UNIVERSIDAD DE COSTA RICA
OFICINA DE PLANIFICACIÓN UNIVERSITARIA
SISTEMA INSTITUCIONAL PLAN PRESUPUESTO

INFORMACION GENERAL DEL PROYECTO

Unidad:	02023500-CENTRO DE INVESTIGACIONES EN MATEMATICA PURA Y APLICADA	Proyecto:	Pry01-1261-2024-Funciones potenciales no regulares y leyes de Tracy-Widom de orden superior
Código de Inscripción:		Estado:	Registrado
Fecha de Inicio:	01/08/2023	Ampliación / Renovación:	
Fecha de Finalización:	31/07/2026	Usuario:	alexander.ramirez_g

Actividad / Subactividad

Investigación — Investigación Básica

Descripción:

Este proyecto de investigación pretende estudiar el comportamiento en el borde de beta-ensembles invariantes de matrices con función potencial no convexa donde las leyes estándar de Tracy-Widom no se cumplen. Su propósito es encontrar descripciones alternativas para leyes ya descubiertas en el caso beta = 2 así como encontrar nuevas distribuciones para beta general. El plan es aplicar métodos desarrollados previamente en conjunto con otros nuevos para tales propósitos. Existen conjeturas sobre estas leyes que esperamos poder probar como ciertas o falsas.

This research project aims to study the edge behavior of invariant matrix beta- ensembles with non convex potential where standard Tracy-Widom behavior is not expected at the edges. Its purpose is to find alternate descriptions for laws already found in the beta = 2 case and also find new laws for general beta. The plan is to apply previously developed methos in conjunction with new ones for those purposes. There exist conjectures on the limiting laws and we expect to prove or disprove them.

Observaciones:

Costo Total del proyecto:

Monto estimado UCR:	0.00
Entes externos:	0.00
Total:	0.00

Entes externos:

No hay información registrada

Unidades participantes en el proyecto:

01030205	ESCUELA DE MATEMATICA
020235	CENTRO DE INVESTIGACIONES EN MATEMATICA PURA Y APLICADA

Instituciones participantes en el proyecto:

No hay información registrada

Adscripciones con programas inscritos en las Vicerrectorías:

No hay información registrada

Información acerca de los encargados del proyecto

Tipo Participación	Identificación	Nombre	Grado	Unidad/Inst. Pertenece	Estado en Régimen	Nombramiento	Horas Propias	Horas Adicional
Investigador principal / Responsable	401570340	ALEXANDER RAMIREZ GONZALEZ	Doctorado académico	ESCUELA DE MATEMÁTICA	CATEDRÁTICO(A)	N/A	N/A	N/A

OBSERVACIONES

Temáticas y Sub Temáticas Equidad e Igualdad de Genéro asociadas al proyecto

ESTRUCTURA DEL PROYECTO

ANTECEDENTES DEL PROYECTO

Ver PDF

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\title{A general beta crossover ensemble}

\author{
 {Jose A.~Ram\'irez}\footnote{CIMPA, Universidad de Costa Rica. e-mail: {\tt{alexander.ramirez$\_$g@ucr.ac.cr}}}
 }


\date{}
 % Activate to display a given date or no date

\begin{document}
 \maketitle
 This proposal is for a joint work with Brian Rider from Temple University.

Ever since the introduction of Tracy-Widom laws in the context of random matrix theory \cite{TW}, their use has expanded to permeate many different fields of mathematics and physics \cite{D}. That first result characterized the law $TW_2$ of the first eigenvalue of a GUE matrix as
 \begin{equation}
 F_2(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right)
 \end{equation}
 where $q$ is the solution to a Painlevé II equation.

There have been many generalizations and different characterizations of these laws. One class of results that play both roles are the $\beta$-ensembles introduced by Sutton in his thesis \cite{S} (proved rigorously in \cite{RRV}). For example, the $TW_2$ law may be viewed as the law of the smallest eigenvalue of the random operator $\mathcal{H}$ acting on functions $f \in L^2[ [0,\infty), \mathbb{C}]$ defined by
 \begin{equation}
 \label{SAOr}
  \mathcal{H} = - \frac{d^2}{dx^2} + x + \sqrt{2} \mathcal{B}_x^{\prime}, \qquad f(0) = 0.
 \end{equation}
 This result stands, among others, because of the novelty of the description but also because of the methodology employed in its proof: it relies on a matrix ensemble composed of sparse matrices (tridiagonal matrices, in this case). Note that, the limiting density for this models has a square-root like behavior at the edges (in the case of GUE, for instance, the density is a semicirle), this is going to be important in what follows.
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A different but related direction is provided by the study of matrix ensembles where the density has the form $e^{-n \tr V(M)} dM$. Under assumptions of convexity for the potential V, this was studied in \cite{KRV} (also see \cite{BEY} and \cite{BFG}). As in the case of GUE the limiting density of eigenvalues ψ_V for the matrix model $e^{-n \tr V(M)} dM$ generically vanishes like a square-root at its endpoints. Taking the rightmost edge of support in particular this leads to Airy asymptotics for the corresponding OPs (thinking first of $\beta = 1, 2, 4$) and Tracy-Widom fluctuations for the maximal eigenvalue $\lambda_{max}(V)$. That is ``regular'' behavior is indeed generic in a suitable sense is proved in \cite{KM}. Of course, the simplest criteria for a potential V to be regular is convexity (which also implies that ψ_V has one band of support).

However, it is possible to find V's for which $\psi_V(x)$, supported say on $[L, R]$ vanishes like $(R -x)^{\frac{4k+1}{2}}$ for $k=1,2, \dots$ ($k=0$ is regular). For such V's you can't get Airy behavior is a vicinity of R, and should instead see ``higher order Tracy-Widom laws'' indexed by k (and with different fluctuation exponents - expect $n^{2/(4k+3)}$ fluctuations, expanding upon the usual $n^{2/3}$). For $\beta =2$ this problem was treated in \cite{CIK}.

In \cite{KRV} a conjecture is put forward for the general beta higher order Tracy-Widom law. We have some preliminary evidence that the conjecture is wrong, but no rigorous results. The main purpose of this project is to analize this type of behaviour, find out if the conjecture is wrong, find the right conjecture and prove rigorous results in that respect.

We will start the projecto with a particular potential, namely,

\begin{equation}
\label{quarticpot}
V(x) = \frac{1}{20} x^4 - \frac{4}{15} x^3 + \frac{1}{5} x^2 + \frac{8}{5} x
\end{equation}

for which $\psi_V(x) = \frac{1}{10 \pi } (x+2)^{1/2} (2-x)^{5/2}$.

In this case the fluctuations of λ_{max} in the corresponding matrix model with weight $e^{-n \frac{\beta}{2} \tr V(M)}$ are of order $n^{-2/7}$ (at least this is known for $\beta =2$ \cite{CIK}) and so, in any operator limit, the continuum scale will be $\Delta x = c n^{-1/7}$. We expect to find a stochastic operator representation for the limiting spectrum of the corresponding ensemble on the edge.

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{F. \ Bekerman, A. \ Figalli, A. \ Guionnet}. (2015)
Transport maps for β-matrix models and universality.
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{P. \ Bourgade, L. \ Erd\"os, H.T. \ Yau}. (2014)
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{T.\ Claeys, A.\ Its, I.\ Krasovsky}. (2010)
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{M.\ Krishnapur, B.\ Rider, B.\ Vir\'ag} (2016)
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{\em Comm. Math. Phys.} {\bf 159}, no. 1.

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\end{thebibliography}

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Justificación del proyecto:

Se conoce que el objeto de estudio del proyecto explica diversos fenómenos en física matemática a través del Hamiltoniano de sistemas cuánticos. También es relevante en el estudio de matrices de covarianza tales como las que se encuentran en la manipulación de big data. La motivación original de las matrices aleatorias era el estudio de sistemas cuánticos complejos. Es en esta área que las leyes de Tracy Widom tienen relevancia directa, dado que modelan el comportamiento de los estados de energía más alta del sistema. En estadística, las matrices de covarianza donde hay un número grande de variables es de magnitud similar al de observaciones provee un caso típico en los problemas de ciencia de datos modernos. Las leyes de Tracy Widom tienen aplicación directa en ese caso.

The object of study of the project has been found to explain diverse phenomena in mathematical physics through the Hamiltonians of quantum systems. Also it is of relevance in the study of covariance matrices such as the ones used in the manipulation of big data. The original motivation for random matrices was the study of complex quantum systems. It is in this area that the Tracy Widom Laws have direct relevance, since they model the behaviour of the highest energy states in the system. For the case of statistics, the covariance matrices of a large number of variables where the number of observations is of similar magnitud provides a typical case in today's data science problems. The laws of Tracy Widom also have direct application in that situation.

Descriptores:

16366 - Teoría de las probabilidades

Áreas de Impacto:

DESARROLLO CIENTÍFICO

DESARROLLO TECNOLÓGICO

¿En qué consiste el impacto?:

These project will add to the understanding of the natural probabilistic laws that appear in certain phenomena that is of relevance in mathematics, physics and statistics.

Población Beneficiada Directa:

¿Quién o quiénes se benefician?:

The scientific community are the direct beneficiaries of the project, but its applications might reach the general community through its use in other applied fields.

Beneficios para la población:

Better understanding of natural phenomena and, through applications, technological development.

Beneficios para la Universidad:

Increased reputation. Impact on technological development which, in turn, benefits society in general, improving the university's standing in the country.

Objetivos y Metas de Desarrollo Sostenible

Objetivo: 9 : Industria, innovación e infaestructura

Metas seleccionadas del objetivo número: 9

Meta 9.b - Apoyar el desarrollo de tecnologías, la investigación y la innovación nacionales en los países en desarrollo, incluso garantizando un entorno normativo propicio a la diversificación industrial y la adición de valor a los productos básicos, entre otras cosas.

Objetivos y Metas

Objetivo general:

To describe the laws that model the behavior of edge eigenvalues for irregular potentials

Objetivo específico 1 : Investigación

Find and prove a stochastic operator description for the limiting edge law in one particular case

Meta 1 - Cualitativa

Find description, through the theory of stochastic differential operators, of the edge law for a beta ensemble with a particular potential.

Indicador 1

Existence of a reasonable conjecture for the needed description.

Meta 2 - Cualitativa

Prove the convergence to the conjectured limiting laws

Indicador 1

Existence of the proof

Objetivo específico 2 : Investigación

Analize the properties of the limiting law found for the edge spectrum

Meta 1 - Cualitativa

Prove a theorem stating the main asymptotic behavior of the limiting laws

Indicador 1

Existence of the theorem

Objetivo específico 3 : Investigación

Conjecture a heuristic description for higher order Tracy Widom laws that explain the unusual behavior of the limiting laws under irregular potentials

Meta 1 - Cualitativa

State a set of equations that would describe a reasonable conjecture for the higher order Tracy Widom laws associated to irregular potentials

Indicador 1

Existence of the conjectured hierarchy

Objetivo específico 4 : Investigación

Prove convergence of the edge limit for invariant ensembles to the conjectured hierarchy of limit laws

Meta 1 - Cualitativa

Prove that, under suitable conditions, the edge spectrum converges, in the appropriate sense, to the limit laws that were put forward in the previous objective.

Indicador 1

Existence of the proof

Ubicación geográfica del proyecto

País	Provincia	Cantón	Distrito	Región
COSTA RICA	SAN JOSÉ	MONTES DE OCA	MERCEDES	CENTRAL

No se realizan actividades en ninguna área protegida, o no se especificaron

Objetivos y políticas asociadas al proyecto, según catálogo de Políticas Institucionales

Objetivo asociado

1.3.1 - Facilitar a la sociedad el acceso a la producción científica, artística y cultural, generada por la vinculación remunerada y no remunerada, de manera que se contribuya activamente a la transformación social.

Políticas según objetivo asociado: Eje/Política: 1.3 - Fomentará que todas las actividades de vinculación con el sector externo respondan a los
principios, propósitos y valores institucionales, así como a las necesidades del país.

Objetivo asociado

10.3.3 - Potenciar la investigación que fortalezca las economías locales y nacionales, fundamentadas en los sistemas ambientales y agroalimentarios.

Políticas según objetivo asociado: Eje/Política: 10.3 - Fortalecerá el quehacer universitario para contribuir con el análisis de la seguridad y soberanía alimentaria y nutricional.

Objetivo asociado

2.1.3 - Fomentar en el quehacer académico del profesorado la honestidad intelectual y el rigor científico.

Políticas según objetivo asociado: Eje/Política: 2.1 - Desarrollará el personal docente idóneo para responder a las necesidades y retos de la sociedad,
comprometido con el mejoramiento académico y la integridad individual, con el fin de formar
personas profesionales críticas y humanistas.

METODOLOGÍA DEL PROYECTO

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The proposed approach is based on, for the particular case to be considered:

\begin{enumerate}
\item The ideas of the paper \cite{KRV} will be used to find a sparse matrix representation for the invariant laws.
\item A degeneracy in the calculation will have to be exploited in order to go past the standard TW laws and into higher order. This will require, in all likelihood, higher order stochastic differential equations.
\item Ideas from a previous project will have to be included to deal with the tail of the matrix.
\item Some type of concentration will be needed in order to finish the convergence. This is likely the most difficult part because of the non convexity.
\end{enumerate}
For the general case, the general description of the steps is basically the same, except there is a great deal of extra work in the following:
\begin{enumerate}
\item The degeneracy will have to be deeply understood because this is the basis for the generalization to other models. Preliminary studies show that this is related to the nature of the cancellations in the Hamiltonian for the matrix models.
\end{enumerate}

Recursos con que cuenta el proyecto:

Information systems in the university (databases), computing systems (Matlab, R, Python, Mathematica), typesetting systems (Latex), travel support, communication systems (Zoom, Teams)

Animales, seres humanos y biodiversidad:

¿Hace uso de animales?	NO
¿El proyecto trabaja con seres humanos?	NO
¿Requiere acceso a elementos de biodiversidad?	NO

Modalidad:

Proyectos de investigación

EVALUACIÓN

Evaluación del impacto:

The publication of at least one article in an indexed journal with a good reputation

PRESUPUESTO

No hay información registrada

Acciones por proyecto y objeto del gasto

No hay información registrada

RÉGIMEN BECARIO

No hay información registrada

CRONOGRAMA DE ACTIVIDADES

Actividades	Fecha Inicial	Fecha Final
Find the description for the limiting laws in the particular case chosen for this	01/08/2023	31/07/2026
Prove the limit for the particular case	01/01/2024	31/12/2024
Write partial report	01/06/2024	30/06/2024
Find the properties for the limit laws	01/01/2025	30/06/2025
Write partial report	01/06/2025	30/06/2025
Find a description for the limit laws, in the general case	01/07/2025	31/12/2025
Prove the limit in the general case	01/01/2026	31/07/2026
Write article	01/06/2026	30/06/2026
Write final report	01/07/2026	31/07/2026