STAT 751: Measure and Probability Theory

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Learning objectives

At the end of this section, learners will be able to:

- Define and explain the concepts of measure, measurable spaces, measure spaces, and probability spaces, including their components and properties
- Illustrate classical examples of measures, such as length, area, volume, counting, and probability measures, and apply them to practical scenarios
- Differentiate between finite additivity, σ-additivity, and σ-subadditivity, and demonstrate their relationships with examples
- Apply the principles of countable additivity and subadditivity to sequences of sets and prove key properties of measures, including length on bounded intervals
- Analyze and evaluate the extension of simple measures to complex or infinite sets, emphasizing the need for a consistent and general approach in measure theory

WHAT IS MEASURE AND WHY MEASURE?

Why measure?

In mathematics, we often ask questions like:

- how long is this set?
- how large is this region?
- what is the chance that an event happens?

All these questions involve measure; long (length), large (area), chance (probability), amount (volume), etc.

To answer these questions, we need a systematic way to assign numbers to sets.

The problem with "size"

For simple sets, "size" is easy.

Examples:

- the length of an interval (a, b) is $b - a$
- the number of elements in a finite set is easy to count

But it is not simple for many sets.

Examples:

- unions of many intervals
- infinite sets
- complicated events in probability

We need a more general idea of size.

Nice Practical Examples

Now take two intervals that do not overlap.

$$A = (1, 2), \qquad B = (3, 5)$$

Their lengths are:

- $\text{length}(A) = 1$
- $\text{length}(B) = 2$

The total length of $A \cup B$ is

$$1 + 2 = 3$$

So far, everything works nicely.

Nice Practical Examples

Half-open interval example:

$$C = [0, 1), \qquad D = [1, 3)$$

Lengths:

- $\text{length}(C) = 1$
- $\text{length}(D) = 2$

The total length of $C \cup D$ is

$$1 + 2 = 3$$

Even with half-open intervals, the additivity of length works as expected.

Limitations of basic ideas

Length works well for single intervals, but what about:

- a countable union of intervals?
- sets with infinitely many pieces?
- random events formed from many outcomes?

Basic formulas are not enough.

So we need a rule that:

- works for simple sets
- extends to complicated sets
- behaves consistently

From size to measure

A measure is a mathematical tool that:

- assigns a size to sets
- works for very general sets
- satisfies the natural properties we expect

Length, area, volume, and probability are all special cases of measures.

This motivates the formal definition of a measure.

Definition of Measure

Let Ω be a set and $\mathcal{F}$ a collection of subsets of Ω (called a σ-algebra). A function $\mu : \mathcal{F} \to [0, \infty]$ is called a **measure** if it satisfies:

- $\mu(\emptyset) = 0$
- **Countable additivity:** for any countable collection $\{A_1, A_2, A_3, \dots\}$ of disjoint sets in $\mathcal{F}$,

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

Remarks

- μ assigns a non-negative extended real number (can be ∞) to each measurable set.
- The σ-algebra $\mathcal{F}$ ensures that unions, intersections, and complements of sets are measurable.
- Countable additivity is stronger than finite additivity; it works for infinitely many disjoint sets.

Probability Measure

A probability measure P is a measure on a σ-algebra $\mathcal{F}$ of Ω such that:

- $P(A) \geq 0$ for all $A \in \mathcal{F}$
- $P(\Omega) = 1$
- For any countable collection $\{A_1, A_2, \dots\}$ of disjoint events in $\mathcal{F}$,

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

Measure Space

Definition of Measure Space

A measure space is an ordered triple $(\Omega, \mathcal{F}, \mu)$ where:

- Ω is a set, called the sample space.
- $\mathcal{F}$ is a σ-algebra of subsets of Ω,
- $\mu : \mathcal{F} \to [0, \infty]$ is a measure, i.e., a functional satisfying:
  - $\mu(\emptyset) = 0$
  - Countable additivity: for any countable collection $\{A_1, A_2, \dots\}$ of disjoint sets in $\mathcal{F}$,

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i)$$

Measurable Space

Definition of Measurable Space

A **measurable space** is an ordered pair $(\Omega, \mathcal{F})$ where:

- Ω is a set, called the sample space.
- $\mathcal{F}$ is a σ-algebra of subsets of Ω

Definition of Probability Space

A probability space is a triple $(\Omega, \mathcal{F}, P)$ where:

- Ω is the sample space (all possible outcomes)
- $\mathcal{F}$ is a σ-algebra of subsets of Ω (the events)
- P is a probability measure

Properties of a Measure

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

1. **Non-negativity:**
$$\mu(A) \geq 0 \quad \text{for all } A \in \mathcal{F}.$$
2. **Null empty set:**
$$\mu(\emptyset) = 0.$$
3. **Countable additivity (sigma-additivity):** If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint, then
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$
4. **Finite additivity:** If $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$, then
$$\mu(A \cup B) = \mu(A) + \mu(B).$$
5. **Monotonicity:** If $A \subseteq B$, then
$$\mu(A) \leq \mu(B).$$

Properties of a Measure (cont.)

1. **Continuity from below:** If $A_1 \subseteq A_2 \subseteq \cdots$ and
$$A = \bigcup_{n=1}^{\infty} A_n,$$
then
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
2. **Continuity from above:** If $A_1 \supseteq A_2 \supseteq \cdots$, $\mu(A_1) < \infty$, and
$$A = \bigcap_{n=1}^{\infty} A_n,$$
then
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
3. **Inclusion–exclusion (two sets):**
$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

Length measure on $\mathbb{R}$: $\mu((a,b)) = b - a$

$$\text{Example: } \mu([2,5]) = 5 - 2 = 3$$

Counting measure on any set Ω: $\mu(A) = \text{number of elements in set } A$

$$\mu(\{x,y,z\}) = 3$$

Probability measure on a probability space $(\Omega, \mathcal{F}, P)$:
Let the event of rolling an even number be, $A = \{2,4,6\}$.
Since each event A_i of Ω has a probability of $\frac{1}{6}$.
It is easy to prove that

- $P(\Omega) = P(\{1\}) + P(\{2\}) + \cdots + P(\{6\}) = 1$
- $P(A_1 \cup A_2 \cup A_3) = P(A) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

Lengths in $\mathbb{R}$

Length in $\mathbb{R}$: For any interval $I \in \mathbb{R}$ bounded by the endpoints a, b , its length is given as
$\ell(I) = |b - a|$.

Additivity property of length

If an interval I is a disjoint union of a finite family $\{I_k\}_{k=1}^{n}$ of intervals, then $\ell(I) = \sum_{k=1}^{n} \ell(I_k)$.

Classical examples of Measure

Areas in $\mathbb{R}^2$

Given that I, J are the intervals (or lengths) of any rectangle A, then $\text{area}(A) = \ell(I)\ell(J)$.

Additive property of Area

If A is a rectangle of disjoint union of a finite family of rectangles $A_1, A_2, \ldots, A_n$, then $\text{area}(A) = \sum_{k=1}^{n} \text{area}(A_k)$.

Classical examples of Measure

Volumes in $\mathbb{R}^3$

Any box in $\mathbb{R}^3$ of the form $A = I\text{x}J\text{x}K$, where I, J, K are intervals in $\mathbb{R}$, will yield the set $\text{vol}(A) = \ell(I)\ell(J)\ell(K)$.

- The additive property of volume follows similarly.

Probability

If the event $A \subset \Omega$, and A is a disjoint union of a finite sequence of events $A_1, \ldots, A_n$, then $\mathbb{P}(A) = \sum_{k=1}^{n} \mathbb{P}(A_k)$.

Note these similarities

All the above had the following.

- A non-empty set Ω (i.e. $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \Omega$).
- A family of subsets S (i.e. intervals, rectangles, boxes, events).
- A functional $\mu : S \to \mathbb{R}_+ := [0, +\infty)$ (length, area, etc.) with the following property:
  - if $A \in S$ is a dsijoint union of a finite family $\{A_k\}_{k=1}^n$ of sets from S, then $\mu(A) = \sum_{k=1}^n \mu(A_k)$.

Lebesgue Measures

- **Length:** Lebesgue measure on $\mathbb{R}$ (1-dimensional)
- **Area:** Lebesgue measure on $\mathbb{R}^2$ (2-dimensional)
- **Volume:** Lebesgue measure on $\mathbb{R}^3$ (3-dimensional)
- **Probability:** Lebesgue measure on a probability space assigns probabilities to events (for continuous spaces)

All of the above are examples of Lebesgue measures in different contexts. They generalize the concept of “size” — whether length, area, volume, or probability — in a mathematically rigorous way that works even for very irregular sets.

σ-additive measures

Let Ω be a non-empty set and S be a family of subsets of Ω.

Definition of σ-additive measures

A functional $\mu : S \to \mathbb{R}_+$ is called a σ-**additive measure** if whenever

- a set $A \in S$ is a disjoint union of an at most countable sequence $\{A_k\}_{k=1}^N$ (where N is either finite or $N = \infty$), then
- $\mu(A) = \sum_{k=1}^N \mu(A_k)$.

Remark

- μ is a *finitely additive measure* if this property holds for finite values of N.
- Every σ-additive measure is finitely additive, but the converse is not true.

Proof that σ-additivity $\implies$ finite additivity

Let μ be a σ-additive measure on a σ-algebra $\mathcal{F}$.
Take any two disjoint sets $A, B \in \mathcal{F}$.
Consider the countable sequence

$$A_1 = A, \quad A_2 = B, \quad A_3 = \emptyset, \quad A_4 = \emptyset, \ldots$$

Since the sets are disjoint, σ-additivity gives:

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

The left-hand side is

$$\mu(A \cup B),$$

and the right-hand side is

$$\mu(A) + \mu(B) + 0 + 0 + \cdots = \mu(A) + \mu(B).$$

Hence,

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

which is exactly finite additivity.

Theorem

The length is a σ-additive measure on the family of all bounded intervals in $\mathbb{R}$.

σ-subadditive measures

Definition

A functional $\mu : \mathcal{S} \to \mathbb{R}_+$ is called σ-subadditive if whenever $A \subset \bigcup_{k=1}^{N} A_k$ where A and A_k are all elements of $\mathcal{S}$ and N is either finite or infinite, $\mu(A) \leq \sum_{k=1}^{N} \mu(A_k)$.

Note

If this property holds for finite values of N, then μ is called finitely subadditive.

Lemma

The *length* is σ-subadditive.

Proof

Let I, $\{I_k\}_{k=1}^{\infty}$ be intervals such that $I \subset \cup_{k=1}^{\infty} I_k$, we want to prove that $\ell(I) \leq \displaystyle\sum_{k=1}^{\infty} \ell(I_k)$.

Let us fix some $\varepsilon > 0$ and choose a bounded closed interval $I' \subset I$ such that $\ell(I) \leq \ell(I') + \varepsilon$.

For any k, choose an open interval $I_k^{'} \supset I_k$ such that $\ell(I_k') \leq \ell(I_k) + \frac{\varepsilon}{2^k}$.

Then the bounded closed interval I' is covered by a sequence $\{I_k'\}$ of open intervals. There is a finite subfamily $\{I_{k_j}'\}_{j=1}^{n}$ that also covers I'.

Proof

It follows from the finite additivity of length that it is finitely subadditive. That is,

$$\ell(I') \leq \sum_j \ell(I'_{k_j}) \implies \ell(I') \leq \sum_{k=1}^{\infty} \ell(I'_k).$$

This yields

$$\ell(I) \leq \varepsilon + \sum_{k=1}^{\infty} \left(\ell(I_k) + \frac{\varepsilon}{2^k} \right) = 2\varepsilon + \sum_{k=1}^{\infty} \ell(I_k).$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$ finishes the proof. $\square$

Theorem

The length is a σ-additive measure on the family of all bounded intervals in $\mathbb{R}$.

Proof

We need to prove that if $I = \bigsqcup_{k=1}^{\infty} I_k$, then $\ell(I) = \sum_{k=1}^{\infty} \ell(I_k)$.

By the σ-subadditive lemma, we have $\ell(I) \leq \sum_{k=1}^{\infty} \ell(I_k)$, so we need to prove the opposite inequality.

For a fixed $n \in \mathbb{N}$, we have

$$I \supset \bigsqcup_{k=1}^{n} I_k.$$

It follows from the finite additivity of length that

$$\ell(I) \geq \sum_{k=1}^{n} \ell(I_k).$$

Proof

Letting $n \to \infty$, we obtain

$$\ell(I) \geq \sum_{k=1}^{\infty} \ell(I_k)$$

which finishes the proof.