# Assignment 2

# DSECL ZC416 - Mathematical Foundations for Data Science

**Group Name:**

Group132

**Contribution Table:**

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## Q1)

i) Write a code to generate five random vectors a1, a2, a3, a4, a5 ∈R5 and verify that these set of vectors are linearly independent.

## Program

```
import numpy as np
a1 = np.random.rand(5)
a2 = np.random.rand(5)
a3 = np.random.rand(5)
a4 = np.random.rand(5)
a5 = np.random.rand(5)
A = np.array([a1, a2, a3, a4, a5])
print(np.linalg.det(A))
```

## Result

```
-0.001224671544199194
```

ii Write a function that applies Gram-Schmidt Algorithm on the set of vectors a1, a2, a3, a4, a5. Let the output of Gram Schmidt algorithm be the vectors q1, q2, q3, q4, q5.

**Program**

```
import numpy as np
a1 = np.random.rand(5)
a2 = np.random.rand(5)
a3 = np.random.rand(5)
a4 = np.random.rand(5)
a5 = np.random.rand(5)
A = np.array([a1, a2, a3, a4, a5])
def gram_schmidt(A):
  Q = np.zeros_like(A)
  c = np.zeros_like(A[0])
  for i in range(len(A)):
    c = A[i]
    for j in range(i):
      c -= np.dot(Q[j], A[i]) * Q[j]
    Q[i] = c / np.linalg.norm(c)
  return Q
Q = gram_schmidt(A)
print(Q)
```

**Result**

```
[[ 2.30961296e-01  1.27926117e-01  9.40301304e-01  1.97321322e-01
   8.47911707e-02]
 [ 6.09711543e-01  2.92501383e-01 -2.95072975e-01  2.30156979e-01
   6.34550613e-01]
 [-2.22339846e-01 -2.64193327e-01  1.69100116e-01 -6.54175188e-01
   6.51326994e-01]
 [-6.95565808e-02 -7.93326385e-01 -1.30352728e-02  5.62229911e-01
   2.22537048e-01]
 [-7.21550822e-01  4.45996617e-01  7.93851477e-04  4.05023965e-01
   3.41183801e-01]]
```

iii Write a code to create a matrix Q ∈R5×5 with q1, q2, q3, q4, q5 as the five columns. Then create a matrix B = QTQ.

**Program**

```
import numpy as np
a1 = np.random.rand(5)
a2 = np.random.rand(5)
a3 = np.random.rand(5)
a4 = np.random.rand(5)
a5 = np.random.rand(5)
A = np.array([a1, a2, a3, a4, a5])
def gram_schmidt(A):
  Q = np.zeros_like(A)
  c = np.zeros_like(A[0])
  for i in range(len(A)):
    c = A[i]
    for j in range(i):
      c -= np.dot(Q[j], A[i]) * Q[j]
    Q[i] = c / np.linalg.norm(c)
  return Q
Q = gram_schmidt(A)
B= Q.T.dot(Q)
print(B)
```

**Result**

```
[[ 1.00000000e+00  3.62025709e-16  7.26214698e-17  9.87990047e-16
   1.20515991e-16]
 [ 3.62025709e-16  1.00000000e+00 -7.55850708e-16  1.48475619e-16
  -1.97811678e-16]
 [ 7.26214698e-17 -7.55850708e-16  1.00000000e+00 -1.15595609e-15
  -6.95194381e-16]
 [ 9.87990047e-16  1.48475619e-16 -1.15595609e-15  1.00000000e+00
  -2.00867220e-16]
 [ 1.20515991e-16 -1.97811678e-16 -6.95194381e-16 -2.00867220e-16
   1.00000000e+00]]
```

iv) Write a code to calculate the value of ‖B −I5‖F where ‖·‖F is the matrix Frobenius norm.

**Program**

```
import numpy as np
a1 = np.random.rand(5)
a2 = np.random.rand(5)
a3 = np.random.rand(5)
a4 = np.random.rand(5)
a5 = np.random.rand(5)
A = np.array([a1, a2, a3, a4, a5])
def gram_schmidt(A):
  Q = np.zeros_like(A)
  c = np.zeros_like(A[0])
  for i in range(len(A)):
    c = A[i]
    for j in range(i):
      c -= np.dot(Q[j], A[i]) * Q[j]
    Q[i] = c / np.linalg.norm(c)
  return Q
Q = gram_schmidt(A)
B= Q.T.dot(Q)
IIB=np.linalg.inv(B)
print(IIB)
print(np.linalg.norm(IIB-np.identity(5),'fro'))
```

**Result**

```
[[ 1.00000000e+00  3.65754321e-16 -1.05301682e-16 -1.30509378e-17
  -5.61915161e-17]
 [ 3.65754321e-16  1.00000000e+00  1.80053116e-16  2.68374791e-16
   1.69815156e-16]
 [-1.05301682e-16  1.80053116e-16  1.00000000e+00 -2.65537810e-17
  -5.08556100e-17]
 [-1.30509378e-17  2.68374791e-16 -2.65537810e-17  1.00000000e+00
   7.74237284e-17]
 [-5.61915161e-17  1.69815156e-16 -5.08556100e-17  7.74237284e-17
   1.00000000e+00]]
1.0125480534528262e-15
```

**Q2)** A carpenter makes tables and chairs. Each table can be sold for a profit of $30 and each chair for a profit of $10. The carpenter can afford to spend up to 40 hours per week working and takes six hours to make a table and three hours to make a chair. Customer demand requires that he makes at least three times as many chairs as tables.Tables take up four times as much storage space as chairs and there is room for at most four tables each week. Formulate this problem as a linear programming problem and solve it graphically.

Q2) Solution :-

Let $x$ = Number of tabels made per week

Let $y$ = Number of Chairs made per week.

Constraints :-

a) Total work time

$$6x + 3y <= 40$$

b) Customer demand

$$y >= 3x$$

c) Storage Space

$$x + \left(\frac{y}{4}\right) <= 4$$

d) all variables $>= 0$

Objective maximise - $30x + 10y$

Graphical representation,

Intersection of

$$x + \left(\frac{y}{4}\right) = 4 \quad — (1)$$

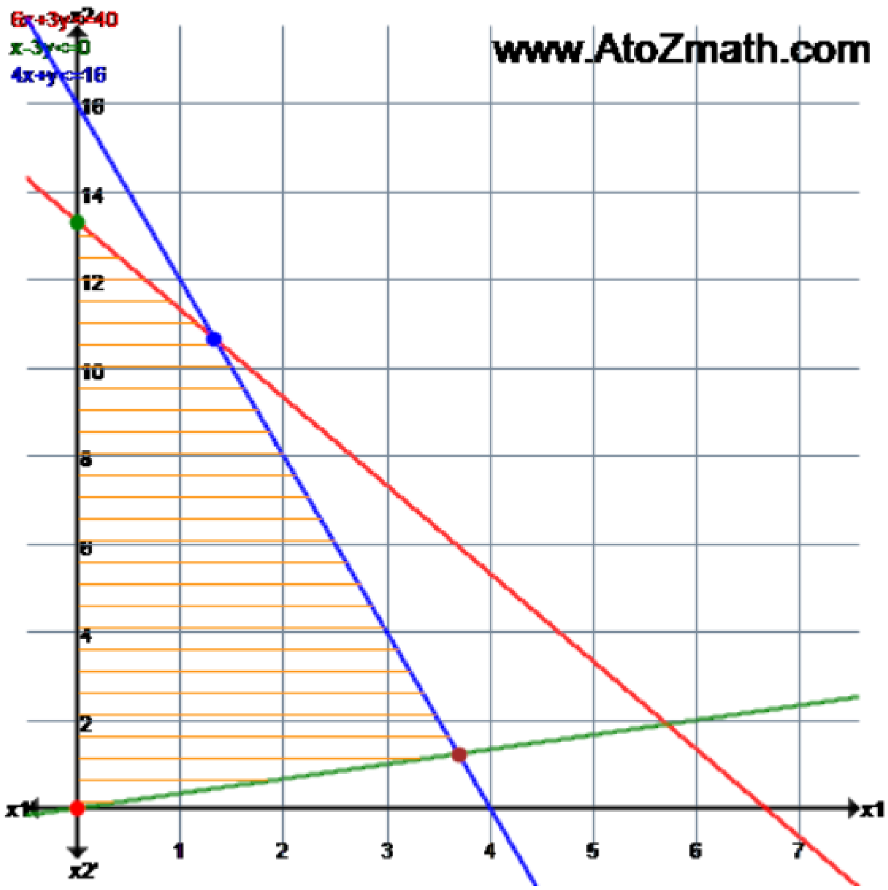
$$6x + 3y = 40 \quad — (2)$$

Solving equation,

We get $x = 1.3333$

$y = 10.6667$

Carpenter Profit => 146.667 #



6x+3y<=40
x-3y<=0
4x+y<=16
www.AtoZmath.com
x2
16
14
12
10
8
6
4
2
x1
x1
x2'
1
2
3
4
5
6
7

The value of the objective function at each these extreme points are,

$x_1 = X$
$x_2 = Y$

| Extreme Point Co-ordinates $(x_1, x_2)$ | Lines through extreme points | Objective function Value $Z = 30X_1 + 10X_2$ |
|---|---|---|
| $O(0,0)$ | $2 \rightarrow X_1 - 3X_2 \leq 0$<br>$\cancel{4 \rightarrow 6X_1 + 3X_2 \leq 40}$<br>$4 \rightarrow X_1 \geq 0$ | $30(0) + 10(0) = 0$ |
| $A(0, 13.33)$ | $1 \rightarrow 6X_1 + 3X_2 \leq 40$<br>$4 \rightarrow X_1 \geq 0$ | $30(0) + 10(13.33)$<br>$= 133.33$ |
| $B(1.33, 10.67)$ | $1 \rightarrow 6X_1 + 3X_2 \leq 0$<br>$3 \rightarrow 4X_1 + X_2 \leq 6$ | $30(1.33) + 10(10.67)$<br>$= 146.67$ |
| $C(3.69, 1.23)$ | $2 \rightarrow X_1 - 3X_2 \leq 0$<br>$3 \rightarrow 4X_1 + X_2 \leq 16$ | $30(3.69) + 10(1.23)$<br>$= 123.08$ |

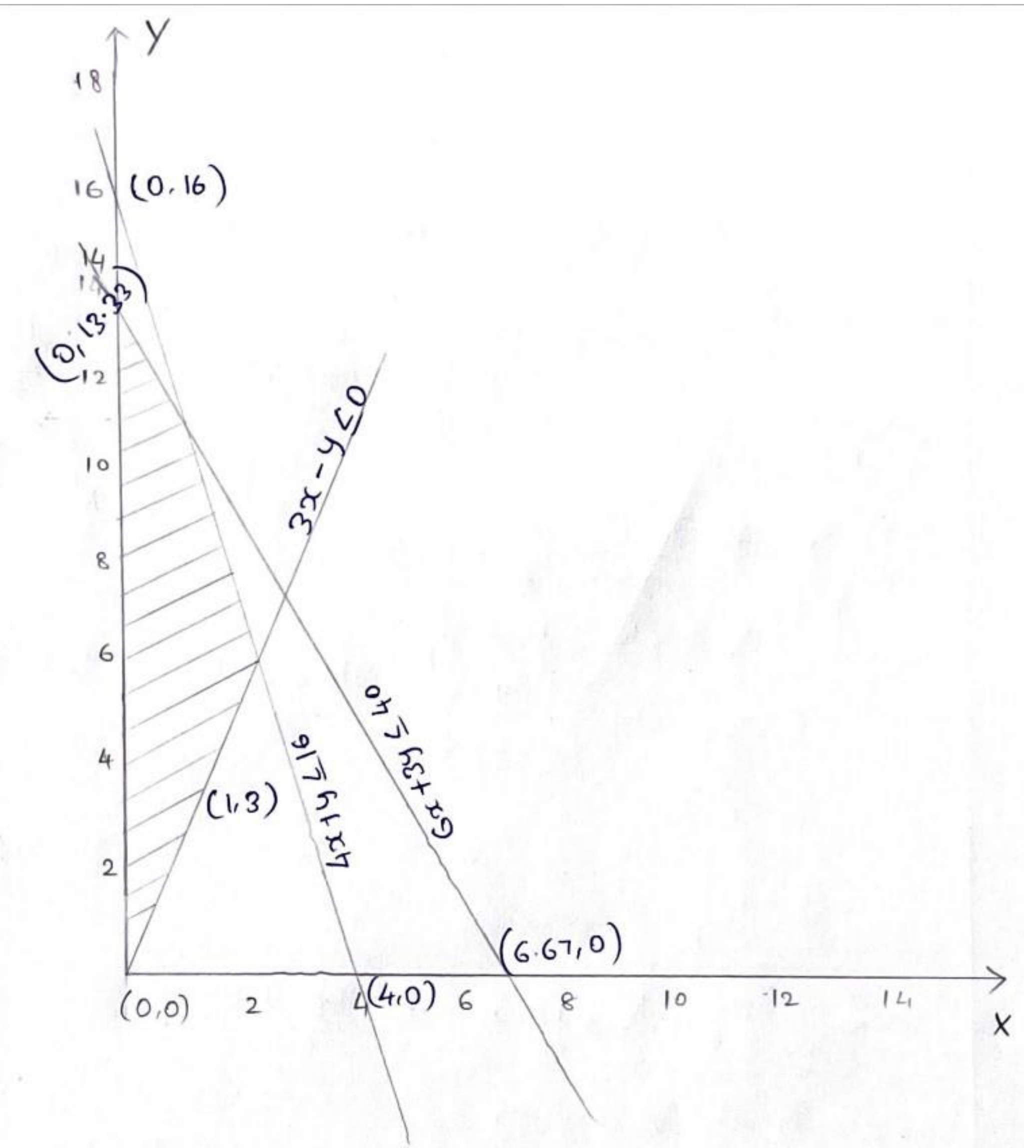
The maximum value of the objective function $Z = 146.67$ occurs at the extreme point $(1.33, 10.67)$

Hence,

the optimal solution to the given LP problem is:

$x_1 = 1.33$
$X_2 = 10.67$

Max $Z = 146.67$



Y
(0, 16)
(0, 13.33)
$3x - y \leq 0$
$4x + y \leq 16$
$6x + 3y \leq 40$
(1,3)
(6.67, 0)
(0,0)
(4,0)
X

**Q3)** For X≠ 0 let f (x) = 1/x where [t] denotes the greatest integer ≤t.

**(a) Sketch the graph off over the intervals [−2, −1/5] and [1/5, 2].**

Q3) a) Solution:-

For $x<0$;

In the interval $(-\infty, -1)$, $x \leq -1$

$\frac{1}{x} \geq -1$ & $\frac{1}{x} < 0$

$f(x) = -1$

For in the interval $(-1, -\frac{1}{2})$, $x \leq \frac{1}{2}$

$\frac{1}{x} \geq -2$ & $\frac{1}{x} \leq -1$

$f(x) = -2$

Likewise $-3, -4, -5$ respectively

For $x=0$;

let $f(x) = \frac{1}{x}$

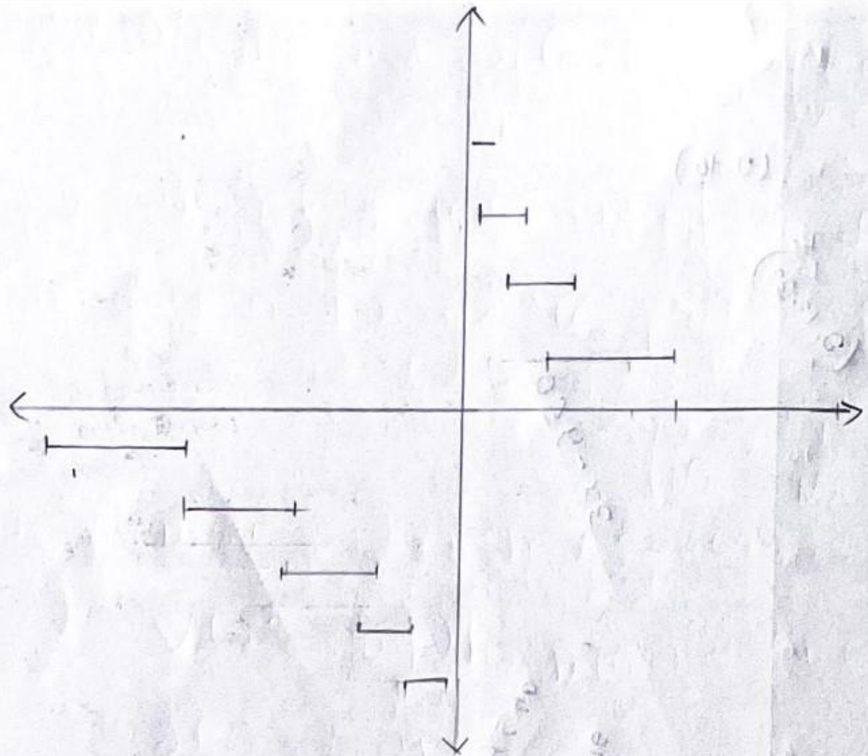
Where $[t]$ denotes the greatest integer $\leq t$

In general we have

$\left(\frac{1}{n+1}, \frac{1}{n}\right)$, $x > \frac{1}{n+1} \Rightarrow$

$\frac{1}{x} < n+1$ & $\frac{1}{x} \geq n$

$f(x) = \frac{1}{x} = n \;\forall x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$



**(b) What happens to f (x) as x →0 through positive values?**

Q3) b) Solution:-

f(x) as x → 0 through Positive valus.

$$\lim_{x\to 0^+} \Rightarrow f(x) = \lim_{x\to 0^+} \frac{1}{x}$$

$$f(x) = \frac{1}{x} \;:\; x > 0$$

Now,

if f is monotone decreasing function on $(0,\infty)$, f is bounded below on $(0,\infty)$; therefore

$$\lim_{x\to\infty} f(x) = \inf_{x\in(0,\infty)} f(x) = 0$$

f is unbounded on above $(0,\infty)$

therefore $\lim_{x\to 0^+} f(x) = +\infty$.

$$f(x) = +\infty$$

**(c) Through negative values?**

Q3) c) Solution :-

through negative values

So, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x}$

let $f(x) = \frac{1}{x}$ ; $x < 0$

f is monotone decreasing function on $(0, -\infty)$

hence f is bounded above on $(-\infty, 0)$

therefore $f(x) = 0$

$x \in (-\infty, 0)$

hence $\lim_{x \to -\infty} f(x) = \sup_{x \in (-\infty, 0)} f(x) = 0$

$\therefore$ f is unbounded below on $(-\infty, 0)$

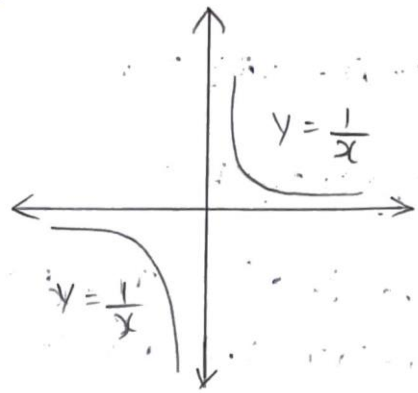
therefore $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$

So,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

**(d) Can you define f (0) so that f is continuous at 0?**

Q3) d) Solution :-




we see that when $x$ approaches 'Zero' the graph $f(x)$ approaches $\infty$,

hence we cannot define $f(0)$,

$\therefore$ f is continuous at 0.

$f(x)$ has an infinite discontinuity at $x=0$ as $\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to 0^-} f(x) = -\infty$

by defining $x^+ \to 0$, $\frac{1}{x} \to +\infty \Rightarrow \frac{1}{x} \to +\infty$

$\therefore$

a) $f(x) \to +\infty$ as $x \to 0$ through +ve values

b) $f(x) \to -\infty$ as $x \to 0$ through -ve values

c) Since f has an discontinuity at $x=0$,

f can be made continuously by defining $f(0)$.

**Q4) Let F (x, y) = x2 + y2 −xy. Compute:(a) Compute ∇f (1, 3)**

Q4) a) Solution:-

$$F(x,y) = x^2+y^2-xy$$

hence, $\nabla f(1,3)$ be;

To compute the gradient of a function we just simply differentiate the function partially

$$F(x,y) = x^2+y^2-xy$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x} \Rightarrow \frac{d}{dx}(x^2-xy+y^2) = \left(\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)-\frac{d}{dx}(xy)\right)$$

$$\doteq 2x-\left(\frac{d}{dx}(xy)\right) = 2x-\left(y\frac{d}{dx}(x)\right)$$

$$\frac{\partial f}{\partial x} = 2x-y$$

$$\frac{\partial f}{dy} \Rightarrow \frac{d}{dy}(x^2)-\frac{d}{dy}(xy)+\frac{d}{dy}(y^2)$$

$$\doteq -x\left(\frac{d}{dy}(y)\right)+2y = -x(1)+2y$$

$$\frac{\partial f}{\partial y} \Rightarrow -x+2y \Rightarrow x-2y$$

equating $\nabla(x^2-xy+y^2)(x,y) = (2x-y, -x+2y)$

$$\nabla(x^2-xy-y^2)\Big|_{(x,y)=(1,3)} = (-1, 5)$$

$$\therefore \nabla f(1,3) = (-1,5)$$

**(b) Compute the directional derivative off at (1, 3) in the direction of 1√2ˆi − 1√2ˆj**

Q4) b) Solution :-

Given ⇒ $x^2 + y^2 - xy$

$x^2 + y^2 - xy$ at $(x, y) = (1, 3)$

In the direction of vector $\vec{u} = \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$

$= \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$

the Gradient ⇒

$$\nabla (x^2 - xy + y^2) \big|_{(x,y) = (1,3)} = (-1, 5)$$

$$\frac{\partial f}{\partial x} = 2x - y$$

$$\frac{\partial f}{\partial y} = x - 2y$$

$$\therefore \nabla(1,3) = (-1, 5)$$

Length of the Vector ⇒

$$|\vec{u}| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2}$$

$$|\vec{u}| = 1$$

Since Length of Vector ⇒ 1

the Vector is already Normalized.

Q4) b) Directional derivative => (Gradient) * (Normalised vector)

$$D(x^2-xy+y^2)_{\vec{u}}(1,3) = (-1,5)\cdot\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$= -3\sqrt{2}$$

$$\therefore D(x^2-xy+y^2)_{\vec{u}}(1,3) = -4.242640$$

$$\therefore D(x^2-xy+y^2)_{\vec{u}}(1,3) = -4.24 \quad \text{(round off)}$$

**Q5) (a) Find and classify the critical points of f (x, y) = x2y + 3xy −3x2 − 4x + 2y + 1**

Q5) a) Solution:-

$$f(x,y) = x^2y + 3xy - 3x^2 - 4x + 2y + 1$$

First order Partial derivatives

$$\frac{\partial}{\partial x}\left(x^2y - 3x^2 + 3xy - 4x + 2y + 1\right) = 2xy - 6x + 3y - 4$$

$$f(x) = 2xy - 6x + 3y - 4$$

$$f(y) = x^2 + 3x + 2$$

$$\frac{\partial}{\partial y}\left(x^2y - 3x^2 - 3xy - 4x + 2y + 1\right) = (x+1)(x+2)$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}, \text{(or)} \begin{cases} 2xy - 6x + 3y - 4 = 0 \\ x^2 + 3x + 2 = 0 \end{cases}$$

for Critical point

$$f_x = 0 = f_y$$

$$2xy + 3y - 6x - 4 = 0$$

$$x^2 + 2x + x + 2 = 0$$

$$(x+1)(x+2) = 0$$

$$(x,y) = (-1, -2)$$

if $x = -1$

$$-2y + 3y + 6 - 4 = 0$$

$$y = -2$$

25) a) if $x = -2$

$$2xy + 3y - 6x - 4 = 0$$

$$-4y + 3y + 12 - 4 = 0$$

$$-y = -8$$

$$y = 8$$

$(x, y) = (-2, 8)$ & $(-1, -2) \Rightarrow$ Critical Points

Second order Partial derivatives;

$$\frac{\partial^2}{\partial x^2}(x^2y - 3x^2 + 3xy - 4x + 2y + 1) = 2y - 6 =$$

$$\Rightarrow 2(y-3)$$

$$\frac{\partial^2}{\partial y^2}(x^2y - 3x^2 + 3xy - 4x + 2y + 1) = \cancel{2x+3}\ 0$$

$$\frac{\partial^2}{\partial x \partial y}(x^2y - 3x^2 + 3xy - 4x + 2y + 1) = 2x + 3$$

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -(2x+3)^2$$

① Since $D(-2, 8) = -1 \Rightarrow f(-2, 8) \Rightarrow -3_A$

which is less than 0, Can be stated that $(-2, 8)$ is Saddle point.

② Since $D(-1, -2) = -1 \Rightarrow f(-1, -2) \Rightarrow 2_A$

which is less than 0, Can be stated that $(-1, -2)$ is Saddle Point.

**(b) Suppose the function f : ℜm → ℜ is even, i.e. f (x) = f (−x) for every x ∈ ℜm, and suppose f is differentiable. Find ∇f at the origin.**

Q5) b) Solution:-

$$f(x) = f(-x)$$

differentiating both sides,

$$f'(x) = f'(-x) \frac{d(-x)}{dx}$$

(i,e) $f'(x) = -f'(-x)$

$\nabla f$ at origin $= f'(x)$ as $(x=0)$.

$$f'(0) = -f'(0)$$

$$2f'(0) = 0$$

$$f'(0) = 0$$

$\therefore$ Gradient of $f(x)$ at origin $(x=0)$ $\Big\} = \nabla_0 f(x) = f(0) = 0$

$$\therefore \nabla_0 f(x) \Rightarrow f(0) = 0$$

**Q6) What is the volume of the largest box with edges parallel to the coordinate axes that fits the ellipsoid**
**x2/a2 + y2/b2 + z2/c2 = 1**

Q6) Solution:-

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

the equation of ellipsoid is $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Volume of the box $V = 2x \cdot 2y \cdot 2z \Rightarrow 8xyz$

$\therefore$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$z^2 = c^2 - \frac{c^2x^2}{a^2} - \frac{c^2y^2}{b^2}$$

$$z^2 = \frac{a^2b^2c^2 - c^2b^2x^2 - c^2a^2y^2}{a^2b^2}$$

$$z = \pm \frac{c\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{ab}$$

hence $V = 8xyz$

$$= 8xy\left(\frac{c\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{ab}\right)$$

$$= \frac{8cxy}{ab}\sqrt{a^2b^2 - b^2x^2 - a^2y^2}$$

$$\therefore V_x = \frac{8cy}{ab}\sqrt{a^2b^2 - b^2x^2 - a^2y^2} + \frac{8cxy}{ab} \cdot \frac{(-2b^2x)}{2\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}$$

26)

$$\therefore V_x = \frac{8cy}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2} + \frac{8cxy}{ab}\frac{(-b^2x)}{\sqrt{a^2b^2-b^2x^2-a^2y^2}}$$

$$\therefore V_y = \frac{8cx}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2} + \frac{8cxy}{ab}\frac{(-a^2y)}{\sqrt{a^2b^2-b^2x^2-a^2y^2}}$$

$V_x \Rightarrow 0$

$$\frac{8cy}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2} + \frac{8cxy}{ab}\frac{-b^2x}{\sqrt{a^2b^2-b^2x^2-a^2y^2}} = 0$$

$$\Rightarrow a^2b^2 - b^2x^2 - a^2y^2 - b^2x^2 = 0$$

$$a^2b^2 - a^2y^2 - 2b^2x^2 = 0 \quad \text{--- (1)}$$

$V_y \Rightarrow 0$

$$a^2b^2 - b^2x^2 - a^2y^2 - a^2y^2 = 0$$

$$a^2b^2 - 2a^2y^2 - b^2x^2 = 0 \quad \text{--- (2)}$$

Solving (1) & (2) we get

$$x = \pm\frac{a}{\sqrt{3}}$$

$$y = \pm\frac{b}{\sqrt{3}}$$

So

$$z = \pm c \sqrt{a^2b^2 - b^2\left(\frac{a^2}{3}\right) - a^2\left(\frac{b^2}{3}\right)}$$

$$= \pm \frac{c}{\sqrt{3}} \frac{ab}{ab}$$

$$z = \pm \frac{c}{\sqrt{3}}$$

Since volume is not negative

$$v = 8xyz$$

$$= 8\left(\frac{a}{\sqrt{3}}\right) \times \left(\frac{b}{\sqrt{3}}\right) \times \left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}}$$

$$= \frac{8\sqrt{3}}{9} abc$$

$\therefore$ Volume of the largest box is $\Rightarrow \frac{8\sqrt{3}}{9} abc$ //