ZYZ Decomposition

$$U : \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} e^{i\delta + \alpha/2 + \beta/2} . \cos \theta/2 & e^{i\delta + \alpha/2 - \beta/2} . \sin \theta/2 \\ -e^{i\delta - \frac{\alpha}{2} + \beta/2} . \sin \theta/2 & e^{i\delta - \alpha/2 - \beta/2} . \cos \theta/2 \end{bmatrix}$$

If U is Unitary i.e. $U . U^{\dagger} = I$

$$\frac{B}{C} = -e^{\alpha - \beta} \qquad \alpha - \beta = \log_e(-B/C)$$

$$\frac{A}{D} = +e^{\alpha + \beta} \qquad \alpha + \beta = \log_e(A/D)$$

$$2\alpha = \log_e(-B/C) + \log_e(A/D)$$
$$2\beta = \log_e(A/D) - \log_e(-B/C)$$

$$\alpha = \frac{\log_e\left(-\frac{AB}{CD}\right)}{2} \qquad \beta = \frac{\log_e\left(-\frac{AC}{BD}\right)}{2}$$

$$\frac{A}{B} = \frac{e^{\beta}}{\tan \theta/2} \qquad \theta = 2\tan^{-1}\left(\frac{B . e^{\beta}}{A}\right) \pm n\pi$$

$$A . D = e^{2i\delta} \cos^2 \theta/2 \qquad \delta = \frac{\log_e\left(\frac{AD}{\cos^2 \theta/2}\right)}{2i}$$

$$SU : \begin{bmatrix} e^{\alpha/2 + \beta/2} . \cos \theta/2 & e^{\alpha/2 - \beta/2} . \sin \theta/2 \\ -e^{\alpha/2 + \beta/2} . \sin \theta/2 & e^{-\alpha/2 - \beta/2} . \cos \theta/2 \end{bmatrix}$$

If SU is Special Unitary i.e. $SU . SU^{\dagger} = I$ and $\det(SU) = +1$

$U : e^{i\delta} . SU$ since Global Arbitrary Phase is unmeasurable use SU.

$$\det(U) = AD - BC = e^{i\delta}\left(\cos^2 \theta/2 + \sin^2 \theta/2\right) = 1 \quad \text{if } SU$$

$$\therefore \delta = \frac{\log_e(1)}{i} = 0 \quad \text{for } SU$$

$W \in SU(2)$

$W = R_z(\alpha) . R_y(\theta) . R_z(\beta)$

$A = R_z(\alpha) . R_y(\theta/2)$

$B = R_y(-\theta/2) . R_z\left(-\frac{\alpha+\beta}{2}\right)$

$C = R_z\left(\frac{\beta-\alpha}{2}\right)$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

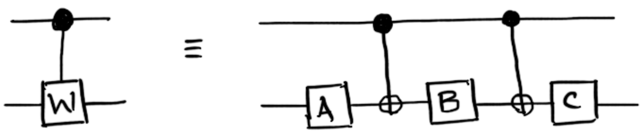
$R_z(\alpha) = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}$

$R_y(\theta) = \begin{bmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{bmatrix}$

$A . B . C = I$

$A . X . B . X . C = W$

Controlled - W

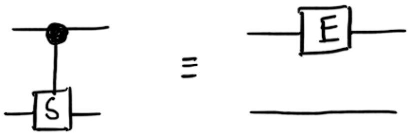


$S = Ph(\delta)$

$Ph(\delta) = \begin{bmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{bmatrix}$

$E = R_z(-\delta) . Ph(\delta/2)$

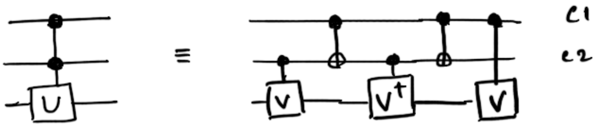
Controlled - S



$\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\delta} & 0 \\ 0 & 0 & 0 & e^{i\delta} \end{bmatrix}$

$\therefore$ Arbitrary Unitary Matrix U :

Controlled - U = Controlled - S * Controlled - W

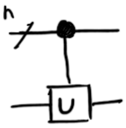


			c1	c2
✗	✗	✗	0	0
✓	✓	✗	0	1
✗	✓	✓	1	0
✓	✗	✓	1	1

$V \cdot V^{\dagger} = I$

$V^{\dagger} \cdot V = I$

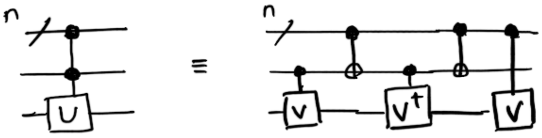
$V \cdot V = U$



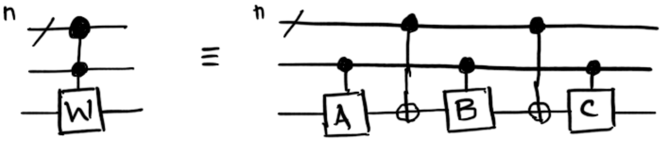
$$V^{2^{n-1}} = U$$

$$x_1 - (x_1 \oplus x_2) + x_2 - (x_2 \oplus x_3) + (x_1 \oplus x_2 \oplus x_3) - (x_1 \oplus x_3) + x_3 = 4 \cdot (x_1 \wedge x_2 \wedge x_3)$$

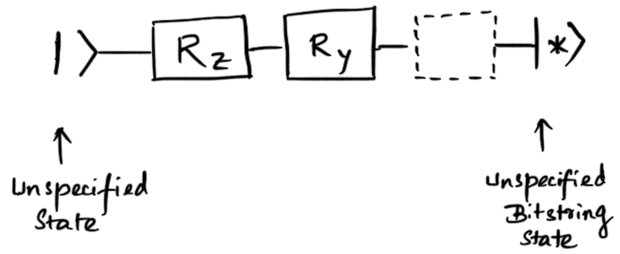
$$\left.\begin{array}{l}\text{Zeroed Ancilla}\\ \text{Borrowed Ancilla}\end{array}\right\} \text{for } \Lambda_n(\sigma_x)$$

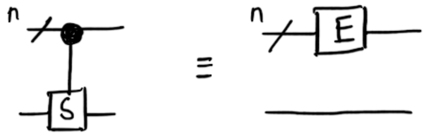


$V^2 = U$



$W \in SU(2)$



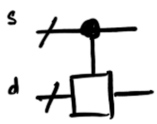


E : Diagonal Matrix $\boxed{\Delta}$

S : $Ph(\delta)$

QMUX

$$U_m = \begin{bmatrix} U_0 & & & 0 \\ & U_1 & & \\ & & \ddots & \\ 0 & & & U_n \end{bmatrix}$$ Block Diagonal



$\equiv U_m \in \mathcal{H}^{\otimes s+d}$

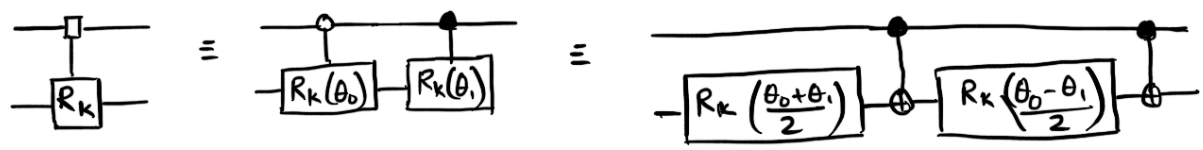
s : select bits
d : data bits

U_i size is $2^d \times 2^d$
n size is 2^s

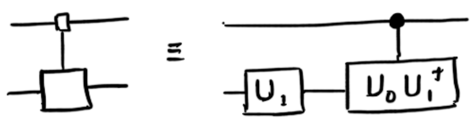
e.g. CNOT $s=1$ $d=1$ $U = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}$

$$R = \begin{bmatrix} R_k(\theta_0) & 0 \\ 0 & R_k(\theta_1) \end{bmatrix} \quad \begin{matrix} s=1 \\ d=1 \end{matrix}$$

R $\equiv$

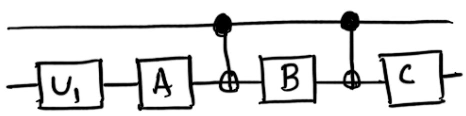


$U_m = \begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix} \equiv$



$$W = U_0 U_1^{\dagger} = A.X.B.X.C$$

$U_m \equiv$



Multiplexer Extension Property

$s\,?(A_0.B_0):(A_1.B_1) \equiv (s\,?A_0:A_1).(s\,?B_0:B_1)$