# MSTAT 551: Probability and Measure Theory

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# Learning objectives

## Learning Outcomes

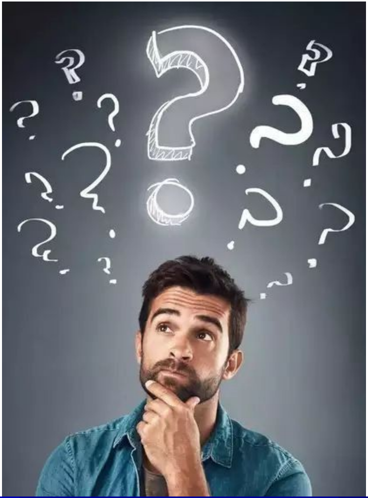
At the end of this unit, students are expected to:

- Define and explain fundamental concepts in set theory, including sets, sample spaces, events, and set operations
- Classify sets as finite, countably infinite, or uncountable, providing clear examples for each type
- Apply set operations, De Morgan's Laws, and the principles of mutually exclusive and exhaustive events in probability scenarios
- Construct and analyze sequences of sets—including increasing, decreasing, and disjointized sequences—and describe their convergence behavior
- Compare and evaluate key concepts such as partitions versus disjointization and bounded versus unbounded intervals, highlighting their relevance in measure theory and probability

# Sets

## What do you remember about Sets in high school?

Consider examples of sets like numbers, letters, or objects you learned.



# Sets

## Definition of Sets

A set is a collection of distinct objects, called elements.

## Examples of sets

- $A = \{1, 2, 3\}$
- $B = \{\text{names of students in a class}\}$
- $C = [0, 3]$
- $D = \{-3 \leq x \leq 5\}$

# Sets

## Definition (Sample space)

The sample space, written as $\Omega$, is the set of all possible outcomes of an experiment.
Example:

- Tossing a coin: $\Omega = \{H, T\}$
- Throwing a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

## Event

An event is any subset of the sample space.
Example: Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space of rolling a fair six-sided die. Let $A$ be the event "the outcome is an even number." Then $A = \{2, 4, 6\}$.

# Sets

## Notations (Set Operations)

Let $\Omega$ denote an abstract space. For $A, B, x \subset \Omega$, we denote

- $A \cup B =: \{x \in A \text{ or } x \in B\}$
- $A \cap B =: \{x \in A \text{ and } x \in B\}$
- $A^c =: \{x \notin A\}$
- $A \setminus B = A - B = \{x \in A : x \notin B\}$
- $A \triangle B =: \{x \in (A \cap B^c) \text{ or } x \in (A^c \cap B) \text{ but } x \notin (A \cap B)\}$

# Sets

## Empty set

The empty set, denoted by $\emptyset$ or $\{\}$, is the set with no elements.

## Subset

A set $A$ is a subset of $B$, written as $A \subseteq B$, if every element of $A$ is also in $B$.

- Proper subset: $A \subset B$ if $A \subseteq B$ and $A \neq B$.

## Power Set

The power set of a set $A$, denoted by $\mathcal{P}(A)$, is the set of all subsets of $A$, including the empty set and the set $A$ itself.

- $\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$
- If $A$ has $n$ elements, then $\mathcal{P}(A)$ has $2^n$ elements.

## Intervals

An interval in $\mathbb{R}$ is set of real numbers such that whenever $x$ and $y$ are in the set $x < z < y$, then $z$ is also in the set.

## Open Interval

An open interval is an interval that does not include its endpoints.

- Notation: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

## Closed Interval

A closed interval is an interval that includes both endpoints.

- Notation: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

### Half-open or Half-closed or Clopen sets

A half-open interval includes exactly one endpoint.

- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

### Bounded Interval

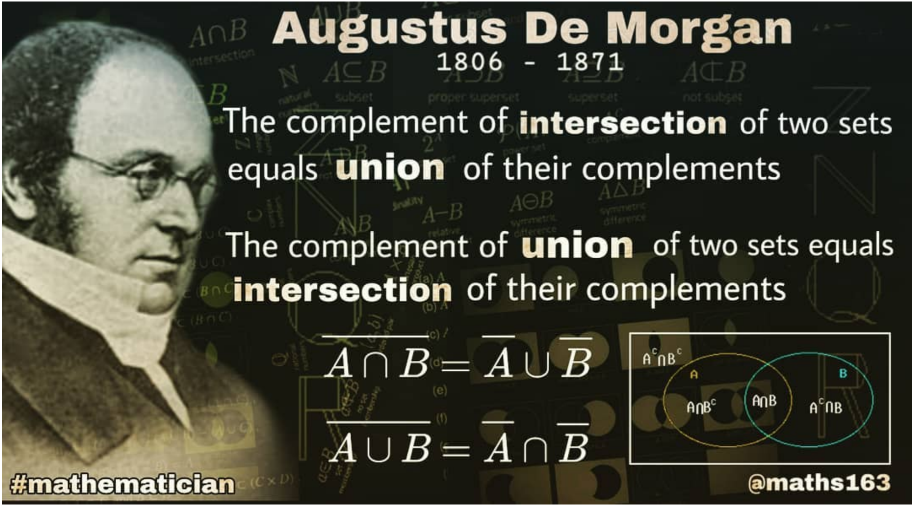
An interval is bounded if it has finite endpoints on both sides.

- Example: $[a, b]$, where $a < b$ and both are finite numbers.
- Other forms of bounded intervals: $(a, b], [a, b), (a, b)$

### Question

Can we then say that every interval is bounded?

# Augustus De Morgan



## Unbounded Interval

An interval is unbounded if at least one endpoint is infinite.

- Example: $(-\infty, a], (b, +\infty), (-\infty, +\infty)$.

## De Morgan's Laws

For any sets $A$ and $B$:

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

## Indexed Families of Sets

A collection of sets $\{A_k\}_{k\in I}$ indexed by a set $I$ allows us to define:

## Union of an Indexed Family

$$\bigcup_{k\in I} A_k = \{\, x : x \in A_k \text{ for some } k \in I \,\} \tag{1}$$

## Intersection of an Indexed Family

$$\bigcap_{k\in I} A_k = \{\, x : x \in A_k \text{ for all } k \in I \,\} \tag{2}$$

## Sequence of sets

A sequence of sets is an ordered collection of sets indexed by the natural numbers. It is written as
$\{A_n\}_{n=1}^{\infty}$ or $A_1, A_2, A_3, \ldots$

## Note

Every sequence of sets is an indexed family.
But not every indexed family is a sequence.

# Sequences of Sets vs Indexed Families

- **Example of a sequence of sets (hence an indexed family):**
  Let
  $$A_1 = (0,1), \quad A_2 = (0,2), \quad A_3 = (0,3), \ \ldots$$
  Then $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of sets, indexed by the natural numbers. Since it is indexed, it is also an indexed family.
- **Example of an indexed family that is not a sequence:**
  Let the index set be $\mathbb{R}$, and define
  $$A_t = (-t, t) \quad \text{for } t \in \mathbb{R}.$$
  Then $\{A_t\}_{t\in\mathbb{R}}$ is an indexed family of sets, but it is not a sequence, because the index set is not $\mathbb{N}$.

## Increasing Sequence of sets

A sequence of sets $A_1, A_2, A_3, \ldots$ is increasing if each set is contained in the next one:
$A_1 \subset A_2 \subset A_3 \subset \ldots.$

## Decreasing Sequence of sets

A sequence of sets $A_1, A_2, A_3, \ldots$ is decreasing if each set contains the next one:
$A_1 \supset A_2 \supset A_3 \supset \ldots$

# Monotone Convergence of Sets

## Increasing case

A sequence of sets $(A_n)_{n=1}^{\infty}$ converges increasingly to a set $A$ if:

- the sets are increasing (as shown above)
- the limit set $A$ is the union of all sets: i.e. $A = \bigcup_{n=1}^{\infty} A_n$
- We write: $A_n \uparrow A$.

## Decreasing case

A sequence of sets $(A_n)_{n=1}^{\infty}$ converges decreasingly to a set $A$ if:

- the sets are decreasing (as shown earlier)
- the limit set $A$ is the intersection of all sets: i.e. $A = \bigcap_{n=1}^{\infty} A_n$
- We write: $A_n \downarrow A$.

## Partition of a Set

The partition of a set $A$ is a collection of disjoint subsets $\{A_i\}$ such that:

- $A_i \cap A_j = \emptyset$ for $i \neq j$
- $\bigcup_i A_i = A$

# Disjointization

## Definition

Given a sequence of sets $A_1, A_2, A_3, \ldots,$ disjointization creates a sequence of disjoint sets $B_1, B_2, B_3 \ldots,$ such that:

- $B_i \cap B_j = \emptyset$ for $i \neq j$ (disjoint).
- The union is preserved:
$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

# Disjointization of an Increasing Sequence of Sets

## Definition

- Let $(A_n)$ be an increasing sequence of sets, meaning $A_1 \subset A_2 \subset A_3 \subset \dots$.
- The disjointization of $(A_n)$ is the sequence of disjoint sets $(B_n)$ defined by
  $B_1 = A_1, \quad B_n = A_n \setminus A_{n-1} \quad (n \geq 2)$
- These sets satisfy:
  $B_i \cap B_j = \emptyset$ for $i \neq j$ (they are pairwise disjoint).
- Their union equals the union of the original sequence:
  $$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

# Examples of Disjointization

**Example 1: Overlapping intervals on the real line**
Let

$$A_1 = (0,2), \quad A_2 = (1,3), \quad A_3 = (2,4).$$

Disjointization gives

$$\begin{aligned} B_1 &= (0,2), \\ B_2 &= (1,3) \setminus (0,2) = [2,3), \\ B_3 &= (2,4) \setminus \big((0,2) \cup (1,3)\big) = [3,4). \end{aligned}$$

Then the sets $B_1, B_2, B_3$ are disjoint and

$$\bigcup_{n=1}^{3} B_n = (0,4).$$

**Example 2: Repeated events in probability**
Let $(\Omega, \mathcal{F}, P)$ be a probability space, and define

$$A_1 = \{\text{first toss is a head}\},$$

$$A_2 = \{\text{at least one head in two tosses}\},$$
$$A_3 = \{\text{at least one head in three tosses}\}.$$

Disjointization gives

$$B_1 = A_1,$$
$$B_2 = A_2 \setminus A_1 = \{\text{first tail, second head}\},$$
$$B_3 = A_3 \setminus (A_1 \cup A_2) = \{\text{first two tails, third head}\}.$$

Each $B_n$ represents the event that the first head occurs exactly at time $n$.

**Example 3: Measurable sets in integration**
Let $f : \Omega \to [0, \infty)$ be a measurable function and define

$$A_n = \{\omega \in \Omega : f(\omega) > n\}, \quad n \in \mathbb{N}.$$

The disjointization is

$$B_1 = A_1,$$

# Examples of disjointization

$$B_n = A_n \setminus A_{n-1} = \{\omega : n < f(\omega) \leq n+1\}.$$

The sets $B_n$ are disjoint and partition the values of $f$ into non-overlapping levels.

# Discussion

Compare and contrast Partition and Disjointization.

## Countability

- A set is finite if it has a limited number of elements. Eg. $A = \{2, 4, 6, 8, 10\}$

### Definition of Infinite Sample Space

A **sample space** $\Omega$ in probability is called **infinite** if it contains infinitely many outcomes. That is,

$$|\Omega| = \infty.$$

- **Countably infinite:** the outcomes can be listed as $\omega_1, \omega_2, \omega_3, \ldots$ (e.g., tossing a coin until the first head)
- **Uncountably infinite:** the outcomes cannot be listed in a sequence (e.g., choosing a real number in the interval $[0, 1]$)

**Plain explanation:** An infinite sample space means there are endlessly many possible outcomes, either in a way we can count one by one (countable) or not (uncountable).

## Mutually Exclusive Events

Two events $A$ and $B$ are mutually exclusive if they cannot occur at the same time.

- Formally: $A \cap B = \emptyset$.

## Exhaustive events

A collection of events $\{A_i\}$ is exhaustive if at least one of them must occur.

- Formally: $\bigcup_i A_i = \Omega$

# Assignment

With the aid of examples, discuss the similarities and differences between finite, countably infinite, and uncountable sets.

# References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- K. B. Athreya and S. N. Lahiri, *Measure and Probability Theory*, Springer, 2006.
- P. Billingsley, *Probability and Measure*, 3rd ed., Wiley, 1995.
- A Probability Space, *Measure Theory and Probability*, YouTube lecture series, 2023.
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