STAT 751: Measure and Probability Theory

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Learning objectives

Learning Outcomes

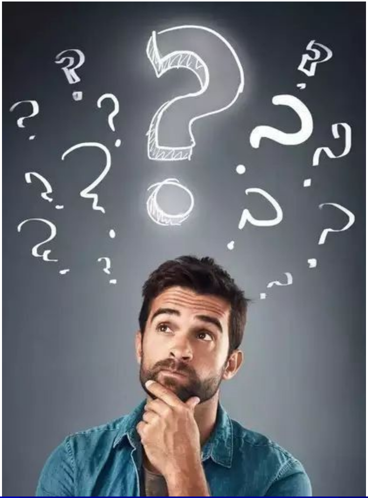
At the end of this unit, students are expected to:

- Define and explain fundamental concepts in set theory, including sets, sample spaces, events, and set operations
- Classify sets as finite, countably infinite, or uncountable, providing clear examples for each type
- Apply set operations, De Morgan's Laws, and the principles of mutually exclusive and exhaustive events in probability scenarios
- Construct and analyze sequences of sets—including increasing, decreasing, and disjointized sequences—and describe their convergence behavior
- Compare and evaluate key concepts such as partitions versus disjointization and bounded versus unbounded intervals, highlighting their relevance in measure theory and probability

Sets

What do you remember about Sets in high school?

Consider examples of sets like numbers, letters, or objects you learned.



Sets

Definition of Sets

A set is a collection of distinct objects, called elements.

Examples of sets

- $A = \{1, 2, 3\}$
- $B = \{\text{names of students in a class}\}$
- $C = [0, 3]$
- $D = \{-3 \leq x \leq 5\}$

Sets

Definition (Sample space)

The sample space, written as Ω, is the set of all possible outcomes of an experiment.
Example:

- Tossing a coin: $\Omega = \{H, T\}$
- Throwing a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Event

An event is any subset of the sample space.
Example: Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space of rolling a fair six-sided die. Let A be the event "the outcome is an even number." Then $A = \{2, 4, 6\}$.

Sets

Notations (Set Operations)

Let Ω denote an abstract space. For $A, B, x \subset \Omega$, we denote

- $A \cup B =: \{x \in A \text{ or } x \in B\}$
- $A \cap B =: \{x \in A \text{ and } x \in B\}$
- $A^c =: \{x \notin A\}$
- $A \setminus B = A - B = \{x \in A : x \notin B\}$
- $A \triangle B =: \{x \in (A \cap B^c) \text{ or } x \in (A^c \cap B) \text{ but } x \notin (A \cap B)\}$

Sets

Empty set

The empty set, denoted by $\emptyset$ or $\{\}$, is the set with no elements.

Subset

A set A is a subset of B, written as $A \subseteq B$, if every element of A is also in B.

- Proper subset: $A \subset B$ if $A \subseteq B$ and $A \neq B$.

Power Set

The power set of a set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A, including the empty set and the set A itself.

- $\mathcal{P}(A) = \{ B : B \subseteq A \}$
- If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Sets

Intervals

An interval in $\mathbb{R}$ is set of real numbers such that whenever x and y are in the set $x < z < y$, then z is also in the set.

Open Interval

An open interval is an interval that does not include its endpoints.

- Notation: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

Closed Interval

A closed interval is an interval that includes both endpoints.

- Notation: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Sets

Half-open or Half-closed or Clopen sets

A half-open interval includes exactly one endpoint.

- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

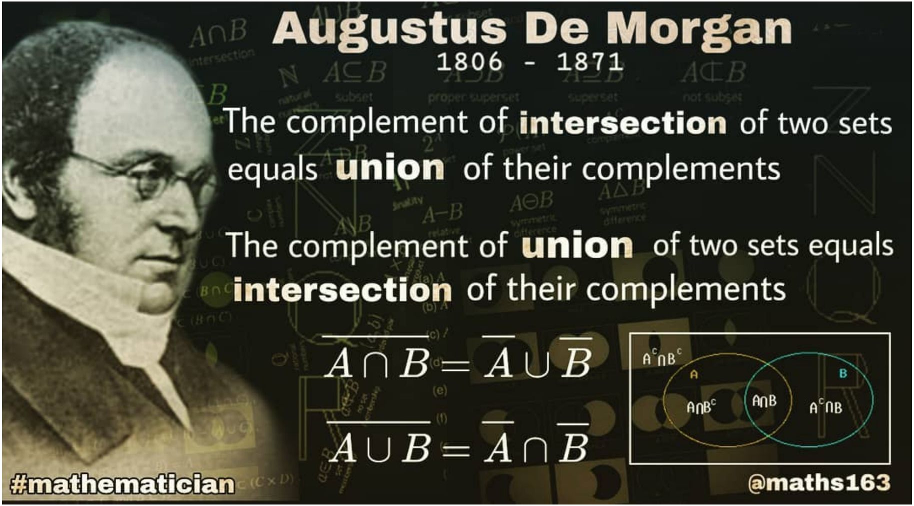
Bounded Interval

An interval is bounded if it has finite endpoints on both sides.

- Example: $[a, b]$, where $a < b$ and both are finite numbers.
- Other forms of bounded intervals: $(a, b], [a, b), (a, b)$

Question

Can we then say that every interval is bounded?


Augustus De Morgan
1806 - 1871
The complement of intersection of two sets equals union of their complements
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$\overline{A \cap B} = \overline{A} \cup \overline{B}$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$
$A^c \cap B^c$
A
B
$A \cap B^c$
$A \cap B$
$A^c \cap B$
#mathematician
@maths163

Unbounded Interval

An interval is unbounded if at least one endpoint is infinite.

- Example: $(-\infty, a], (b, +\infty), (-\infty, +\infty)$.

De Morgan's Laws

For any sets A and B:

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Indexed Families of Sets

A collection of sets $\{A_k\}_{k\in I}$ indexed by a set I allows us to define:

Union of an Indexed Family

$$\bigcup_{k\in I} A_k = \{\, x : x \in A_k \text{ for some } k \in I \,\} \tag{1}$$

Intersection of an Indexed Family

$$\bigcap_{k\in I} A_k = \{\, x : x \in A_k \text{ for all } k \in I \,\} \tag{2}$$

Sets

Sequence of sets

A sequence of sets is an ordered collection of sets indexed by the natural numbers. It is written as
$\{A_n\}_{n=1}^{\infty}$ or $A_1, A_2, A_3, \ldots$

Note

Every sequence of sets is an indexed family.
But not every indexed family is a sequence.

Sequences of Sets vs Indexed Families

- **Example of a sequence of sets (hence an indexed family):**
  Let
  $$A_1 = (0,1), \quad A_2 = (0,2), \quad A_3 = (0,3), \; \ldots$$
  Then $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of sets, indexed by the natural numbers. Since it is indexed, it is also an indexed family.
- **Example of an indexed family that is not a sequence:**
  Let the index set be $\mathbb{R}$, and define
  $$A_t = (-t, t) \quad \text{for } t \in \mathbb{R}.$$
  Then $\{A_t\}_{t\in\mathbb{R}}$ is an indexed family of sets, but it is not a sequence, because the index set is not $\mathbb{N}$.

Increasing Sequence of sets

A sequence of sets $A_1, A_2, A_3, \ldots$ is increasing if each set is contained in the next one:
$A_1 \subset A_2 \subset A_3 \subset \ldots .$

Decreasing Sequence of sets

A sequence of sets $A_1, A_2, A_3, \ldots$ is decreasing if each set contains the next one:
$A_1 \supset A_2 \supset A_3 \supset \ldots$

Monotone Convergence of Sets

Increasing case

A sequence of sets $(A_n)_{n=1}^{\infty}$ converges increasingly to a set A if:

- the sets are increasing (as shown above)
- the limit set A is the union of all sets: i.e. $A = \bigcup_{n=1}^{\infty} A_n$
- We write: $A_n \uparrow A$.

Monotone Convergence of Sets

Decreasing case

A sequence of sets $(A_n)_{n=1}^{\infty}$ converges decreasingly to a set A if:

- the sets are decreasing (as shown earlier)
- the limit set A is the intersection of all sets: i.e. $A = \bigcap_{n=1}^{\infty} A_n$
- We write: $A_n \downarrow A$.

Partition of a Set

The partition of a set A is a collection of disjoint subsets $\{A_i\}$ such that:

- $A_i \cap A_j = \emptyset$ for $i \neq j$
- $\bigcup_i A_i = A$

Disjointization

Definition

Given a sequence of sets $A_1, A_2, A_3, \ldots,$ disjointization creates a sequence of disjoint sets $B_1, B_2, B_3 \ldots,$ such that:

- $B_i \cap B_j = \emptyset$ for $i \neq j$ (disjoint).
- The union is preserved:
$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Disjointization of an Increasing Sequence of Sets

Definition

- Let (A_n) be an increasing sequence of sets, meaning $A_1 \subset A_2 \subset A_3 \subset \dots$.
- The disjointization of (A_n) is the sequence of disjoint sets (B_n) defined by
  $B_1 = A_1, \quad B_n = A_n \setminus A_{n-1} \quad (n \geq 2)$
- These sets satisfy:
  $B_i \cap B_j = \emptyset$ for $i \neq j$ (they are pairwise disjoint).
- Their union equals the union of the original sequence:
  $$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

Examples of Disjointization

Example 1: Overlapping intervals on the real line
Let

$$A_1 = (0,2), \quad A_2 = (1,3), \quad A_3 = (2,4).$$

Disjointization gives

$$\begin{aligned} B_1 &= (0,2), \\ B_2 &= (1,3) \setminus (0,2) = [2,3), \\ B_3 &= (2,4) \setminus \big((0,2) \cup (1,3)\big) = [3,4). \end{aligned}$$

Then the sets B_1, B_2, B_3 are disjoint and

$$\bigcup_{n=1}^{3} B_n = (0,4).$$

Example 2: Repeated events in probability
Let $(\Omega, \mathcal{F}, P)$ be a probability space, and define

$$A_1 = \{\text{first toss is a head}\},$$

$$A_2 = \{\text{at least one head in two tosses}\},$$
$$A_3 = \{\text{at least one head in three tosses}\}.$$

Disjointization gives

$$B_1 = A_1,$$
$$B_2 = A_2 \setminus A_1 = \{\text{first tail, second head}\},$$
$$B_3 = A_3 \setminus (A_1 \cup A_2) = \{\text{first two tails, third head}\}.$$

Each B_n represents the event that the first head occurs exactly at time n.

Example 3: Measurable sets in integration

Let $f : \Omega \to [0, \infty)$ be a measurable function and define

$$A_n = \{\omega \in \Omega : f(\omega) > n\}, \quad n \in \mathbb{N}.$$

The disjointization is

$$B_1 = A_1,$$

Examples of disjointization

$$B_n = A_n \setminus A_{n-1} = \{\omega : n < f(\omega) \leq n+1\}.$$

The sets B_n are disjoint and partition the values of f into non-overlapping levels.

Discussion

Compare and contrast Partition and Disjointization.

Sets

Countability

- A set is finite if it has a limited number of elements. Eg. $A = \{2, 4, 6, 8, 10\}$

Definition of Infinite Sample Space

A **sample space** Ω in probability is called **infinite** if it contains infinitely many outcomes. That is,

$$|\Omega| = \infty.$$

- **Countably infinite:** the outcomes can be listed as $\omega_1, \omega_2, \omega_3, \dots$ (e.g., tossing a coin until the first head)
- **Uncountably infinite:** the outcomes cannot be listed in a sequence (e.g., choosing a real number in the interval $[0, 1]$)

Plain explanation: An infinite sample space means there are endlessly many possible outcomes, either in a way we can count one by one (countable) or not (uncountable).

Sets

Mutually Exclusive Events

Two events A and B are mutually exclusive if they cannot occur at the same time.

- Formally: $A \cap B = \emptyset$.

Exhaustive events

A collection of events $\{A_i\}$ is exhaustive if at least one of them must occur.

- Formally: $\bigcup_i A_i = \Omega$

Assignment

With the aid of examples, discuss the similarities and differences between finite, countably infinite, and uncountable sets.