begin{cases} Z_1=\frac{X_1}{\sqrt{X_1^2+X_2^2}} \\ Z_2=\sqrt{X_1^2+X_2^2}\end{cases}$ $u_2^{-1}:\begin{cases} X_1=Z_1Z_2 \\ X_2=-Z_2\sqrt{1-Z_1^2}\end{cases}$

$|\det(J_{u_1^{-1}})|=|\det(J_{u_2^{-1}})|=\frac{Z_2}{\sqrt{1-Z_1^2}}>0$

$pdf_{Z_1,Z_2}(z_1,z_2)=\frac{z_2}{\pi\sqrt{1-z_1^2}}e^{-\frac{z_2^2}{2}}I\{0\le z_1<1, z_2\ge0\}$

$+\frac{z_2}{\pi\sqrt{1-z_1^2}}e^{-\frac{z_2^2}{2}}I\{-1<z_1<0, z_2\ge0\}$

$=\frac{2z_2}{\pi\sqrt{1-z_1^2}}e^{-\frac{z_2^2}{2}}I_{(-1,1)}(z_1)I_{(0,\infty)}(z_2)$에서

$pdf_{Z_1}(z_1)=\frac{I_{(-1,1)}(z_1)}{\pi\sqrt{1-z_1^2}}$, $pdf_Z(z)=\frac{I_{(-1,1)}(z)}{\pi\sqrt{1-z^2}}$

(c) 우선 $U=\frac{X_1X_2}{\sqrt{X_1^2+X_2^2}}$, $V=\frac{X_1^2-X_2^2}{\sqrt{X_1^2+X_2^2}}$은 2-to-1 대응이다.

즉 (X_1,X_2)와 $(-X_1,-X_2)$ 각각에서 대응을 갖는다.

이때 $F_1=\begin{cases}R=\sqrt{X_1^2+X_2^2}\\ \theta=\arctan\left(\frac{X_2}{X_1}\right)\end{cases}\Leftrightarrow F_1^{-1}=\begin{cases}X_1=R\cos\theta\\X_2=R\sin\theta\end{cases}$

$F_2=\begin{cases}R=\sqrt{X_1^2+X_2^2}\\ \theta=\arctan\left(\frac{X_2}{X_1}\right)\end{cases}\Leftrightarrow F_2^{-1}=\begin{cases}X_1=-R\cos\theta\\X_2=-R\sin\theta\end{cases}$

$G_1=\begin{cases}U=\frac{R}{2}\sin2\theta\\V=\frac{R}{2}\cos2\theta\end{cases}\Longleftrightarrow G_1^{-1}=\begin{cases}R=2\sqrt{U^2+V^2}\\ \theta=\frac{1}{2}\arctan\left(\frac{U}{V}\right)\end{cases}$

즉 $|J_{F_1^{-1}}|=|J_{F_2^{-1}}|=R$, $\|J_{G_1^{-1}}\|=\frac{2}{\sqrt{U^2+V^2}}$이므로

$pdf_{R,\theta}(r,\theta)=\frac{r}{2\pi}e^{-\frac{r^2}{2}}$ (x_1, x_2 각각에서)

$pdf_{U,V}(u,v)=\frac{r}{2\pi}\times\frac{2}{r}\times e^{-2(u^2+v^2)}\times2I_{(-\infty<u,v<\infty)}$

$=\frac{1}{2\pi\times\frac{1}{4}}e^{-\frac{u^2+v^2}{2\times\frac{1}{4}}}I_{(-\infty<u,v<\infty)}$

$=\frac{1}{\sqrt{2\pi}\times\frac{1}{2}}e^{-\frac{u^2}{2\times(\frac{1}{2})^2}}I_{(-\infty,\infty)}(u)\frac{1}{\sqrt{2\pi}\times\frac{1}{2}}e^{-\frac{v^2}{2\times(\frac{1}{2})^2}}I_{(-\infty,\infty)}(v)$

$\therefore U, V \overset{iid}{\sim} N(0,(\frac{1}{2})^2)$

$pdf_U(u)=\frac{\sqrt{2}}{\sqrt{\pi}}e^{-2u^2}I_{(-\infty,\infty)}(u)$