MSTAT 551: Probability and Measure Theory

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Learning objectives

At the end of this section, learners will be able to:

- Explain the need for structured collections of sets in measure and probability theory.
- Construct rings and σ-algebras generated by given families of sets.
- Illustrate semi-rings, rings, algebras, and σ-algebras with concrete examples.
- Distinguish between semi-rings, rings, algebras, and σ-algebras using their closure properties.
- Analyze how increasing closure requirements lead from semi-rings to σ-algebras.

Towards Algebra – The Idea

So far, we have worked with individual sets such as intervals and simple events.
To define length, probability, or measure, we need to work with collections of sets, not just single sets.
We want a collection of sets with the following properties:

- the empty set should be included
- complements should remain inside the collection
- unions of sets should stay inside the collection

However, requiring all these properties at once is often too strong at the beginning.
Instead, we start with very simple building blocks, such as intervals, and gradually add structure.

Semi-ring of Sets

Definition (Semi-ring)

A family $\mathcal{S}$ of subsets of Ω is called a semi-ring if:

- $\emptyset \in \mathcal{S}$
- If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$
- If $A, B \in \mathcal{S}$, then $A \setminus B$ is a disjoint union of a finite family of sets from $\mathcal{S}$

Semi-Ring

Example

Consider the set of half-open intervals in $\mathbb{R}$:

$$\mathcal{S} = \{[a, b) \mid a < b,\ a, b \in \mathbb{R}\}.$$

Semi-ring

Proof that $\mathcal{S}$ is a semi-ring.

- Take $[a, a) = \emptyset$, so $\emptyset \in \mathcal{S}$.
- The intersection $[a, b) \cap [c, d) = [\max(a, c), \min(b, d)) \in \mathcal{S}$.
- Suppose that $[a, b) \subseteq [c, d)$. The difference $[a, b) \setminus [c, d)$ is a finite union of disjoint sets from $\mathcal{S}$.
  That is, $[a, b) \setminus [c, d) = [c, a) \cup [b, d)$.

Other examples of semi-ring

- All bounded intervals in $\mathbb{R}$ are semi-rings.
- Finite unions of half-open intervals in $\mathbb{R}$.
- Finite subsets of any set.

Prove the 3 examples above.

Definition of a Ring

A family $\mathcal{R}$ of subsets of Ω is called a ring if

- $\mathcal{R}$ contains $\emptyset$
- $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$, and $A \setminus B \in \mathcal{R}$.

It follows that the intersection $A \cap B$ belongs to $\mathcal{R}$ because $A \cap B = B \setminus (B \setminus A)$ is also in $\mathcal{R}$.
Also, it follows that a ring is a semi-ring.

Ring

Example

Let $\Omega = \{1, 2, 3, 4, 5, \dots\}$ and let $\mathcal{R} = \{\text{All finite subsets of } \Omega\}$.

Proof that $\mathcal{R}$ is a ring

- $\emptyset \subseteq \Omega$ and is finite, therefore $\emptyset \in \mathcal{R}$
- $\{1, 3\} \cup \{2, 4, 5\} = \{1, 2, 3, 4, 5\} \in \mathcal{R}$.
- $\{1, 2, 3\} \setminus \{2\} = \{1, 3\} \in \mathcal{R}$.

Other examples of a ring

- $\mathcal{R} = \{\text{all finite subsets of } \mathbb{N}\}$ is a ring.
- $\mathcal{R} = \{[a, b) \subseteq \mathbb{R} | a, b \in \mathbb{Q}, a \leq b\}$. That is, all intervals of this form on the real line with rational endpoints.

σ-ring

Definition of σ-ring

A ring $\mathcal{R}$ is called a σ-ring if the union of any countable family $\{A_k\}_{k=1}^{\infty}$ of sets from $\mathcal{R}$ is also in $\mathcal{R}$. That is:

- Closure under unions: If $A_1, A_2, A_3, \cdots \in \mathcal{R}$, then
$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}.$$
- Closure under difference: If $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$.

- It follows that the intersection $A = \bigcap_k A_k$ is also in $\mathcal{R}$.
- Let B be any of the sets A_k so that $B \supset A$, then
$A = B \setminus (B \setminus A) = B \setminus (\bigcup_k (B \setminus A_k)) \in \mathcal{R}$.

Examples of σ-ring

- The power set of a set is a σ-ring.

Proof:

i. Empty set

By definition of the power set, every subset of Ω belongs to $\mathcal{P}(\Omega)$.
Since $\emptyset \subseteq \Omega$, we have

$$\emptyset \in \mathcal{P}(\Omega).$$

ii. Closed under set difference

Let $A, B \in \mathcal{P}(\Omega)$. Then $A \subseteq \Omega$ and $B \subseteq \Omega$.
The difference $A \setminus B$ is also a subset of Ω. Therefore,

$$A \setminus B \in \mathcal{P}(\Omega).$$

iii. Closed under countable unions

Let $A_1, A_2, A_3, \ldots$ be any sequence of sets in $\mathcal{P}(\Omega)$. Each A_n is a subset of Ω.

The union $\bigcup_n A_n$ is also a subset of Ω. Hence,

$$\bigcup_n A_n \in \mathcal{P}(\Omega).$$

Other examples of a σ-ring

1. All countable subsets of a set.
2. All finite subsets of a set.
3. The σ-ring generated by a semi-ring.

A ring generated by a family of sets

Definition

Let $S \subseteq 2^{\Omega}$ be a family of subsets of Ω. Define

$$\mathcal{R}(S) = \bigcap \left\{ \mathcal{R} : \mathcal{R} \text{ is a ring of subsets of } \Omega \text{ and } S \subseteq \mathcal{R} \right\}.$$

Then $\mathcal{R}(S)$ is called the *ring generated by* S. It is the *smallest ring* containing S.

Intuition

If $\Omega = \mathbb{R}$ and S is the family of all intervals $[a, b]$, then $\mathcal{R}(S)$ is the ring of *finite unions of intervals*.
The minimal ring contains exactly what you need to satisfy the ring properties, and nothing extra.

Construction

Setup: Let $\Omega = \{1,2,3,4\}$ and let the family of sets be $S = \{\{1\},\{2\}\}$.

Step 1: Build a ring

A ring must be closed under *union* and *difference*. Starting with S:

Unions:

$$\{1\} \cup \{2\} = \{1,2\}$$

Differences:

$$\{1,2\} \setminus \{1\} = \{2\}, \quad \{1,2\} \setminus \{2\} = \{1\}$$

Step 2: Include empty set

Every ring contains the empty set:

$$\emptyset \in \mathcal{R}(S)$$

Step 3: Minimal ring

After including all unions and differences, the smallest ring containing S is:

$$\mathcal{R}(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

Group Assignment

Analyze the difference and similarities between a semi-ring, ring, and a σ-ring. Add examples.

Definition of a Semi-Algebra of Sets

Semi-Algebra of Sets

Let Ω be a nonempty set. A collection $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ is called a **semi-algebra** if it satisfies:

1. $\varnothing \in \mathcal{S}$
2. **Closure under finite intersections:** if $A, B \in \mathcal{S}$, then
$$A \cap B \in \mathcal{S}$$
3. **Complement is a finite disjoint union:** for every $A \in \mathcal{S}$, there exist pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{S}$ such that
$$A^c = \bigcup_{k=1}^{n} A_k$$

Example of a Semi-Algebra

Semi-Algebra of Intervals on $\mathbb{R}$

Let

$$\mathcal{S} = \{[a, b) \mid a \leq b, \; a, b \in \mathbb{R}\}.$$

Claim: $\mathcal{S}$ is a semi-algebra on $\mathbb{R}$.
Proof:

1. **Empty set:**
$$[a, a) = \varnothing \in \mathcal{S}.$$
2. **Closure under finite intersections:** Let $[a, b), [c, d) \in \mathcal{S}$. Then
$$[a, b) \cap [c, d) = [\max(a, c), \min(b, d)),$$
which is either empty or an interval in $\mathcal{S}$.
3. **Complement as a finite disjoint union:** For any $[a, b) \in \mathcal{S}$,
$$[a, b)^c = (-\infty, a) \cup [b, \infty),$$

Example of a Semi-algebra

which is a union of two disjoint sets that can be expressed as intervals of the same type.

Therefore, $\mathcal{S}$ satisfies all conditions of a semi-algebra.

Plain explanation: a semi-algebra allows intersections, but not necessarily complements. However, the complement of any set can be written as a finite union of disjoint sets in the collection.

Simple Examples of Semi-Algebras

1. **Intervals starting from 0 on the real line:**

$$\mathcal{S} = \{[0, a) \mid a \geq 0\}.$$

   - Finite intersections: $[0, a) \cap [0, b) = [0, \min(a, b)) \in \mathcal{S}$
   - Complement: $[0, a)^c = [a, \infty)$, a single interval

2. **Single-element and empty sets in a finite set:** Let $\Omega = \{1, 2, 3\}$ and

$$\mathcal{S} = \{\varnothing, \{1\}, \{2\}, \{3\}\}.$$

   - Finite intersections: intersection of any sets in $\mathcal{S}$ is still in $\mathcal{S}$
   - Complement: e.g., $\{1\}^c = \{2, 3\}$, which can be written as a union of disjoint singletons

Algebra

Definition of an Algebra of Sets

Let Ω be a non-empty set.
A collection of subsets of Ω is called an algebra $\mathcal{A}$, if it satisfies the following conditions:

1. $\Omega \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$
3. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$

Note

A ring containing Ω is called an algebra.

Example

Given $\Omega = \{1, 2, 3\}$. Construct the algebra $\mathcal{A}$.

Solution

- $\Omega = \{1, 2, 3\}$
- $\mathcal{A} = \{\emptyset, \Omega, \{2\}, \{1, 3\}\}$
- $\mathcal{A} = \{\emptyset, \Omega, \{1\}, \{2, 3\}\}$
- $\mathcal{A} = \{\emptyset, \Omega, \{3\}, \{1, 2\}\}$

Clearly, any of the $\mathcal{A}$ satisfies all three properties of an algebra.

Note

The power set of Ω, denoted by $\mathcal{P}(\Omega)$ or 2^{Ω} is an algebra.

σ-Algebra

Definition of σ-Algebra

A collection of sets $\mathcal{F}$ is called a σ-algebra if it is an algebra and satisfies the property:

- If $A_n \in \mathcal{F}$ for $n \geq 1$, then $\bigcup_{n \geq 1} A_n \in \mathcal{F}$.

That is, a σ-algebra is an algebra and is closed under countable unions.

(i) $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,

(iii) $A_1, A_2, \cdots \in \mathcal{F} \implies \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$

σ-Algebra

Some general, simple examples of σ-algebras

1. $\mathcal{F} = \{\emptyset, \Omega\}$ — *trivial σ-algebra*
2. $\mathcal{F} = \{\text{all subsets of } \Omega\}$ — The largest σ-algebra
3. Let $\mathcal{A} = \{A\} \subset \Omega$, then
   $\sigma(\mathcal{A}) = \{\emptyset, A, A^c, \Omega\}$

A σ-algebra generated by a set

Definition

Let $S \subseteq P(\Omega)$ be a family of subsets of Ω. Define

$$\sigma(S) = \bigcap \left\{ \mathcal{F} \subseteq 2^{\Omega} : \mathcal{F} \text{ is a sigma-algebra and } S \subseteq A \right\}.$$

Then $\sigma(S)$ is called the *sigma-algebra generated by* S. It is the *smallest sigma-algebra* containing S.

Construction of a σ-algebra generated by a set

Step 1: Start with the set

Let $\Omega = \{1, 2, 3\} \quad \text{and} \quad S = \{\{1\}\}$

as your starting set.

Step 2: Include complements

The complement of $\{1\}$ in Ω is $\{2, 3\}$.

Always include the empty set $\emptyset$ and the whole set Ω.

So now we have: $\emptyset, \{1\}, \{2, 3\}, \Omega$

Step 3: Include countable unions and intersections

Check unions and intersections of all included sets:

$$\{1\} \cup \{2, 3\} = \Omega \quad \text{(already included)}$$

$$\{1\} \cap \{2, 3\} = \emptyset \quad \text{(already included)}$$

All other unions and intersections are also already included.

The sigma-algebra generated by S is:

$$\sigma(S) = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}.$$

σ-Algebra

Discussion

Given $\Omega = \{HH, TH, HT, TT\}$. Construct the following on Ω.

- $\sigma(\{HH, TT\})$
- $\sigma(\{TH\})$
- $\sigma(\{\emptyset\})$
- $\sigma(\{TH\}, \{HT\})$

Note

A σ-ring on Ω containing Ω is called a σ-algebra.

Is every *Algebra* a σ-Algebra? Is the converse true?

Hierarchy of Set Systems

Important Takeaway

Every semi-algebra can generate an algebra, and every algebra is a σ-algebra if it is closed under countable unions.

Thus, a σ-algebra is stronger than an algebra, which is stronger than a semi-algebra, because each adds more closure properties:

$$\begin{aligned}&\text{Semi-Algebra: finite intersections}\\ &\to \text{Algebra: complements} + \text{finite unions}\\ &\to \sigma\text{-Algebra: countable unions}\end{aligned}$$