다항분포

$$f(x_1,\ldots,x_k;n,p_1,\ldots,p_k) = \Pr(X_1 = x_1 \text{ and } \ldots \text{ and } X_k = x_k)$$
$$= \begin{cases} \dfrac{n!}{x_1!\cdots x_k!} p_1^{x_1} \times \cdots \times p_k^{x_k}, & \text{when } \sum_{i=1}^k x_i = n \\ \\ 0 & \text{otherwise,} \end{cases}$$

$$E(X_i) = \sum_{x_1+\cdots+x_k=n} (x_i) \cdot \frac{n!}{x_1!\cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

$$= \underbrace{\sum \frac{(n-1)!}{x_1!\cdots(x_i-1)!\cdots x_k!} p_1^{x_1} \cdots p_i^{x_i-1} \cdots p_k^{x_k}}_{=1} \cdot np_i$$

$$= np_i$$

$$E(X_i(X_i-1)) = \sum_{x_1+\cdots+x_k=n} x_i(x_i-1) \cdot \frac{n!}{x_1!\cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

$$= \cdots = n(n-1)p_i^2$$

$$\therefore E(X_i^2) = n(n-1)p_i^2 + np_i$$

$$\therefore Var(X_i) = E(X_i^2) - E(X_i)^2$$

$$= n(n-1)p_i^2 + np_i - n^2p_i^2$$

$$= np_i(1-p_i)$$

$$E(X_iX_j) = \sum_{x_1+\cdots+x_k=n} (x_i x_j) \frac{n!}{x_1!\cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

$$= \sum \frac{(n-2)!}{x_1!\cdots(x_i-1)!\cdots(x_j-1)!\cdots x_k!} p_1^{x_1}\cdots p_i^{x_i-1}\cdots p_j^{x_j-1}\cdots p_k^{x_k}\, n(n-1)p_ip_j$$

$$= n(n-1)p_ip_j$$

$$\therefore \mathrm{Cor}(X_i, X_j)$$

$$= E[(X_i - E(X_i))(X_j - E(X_j))]$$

$$= E(X_iX_j) - E(X_i)E(X_j)$$

$$= n(n-1)p_ip_j - (np_i)(np_j)$$

$$= -np_ip_j$$