2. p. 68 # 3. ("Superstars" and the value of life)

An individual with endowed income $\bar{c}$ has a concave preference-scaling function $v(c)$. He has contracted a disease which, if not treated, will be fatal with probability $1-p_0$ and will spontaneously cure itself with probability p_0. His "bequest utility" in the event of death is zero everywhere.

(a) Suppose that, when treated by a physician who charges z, his probability of survival rises to p. If z is his maximum willingness to pay for that treatment, show that

$$pv(\bar{c}-z)=p_0v(\bar{c})$$

$$\text{Expected utility}\begin{cases}\text{when not treated}: & P_0V(\bar{c})+(1-P_0)\times 0 = P_0V(\bar{c})\\ \text{when treated}: & PV(\bar{c}-z)+(1-P)\times 0 = PV(\bar{c}-z)\end{cases}$$

For $z\in\mathbb{R}$,

(i) z is sufficiently large : $PV(\bar{c}-z)<P_0V(\bar{c})$

(ii) z is sufficiently small : $PV(\bar{c}-z)>P_0V(\bar{c})$

(i), (ii) $\Rightarrow \exists z\in\mathbb{R}$ s.t. $PV(\bar{c}-z)=P_0V(\bar{c})$

(b) Hence show that

$$\frac{dp}{dz}=\frac{p_0v(\bar{c})v'(\bar{c}-z)}{v(\bar{c}-z)^2}$$

Depict the relationship between p and z in a figure. Interpret its shape.

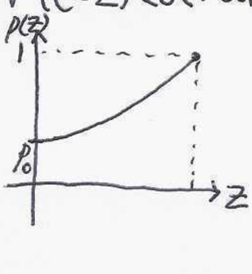
From (a), $P(z)=\dfrac{P_0V(\bar{c})}{V(\bar{c}-z)}$, $\dfrac{dP(z)}{dz}=\dfrac{-P_0\cdot V(\bar{c})\cdot(-1)\cdot V'(\bar{c}-z)}{\{V(\bar{c}-z)\}^2}=\dfrac{P_0V(\bar{c})V'(\bar{c}-z)}{\{V(\bar{c}-z)\}^2}$

$$\frac{d^2P(z)}{dz^2}=P_0V(\bar{c})\frac{-V''(\bar{c}-z)\{V(\bar{c}-z)\}^2+2\{V'(\bar{c}-z)\}^2V(\bar{c}-z)}{\{V(\bar{c}-z)\}^4}=\frac{P_0V(\bar{c})}{\{V(\bar{c}-z)\}^3}\{2V'(\bar{c}-z)-V(\bar{c}-z)V''(\bar{c}-z)\}$$

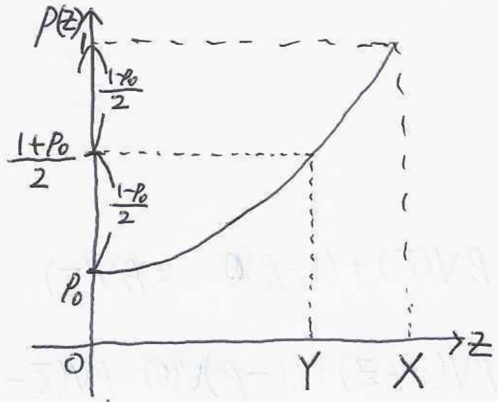
$P_0>0$, $V(\bar{c})>V(\bar{c}-z)>0$, $V'(\bar{c}-z)>0$ ($\because$ increasing V), $V''(\bar{c}-z)<0$ ($\because$ concave V)

$\therefore P(z)>0$, $P''(z)>0 \Rightarrow$ increasing, convex $P(z)$.

If $z=0$, $P(0)=P_0$.



(c) Suppose a "superstar" physician can increase the individual's probability of survival by $1-p_0$, so that he is sure to live, while another physician can increase this probability by only $\frac{1-p_0}{2}$. Indicate in the figure the maximum amount X and Y that the two physicians could charge.



X : the superstar physician

Y : another physician

(d) It has been asserted that "superstars" tend to receive disproportionally high incomes. In this context, this means that the ratio of physicians' fees would exceed the ratio of the survival rate increments that they provide. Assuming that both physicians can charge the maximum, is the assertion $\frac{X}{Y} > 2$ valid here?

$\frac{X}{Y} > 2$ contradicts convexity of $P(z)$.

$P(0) = P_0$, $P(X) = 1$, $P(Y) = \frac{1+P_0}{2}$. Then, $P\left(\frac{0}{2}+\frac{X}{2}\right) = P\left(\frac{X}{2}\right) < \frac{1}{2}P(0) + \frac{1}{2}P(X) = P(Y)$

$\therefore P\left(\frac{X}{2}\right) < P(Y) \Rightarrow \frac{X}{2} < Y$, $\frac{X}{Y} < 2$ ($\because P(z)$: increasing)

(e) The maximum that the individual would be willing to pay a physician, who (in effect) provides him with "a fraction $p-p_0$ of his life," is $z(p)$ where this function is defined implicitly in part (a). Then it can be argued that his implied valuation of his own life is $\dfrac{z(p)}{p-p_0}$. For example, if $p_0 = p-p_0 = 0.5$ and $X = \$100{,}000$, the value of his life would be \$200,000. Carry this argument to the limit in which p is small and show that, at this limit, the value he places upon his life is $\dfrac{v(\bar{c})}{p_0 v'(\bar{c})}$. Compare this conclusion with that reached at the end of the discussion in the text.

$$\lim_{P\to P_0^+} \frac{Z(P)}{P-P_0} = \lim_{P\to P_0^+} \frac{Z(P)-Z(P_0)}{P-P_0} \quad (\because Z(P_0)=0) \quad = \left.\frac{dZ(P)}{dP}\right|_{P=P_0} = \frac{1}{\left.\frac{dP(Z)}{dZ}\right|_{Z=0}}$$

$$= \left.\left(\frac{P_0 V(\bar{C}) V'(\bar{C}-Z)}{\{V(\bar{C}-Z)\}^2}\right)^{-1}\right|_{Z=0} = \frac{V(\bar{C})}{P_0 V'(\bar{C})}$$

8. R is the interest rate, the rate of return of the riskless asset. A specific risky asset 'a' has a yield vector $\tilde{z}_a$, where $E(\tilde{z}_a) \equiv \mu_a$ and $\sigma(\tilde{z}_a) \equiv \sigma_a$, and its equilibrium price is given by p_a^A. Also, the market portfolio has a yield vector $\tilde{z}_F$, where $E(\tilde{z}_F) \equiv \mu_F$ and $\sigma(\tilde{z}_F) \equiv \sigma_F$, and its price is given by p_F.

The covariance between $\tilde{z}_a$ and $\tilde{z}_F$ is denoted as σ_{aF}. Then, in CAPM, we have derived that

$$p_a^A = \frac{1}{1+R}[\mu_a - \frac{\sigma_{aF}}{\sigma_F}\theta], \tag{1}$$

where $\theta = \dfrac{\mu_F - p_F(1+R)}{\sigma_F}$ denotes the market price for risk reduction.

(a) From (1), derive that

$$\mu(\tilde{R}_a) = R + \theta_{\rho}(\tilde{R}_a, \tilde{R}_F)\sigma(\tilde{R}_a), \tag{2}$$

where $\tilde{R}_a$ is the rate of return of asset 'a', $\dfrac{\tilde{z}_a}{p_a^A} = 1 + \tilde{R}_a$, and $\tilde{R}_F$ is the rate of return of the market portfolio, $\dfrac{\tilde{z}_F}{p_F} = 1 + \tilde{R}_F$. Also, $\mu(\tilde{R}_a)$ and $\sigma(\tilde{R}_a)$ denote the mean value and the standard deviation of $\tilde{R}_a$, respectively, and $\rho(\tilde{R}_a, \tilde{R}_F)$ denotes the correlation coefficient between $\tilde{R}_a$ and $\tilde{R}_F$.

Also explain the implication of (2).

$$(1) \Leftrightarrow \frac{\mu_a}{p_a^A} - 1 = R + \theta \cdot \frac{\sigma_{aF}}{\sigma_F p_a^A} \ (*), \quad (\tilde{R}_a, \tilde{R}_F) = \left(\frac{\tilde{z}_a}{p_a^A} - 1, \frac{\tilde{z}_F}{p_F} - 1\right)$$

$$\left(\mu(\tilde{R}_a), \mu(\tilde{R}_F)\right) = \left(\mu\left(\frac{\tilde{z}_a}{p_a^A} - 1\right), \mu\left(\frac{\tilde{z}_F}{p_F} - 1\right)\right) = \left(\frac{\mu_a}{p_a^A} - 1, \frac{\mu_F}{p_F} - 1\right)$$

$$\left(\sigma(\tilde{R}_a), \sigma(\tilde{R}_F)\right) = \left(\sigma\left(\frac{\tilde{z}_a}{p_a^A} - 1\right), \sigma\left(\frac{\tilde{z}_F}{p_F} - 1\right)\right) = \left(\frac{\sigma_a}{p_a^A}, \frac{\sigma_F}{p_F}\right)$$

$$\mathrm{cov}(\tilde{R}_a, \tilde{R}_F) = \sigma(\tilde{R}_a, \tilde{R}_F) = E[(\tilde{R}_a - E\tilde{R}_a)(\tilde{R}_F - E\tilde{R}_F)] = E\left[\left(\frac{\tilde{z}_a - \mu_a}{p_a^A}\right)\left(\frac{\tilde{z}_F - \mu_F}{p_F}\right)\right] = \frac{\sigma_{aF}}{p_a^A p_F}$$

$$\frac{\sigma_{aF}}{\sigma_F p_a^A} = \frac{\sigma(\tilde{R}_a \tilde{R}_F)}{\sigma(\tilde{R}_F)} = \rho(\tilde{R}_a \tilde{R}_F)\sigma(\tilde{R}_a), \quad \frac{\mu_a}{p_a^A} - 1 = \mu(\tilde{R}_a)$$

$$\therefore (*) \Leftrightarrow \mu(\tilde{R}_a) = R + \sigma(\tilde{R}_a)\rho(\tilde{R}_a \tilde{R}_F)\theta$$

(b) From (2), derive that

$$\mu(\widetilde{R}_a) = R + \beta_a[\mu(\widetilde{R}_F) - R], \tag{3}$$

where $\beta_a \equiv \dfrac{\sigma(\widetilde{R}_a, \widetilde{R}_F)}{\sigma^2(\widetilde{R}_F)}$.

Also, explain the implication of (3).

$$\frac{\sigma(\widetilde{R}_a \widetilde{R}_F)}{\sigma(\widetilde{R}_F)} = \frac{\sigma(\widetilde{R}_a \widetilde{R}_F)}{\sigma^2(\widetilde{R}_F)} \sigma(\widetilde{R}_F) = \beta_a \cdot \sigma(\widetilde{R}_F)$$

$$\sigma(\widetilde{R}_F) \cdot \theta = \frac{\mu_F - P_F(1+R)}{\sigma_F} \times \frac{\sigma_F}{P_F} = \frac{\mu_F}{P_F} - 1 - R = \mu(\widetilde{R}_F) - R$$

$$(*) \Leftrightarrow \mu(\widetilde{R}_a) = R + \beta_a \sigma(\widetilde{R}_F)\theta$$

$$\Leftrightarrow \mu(\widetilde{R}_a) = R + \beta_a(\mu(\widetilde{R}_F) - R)$$