Hermitian and skew-Hermitian matrices

Definition 1

A square matrix A = $[a_{ij}]$ is said to be Hermitian if the (i, j)th element of A is equal to the conjugate complex of the (j, i)th element of A.

i.e. $a_{ij} = \overline{a_{ji}}$ for all i and j.

Defintion 2

A square matrix A = $[a_{ij}]$ is said to be skew-Hermitian if the (i, j)th element of A is equal to the negative of the conjugate complex of the (j, i)th element of A.

i.e. if $a_{ij} = -\overline{a_{ji}}$ for all i and j.

Example (i)

$$A = \begin{bmatrix} 3 & 4+5i \\ 4-5i & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 6+4i & 3-7i \\ 6-4i & 7 & 2+5i \\ 3+7i & 2-5i & 0 \end{bmatrix}$$

are Hermitian matrices.

Example (ii)

$$\begin{bmatrix} 0 & 3-4i \\ -3-4i & 0 \end{bmatrix}, \begin{bmatrix} -2i & 4-7i \\ -4-7i & 0 \end{bmatrix}$$ are skew-hermitian matrices

Note 1:

If A is Hermitian, then

$$a_{ii} = \overline{a_{ii}}$$

This is possible only if a_{ii} is real. Hence in a Hermitian matrix, all the diagonal elements are real.

Note 2:

If A is a skew Hermitian matrix,

$$a_{ij} = -\overline{a_{ji}}$$

$$\therefore a_{ii} = -\overline{a_{ii}}$$

$$\therefore a_{ii} + \overline{a_{ii}} = 0$$

$\therefore a_{ii}$ is either purely imaginery or zero.

$\therefore$ the diagonal elements of a skew Hermitian matrix is purely imaginary or zero.

Theorem

If A and B are Hermitian, show that AB + BA is Hermitian and AB−BA is skew hermitian.

Solution:

Let A and B be Hermitian matrices of the same order. Then

$$A^* = A \text{ and } B^* = B.$$

$$\begin{aligned}(AB + BA)^* &= (AB)^* + (BA)^* \\ &= B^* A^* + A^* B^* \\ &= BA + AB \\ &= AB + BA\end{aligned}$$

$\therefore$ AB + BA is Hermitian.

$$\begin{aligned}\text{Also} \quad (AB - BA)^* &= (AB)^* - (BA)^* \\ &= B^* A^* - A^* B^* \\ &= BA - AB\end{aligned}$$

$$= -(AB - BA)$$

$\therefore$ $AB + BA$ is skew-hermitian.

Theorem

If A is a square matrix, show that $A - A^*$, A^*A are all Hermitian and $A - A^*$ is skew-Hermitian.

Solution

We know that a square matrix A is Hermitian if $A^* = A$

(i) $(A + A^*)^* = A^* + (A^*)^*$

$$= A^* + A = A + A^*$$

$\therefore$ $A + A^*$ is Hermitian.

(ii) $(AA^*)^* = (A^*)^* A^*$

$$= AA^*$$

$\therefore$ AA^* is Hermitian.

(iii) $(A^* A)^* = A^* \cdot (A^*)^*$

$$= A^* A$$

$\therefore$ $A^* A$ is Hermitian.

(iv) A square matrix A is skew Hermitian if $A^* = -A$.

$$(A - A^*)^* = A^* - (A^*)^*$$

$$= A^* - A$$

$$-(A - A^*)$$

$\therefore$ $A - A^*$ is skew Hermitian.

Theorem

Show that $B^* AB$ is Hermitian or skew Hermitian according as A is Hermitian or skew Hermitian.

Proof

Let A be a Hermitian matrix

Then $A^* = A$

Now $(B^* AB)^* = B^* A^* B$

$$= B^* AB$$

$\therefore$ $B^* AB$ is Hermitian.

Suppose A is skew Hermitian matrix

Then $A^* = -A$

$$(B^* AB)^* = B^* A B^*$$

$$= B^* (-A) B$$

$$= -(B^* AB)$$

$\therefore$ $B^* AB$ is skew Hermitian.

Theorem:

Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew Hermitian matrix.

Solution

Let A be a square matrix.

Then $A + A^*$ is a Hermitian matrix and $A - A^*$ is a skew Hermitian matrix.

Therefore $\frac{1}{2}(A + A^*)$ is a Hermitian matrix and $\frac{1}{2}(A - A^*)$ is a skew Hermitian matrix.

But $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = P + Q$

Here P is Hermitian and Q is skew-Hermitian. Therefore, every square matrix can be expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Suppose $A = R + S$ be another representation of A where R is a Hermitian matrix and S is a skew Hermitian matrix.

$$\text{Then } A^* = (R + S)^*$$
$$= R^* + S^*$$
$$= R - S$$
$$\therefore R = \frac{1}{2}(A + A^*) = P \text{ and}$$
$$S = \frac{1}{2}(A - A^*) = Q$$

$\therefore$ the representation of A is unique.

Theorem:

If A is Hermitian, show that $\overline{A}$ is Hermitian and if A is skew Hermitian show that $\overline{A}$ is a skew Hermitian.

Proof:

Case 1

Let A be a Hermitian matrix.

Then $A^* = A$

We will prove that $\overline{A}$ is a Hermitian.

$$(\overline{A})^* = [(\overline{\overline{A}})]' = A' \quad \text{since } (\overline{\overline{A}}) = A$$
$$= (A^*)' \text{ since A is Hermitian } A = A^*$$
$$= [(\overline{A})']' \text{ since } A^* = (\overline{A})'$$
$$= \overline{A} \quad [\because (A')' = A]$$

Case 2

Suppose A is skew-Hermitian

Then $A^* = -A$

Now $(\overline{A})^* = [(\overline{A})]' = (A)' = (-A^*)'$

$= A^\wedge = -[(\overline{A})] = -A$

$\therefore \overline{A}$ is skew-Hermitian.

Theorem:

Show that every square matrix A can be uniquely expressed as P + iQ where P and Q are Hermitian matrices.

Solution

Let $P = \frac{1}{2}(A + A^*)$ and $Q = \frac{1}{2i}(A - A^*)$

Then $A = P + iQ$

Then $P^* = \frac{1}{2}(A + A^*)^*$

$$= \frac{1}{2}(A^* + A)$$

$$= \frac{1}{2}(A + A^*) = P$$

$\therefore$ P is Hermitian.

Also $Q^* = \left[\frac{1}{2i}(A - A^*)\right]^*$

$$= \frac{1}{2i}(A - A^*)^*$$

$$= -\frac{1}{2i}\left[(A^* - (A^*)^*\right]$$

$$= -\frac{1}{2i}(A^* - A)$$

$$= \frac{1}{2i}(A - A^*) = Q$$

$\therefore$ Q is Hermitian.