STAT ***: Measure and Probability Theory

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Learning objectives

At the end of this unit, students are expected to:

- Define fundamental concepts of sets, algebra, and measure theory.
- Explain the properties and relationships between measurable sets, spaces, and measures.
- Construct measurable spaces and apply measure theory in advanced scenarios.
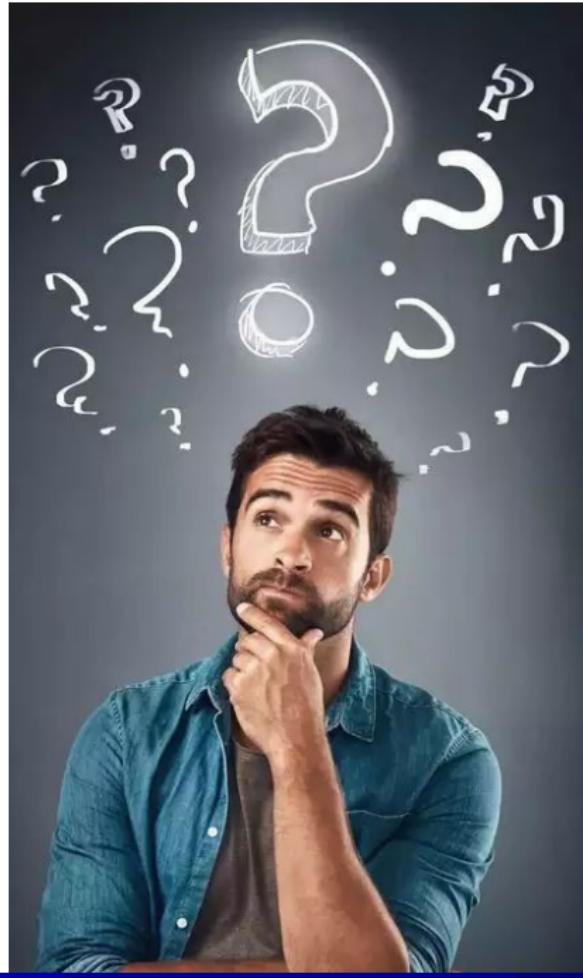- Understand set sequences, outer measures, and measures.

Sets

What do you remember about Sets in high school?

Think about examples like numbers, letters, or objects you learned.

Sets

Definition (Sets)

A set is a collection of distinct objects, called elements.

Examples of sets

- $A = \{1, 2, 3\}$
- $B = \{\text{names of students in a class}\}$

Sets

Definition (Sample space)

The sample space, written as Ω, is the set of all possible outcomes of an experiment.

Example:

- Tossing a coin: $\Omega = \{H, T\}$
- Throwing a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Event

An event is any subset of the sample space.

Example: Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space of rolling a fair six-sided die. Let A be the event "the outcome is an even number." Then $A = \{2, 4, 6\}$.

Sets

Definitions of Notations (Set Operations)

Let Ω denote an abstract space. For $A, B, x \subset \Omega$, we denote

- $A \cup B =: \{x \in A \text{ or } x \in B\}$
- $A \cap B =: \{x \in A \text{ and } x \in B\}$
- $A^c =: \{x \notin A\}$
- $A \setminus B = A - B = \{x \in A : x \notin B\}$
- $A \triangle B =: \{x \in (A \cap B^c) \text{ or } x \in (A^c \cap B) \text{ but } x \notin (A \cap B)\}$

Sets

Empty set

The empty set, denoted by $\emptyset$ or $\{\}$, is the set with no elements.

Subset

A set A is a subset of B, written as $A \subseteq B$, if every element of A is also in B.

- Proper subset: $A \subset B$ if $A \subseteq B$ and $A \neq B$.

Power Set

The power set of a set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A, including the empty set and the set A itself.

- $\mathcal{P}(A) = \{ B : B \subseteq A \}$
- If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Sets

Intervals

An interval in $\mathbb{R}$ is set of real numbers such that whenever x and y are in the set $x < z < y$, then z is also in the set.

Open Interval

An open interval is an interval that does not include its endpoints.

- Notation: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

Closed Interval

A closed interval is an interval that includes both endpoints.

- Notation: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Sets

Half-open or Half-closed or Clopen sets

A half-open interval includes exactly one endpoint.

- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

Bounded Interval

An interval is bounded if it has finite endpoints on both sides.

- Example: $[a, b]$, where $a < b$ and both are finite numbers.
- Other forms of bounded intervals: $(a, b], [a, b), (a, b)$

Unbounded Interval

An interval is unbounded if at least one endpoint is infinite.

- Example: $(-\infty, a], (b, +\infty), (-\infty, +\infty)$.

Sets

De Morgan's Laws

For any sets A and B:

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Sets

Indexed Families of Sets

A collection of sets $\{A_k\}_{k \in I}$ indexed by a set I allows us to define:

Union of an Indexed Family

$$\bigcup_{k \in I} A_k = \{\, x : x \in A_k \text{ for some } k \in I \,\} \tag{1}$$

Intersection of an Indexed Family

$$\bigcap_{k \in I} A_k = \{\, x : x \in A_k \text{ for all } k \in I \,\} \tag{2}$$

Sets

Sequence of sets

A sequence of sets is an ordered collection of sets indexed by the natural numbers. It is written as
$\{A_n\}_{n=1}^{\infty}$ or $A_1, A_2, A_3, \ldots$

Remark

Every sequence of sets is an indexed family.
But not every indexed family is a sequence.

Sets

Increasing Sequence of sets

A sequence of sets $A_1, A_2, A_3, \ldots$ is increasing if each set is contained in the next one:
$A_1 \subset A_2 \subset A_3 \subset \ldots$.

Decreasing Sequence of sets

A sequence of sets $A_1, A_2, A_3, \ldots$ is decreasing if each set contains the next one:
$A_1 \supset A_2 \supset A_3 \supset \ldots$

Monotone Convergence of Sets

Increasing case

A sequence of sets $(A_n)_{n=1}^{\infty}$ converges increasingly to a set A if:

- the sets are increasing (as shown above)

- the limit set A is the union of all sets: i.e. $A = \bigcup\limits_{n=1}^{\infty} A_n$

- We write: $A_n \uparrow A$.

Monotone Convergence of Sets

Decreasing case

A sequence of sets $(A_n)_{n=1}^{\infty}$ converges decreasingly to a set A if:

- the sets are decreasing (as shown earlier)

- the limit set A is the intersection of all sets: i.e. $A = \bigcap\limits_{n=1}^{\infty} A_n$

- We write: $A_n \downarrow A$.

Sets

Partition of a Set

A partition of a set A is a collection of disjoint subsets $\{A_i\}$ such that:

- $A_i \cap A_j = \emptyset$ for $i \neq j$
- $\bigcup_i A_i = A$

Disjointization

Definition

Given a sequence of sets $A_1, A_2, A_3, \ldots$, disjointization creates a sequence of disjoint sets $B_1, B_2, B_3 \ldots$, such that:

- $B_i \cap B_j = \emptyset$ for $i \neq j$ (disjoint).
- The union is preserved:
$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Disjointization of an Increasing Sequence of Sets

Definition

- Let (A_n) be an increasing sequence of sets, meaning
  $A_1 \subset A_2 \subset A_3 \subset \dots$.
- The disjointization of (A_n) is the sequence of disjoint sets (B_n) defined by
  $B_1 = A_1, \quad B_n = A_n \setminus A_{n-1} \quad (n \geq 2)$
- These sets satisfy:
  $B_i \cap B_j = \emptyset$ for $i \neq j$ (they are pairwise disjoint).
- Their union equals the union of the original sequence:
  $$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

Compare and contrast Partition and Disjointization.

Countability

Countability

- A set is finite if it has a limited number of elements. Eg.
  $A = \{2, 4, 6, 8, 10\}$
- Countably infinite: elements can be listed as $a_1, a_2, a_3, \ldots$ (e.g., $\mathbb{N}, \mathbb{Q}$)
- Uncountable: cannot be listed in a sequence (e.g., $\mathbb{R}$)

Sets

Mutually Exclusive Events

Two events A and B are mutually exclusive if they cannot occur at the same time.

- Formally: $A \cap B = \emptyset$.

Exhaustive events

A collection of events $\{A_i\}$ is exhaustive if at least one of them must occur.

- Formally: $\bigcup_i A_i = \Omega$

Towards Algebra – The Idea

So far, we have worked with individual sets such as intervals and simple events.

To define length, probability, or measure, we need to work with collections of sets, not just single sets.

We want a collection of sets with the following properties:

- the empty set should be included
- complements should remain inside the collection
- unions of sets should stay inside the collection

However, requiring all these properties at once is often too strong at the beginning.

Instead, we start with very simple building blocks, such as intervals, and gradually add structure.

Semi-ring of Sets

Definition (Semi-ring)

A family $\mathcal{S}$ of subsets of Ω is called a semi-ring if:

- $\emptyset \in \mathcal{S}$
- If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$
- If $A, B \in \mathcal{S}$, then $A \setminus B$ is a disjoint union of a finite family of sets from $\mathcal{S}$

Semi-Ring

Example

Consider the set of intervals in $\mathbb{R}$:

$$\mathcal{S} = \{[a, b) \mid a < b, \ a, b \in \mathbb{R}\}.$$

Semi-ring

Proof that $\mathcal{S}$ is a semi-ring.

- Take $[a, a) = \emptyset$, so $\emptyset \in \mathcal{S}$.
- The intersection $[a, b) \cap [c, d) = [\max(a, c), \min(b, d)) \in \mathcal{S}$.
- Suppose that $[a, b) \subseteq [c, d)$. The difference $[a, b) \setminus [c, d)$ is a finite union of disjoint sets from $\mathcal{S}$.
  That is, $[a, b) \setminus [c, d) = [c, a) \cup [b, d)$.

Other examples of semi-ring

- The family of all intervals in $\mathbb{R}$ is a semi-ring.

Ring

Definition

A family S of subsets of Ω is called a ring if

- S contains $\emptyset$
- $A, B \in S \implies A \cup B \in S$, and $A \setminus B \in S$.

It follows also that the intersection $A \cap B$ belongs to S because $A \cap B = B \setminus (B \setminus A)$ is also in S.

Also, it follows that a ring is a semi-ring.

Ring

Example

Let $\Omega = \{1, 2, 3, 4, 5, \dots\}$ and let $\mathcal{R} = \{$All finite subsets of $\Omega\}$.

Proof that $\mathcal{R}$ is a ring

- $\emptyset \subseteq \Omega$ and is finite, therefore $\emptyset \in \mathcal{R}$
- $\{1, 3\} \cup \{2, 4, 5\} = \{1, 2, 3, 4, 5\} \in \mathcal{R}$.
- $\{1, 2, 3\} \setminus \{2\} = \{1, 3\} \in \mathcal{R}$.

Other examples of a ring

- $\mathcal{R} = \{$all finite subsets of $\mathbb{N}\}$ is a ring.
- $\mathcal{R} = \{[a, b) \subseteq \mathbb{R} | a, b \in \mathbb{Q}, a \leq b\}$. That is, all intervals of this form on the real line with rational endpoints.

σ-ring

Definition

A ring S is called a σ-ring if the union of any countable family $\{A_k\}_{k=1}^{\infty}$ of sets from S is also in S. That is:

- Closure under unions: If $A_1, A_2, A_3, \cdots \in \mathcal{R}$, then
$\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$.
- Closure under difference: If $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$.

- It follows that the intersection $A = \bigcap_k A_k$ is also in S.
- Let B be any of the sets A_k so that $B \supset A$, then
$A = B \setminus (B \setminus A) = B \setminus (\bigcup_k (B \setminus A_k)) \in S$.

Algebra

Definition (Algebra)

Let Ω be a non-empty set.
A collection of subsets of Ω is called an algebra $\mathcal{F}$, if it satisfies the following conditions:

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. If $A_1, A_2, \ldots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$

Note

A ring containing Ω is called an algebra.

Algebra

Example

Given $\Omega = \{1, 2, 3\}$. Construct the algebra $\mathcal{F}$.

Algebra

Solution

- $\Omega = \{1, 2, 3\}$
- $\mathcal{F} = \{\emptyset, \Omega, \{2\}, \{1, 3\}\}$

Clearly, $\mathcal{F}$ satisfies all three properties of an algebra.

Note

The power set of Ω, denoted by $\mathcal{P}(\Omega)$ or 2^{Ω} is an algebra.

σ-Algebra

Definition

A collection of sets $\mathcal{F}$ is called a σ-algebra if it is an algebra and satisfies the property:

- If $A_n \in \mathcal{F}$ for $n \geq 1$, then $\bigcup_{n \geq 1} A_n \in \mathcal{F}$.

That is, a σ-algebra is an algebra and is closed under countable unions.

(i) $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,

(iii) $A_1, A_2, \cdots \in \mathcal{F} \implies \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$

Some general, simple examples of σ-algebras

Examples

1. $\mathcal{F} = \{\emptyset, \Omega\}$ — *trivial σ-algebra*

2. $\mathcal{F} = \{$all subsets of $\Omega\}$ — The largest σ-algebra

3. Let $\mathcal{A} = \{A\} \subset \Omega$, then
   $\sigma(\mathcal{A}) = \{\emptyset, A, A^c, \Omega\}$

Example

Given $\Omega = \{HH, TH, HT, TT\}$. Construct the following on Ω.

- $\sigma(\{HH, TT\})$
- $\sigma(\{TH\})$
- $\sigma(\{\emptyset\})$
- $\sigma(\{TH\}, \{HT\})$

Note

A σ-ring on Ω containing Ω is called a σ-algebra.

WHAT IS MEASURE AND WHY DO YOU THINK WE NEED
MEASURE IN THIS CONTEXT?

Why measure?

In mathematics, we often ask questions like:

- how long is this set?
- how large is this region?
- what is the chance that an event happens?

To answer these questions, we need a systematic way to assign numbers to sets.

This is the motivation for measure theory.

The problem with "size"

For simple sets, "size" is easy.

Examples:

- the length of an interval (a, b) is $b - a$
- the number of elements in a finite set is easy to count

But it is not simple for many sets.

Examples:

- unions of many intervals
- infinite sets
- complicated events in probability

We need a more general idea of size.

Making it slightly more complicated

Now take two intervals that do not overlap.

$$A = (1, 2), \qquad B = (3, 5)$$

Their lengths are:
- length$(A) = 1$
- length$(B) = 2$

The total length of $A \cup B$ is

$$1 + 2 = 3$$

So far, everything works nicely.

Limitations of basic ideas

Length works well for single intervals, but what about:

- a countable union of intervals?
- sets with infinitely many pieces?
- random events formed from many outcomes?

Basic formulas are not enough.

So we need a rule that:

- works for simple sets
- extends to complicated sets
- behaves consistently

From size to measure

A measure is a mathematical tool that:

- assigns a size to sets
- works for very general sets
- satisfies the natural properties we expect

Length, area, volume, and probability are all special cases of measures.

This motivates the formal definition of a measure.

Measure

Definition of a Measure

Let Ω be a set and $\mathcal{F}$ a collection of subsets of Ω (called a σ-algebra).
A function $\mu : \mathcal{F} \to [0, \infty]$ is called a **measure** if it satisfies:

- $\mu(\emptyset) = 0$
- **Countable additivity:** for any countable collection $\{A_1, A_2, A_3, \dots\}$ of disjoint sets in $\mathcal{F}$,

$$\mu\Big(\bigcup_i A_i \Big) = \sum_i \mu(A_i)$$

Measure

Remarks

- μ assigns a non-negative extended real number (can be ∞) to each measurable set.
- The σ-algebra $\mathcal{F}$ ensures that unions, intersections, and complements of sets are measurable.
- Countable additivity is stronger than finite additivity—it works for infinitely many disjoint sets.

Probability Measure

A probability measure P is a measure on a σ-algebra $\mathcal{F}$ of Ω such that:

- $P(A) \geq 0$ for all $A \in \mathcal{F}$
- $P(\Omega) = 1$
- For any countable collection $\{A_1, A_2, \dots\}$ of disjoint events in $\mathcal{F}$,

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

Measure Space

A measure space is an ordered triple $(\Omega, \mathcal{F}, \mu)$ where:

- Ω is a set, called the sample space.
- $\mathcal{F}$ is a σ-algebra of subsets of Ω, i.e., a collection of subsets that satisfies:
    - $\Omega \in \mathcal{F}$
    - if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (complement is measurable)
    - if $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (countable union is measurable)
- $\mu : \mathcal{F} \to [0, \infty]$ is a measure, i.e., a function satisfying:
    - $\mu(\emptyset) = 0$
    - Countable additivity: for any countable collection $\{A_1, A_2, \dots\}$ of disjoint sets in $\mathcal{F}$,

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i \Big) = \sum_{i=1}^{\infty} \mu(A_i)$$

Definition (Measurable Space)

A **measurable space** is an ordered pair $(\Omega, \mathcal{F})$ where:

- Ω is a set, called the sample space.
- $\mathcal{F}$ is a σ-algebra of subsets of Ω, i.e., a collection of subsets that satisfies:
    - $\Omega \in \mathcal{F}$
    - if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (complement is measurable)
    - if $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (countable union is measurable)
- The elements of $\mathcal{F}$ are called **measurable sets**.

Probability Space

A probability space is a triple $(\Omega, \mathcal{F}, P)$ where:

- Ω is the sample space (all possible outcomes)
- $\mathcal{F}$ is a σ-algebra of subsets of Ω (the events)
- P is a probability measure

Examples of Measures

Length measure on $\mathbb{R}$:

$$\mu((a, b)) = b - a$$

Counting measure on any set Ω:

$$\mu(A) = \text{number of elements in } A$$

Probability measure on a probability space $(\Omega, \mathcal{F}, P)$:

- $P(\Omega) = 1$
- P satisfies countable additivity

Classical examples of Measures

Lengths in $\mathbb{R}$

Length in $\mathbb{R}$: For any interval $I \in \mathbb{R}$ bounded by the endpoints a, b , its length is given as
$\ell(I) = |b - a|$.

Additivity property of length

If an interval I is a disjoint union of a finite family $\{I_k\}_{k=1}^n$ of intervals, then $\ell(I) = \sum_{k=1}^n \ell(I_k)$.

Classical examples of Measure

Areas in $\mathbb{R}^2$

Given that I, J are the intervals (or lengths) of any rectangle A, then area$(A) = \ell(I)\ell(J)$.

Additive property of Area

If A is a rectangle of disjoint union of a finite family of rectangles $A_1, A_2, \ldots, A_n$, then area$(A) = \sum_{k=1}^{n}$ area(A_k).

Volumes in $\mathbb{R}^3$

Any box in $\mathbb{R}^3$ of the form $A = I \times J \times K$, where I, J, K are intervals in $\mathbb{R}$, will yield the set $\text{vol}(A) = \ell(I)\ell(J)\ell(K)$.

- The additive property of the volume is proved similarly.

Probability

If the event $A \subset \Omega$, and A is a disjoint union of a finite sequence of events $A_1, \ldots, A_n$, then $\mathbb{P}(A) = \sum_{k=1}^{n} \mathbb{P}(A_k)$.

Note these similarities

All the above had the following.

- A non-empty set M (i.e. $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \Omega$).
- A family of subsets S (i.e. intervals, rectangles, boxes, events).
- A functional $\mu : S \to \mathbb{R}_+ := [0, +\infty)$ (length, area, etc.) with the following property:
  - if $A \in S$ is a dsijoint union of a finite family $\{A_k\}_{k=1}^n$ of sets from S, then $\mu(A) = \sum_{k=1}^n \mu(A_k)$.

σ-additive measures

Let M be a non-empty set and S be a family of subsets of M.

Definition

A functional $\mu : S \rightarrow \mathbb{R}_+$ is called a σ-**additive measure** if whenever

- a set $A \in S$ is a disjoint union of an at most countable sequence $\{A_k\}_{k=1}^N$ (where N is either finite or $N = \infty$), then
- $\mu(A) = \sum_{k=1}^N \mu(A_k)$.

Remark

- μ is a *finitely additive measure* if this property holds for finite values of N.
- Every σ-additive measure is finitely additive, but the converse is not true.

σ-subadditive measures

Definition

A functional $\mu : \mathcal{S} \to \mathbb{R}_+$ is called σ-subadditive if whenever $A \subset \bigcup_{k=1}^{N} A_k$ where A and A_k are all elements of $\mathcal{S}$ and N is either finite or infinite, $\mu(A) \leq \sum_{k=1}^{N} \mu(A_k)$.

Note

If this property holds for finite values of N, then μ is called finitely subadditive.

Lemma 1.1

The *length* is σ-subadditive.

Proof

Let I, $\{I_k\}_{k=1}^{\infty}$ be intervals such that $I \subset \cup_{k=1}^{\infty} I_k$, we want to prove that $\ell(I) \leq \sum_{k=1}^{\infty} \ell(I_k)$.

Let us fix some $\varepsilon > 0$ and choose a bounded closed interval $I' \subset I$ such that $\ell(I) \leq \ell(I') + \varepsilon$.

For any k, choose an open interval $I_k' \supset I_k$ such that $\ell(I_k') \leq \ell(I_k) + \frac{\varepsilon}{2^k}$.

Then the bounded closed interval I' is covered by a sequence $\{I_k'\}$ of open intervals. By the Borel-Lebesgue lemma, there is a finite subfamily $\{I_{k_j}'\}_{j=1}^{n}$ that also covers I'.

Proof

It follows from the finite additivity of length that it is finitely subadditive. That is,

$$\ell(I') \leq \sum_j \ell(I'_{k_j}) \implies \ell(I') \leq \sum_{k=1}^{\infty} \ell(I'_k).$$

This yields

$$\ell(I) \leq \varepsilon + \sum_{k=1}^{\infty} \left(\ell(I_k) + \frac{\varepsilon}{2^k} \right) = 2\varepsilon + \sum_{k=1}^{\infty} \ell(I_k).$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$ finishes the proof. $\square$

Theorem

The length is a σ-additive measure on the family of all bounded intervals in $\mathbb{R}$.

Proof

We need to prove that if $I = \bigsqcup_{k=1}^{\infty} I_k$, then $\ell(I) = \sum_{k=1}^{\infty} \ell(I_k)$.

By the σ-subadditive lemma, we have $\ell(I) \leq \sum_{k=1}^{\infty} \ell(I_k)$, so we need to prove the opposite inequality.

For a fixed $n \in \mathbb{N}$, we have

$$I \supset \bigsqcup_{k=1}^{n} I_k.$$

It follows from the finite additivity of length that

$$\ell(I) \geq \sum_{k=1}^{n} \ell(I_k).$$

Proof

Letting $n \to \infty$, we obtain
$$\ell(I) \geq \sum_{k=1}^{\infty} \ell(I_k)$$
which finishes the proof.

Outer Measure

Let Ω be a set and R be an algebra of sets of Ω. Suppose μ is a σ-additive measure on R. From the algebra property: if $A \in R$, then $A^c := \Omega \setminus A \in R$.

Definition

For any set $A \subset \Omega$, define its outer measure $\mu^*(A)$ by:

$$\mu^*(A) = \inf\left\{ \sum_{k=1}^{\infty} \mu(A_k) : A_k \in R \text{ and } A \subset \bigcup_{k=1}^{\infty} A_k \right\} \ldots\ldots\ldots\ldots (1)$$

In other words, we consider all coverings $\{A_k\}_{k=1}^{\infty}$ of A by a sequence from the algebra R and define $\mu^*(A)$ as the infimum of the sum of all $\mu(A_k)$, taken over all such coverings.

Outer Measure

Properties of outer measure

- Null (empty) set: $\mu^*(\emptyset) = 0$
- Monotonicity: if $A \subset B \subset \Omega$, then $\mu^*(A) \leq \mu^*(B)$.
- Countable subadditivity (σ-subadditivity): For any countable collection of sets $\{A_n\}_{n=1}^{\infty} \subset \Omega$:
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Outer measure

Lemma 1.2

For any set $A \subset \Omega$, $\mu^*(A) < \infty$ and if $A \in R$, then $\mu^*(A) = \mu(A)$.

Outer measure

Proof

Note that $\emptyset \in R$ and $\mu(\emptyset) = 0$ because $\mu(\emptyset) = \mu(\emptyset \sqcup \emptyset) = \mu(\emptyset) + \mu(\emptyset)$.

For any set $A \subset \Omega$, consider a covering $\{A_k\} = \{\Omega, \emptyset, \emptyset, \dots\}$ of A.

Since $\Omega, \emptyset \in R$, it follows from (1) that

$$\mu^*(A) \leq \mu(\Omega) + \mu(\emptyset) + \mu(\emptyset) + \cdots = \mu(\Omega) < \infty.$$

Assume now $A \in R$. Considering a covering $\{A_k\} = \{A, \emptyset, \emptyset, \dots\}$

and using that $A, \emptyset \in R$, we obtain in the same way that

$$\mu^*(A) \leq \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \cdots = \mu(A).$$

Outer measure

Continuation of Proof

On the other hand, for any sequence $\{A_k\}$ as in (1), we have by the

σ-subadditivity of μ that

$\sum_{k=1}^{\infty} \mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k).$

Taking the infimum over all such sequences $\{A_k\}$, we obtain

$\mu(A) \leq \mu^*(A)$, which together with the previous inequality yields

$\mu^*(A) = \mu(A).$ ∎

Outer measure

Lemma 1.3

The outer measure μ^* is σ-subadditive on 2^Ω.

Outer measure

Proof

We need to prove that if $A \subset \bigcup_{k=1}^{\infty} A_k$ where A and A_k are subsets of Ω,

then $\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$.

By the definition of μ^*, for any set A_k and any $\varepsilon > 0$

there exists a sequence $\{A_{kn}\}_{n=1}^{\infty}$ of sets from R such that

$A_k \subset \bigcup_{n=1}^{\infty} A_{kn}$ and $\mu^*(A_k) \geq \sum_{n=1}^{\infty} \mu(A_{kn}) - \frac{\varepsilon}{2^k}$.

Adding these inequalities over all k, we obtain

$\sum_{k=1}^{\infty} \mu^*(A_k) \geq \sum_{k,n=1}^{\infty} \mu(A_{kn}) - \varepsilon$.

Outer measure

Continuation of proof

On the other hand, by the inclusions $A \subset \bigcup_{k=1}^{\infty} A_k$ and $A_k \subset \bigcup_{n=1}^{\infty} A_{kn}$,

we get $A \subset \bigcup_{k,n=1}^{\infty} A_{kn}$. Since $A_{kn} \in R$, it follows from (1) that

$\mu^*(A) \leq \sum_{k,n=1}^{\infty} \mu(A_{kn})$.

Comparing with the previous inequality gives

$\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(A_k) + \varepsilon$.

Since this holds for any $\varepsilon > 0$, it also holds for $\varepsilon = 0$, which completes the proof. $\blacksquare$

Symmetric Difference

Definition

The symmetric difference of two sets $A, B \subset \Omega$ is the set
$A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

- Clearly, $A \triangle B = B \triangle A$
- Also, $x \in A \triangle B$ if and only if x belongs to exactly one of the sets A, B. That is, either $x \in A$ and $x \notin B$ or $x \notin A$ and $x \in B$.

Symmetric Difference

Lemma 1.4(a)

For arbitrary sets $A_1, A_2, B_1, B_2 \subset \Omega$,

$(A_1 \circ A_2) \triangle (B_1 \circ B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$,

where $\circ$ denotes any of the operations $\cup, \cap, \setminus$.

Lemma 1.4(b)

If μ^* is an outer measure on Ω, then $|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \triangle B)$, for arbitrary sets $A, B \subset \Omega$.

Measurable Sets

We still consider R to be an algebra on Ω and μ is a σ-additive measure on R, and holding the definition of μ^* by (1).

Definition of Measurable sets

A set $A \subset \Omega$ is called measurable (with respect to the algebra R and the measure μ) if, for any $\varepsilon > 0$, there exist $B \in R$ such that $\mu^*(A \triangle B) < \varepsilon$.

Definition

Let μ^* be an outer measure on a set Ω. A subset $E \subseteq \Omega$ is called **measurable** (with respect to μ^*) if for every subset $A \subseteq \Omega$:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

where $E^c = \Omega \setminus E$ is the complement of E.

Measurable sets

Examples of measurable sets

1. Intervals in the real line. i.e. $(a, b), [a, b), [a, b], (-\infty, a), \text{or}(b, +\infty)$.

2. Finite sets and countable sets. i.e. $\{a, b, c\}, \mathbb{N}, \mathbb{Q}$, etc.

3. Complements of measurable sets. i.e. $\mathbb{R} \setminus \mathbb{Q}$, etc

Measurable sets

Are all sets Measurable? Why?

Why All Sets Are Not Measurable

- **Length, area, volume in real life**
  - When measuring objects (a rope, a piece of land, a container), we only deal with well-behaved sets: intervals, rectangles, polygons, or unions of them
  - These are all measurable because their size can be consistently determined

- **Constructed 'pathological' sets**
  - Some sets are created using the axiom of choice and are so irregular that you cannot assign a meaningful size
  - Example: Vitali set—choosing one number from each equivalence class modulo rationals between 0 and 1. No consistent length can be assigned

- **3D paradoxical sets**
  - Banach–Tarski paradox shows you can partition a ball into strange pieces that can be reassembled into two identical ctc

Figure: examples of non-measurable sets.

Illustrative Examples of Non-Measurable Sets

Vitali set ($\mathbb{R}$):

- Consider all real numbers between 0 and 1.
- Partition them into equivalence classes where numbers differ by a rational number.
- Pick one number from each class.
- The resulting set is the Vitali set.
- You cannot assign a consistent length to it, even though each piece "looks like a number."

Banach–Tarski paradox (3D):

- Start with a solid 3D ball (like a basketball).
- Using very strange, non-physical pieces, it is possible to cut it into finitely many pieces and reassemble them into two balls of the same size.
- Each piece is non-measurable, meaning no volume can be consistently assigned.

Assignment

Illustrate the non-measurable Bernstein set of real numbers.

Theorem

Carathéodory's extension theorem

Let R be an algebra on a set Ω and μ be a σ-additive measure on R. Denote by $\mathcal{M}$ the family of all measurable subsets of Ω. Then the following is true:

1. $\mathcal{M}$ is a σ-algebra containing R.

2. The restriction of μ^* on $\mathcal{M}$ is a σ-additive measure (that extends measure μ from R to $\mathcal{M}$).

3. If $\tilde{\mu}$ is a σ-additive measure defined on a σ-algebra $\mathcal{F}$ such that $R \subset \mathcal{F} \subset \mathcal{M}$, then $\tilde{\mu} = \mu^*$ on $\mathcal{M}$.

Caratheodory Extension Theorem Proofs

Claim 1: *The family Ω of all measurable sets is an algebra containing R.*

- If $A \in R$, then A is measurable because:

$$\mu^*(A \triangle A) = \mu^*(\emptyset) = \mu(\emptyset) = 0,$$

  where $\mu^*(\emptyset) = \mu(\emptyset)$ by Lemma 1.2.

- Hence, $R \subset \Omega$ and the entire space Ω is a measurable set.

To verify $\mathcal{M}$ is an algebra: Show that for $A_1, A_2 \in \mathcal{M}$, both $A_1 \cup A_2$ and $A_1 \setminus A_2$ are measurable.

- Let $A = A_1 \cup A_2$. For any $\varepsilon > 0$, there exist $B_1, B_2 \in R$ such that

$$\mu^*(A_1 \triangle B_1) < \varepsilon, \quad \mu^*(A_2 \triangle B_2) < \varepsilon$$

- Set $B = B_1 \cup B_2 \in R$.

Proof

- Then by Lemmas 1.4 and 1.3 respectively:

$$A \triangle B \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$$

and

$$\mu^*(A \triangle B) \leq \mu^*(A_1 \triangle B_1) + \mu^*(A_2 \triangle B_2) < 2\varepsilon$$

- Since $\varepsilon > 0$ is arbitrary and $B \in R$ A is measurable. Similarly, $A_1 \setminus A_2 \in \mathcal{M}$.

Caratheodory Extension Theorem Proofs

Claim 2: σ-additivity of μ^* on $\mathcal{M}$ μ^* is σ-additive on $\mathcal{M}$.

Since $\mathcal{M}$ is an algebra and μ^* is σ-subadditive by Lemma 1.5, it suffices to prove that μ^* is finitely additive on $\mathcal{M}$ (see Exercise 9).

Let us prove that, for any two disjoint measurable sets A_1 and A_2, we have

$$\mu^*(A) = \mu^*(A_1) + \mu^*(A_2)$$

where $A = A_1 \cup A_2$. By Lemma 1.5, we have the inequality

$$\mu^*(A) \leq \mu^*(A_1) + \mu^*(A_2)$$

so that we are left to prove the opposite inequality

$$\mu^*(A) \geq \mu^*(A_1) + \mu^*(A_2).$$

Proof

For any $\varepsilon > 0$, there are sets $B_1, B_2 \in \mathbb{R}$ such that (1.19) holds. Set $B = B_1 \cup B_2 \in \mathbb{R}$ and apply Lemma 1.6, which says that

$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \triangle B) < 2\varepsilon,$$

where in the last inequality we have used (1.20). In particular,

$$\mu^*(A) \geq \mu^*(B) - 2\varepsilon.$$

On the other hand, since $B \in \mathbb{R}$, we have by Lemma 1.4 and the additivity of μ on $\mathbb{R}$ that

$$\mu^*(B) = \mu(B) = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) - \mu(B_1 \cap B_2).$$

Proof

Next, we estimate $\mu(B_i)$ from below via $\mu^*(A_i)$ and show that $\mu(B_1 \cap B_2)$ is small enough.

Indeed, using (1.19) and Lemma 1.6, we obtain, for any $i = 1, 2$,

$$|\mu^*(A_i) - \mu^*(B_i)| \leq \mu^*(A_i \triangle B_i) < \varepsilon,$$

whence

$$\mu(B_1) \geq \mu^*(A_1) - \varepsilon \quad \text{and} \quad \mu(B_2) \geq \mu^*(A_2) - \varepsilon.$$

On the other hand, by Lemma 1.6 and using $A_1 \cap A_2 = \emptyset$, we obtain

$$B_1 \cap B_2 = (A_1 \cap A_2) \triangle (B_1 \cap B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2),$$

whence by (1.20)

$$\mu(B_1 \cap B_2) = \mu^*(B_1 \cap B_2) \leq \mu^*(A_1 \triangle B_1) + \mu^*(A_2 \triangle B_2) < 2\varepsilon.$$

It follows from (1.21)–(1.24) that

$$\mu^*(A) \geq (\mu^*(A_1) - \varepsilon) + (\mu^*(A_2) - \varepsilon) - 2\varepsilon - 2\varepsilon = \mu^*(A_1) + \mu^*(A_2) - 6\varepsilon.$$

Letting $\varepsilon \to 0$, we finish the proof.

Prove the following claims:

1. $\mathcal{M}$ is σ-algebra

2. If Σ is a σ-algebra such that

$$\mathbb{R} \subset \Sigma \subset \Omega,$$

and $\tilde{\mu}$ is a σ-additive measure on Σ such that $\tilde{\mu} = \mu$ on $\mathbb{R}$. Then $\tilde{\mu} = \mu^*$ on Σ.
Prove that

$$\tilde{\mu}(A) = \mu^*(A) \quad \text{for any } A \in \Sigma.$$

σ-finite measures

Definition

Let Ω be a non-empty set and S be a family of subsets of Ω.
A functional $\mu : S \to [0, +\infty]$ is called a *measure* if, for all sets $A, A_k \in S$ such that $A = \bigsqcup_{k=1}^{N} A_k$ (where N is either finite or infinite), we have

$$\mu(A) = \sum_{k=1}^{N} \mu(A_k).$$

Hence, a measure is always σ-additive.

Examples of σ-finite Measures

Lebesgue measure on the real line. Let μ be the usual length on $\mathbb{R}$. The measure of $\mathbb{R}$ is infinite, but

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n].$$

Each interval $[-n, n]$ has finite length $2n$. Therefore, Lebesgue measure on $\mathbb{R}$ is σ-finite.

Counting measure on the integers. Let $X = \mathbb{Z}$ and define $\mu(A)$ to be the number of elements in A. The set $\mathbb{Z}$ is infinite, but

$$\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} \{n\}.$$

Each singleton $\{n\}$ has measure 1, which is finite. Hence, the counting measure on $\mathbb{Z}$ is σ-finite.

Area measure on the plane. Let μ be the usual area measure on $\mathbb{R}^2$. The plane has infinite area, but

$$\mathbb{R}^2 = \bigcup_{n=1}^{\infty} [-n, n] \times [-n, n].$$

Each square has finite area $(2n)^2$. Thus, the area measure on $\mathbb{R}^2$ is σ-finite.