mshw 9 2020-15709 나승환

1.4. $P(Y_n \le y) = \prod_{i=1}^{n} P(X_i \le y) = \{P(X_1 \le y)\}^n$

$= \{1-(1-y)^\alpha\}^n$

$P(n^{\frac{1}{\alpha}}(1-Y_n) \le k) = P(Y_n \ge 1 - k \cdot n^{-\frac{1}{\alpha}})$

$= 1 - P(Y_n < 1 - k \cdot n^{-\frac{1}{\alpha}})$

$= 1 - \{1-(1-1+k\cdot n^{-\frac{1}{\alpha}})^\alpha\}^n$

$= 1 - \left(1 - \frac{k^\alpha}{n}\right)^n$

이때 $\lim_{n\to\infty} P(n^{\frac{1}{\alpha}}(1-Y_n) \le k)$

$= \lim_{n\to\infty}\left\{1 - \left(1-\frac{k^\alpha}{n}\right)^n\right\}$

$= 1 - \left\{\lim_{n\to\infty}\left(1-\frac{k^\alpha}{n}\right)^{\frac{n}{k^\alpha}}\right\}^{k^\alpha} = 1 - e^{-k^\alpha}$ 이다.

$\therefore n^{\frac{1}{\alpha}}(1-Y_n)$의 극한분포의 $cdf(x) = \begin{cases} 1-e^{-x^\alpha} & (x \ge 0) \\ 0 & (x < 0) \end{cases}$

1.6. $P(Y_n \le y) = (1-e^{-y})^n$

$P(Y_n - \log n \le z) = P(Y_n \le z + \log n)$

$= (1-e^{-z-\log n})^n = \left(1-\frac{e^{-z}}{n}\right)^n$

$\lim_{n\to\infty} P(Y_n - \log n \le z) = \lim_{n\to\infty}\left(1-\frac{e^{-z}}{n}\right)^n = e^{-e^{-z}}$

$\therefore Y_n - \log n$의 극한분포의 $cdf(z) = \begin{cases} e^{-e^{-z}} & (x \ge 0) \\ 0 & (x < 0) \end{cases}$

1.8.

(a) $\operatorname{plim}_{n\to\infty}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow \operatorname{plim}_{n\to\infty} X_n = 0,\ \operatorname{plim}_{n\to\infty} Y_n = 1.$

$P(|X_n| \ge \varepsilon) = P(X_n \ge \varepsilon) = (1-\varepsilon)^n$

$\to 0$ as $n\to\infty$. $\therefore \operatorname{plim}_{n\to\infty} X_n = 0.$

$P(|Y_n - 1| \ge \varepsilon) = P(Y_n \le 1-\varepsilon) = (1-\varepsilon)^n$

$\to 0$ as $n\to\infty$. $\operatorname{plim}_{n\to\infty} Y_n = 1.$

(b) $pdf_{X_n, Y_n}(x, y) = n(n-1)(y-x)^{n-2} I\{0<x<y<1\}$

$u: \begin{cases} R_n = Y_n - X_n \\ W_n = X_n \end{cases}$ $u^{-1}: \begin{cases} X_n = W_n \\ Y_n = R_n + W_n \end{cases}$ $|\det(J_{u^{-1}})| = 1$

$pdf_{R_n, W_n}(r, w) = n(n-1) r^{n-2} I\{r>0, w>0, r+w<1\}$

$pdf_{R_n}(r) = n(n-1) r^{n-2}(1-r) I_{(0,1)}(r)$

$cdf_{R_n}(r) = r^{n-1}(r - nr + n)$

이제 $P(1-R_n \le \frac{z}{n}) = P(R_n \ge 1 - \frac{z}{n}) = 1 - P(R_n < 1 - \frac{z}{n})$

$= 1 - \left(1-\frac{z}{n}\right)^{n-1}\left(1-\frac{z}{n}+z\right) = 1 - \left(1-\frac{z}{n}\right)^n - z\left(1-\frac{z}{n}\right)^{n-1}$

$\lim_{n\to\infty} P(n(1-R_n) \le z) = 1 - e^{-z} - ze^{-z}$ 이다.

즉 $n(1-R_n)$은 $cdf(z) = \begin{cases} 1-e^{-z}-ze^{-z} & (z \ge 0) \\ 0 & (z < 0) \end{cases}$ 인

확률변수를 극한분포로 가진다.

1.12. by CLT, $\sqrt{n}(\bar{X}_n - \lambda) \xrightarrow{d} N(0,\lambda)$

let $g(x) := 2\sqrt{x}$. then $\lambda > 0$에서

$g'(\lambda) = \frac{2}{2\sqrt{\lambda}} \neq 0$이고 $x>0$에서 $g(x)$는 미분가능

Delta method에 의해

$\sqrt{n}(g(\bar{X}_n) - g(\lambda)) \xrightarrow{d} N(0, \frac{\lambda}{\lambda}=1)$

λ에 관한 $100(1-\alpha)\%$ 신뢰구간은 점근적으로

$P(Z_{\frac{\alpha}{2}} < 2\sqrt{n}(\sqrt{\bar{X}_n} - \sqrt{\lambda}) < Z_{1-\frac{\alpha}{2}}) = 1-\alpha$를

정리하여 $P\left(\sqrt{\bar{X}_n} - \frac{Z_{1-\frac{\alpha}{2}}}{2\sqrt{n}} < \sqrt{\lambda} < \sqrt{\bar{X}_n} - \frac{Z_{\frac{\alpha}{2}}}{2\sqrt{n}}\right)$

1.14. $V \sim Geo(p)$인 V로부터의 랜덤샘플 $\{V_i\}_{1\le i\le r}$에서

$W_r = W_1 \oplus (W_2 - W_1) \oplus \cdots \oplus (W_r - W_{r-1})$

$\overset{d}{=} V_1 \oplus V_2 \oplus \cdots \oplus V_r$이다.

(a) By CLT, $\sqrt{r}\left(\frac{\sum_{i=1}^{r} V_r}{r} - \frac{1}{p}\right) \xrightarrow{d} N(0, \frac{1-p}{p^2})$

let $g(x) := \frac{1}{x}$. then $p>0$에서 $g'(\frac{1}{p}) = -p^2 \neq 0$.

Delta method에 의해

$\sqrt{r}\left(\frac{r}{W_r} - p\right) \xrightarrow{d} N(0, p^2(1-p))$

(b) $\sqrt{r}(\hat{p}_r - p) \xrightarrow{d} N(0, p^2(1-p))$에 대해

$\{g'(p)\}^2 = \left\{\frac{1}{p\sqrt{1-p}}\right\}^2 > 0$인 $g(\theta)$를 찾자.

$g(\theta) = \int \frac{1}{\theta\sqrt{1-\theta}} d\theta \Leftrightarrow \int \frac{1}{t(1-t^2)}(-2t)\,dt$ ($1-\theta = t^2$ 치환)

$= \int \frac{2}{(t+1)(t-1)} dt = \log \frac{t-1}{t+1}$

$= \log \frac{\sqrt{1-\theta}-1}{\sqrt{1-\theta}+1}$

이제 $100(1-\alpha)\%$ 신뢰구간:

$P(-Z_{1-\frac{\alpha}{2}} \times \frac{1}{\sqrt{r}} + \hat{p}_r < p < \hat{p}_r - Z_{\frac{\alpha}{2}} \times \frac{1}{\sqrt{r}}) = 1-\alpha$

1.16. let $Y_n := \begin{pmatrix} \frac{1}{n}Z_1 + \cdots + \frac{1}{n-r_n+1}Z_{r_n} \\ \frac{1}{n}Z_1 + \cdots + \frac{1}{n-s_n+1}Z_{s_n} \end{pmatrix} = \begin{pmatrix} R_n \\ S_n \end{pmatrix}$

$E(R_n) = \frac{1}{n} + \cdots + \frac{1}{n-r_n+1} \approx -\log(1-\alpha)$

$E(S_n) = \frac{1}{n} + \cdots + \frac{1}{n-s_n+1} \approx -\log(1-\beta)$

$Var(R_n) = \frac{1}{n^2} + \cdots + \frac{1}{(n-r_n+1)^2} \approx \frac{\alpha}{n(1-\alpha)}$

$Var(S_n) \approx \frac{\beta}{n(1-\beta)}$

$Cov(R_n, S_n)$

2.

(a) $V_i := X_i^2$으로 두자. $(0,a)$에서 $V=X^2$은 one-to-one이고 $pdf_X(x)=\frac{1}{a}I_{(0,a)}(x)$이므로 $pdf_V(v)=\frac{1}{2a\sqrt{v}}I_{(0,a^2)}(v)$이다. 즉

$V_i \overset{iid}{\sim} V : pdf_V(v)=\frac{1}{2a\sqrt{v}}I_{(0,a^2)}(v)$이다.

이제 $W_n=\frac{1}{n}\sum_{i=1}^{n}X_i^2=\frac{1}{n}\sum_{i=1}^{n}V_i$이므로

Weak Law of Large Number에 의해

$W_n \xrightarrow{P} E(V)=\frac{a^2}{3}=\mu$이다.

(b) $\sqrt{n}(W_n^2-\mu^2)=\sqrt{n}(W_n-\mu)(W_n+\mu)$

$W_n \xrightarrow{P} \mu$, $\mu \xrightarrow{P} \mu$이므로 $W_n+\mu \xrightarrow{P} 2\mu=\frac{2}{3}a^2$ (CR)

중심극한정리에 의해 $\sqrt{n}(W_n-\mu) \xrightarrow{d} N(0, Var(V))$

이때 $Var(V)=\frac{a^4}{5}-\frac{a^4}{9}=\frac{4}{45}a^4$이다.

또한 슬러츠키 정리에 의해

$\sqrt{n}(W_n-\mu)(W_n+\mu) \xrightarrow{d} \frac{2}{3}a^2\sqrt{n}(W_n-\mu)$

$Var(\sqrt{n}(W_n+\mu)(W_n-\mu))=\frac{4}{9}a^4\times\frac{4}{45}a^4$

$\therefore \sqrt{n}(W_n^2-\mu^2) \xrightarrow{d} N\left(0, \frac{16}{405}a^8\right)$

3. $W_{r_1} \xrightarrow{d} N(r_1, 2r_1)$ $(W_{r_1}\sim\chi^2(r_1))$

$W_{r_2} \xrightarrow{d} N(r_2, 2r_2)$ $(W_{r_2}\sim\chi^2(r_2))$

$W_{r_1} \perp W_{r_2}$이면

$$\begin{pmatrix} W_{r_1} \\ W_{r_2} \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \begin{pmatrix} 2r_1 & 0 \\ 0 & 2r_2 \end{pmatrix}\right)$$