4.24.

(a) $A=\begin{pmatrix}1&-1&1\\2&1&-1\end{pmatrix}$에 대해 $Y=AX$이므로

$EY = AEX = \begin{pmatrix}1&-1&1\\2&1&-1\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

$Var(Y)=Var(AX)=AVar(X)A^T$

$=A\Sigma A^T=\begin{pmatrix}1&-1&1\\2&1&-1\end{pmatrix}\begin{pmatrix}1&-1&0\\-1&2&1\\0&1&2\end{pmatrix}\begin{pmatrix}1&2\\-1&1\\1&-1\end{pmatrix}$

$=\begin{pmatrix}5&1\\1&2\end{pmatrix}$. $Y\sim N\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}5&1\\1&2\end{pmatrix}\right)$

(b) $\mu_1:=\begin{pmatrix}1\\2\end{pmatrix}$, $\mu_2:=3$

$\Sigma:=\begin{pmatrix}\Sigma_{11}&\Sigma_{12}\\\Sigma_{21}&\Sigma_{22}\end{pmatrix}$ where $\Sigma_{22}=(2)$

$\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(x_3-\mu_2)=\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}0\\\frac{1}{2}(x_3-3)\end{pmatrix}$

$\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}=\begin{pmatrix}1&-1\\-1&2\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\times\frac{1}{2}\times(0\ \ 1)$

Thm 4.29(b)에 의해

$X_1,X_2|X_3=x_3\sim N\left(\begin{pmatrix}1\\\frac{x_3}{2}+\frac{1}{2}\end{pmatrix},\begin{pmatrix}1&-1\\-1&\frac{3}{2}\end{pmatrix}\right)$

5.10. $\vee$: or $\wedge$: and

(a) $|X_n+Y_n|>R\Rightarrow|X_n|>\frac{R}{2}\vee|Y_n|>\frac{R}{2}$

$P(|X_n+Y_n|>R)\le P(|X_n|>\frac{R}{2})+P(|Y_n|>\frac{R}{2})$

$\limsup_{n\to\infty}$을 씌우면 $0\le\lim_{n\to\infty}P(|X_n+Y_n|>R)\le 0$

$\therefore X_n+Y_n=O_p(1)$

(b) $R>0$인 범위에서 R^2과 R는 one-to-one.

$|X_nY_n|>R^2\Rightarrow|X_n|>\sqrt{R}$ or $|Y_n|>\sqrt{R}$

$P(|X_nY_n|>R^2)\le P(|X_n|>\sqrt{R})+P(|Y_n|>\sqrt{R})$

$\limsup_{n\to\infty}P(|X_nY_n|>R^2)\le 0$이므로

$\therefore X_nY_n=O_p(1)$

(c) $\epsilon>0$, $\alpha>0$에 대해

$(|Z_nX_n|>\epsilon)\Rightarrow$

$(|X_n|\le\alpha\wedge|Z_nX_n|>\epsilon)\vee(|X_n|>\alpha,|Z_nX_n|>\epsilon)$

$\Rightarrow(|Z_n|>\frac{\epsilon}{\alpha})\vee(|X_n|>\alpha)$

$P(|Z_nX_n|>\epsilon)\le P(|Z_n|>\frac{\epsilon}{\alpha})+\sup_n P(|X_n|>\alpha)$

$\lim_{n\to\infty}P(|Z_nX_n|>\epsilon)\le\underbrace{0}_{Z_n=o_p(1)}+\underbrace{\limsup_{n\to\infty}P(|X_n|>\alpha)}_{X_n=O_p(1)}=0$

$\therefore Z_nX_n\xrightarrow{P}1$

(d) 오차가 임의의 ε 이하가 되는 $n_1,\cdots,n_?$ 찾기.

$\lim_{n\to\infty}P(|Z_n|\ge\delta_0)=0\Leftrightarrow n\ge n_0\Rightarrow P(|Z_n|\ge\delta_0)<\varepsilon$

이제 $n\le n_0$인 n에 대해 $\lim_{R\to\infty}P(|Z_n|>R)=0$이므로

$1\le i\le n_0$에서 $\forall\varepsilon>0$, $\exists\delta_i$ s.t. $P(|Z_i|>\delta_i)<\varepsilon$

$\delta:=\max_i\{\delta_i:0\le i\le n_0\}$에서

$(i\le n_0)$ $\max_i P(|X_i|>\delta)\le\max_i P(|X_i|>\delta_i)<\varepsilon$

$(i>n_0)$ $\sup_i P(|X_i|>\delta)\le\sup_i P(|X_i|>\delta_0)<\varepsilon$

$\therefore x>\delta\Rightarrow\sup_x P(|X_x|>\delta)\le\varepsilon$, $Z_n=O_p(1)$

4.26.

(a)

(d) (a)에 의해

$$\frac{r}{\sqrt{1-r^2}} = \frac{\frac{\rho}{\sqrt{1-\rho^2}} S_{zz} + S_{zw}}{\sqrt{S_{zz}S_{ww} - S_{zw}^2}} = \frac{\frac{\rho}{\sqrt{1-\rho^2}} \frac{S_{zz}}{\sqrt{S_{zz}}} + \frac{S_{zw}}{\sqrt{S_{zz}}}}{\sqrt{S_{ww} - \frac{S_{zw}^2}{S_{zz}}}}$$

이때 (c-i)에서 $\sqrt{S_{ww} - S_{zw}^2/S_{zz}} \overset{d}{=} V_2 \sim \chi^2(n-2)$

(c-ii)에서 $S_{zw}/S_{zz} \overset{d}{=} U \sim N(0, 1^2)$

$S_{zz} = \sum^{n} (Z_i - \bar{Z})^2 \sim \chi^2(n-1) = V_1$

$$\therefore \frac{r}{\sqrt{1-r^2}} \overset{d}{=} \frac{\frac{\rho}{\sqrt{1-\rho^2}}\sqrt{V_1} + U}{\sqrt{V_2}} \quad (\because \text{(iii)} - 독립)$$

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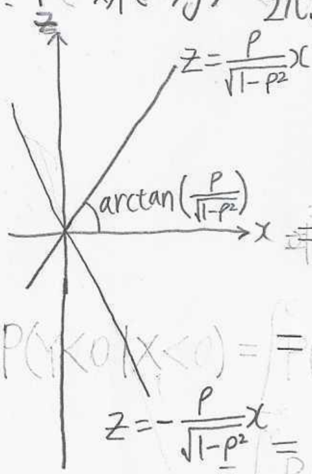
2. $pdf_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}} \quad (x,y\in\mathbb{R})$



$P(X<0, Y<0) =$

$P(X<0)P(Y|X=x<0)$

$= P(X<0)P\left(Z<\frac{-\rho x}{\sqrt{1-\rho^2}}\right)$

$= P\left(X<0, Z<-\frac{\rho x}{\sqrt{1-\rho^2}}\right)$

$= P\left(X>0, Z>-\frac{\rho x}{\sqrt{1-\rho^2}}\right)$

$= \int_0^\infty\int \frac{1}{2\pi} e^{-\frac{x^2+z^2}{2}} dz\,dx$ (극좌표 치환)

$= \int_0^\infty\int_{\arctan(\frac{\rho}{\sqrt{1-\rho^2}})}^{\frac{\pi}{2}} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\theta\,dr$ $\left(-\frac{\rho x}{\sqrt{1-\rho^2}} \to \arctan\left(-\frac{\rho}{\sqrt{1-\rho^2}}\right)\right)$

$= \int_0^\infty \frac{1}{2\pi}\left(\frac{\pi}{2} - \arctan\left(-\frac{\rho}{\sqrt{1-\rho^2}}\right)\right) r e^{-\frac{r^2}{2}} dr$

$= \left(\frac{1}{4} + \arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)\cdot\frac{1}{2\pi}\right)\left[-e^{-\frac{r^2}{2}}\right]_0^\infty$

$= \frac{1}{4} + \frac{1}{2\pi}\cdot\arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)$

3.

(a) $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} := \begin{pmatrix} X+Y \\ X-Y \end{pmatrix} = \overset{A}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}\begin{pmatrix} X \\ Y \end{pmatrix}$ 이다.

즉 $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \sim N_2\left(A\mu, A\Sigma A^T\right)$에서

$A\mu = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1+\mu_2 \\ \mu_1-\mu_2 \end{pmatrix}$

$A\Sigma A^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} \sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2 & \sigma_1^2-\sigma_2^2 \\ \sigma_1^2-\sigma_2^2 & \sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2 \end{pmatrix}$

Prop 4.28 (b)에서 $W_1 \perp W_2 \Leftrightarrow \sigma_1^2-\sigma_2^2=0$

$\therefore (X+Y)\perp(X-Y) \Leftrightarrow \sigma_1^2=\sigma_2^2$

(b) $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix} = \overset{B}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} X \\ Y \end{pmatrix}$

따라서 $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \sim N\left(B\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, B\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}B^T\right)$

Thm 4.29 (b)에서 $V_1|V_2=t$는 이변량 정규분포를 따른다.

4. 우선 $U_{(1)} := \min U_i$, $U_{(n)} := \max U_i$

$$pdf_{U_{(1)}}(x) = \frac{n}{a}\left(1-\frac{x}{a}\right)^{n-1} I_{(0,a)}(x)$$

$$E(U_{(1)}) = \int_0^a x \cdot \frac{n}{a}\left(1-\frac{x}{a}\right)^{n-1} dx = \frac{a}{n+1}$$

$$Var(U_{(1)}) = \int_0^a na\left(\frac{a}{x}\right)^{3-1}\left(1-\frac{a}{x}\right)^{n-1} dx - E(U_{(1)})^2 = na \times \frac{\Gamma(3)\Gamma(n)}{\Gamma(n+3)} - E(U_{(1)})^2$$

$$= \frac{2a^2}{(n+1)(n+2)} - \frac{a^2}{(n+1)^2}$$

$$pdf_{U_{(n)}}(x) = \frac{n}{a}\left(\frac{x}{a}\right)^{n-1} I_{(0,a)}(x)$$

$$E(U_{(n)}) = \int_0^a \frac{n}{a^n} x^n dx = \frac{na}{n+1}$$

$$Var(U_{(n)}) = \int_0^a \frac{n}{a^n} x^{n+1} dx - \frac{n^2a^2}{(n+1)^2}$$

$$= \frac{a^2 n}{(n+1)^2(n+2)}$$

$$\lim_{n\to\infty} E(U_{(1)}) = 0, \quad \lim_{n\to\infty} Var(U_{(1)}) = 0$$

$$\lim_{n\to\infty} E(U_{(n)}) = a, \quad \lim_{n\to\infty} Var(U_{(n)}) = 0$$

$$\operatorname{plim}_{n\to\infty} U_{(1)} = 0, \quad \operatorname{plim}_{n\to\infty} U_{(n)} = a$$

$$\operatorname{plim}_{n\to\infty} X_n = \operatorname{plim}_{n\to\infty} U_{(1)} + \operatorname{plim}_{n\to\infty} U_{(n)} = a$$