수리통계 I HW5

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1.3.12.

(a) $(W_r > t) \Leftrightarrow r$th 현상이 발생하기까지의 시간이 t 이상인 사건 $\Leftrightarrow t$까지의 시간 동안 $(r-1)$회의 이하의 현상이 발생한 사건 $\Leftrightarrow (N_t \le r-1)$

$$P(N_t \le r-1) = \sum_{x=0}^{r-1} \frac{(\lambda t)^x e^{-\lambda t}}{x!} \quad (\text{pois}(\lambda t))$$

$$P(W_r > t) = \int_t^\infty \frac{1}{\Gamma(r)\left(\frac{1}{\lambda}\right)^r} y^{r-1} e^{-\frac{y}{\frac{1}{\lambda}}} dy$$

$$= \int_t^\infty \frac{1}{\Gamma(r)\lambda^{-r}} y^{r-1} e^{-\lambda y} dy \quad \left(\text{Gamma}\left(r, \frac{1}{\lambda}\right)\right)$$

(b) $W_r = W_1 \oplus (W_2 - W_1) \oplus \cdots \oplus (W_r - W_{r-1})$

$\mathrm{cov}(W_1, W_r) = \mathrm{Var}(W_1)$ where $W_1 \sim \exp\left(\frac{1}{\lambda}\right)$

$= \frac{1}{\lambda^2}$

1.3.16.

$$mgf_X(t) = \sum_{i=0}^\infty \frac{E(X^i)}{i!} t^i = \sum_{i=0}^\infty \frac{E(X^{2i})}{(2i)!} t^{2i}$$

$$= \sum_{i=0}^\infty \frac{1}{i!}\left(\frac{t^2}{2}\right)^i = e^{\frac{t^2}{2}} \text{이므로}$$

$$pdf_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} I_{(-\infty,\infty)}(x)$$

1.3.18.

$\log X =: Y$로 두면 $e^Y = X$

(a) $E(X^k) = E(e^{Yk}) = mgf_Y(k) = e^{\frac{\sigma^2}{2}k^2 + \mu k}$

(b) $Z := \log X$, $Z \sim N(0, 1^2)$으로 두자.

$$E(e^{tX}) = \int_{\mathbb{R}} e^{tx} pdf_X(x)\, dx = \int_{\mathbb{R}} e^{te^z} pdf_X(e^z) e^z dz$$

$$= \int_{\mathbb{R}} e^{te^z} \cdot e^z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{1}{e^z} dz$$

$$= \int_{\mathbb{R}} e^{te^z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \ge \int_{z > \sqrt{\frac{12}{t}}} e^{te^z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \infty$$

$\therefore E(e^{tX})$는 존재하지 않는다.

2.

W_i를 ith 승객이 버스를 기다린 시간으로 정의하면 $W = \sum_{i=1}^n W_i$ (승객은 n명)이고 $W_i \sim Exp\left(\frac{1}{\lambda}\right)$ with mean $\frac{1}{\lambda}$이다.

$$E(W) = \sum_{i=1}^n E(W_i) = \frac{n}{\lambda}$$

3.

(a) $pdf_Y(y) = \int_0^\infty pdf_{X,Y}(x,y)\, dx$

$$= \int_0^\infty \frac{1}{(n-1)!\,\beta^n y!} e^{-(1+\frac{1}{\beta})x} x^{n+y-1} dy = \int_0^\infty pdf_{Y|X}(y|x)\, pdf_X(x)\, dx$$

이때 $g(k) := \int_0^\infty x^{k-1} e^{-(1+\frac{1}{\beta})x} dx$로 정의하자. 그러면

$$g(k) = \left[-\frac{x^{k-1} e^{-(1+\frac{1}{\beta})x}}{1+\frac{1}{\beta}}\right]_0^\infty + \frac{k-1}{1+\frac{1}{\beta}} \int_0^\infty x^{k-2} e^{-(1+\frac{1}{\beta})x} dx$$

$$= \frac{\beta}{\beta+1}(k-1) g(k-1) \text{이고,}$$

$$g(1) = \int_0^\infty e^{-(1+\frac{1}{\beta})x} dx = \frac{1}{1+\frac{1}{\beta}} \text{이므로 } g(k) = \left(\frac{\beta}{\beta+1}\right)^k (k-1)!$$

따라서 $P(Y=y) = \frac{g(n+y)}{y!(n-1)!\,\beta^n} = \frac{(n+y-1)!}{y!(n-1)!\,\beta^n (1+\frac{1}{\beta})^{n+y}}$

$$= \binom{n+y-1}{n-1}\left(\frac{\beta}{\beta+1}\right)^y \left(\frac{1}{\beta+1}\right)^n = pdf_Y(Y=y) \quad (y \in \mathbb{N} \cup \{0\})$$

한편, $W := Y + n$, $p := \frac{1}{\beta+1}$로 두면 위의 확률은 $\binom{W-1}{n-1} p^n (1-p)^{W-n}$이므로 $W \sim \mathrm{NegBin}(n, p)$이다.

(b) $E(Y) = E(Y+n) - n = n(\beta+1) - n = n\beta$

$\mathrm{Var}(Y) = \mathrm{Var}(Y+n) = n\beta(\beta+1)$