과제 8 채점 기준

※ 제출 전 모든 문제의 지시사항을 충분히 따랐는지 확인할 것
※ 과정이 합당한 경우, 정답이 아니더라도 점수 인정
※ 과제는 완성도 위주로 평가
※ 코딩을 이용해서 계산한 경우에도 인정
※ 이번 과제는 문제가 많은 관계로 0.5점 단위로 채점

문제 9.1. [0.5점]
- $\rho = LL' + \Psi$임을 계산을 통해 보일 것

문제 9.2. [1점]
(a) : 0.5점, (b) : 0.5점
- (a)에서 $h_i^2 (i=1,2,3)$을 구하고 결과를 해석할 것
- (b)에서 상관계수 3개와 후속 질문에 대한 답을 쓰고, 이유를 설명할 것

문제 9.3. [1점]
(a) : 0.5점, (b) : 0.5점
- (a)에서 L과 Ψ를 구하고 문제 9.1.의 결과와 비교할 것
- (b)에서 비율을 구할 것

문제 9.4. [1점]
- $\tilde{\rho}$와 L을 구할 것 (0.5점)
- 문제 9.1.의 결과와 비교하고, 그 이유를 설명할 것 (0.5점)

문제 9.5. [0.5점]
- 힌트와 대각합의 성질을 이용해서 증명할 것

문제 9.7. [1점]
- σ_{11}, σ_{12}, σ_{22}에 대한 식을 증명할 것 (0.5점)
- 주어진 σ_{11}, σ_{12}, σ_{22}에 대해 L과 Ψ를 무한히 선택할 수 있음을 증명할 것 (0.5점)

문제 9.8. [1점]
- $\Sigma = LL' + \Psi$인 L과 Ψ가 유일하게 선택됨을 증명할 것 (0.5점)
- $\Psi_3 < 0$임을 증명할 것 (0.5점)

문제 9.20. [1.5점]
(a) : 0.5점, (b) : 0.5점, (c) : 0.5점
- (a)에서 표본 공분산 행렬을 구하고, $m=1$과 $m=2$에 대한 주성분 모두 구할 것
- (b)에서 $m=1$과 $m=2$에 대해 L과 Ψ의 최대가능도 추정량을 모두 구할 것
- (c)에서 (a), (b)의 결과를 바탕으로 설명할 것

문제 9.21. [1점]
- $m=2$일 때 문제 9.20.의 (a), (b) 각각에 대해 varimax rotation을 구할 것 (0.5점)
- 두 결과의 일치성에 대해 설명할 것 (0.5점)

문제 9.22. [1.5점]

(a) : 0.5점, (b) : 0.5점, (c) : 0.5점

- (a)에서 (9-50)과 (9-58)을 이용하여 답을 구할 것
- (b)에서 (9-51)을 이용하여 답을 구할 것
- (c)에서 (a), (b)의 결과를 바탕으로 설명할 것

9.1. Show that the covariance matrix

$$\boldsymbol{\rho} = \begin{bmatrix} 1.0 & .63 & .45 \\ .63 & 1.0 & .35 \\ .45 & .35 & 1.0 \end{bmatrix}$$

for the $p = 3$ standardized random variables Z_1, Z_2, and Z_3 can be generated by the $m = 1$ factor model

$$\begin{aligned} Z_1 &= .9F_1 + \varepsilon_1 \\ Z_2 &= .7F_1 + \varepsilon_2 \\ Z_3 &= .5F_1 + \varepsilon_3 \end{aligned}$$

where $\text{Var}(F_1) = 1$, $\text{Cov}(\boldsymbol{\varepsilon}, F_1) = \mathbf{0}$, and

$$\boldsymbol{\Psi} = \text{Cov}(\boldsymbol{\varepsilon}) = \begin{bmatrix} .19 & 0 & 0 \\ 0 & .51 & 0 \\ 0 & 0 & .75 \end{bmatrix}$$

That is, write $\boldsymbol{\rho}$ in the form $\boldsymbol{\rho} = \mathbf{LL}' + \boldsymbol{\Psi}$.

9.2. Use the information in Exercise 9.1.

(a) Calculate communalities h_i^2, $i = 1, 2, 3$, and interpret these quantities.

(b) Calculate $\text{Corr}(Z_i, F_1)$ for $i = 1, 2, 3$. Which variable might carry the greatest weight in "naming" the common factor? Why?

9.3. The eigenvalues and eigenvectors of the correlation matrix $\boldsymbol{\rho}$ in Exercise 9.1 are

$$\begin{aligned} \lambda_1 &= 1.96, & \mathbf{e}_1' &= [.625, .593, .507] \\ \lambda_2 &= .68, & \mathbf{e}_2' &= [-.219, -.491, .843] \\ \lambda_3 &= .36, & \mathbf{e}_3' &= [.749, -.638, -.177] \end{aligned}$$

(a) Assuming an $m = 1$ factor model, calculate the loading matrix $\mathbf{L}$ and matrix of specific variances $\boldsymbol{\Psi}$ using the principal component solution method. Compare the results with those in Exercise 9.1.

(b) What proportion of the total population variance is explained by the first common factor?

9.4. Given $\boldsymbol{\rho}$ and $\boldsymbol{\Psi}$ in Exercise 9.1 and an $m = 1$ factor model, calculate the reduced correlation matrix $\widetilde{\boldsymbol{\rho}} = \boldsymbol{\rho} - \boldsymbol{\Psi}$ and the principal factor solution for the loading matrix $\mathbf{L}$. Is the result consistent with the information in Exercise 9.1? Should it be?

$$\text{Sum of squared entries of } (\mathbf{S} - (\widetilde{\mathbf{L}}\widetilde{\mathbf{L}}' + \widetilde{\boldsymbol{\Psi}})) \le \hat{\lambda}_{m+1}^2 + \cdots + \hat{\lambda}_p^2 \qquad (9\text{-}19)$$

9.5. Establish the inequality (9-19).

Hint: Since $\mathbf{S} - \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}' - \widetilde{\boldsymbol{\Psi}}$ has zeros on the diagonal,

(sum of squared entries of $\mathbf{S} - \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}' - \widetilde{\boldsymbol{\Psi}}$) $\le$ (sum of squared entries of $\mathbf{S} - \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}'$)

Now, $\mathbf{S} - \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}' = \hat{\lambda}_{m+1}\hat{\mathbf{e}}_{m+1}\hat{\mathbf{e}}_{m+1}' + \cdots + \hat{\lambda}_p\hat{\mathbf{e}}_p\hat{\mathbf{e}}_p' = \hat{\mathbf{P}}_{(2)}\hat{\boldsymbol{\Lambda}}_{(2)}\hat{\mathbf{P}}_{(2)}'$, where $\hat{\mathbf{P}}_{(2)} = [\hat{\mathbf{e}}_{m+1} \vdots \cdots \vdots \hat{\mathbf{e}}_p]$ and $\hat{\boldsymbol{\Lambda}}_{(2)}$ is the diagonal matrix with elements $\hat{\lambda}_{m+1}, \ldots, \hat{\lambda}_p$.

Use (sum of squared entries of $\mathbf{A}$) $= \text{tr}\,\mathbf{AA}'$ and $\text{tr}[\hat{\mathbf{P}}_{(2)}\hat{\boldsymbol{\Lambda}}_{(2)}\hat{\boldsymbol{\Lambda}}_{(2)}\hat{\mathbf{P}}_{(2)}'] = \text{tr}[\hat{\boldsymbol{\Lambda}}_{(2)}\hat{\boldsymbol{\Lambda}}_{(2)}]$.

9.7. *(The factor model parameterization need not be unique.)* Let the factor model with $p = 2$ and $m = 1$ prevail. Show that

$$\sigma_{11} = \ell_{11}^2 + \psi_1, \qquad \sigma_{12} = \sigma_{21} = \ell_{11}\ell_{21}$$
$$\sigma_{22} = \ell_{21}^2 + \psi_2$$

and, for given σ_{11}, σ_{22}, and σ_{12}, there is an infinity of choices for $\mathbf{L}$ and $\boldsymbol{\Psi}$.

9.8. *(Unique but improper solution: Heywood case.)*
Consider an $m = 1$ factor model for the population with covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & .4 & .9 \\ .4 & 1 & .7 \\ .9 & .7 & 1 \end{bmatrix}$$

Show that there is a unique choice of $\mathbf{L}$ and $\boldsymbol{\Psi}$ with $\boldsymbol{\Sigma} = \mathbf{LL}' + \boldsymbol{\Psi}$, but that $\psi_3 < 0$, so the choice is not admissible.

Table 1.5 Air-Pollution Data

Wind (x_1)	Solar radiation (x_2)	CO (x_3)	NO (x_4)	NO_2 (x_5)	O_3 (x_6)	HC (x_7)
8	98	7	2	12	8	2
7	107	4	3	9	5	3
7	103	4	3	5	6	3
10	88	5	2	8	15	4
6	91	4	2	8	10	3
8	90	5	2	12	12	4
9	84	7	4	12	15	5
5	72	6	4	21	14	4
7	82	5	1	11	11	3
8	64	5	2	13	9	4
6	71	5	4	10	3	3
6	91	4	2	12	7	3
7	72	7	4	18	10	3
10	70	4	2	11	7	3
10	72	4	1	8	10	3
9	77	4	1	9	10	3
8	76	4	1	7	7	3
8	71	5	3	16	4	4
9	67	4	2	13	2	3
9	69	3	3	9	5	3
10	62	5	3	14	4	4
9	88	4	2	7	6	3
8	80	4	2	13	11	4
5	30	3	3	5	2	3
6	83	5	1	10	23	4
8	84	3	2	7	6	3
6	78	4	2	11	11	3
8	79	2	1	7	10	3
6	62	4	3	9	8	3
10	37	3	1	7	2	3
8	71	4	1	10	7	3
7	52	4	1	12	8	4
5	48	6	5	8	4	3
6	75	4	1	10	24	3
10	35	4	1	6	9	2
8	85	4	1	9	10	2
5	86	3	1	6	12	2
5	86	7	2	13	18	2
7	79	7	4	9	25	3
7	79	5	2	8	6	2
6	68	6	2	11	14	3
8	40	4	3	6	5	2

Source: Data courtesy of Professor G. C. Tiao.

$$\hat{\mathbf{f}}_j = (\hat{\mathbf{L}}'\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{L}})^{-1}\hat{\mathbf{L}}'\hat{\mathbf{\Psi}}^{-1}(\mathbf{x}_j - \hat{\boldsymbol{\mu}})$$
$$= \hat{\mathbf{\Delta}}^{-1}\hat{\mathbf{L}}'\hat{\mathbf{\Psi}}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}}), \qquad j = 1, 2, \ldots, n$$

or, if the correlation matrix is factored (9-50)

$$\hat{\mathbf{f}}_j = (\hat{\mathbf{L}}_z'\hat{\mathbf{\Psi}}_z^{-1}\hat{\mathbf{L}}_z)^{-1}\hat{\mathbf{L}}_z'\hat{\mathbf{\Psi}}_z^{-1}\mathbf{z}_j$$
$$= \hat{\mathbf{\Delta}}_z^{-1}\hat{\mathbf{L}}_z'\hat{\mathbf{\Psi}}_z^{-1}\mathbf{z}_j, \qquad j = 1, 2, \ldots, n$$

where $\mathbf{z}_j = \mathbf{D}^{-1/2}(\mathbf{x}_j - \bar{\mathbf{x}})$, as in (8-25), and $\hat{\boldsymbol{\rho}} = \hat{\mathbf{L}}_z\hat{\mathbf{L}}_z' + \hat{\mathbf{\Psi}}_z$.

$$\hat{\mathbf{f}}_j = \begin{bmatrix} \dfrac{1}{\sqrt{\hat{\lambda}_1}}\hat{\mathbf{e}}_1'(\mathbf{x}_j - \bar{\mathbf{x}}) \\ \dfrac{1}{\sqrt{\hat{\lambda}_2}}\hat{\mathbf{e}}_2'(\mathbf{x}_j - \bar{\mathbf{x}}) \\ \vdots \\ \dfrac{1}{\sqrt{\hat{\lambda}_m}}\hat{\mathbf{e}}_m'(\mathbf{x}_j - \bar{\mathbf{x}}) \end{bmatrix} \qquad (9\text{-}51)$$

$$\hat{\mathbf{f}}_j = \hat{\mathbf{L}}'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}}), \qquad j = 1, 2, \ldots, n$$

or, if a correlation matrix is factored, (9-58)

$$\hat{\mathbf{f}} = \hat{\mathbf{L}}_z'\mathbf{R}^{-1}\mathbf{z}_j, \qquad j = 1, 2, \ldots, n$$

where, see (8-25),

$$\mathbf{z}_j = \mathbf{D}^{-1/2}(\mathbf{x}_j - \bar{\mathbf{x}}) \quad \text{and} \quad \hat{\boldsymbol{\rho}} = \hat{\mathbf{L}}_z\hat{\mathbf{L}}_z' + \hat{\mathbf{\Psi}}_z$$

9.20. Using the air-pollution variables X_1, X_2, X_5, and X_6 given in Table 1.5, generate the sample *covariance* matrix.

(a) Obtain the principal component solution to a factor model with $m = 1$ and $m = 2$.

(b) Find the maximum likelihood estimates of $\mathbf{L}$ and $\mathbf{\Psi}$ for $m = 1$ and $m = 2$.

(c) Compare the factorization obtained by the principal component and maximum likelihood methods.

9.21. Perform a varimax rotation of both $m = 2$ solutions in Exercise 9.20. Interpret the results. Are the principal component and maximum likelihood solutions consistent with each other?

9.22. Refer to Exercise 9.20.

(a) Calculate the factor scores from the $m = 2$ maximum likelihood estimates by (i) weighted least squares in (9-50) and (ii) the regression approach of (9-58).

(b) Find the factor scores from the principal component solution, using (9-51).

(c) Compare the three sets of factor scores.