AN ANALYTICAL APPROACH IN SOLVING A SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

Jerico B. Bacani and Julius Fergy T. Rabago
Department of Mathematics and Computer Science
College of Science, University of the Philippines Baguio
Baguio City 2600, Benguet, Philippines
jicderivative@yahoo.com and jfrabago@gmail.com

ABSTRACT

We re-examine the system of difference equations given by

$$x_{n+1} = \frac{x_{n-(2k-1)}}{\varepsilon+\delta x_{n-(2k-1)}y_{n-(k-1)}}, \qquad y_{n+1} = \frac{y_{n-(2k-1)}}{\rho+\sigma y_{n-(2k-1)}x_{n-(k-1)}},$$

where $\varepsilon, \delta, \rho, \sigma \in \{-1,1\}$ and $k \in \mathbb{N}$, with the real initial values $\{x_n\}_{n=-(2k-1)}^{0}$ and $\{y_n\}_{n=-(2k-1)}^{0}$ such that $\delta x_{m-(k-1)} \neq -\varepsilon$ and $\sigma y_{m-(2k+1)} \neq -\rho$ for all possible values of m and k. We present an analytical approach to derive the closed-form solution of the given system. Our technique uses appropriate substitutions on the phase variables coupled with the application of the method of differences.

INTRODUCTION

Difference equations are discrete analogues of differential equations. They appear naturally in nature and serve as mathematical models of various physical phenomena. For instance, the well-known Fibonacci recurrence given by the difference equation

$F_{n+1} = F_n + F_{n-1}$, with initial conditions $F_0 = 1$ and $F_1 = 1$, describes many biological events, from the arrangement of seeds and petals [7] to branching of tress [20]. It also describes the behavior of fruit sprouts of a pineapple, the flowering of an artichoke, an uncurling fern and the arrangement of a pine cone's bracts [5], and many other things found in nature. The Fibonacci equation is an example of a linear type difference equation. Meanwhile, there are the so-called nonlinear types of difference equations, which, apparently, posses more complex and interesting properties. Their complexity provide richer dynamics and intriguing behavior that are much more suitable to describe some physical events and interactions found in nature.

In a modeling setting, the two-dimensional competitive system of nonlinear rational difference equations

$$(1) \qquad x_{n+1} = \frac{x_n}{a+y_n}, \quad y_{n+1} = \frac{y_n}{a+y_n}, \quad n \in \mathbb{N}_0 := \{0,1,2,\ldots\},$$

represents the rule by which two discrete competitive populations reproduce from one generation to the next. The quantities x_n and y_n describe the population density of two particular beings during the n^{th} generation. The sequence of iterates or orbits $\{(x_n, y_n)\}_{n=0}^{\infty} := \{(x_n, y_n)\}_0^{\infty}$ generated by these recursions describes how the populations evolve over time. Competition between the populations is reflected by the fact that the transition function for each population is a decreasing function of the other population size. For instance, in [18], M. P. Hassell and H. N. Comins studied a discrete (difference) single age-class model for two-species competition and investigated the stability properties of its solutions. Nonlinear types of difference equations also posses numerous applications in pure mathematics and related sciences. For instance, the Newton's method (or Newton-Raphson's method), which is known as a root-finding algorithm, uses the nonlinear difference equation $x_{n+1} = x_n - f(x_n)/f'(x_n)$, $n = 0,1,\ldots$, to approximate roots (or zeros) of real-valued function $f(x) = 0$(cf. [2]).

Here the initial value x_0 is described as the initial guess or initial iterate of the iteration process. Other applications in related sciences, such as in Theoretical Biology, also appear in countless literatures (cf. [6, 13] and the relevant references cited therein).

In most situations, especially in the case of nonlinear difference equations, the solution forms of such equations are keys to examine the dynamics of the systems they model. However, there are instances that the solution of a difference equation or a system of difference equations is not expressible in closed forms. Nevertheless, there are known cases for which the solutions of the systems are in closed-form (cf. [23, 24, 25, 26, 27, 28, 29] and the related references therein). In these papers, numerous techniques were provided to derive the solutions of various systems of difference equations. The works were motivated by several investigations carried out in this area of research. Moreover, in [1, 8, 9, 10, 11, 12, 14, 30, 31], the authors examined different systems of nonlinear difference equations (with higher-order de- lay). However, the solutions were established through a mere application of the induction method. The induction approach surely validates the forms of solutions exhibited in the studies, but its main drawback is that it does not provide further informations on how the solutions have appeared in such structure. To fill this gap, several related systems were re-examined in [23]-[29], extended the studies to larger classes of difference equations, and the previously published results via induction method were explained theoretically. It is interesting to note that the systems previously mentioned are reducible into a class of linear difference equations using appropriate substitutions on the variables. Remarkably, various methods in solving difference equations of linear type are already available in the literature (cf. [3] and [21]). Reducing these non-linear types into linear ones greatly simplifies the solving.

In the present paper, we apply the same idea used in [27]; that is, we employ the method of differences or telescoping sums to arrive at the desired solution form of the system under consideration. In this work, we are particularly concerned with the two-dimensional system of difference equations given by

(2) $$x_{n+1}=\frac{x_{n-(2k-1)}}{\varepsilon+\delta x_{n-(2k-1)}y_{n-(k-1)}},\qquad y_{n+1}=\frac{y_{n-(2k-1)}}{\rho+\sigma y_{n-(2k-1)}x_{n-(k-1)}}$$

where $\varepsilon,\delta,\rho,\sigma\in\{-1,1\}$ and $k\in\mathbb{N}$, with the real initial values $\{x_n\}_{n=-(2k-1)}^{0}$ and $\{y_n\}_{n=-(2k-1)}^{0}$ such that $\delta x_{m-(k-1)}\neq-\varepsilon$ and $\sigma y_{m-(2k+1)}\neq-\rho$ for all $m\in I_1\cup I_2$ where $I_1:=\{-(2k-1),-(2k-2),\ldots,-k\}$ and $I_2:=\{-(k-1),-(k-2),\ldots,0\}$. We emphasize that this system can be viewed as a model of a competitive system analogous to the system of equations (1) but with a higher-order delay on the index. We examined this system in [4], and investigated the boundedness, convergence and periodicity character of its solutions. However, we have just established the closed-form solution of (2) by a mere application of induction method. So, our objective in this work is to provide theoretical explanation of the results exhibited in [4] regarding the solution form of (2). Thus, the novelty of this study is the analytical approach that is used to obtain the closed-form solution of the given system for all possible combinations of (ε,δ) and (ρ,σ). We mentioned that once the closed-form solution of a given system of difference equation is already at hand, the qualitative behavior of its solutions such as boundedness and convergence will be easily examined. Therefore, knowing the closed-form solution of the given system is of great importance in the analysis of its long term dynamics and other interesting properties.

We remark that (2) is, in fact, a certain general form of the system of equations considered in Mansour et al. [22] (see also [14]) from which the system of difference equations of the form

$$x_{n+1}=\frac{x_{n-5}}{\pm1\pm x_{n-5}y_{n-2}},\qquad y_{n+1}=\frac{y_{n-5}}{\pm1\pm y_{n-5}x_{n-2}}$$

have been studied. System (2) does not only provide a generalized form of the above equation but also contains several cases studied in [1] and [15]. Also, the one-dimensional case of the given system also covers those equations investigated in [9], [10] and [12].

MAIN RESULTS

In this section, we shall provide theoretical explanations for the results delivered in [4] regarding the closed-form solution of the system (2). We remark that only ten cases were considered and completely examined in [4]. This is because solving the other remaining cases is similar to solving the ten cases considered; hence, similar proofs were omitted. In the present paper, however, we shall only examine the following cases:

(i) $(\varepsilon, \delta) = (\rho, \sigma) = (-1,1)$,

(ii) $(\varepsilon, \delta) = (\rho, \sigma) = (1,1)$, and

(iii) $(\varepsilon, \delta) = (-1,1)$and $(\rho, \sigma) = (1,-1)$.

We opted to only consider these cases since similar proofs will be done for the rest of the possible cases.

Before we proceed further, we first recall the definition of the periodicity of a sequence.

Definition 1 ([19]). *We say that a solution of a given difference equation, say $x_{n+1} = f(x, x_{n-1}, \ldots, x_{n-k})$ (which is of order $k+1$) is periodic with period $p > 0$ if we can find an integer $N \geq -k$ such that the relation $x_{n+p} = x_n$ holds for all $n \geq N$.*

The following subsections are organized according to the cases mentioned above.

2.1. **System 1**. Consider the system

(S1) $$x_{n+1} = \frac{x_{n-(2k-1)}}{-1+x_{n-(2k-1)}y_{n-(k-1)}}, \qquad y_{n+1} = \frac{y_{n-(2k-1)}}{-1+y_{n-(2k-1)}x_{n-(k-1)}}.$$

Let $w_n \coloneqq x_n y_{n-k}$ and $z_n \coloneqq y_n x_{n-k}$. Then, from system (S1), we have

$$w_{n+1} = \frac{z_{n+1-k}}{-1+z_{n+1-k}} \qquad \text{and} \qquad z_{n+1} = \frac{w_{n+1-k}}{-1+w_{n+1-k}}.$$

Eliminating the phase variable z, we get $w_{n+2k} = w_n$, and so the sequence $\{w_n\}_{-2k}^{\infty}$ is periodic with period $2k$. Similarly, $\{z_n\}_{-2k}^{\infty}$ is periodic with period $2k$, i.e.

$z_{n+2k} = z_n$. These informations imply that

$$x_{n+4k}y_{n+3k} = x_n y_{n-k} \quad \text{and} \qquad y_{n+k}x_n = y_{n-3k}x_{n-4k},$$

from which we get the ratio

$$\frac{x_{n+2k}}{x_n} = \frac{x_n}{x_{n-2k}}.$$

Replacing n by $2kn + i$ and then applying the idea of telescoping for products (see [27]), we obtain

$$\frac{x_{2kn+i}}{x_i} = \prod_{j=0}^{n-1} \frac{x_{2k(j+1)+i}}{x_{2kj+i}} = \prod_{j=0}^{n-1} \frac{x_{2kj+i}}{x_{2k(j-1)+i}} = \frac{1}{(-1+x_i y_{i+k})^n}, \tag{3}$$

for all $n \in \mathbb{N}_0$, where we applied the first equation in (S1), and the fact that the sequence of ratios $\{w_{n+2k}/z_{n+k}\}_{-2k+1}^{\infty}$ in the last equation is $2k$-periodic. Using similar arguments, by eliminating the phase variable w_n, we are led to the following identity

$$\frac{y_{2kn+i}}{y_i} = \frac{1}{(-1+y_i x_{i+k})^n}, \tag{4}$$

for all $n \in \mathbb{N}_0$ and $i \in I_1$. In equations (3) and (4), the index i runs from $-2k+1$ to $-k$ (i.e $i \in I_1$) as the sequence $\{w_{n+2k}/z_{n+k}\}$ starts from $n = -2k+1$. For indices i, running from $-k+1$ to 0 (i.e., $i \in I_2$), we shall proceed as follows. We first consider the case when $i = 0$. Observe from the first equation in (S1) that

$$x_{2k} = \frac{x_0}{-1+x_0 y_k}. \tag{5}$$

However, we want y_k to be expressed in terms of the initial conditions. Then, using the second equation in (S1), we know that

$$y_k = \frac{y_{-k}}{-1+y_{-k}x_0}.$$

Substituting this identity into (5), we obtain

$$x_{2k} = \frac{x_0}{-1+x_0\left(\frac{y_{-k}}{-1+y_{-k}x_0}\right)} = x_0(-1+y_{-k}x_0). \tag{6}$$

Now, in general, by using (S1), we have

(7) $$x_{2kn+i} = \frac{x_{2k(n-1)+i}}{-1+x_{2k(n-1)+i}y_{2kn+i-k}}.$$

Using the fact that the sequence $\{y_n x_{n-k}\}$ is $2k$-periodic, we can write the right side of (7) as

(8) $$x_{2kn+i} = \left(\frac{1}{-1+x_i y_{i+k}}\right) x_{2k(n-1)+i}.$$

On one hand, we express y_{i+k}in terms of y_{i-k}. To do this, we use the fact that

$$y_{i+k} = \frac{y_{i-k}}{-1+y_{i-k}x_i},$$

and substitute this expression for y_{i+k} in (8). Then, we get

$$x_{2kn+i} = (-1 + y_{i-k}x_i)x_{2k(n-1)+i}.$$

Iterating the right side of the this identity yields the desired formula for x_{2k+i} for indices

$i \in I_2$, for all $n \in \mathbb{N}_0$, which can be written as

$$x_{2kn+i} = x_i(-1 + y_{i-k}x_i)^n.$$

By symmetry, and by using the property that $\{z_{2kn+i}\}_0^\infty$is also $2k$-periodic, we derive the formula

$$y_{2kn+i} = y_i(-1 + x_{i-k}y_i)^n$$

for all indices $i \in I_2$ and for all $n \in \mathbb{N}_0$. In summary, the following result is theoretically established (cf. [4, Case (i), Theorem 2.1]).

Theorem 1. *Every solution of (S1) takes the form*

$$x_{2kn+i} = \begin{cases} \dfrac{x_i}{(-1 + x_i y_{i+k})^n}, for i \in I_1 \\ x_i(-1 + x_i y_{i-k})^n, for i \in I_2 \end{cases}$$

and

$$y_{2kn+i} = \begin{cases} \dfrac{y_i}{(-1 + y_i x_{i+k})^n}, for i \in I_1 \\ y_i(-1 + y_i x_{i-k})^n, for i \in I_2 \end{cases}$$

for any set of initial conditions $\{x_n\}_{n=-(2k-1)}^0, \{y_n\}_{n=-(2k-1)}^0 \subset \mathbb{R}$ such that $x_i y_{i+k} \neq 1$ for all $i \in I_1$ and $x_i y_{i-k} \neq 1$for all $i \in I_2$.

2.2. **System 2**. Consider the system

(S2) $$x_{n+1}=\frac{x_{n-(2k-1)}}{1+x_{n-(2k-1)}y_{n-(k-1)}},\quad y_{n+1}=\frac{y_{n-(2k-1)}}{1+y_{n-(2k-1)}x_{n-(k-1)}}.$$

Using the definitions $w_n := x_n y_{n-k}$ and $z_n := y_n x_{n-k}$, we get

$$w_n=\frac{z_{n-k}}{1+z_{n-k}} \qquad \text{and} \qquad z_{n-k}=\frac{w_{n-2k}}{1+w_{n-2k}},$$

and so

$$w_n=\frac{w_{n-2k}}{1+2w_{n-2k}} \Leftrightarrow \frac{1}{u_n}:=w_n=\frac{1}{2+\frac{1}{w_{n-2k}}}=\frac{1}{2+u_{n-2k}}$$

The latter equation implies $u_{2k(j+1)+i}-u_{2kj+i}=2$. By telescoping sums, we get

$$u_{2kn+i}-u_i=\sum_{j=0}^{n-1}\left(u_{2k(j+1)+i}-u_{2kj+i}\right)=2n,$$

or equivalently,

(9) $$x_{2kn+i}y_{2kn+i-k}=w_{2kn+i}=\frac{w_i}{1+2nw_i}=\frac{x_i y_{i-k}}{1+2nx_i y_{i-k}}.$$

Now, using the second equation in (S2), we have

$$\frac{y_{2k(n+1)+i-k}}{y_{2kn+i-k}}=\frac{1}{1+y_{2kn+i-k}x_{2kn+i}}=\frac{1}{1+\left(\frac{x_i y_{i-k}}{1+2nx_i y_{i-k}}\right)}=\frac{1+2nx_i y_{i-k}}{1+(2n+1)x_i y_{i-k}}.$$

By telescoping for products, we get

(10) $$\frac{y_{2kn+i-k}}{y_{i-k}}=\prod_{j=0}^{n-1}\left(\frac{1+2jx_i y_{i-k}}{1+(2j+1)x_i y_{i-k}}\right).$$

Replacing i by $i+k$, and letting i runs from $-2k+1$ to$-k$, we obtain

$$y_{2kn+i}=y_i\prod_{j=0}^{n-1}\frac{1+2jx_{i+k}y_i}{1+(2j+1)x_{i+k}y_i}.$$

In a similar fashion, we can also show that

$$x_{2kn+i}=x_i\prod_{j=0}^{n-1}\frac{1+2jy_{i+k}x_i}{1+(2j+1)y_{i+k}x_i}.$$

Now, to obtain the form of solutions for indices $i\in I_2$, we shall proceed as follows. From (S2), we have

(11) $$y_{2kn+i} = \frac{y_{2k(n-1)+i}}{1+y_{2k(n-1)+i}x_{2kn+i-k}}.$$

From (9), we replace i by $i-k$ to obtain

(12) $$x_{2kn+i-k}y_{2k(n-1)+i} = \frac{x_{i-k}y_{i-2k}}{1+2nx_iy_{i-2k}}.$$

Then, again, using (S2) and after doing some rearrangements, we get

$$y_{i-2k} = \frac{-y_i}{y_ix_{i-k}-1}.$$

Substituting this identity to (12), we get

$$x_{2kn+i-k}y_{2k(n-1)+i} = \frac{x_{i-k}\left(\frac{-y_i}{y_ix_{i-k}-1}\right)}{1+2nx_i\left(\frac{-y_i}{y_ix_{i-k}-1}\right)} = \frac{x_{i-k}y_i}{1+2(n+1)x_iy_i}.$$

Going back to (11) (replacing n by $n+1$), we now have

$$\frac{y_{2k(n+1)+i}}{y_{2kn+i}} = \frac{1}{1+\left(\frac{x_{i-k}y_i}{1+2(j+1)x_iy_i}\right)} = \frac{1+(2n+1)x_iy_i}{1+2(n+1)x_iy_i}.$$

Finally, by telescoping for products, we get

$$y_{2kn+i} = y_i\prod_{j=0}^{n-1}\left(\frac{1+(2j+1)x_{i+k}y_i}{1+2(j+1)x_{i+k}y_i}\right),$$

for all $n \in \mathbb{N}_0$ and indices $i \in I_2$. Using a similar argument, one can also show that a similar form for the phase variable x_{2kn+i} will be obtained. To summarize the forms, we state the following result (cf. [4, Case (ii), Theorem 2.1]).

Theorem 2. *Every solution of (S2) takes the form*

$$x_{2kn+i} = \begin{cases} x_i\prod_{j=0}^{n-1}\left(\frac{1+2jy_{i+k}x_i}{1+(2j+1)y_{i+k}x_i}\right), for i \in I_1 \\ x_i\prod_{j=0}^{n-1}\left(\frac{1+(2j+1)y_{i-k}x_i}{1+2(j+1)y_{i-k}x_i}\right), for i \in I_2 \end{cases}$$

and

$$y_{2kn+i} = \begin{cases} y_i\prod_{j=0}^{n-1}\left(\frac{1+2jx_{i+k}y_i}{1+(2j+1)x_{i+k}y_i}\right), for i \in I_1 \\ y_i\prod_{j=0}^{n-1}\left(\frac{1+(2j+1)x_{i-k}y_i}{1+2(j+1)x_{i-k}y_i}\right), for i \in I_2 \end{cases}$$

for any set of initial conditions $\{x_n\}_{n=-(2k-1)}^{0}, \{y_n\}_{n=-(2k-1)}^{0} \subset \mathbb{R}$ *such that* $x_iy_{i+k} \neq -1$ *for all* $i \in I_1$ *and* $x_iy_{i-k} \neq -1$ *for all* $i \in I_2$.

2.3. **System 3**. We consider the system

$$\text{(S3)} \qquad x_{n+1} = \frac{x_{n-(2k-1)}}{-1+x_{n-(2k-1)}y_{n-(k-1)}}, \qquad y_{n+1} = \frac{y_{n-(2k-1)}}{1-y_{n-(2k-1)}x_{n-(k-1)}}.$$

We start by considering the formulas $w_n := x_n y_{n-k}$ and $z_n := y_n x_{n-k}$ to get

$$w_n = \frac{z_{n-k}}{-1+z_{n-k}} \qquad \text{and} \qquad z_{n-k} = \frac{w_{n-2k}}{1-w_{n-2k}},$$

from which it follows that

$$w_n = \frac{\frac{w_{n-2k}}{1-w_{n-2k}}}{-1+\left(\frac{w_{n-2k}}{1-w_{n-2k}}\right)} = \frac{w_{n-2k}}{-1+2w_{n-2k}} = \frac{\frac{w_{n-4k}}{-1+2w_{n-4k}}}{-1+2\left(\frac{w_{n-4k}}{-1+2w_{n-4k}}\right)} = w_{n-4k}.$$

Apparently, the sequence $\{w_n\}_{-2k}^{\infty}$ is $4k$-periodic. Furthermore, using a similar argument, one also finds that $\{z_n\}_{-2k}^{\infty}$ is also $4k$-periodic, i.e. $z_{n+4k} = z_n$ for all $n \in \mathbb{N}_0$. Now by telescoping, we have $\sum_{j=0}^{n-1}(w_{4k(n+1)+i} - w_{4kn+i}) = w_{4kn+i} - w_i = 0$, or equivalently $x_{4kn+i}y_{4kn-k+i} = x_i y_{i-k}$. Similarly, we have $y_{4kn-2k+i}x_{4kn-2k+i} = y_{i-k}x_{i-2k}$. These latter two identities imply that

$$x_{4kn+i} = \frac{x_i}{x_{i-2k}} x_{4kn-2k+i} = (-1 + x_i y_{i-k})x_{4kn-2k+i}.$$

Also, from (S3) and the fact the sequence $\{x_n y_{n-k}\}$is $4k$-periodic, we get

$$x_{4kn-2k+i} = \frac{x_{4kn-2k+i-2k}}{-1+x_i y_{i+k}} \qquad \text{and} \qquad y_{i-k} = \frac{y_{i+k}}{y_{i+k}x_i+1}.$$

Inserting these identities to the previous one and after some algebra yields

$$x_{4kn+i} = \left(\frac{-1+x_i y_{i-k}}{-1+x_i y_{i+k}}\right) x_{4k(n-1)+i} = \frac{1}{1-x_i^2 y_{i+k}^2} x_{4k(n-1)+i}.$$

Iterating the left side of the above equation gives us

$$x_{4kn+i} = \frac{x_i}{(1-x_i^2 y_{i+k}^2)^n},$$

for all $i \in I_1$ and $n \in \mathbb{N}_0$. Using the same idea used above, we turn our attention to the form of y_{4kn+i}, for $i \in I_2$. Now, note that

Also, from (S3) and the fact the sequence $\{x_n y_{n-k}\}$is $4k$-periodic, we get

$$x_{4kn-2k+i} = \frac{x_{4kn-2k+i-2k}}{-1+x_i y_{i+k}} \qquad \text{and} \qquad y_{i-k} = \frac{y_{i+k}}{y_{i+k}x_i+1}.$$

Inserting these identities to the previous one and after some algebra yields

$$x_{4kn+i} = \left(\frac{-1+x_i y_{i-k}}{-1+x_i y_{i+k}}\right) x_{4k(n-1)+i} = \frac{1}{1-x_i^2 y_{i+k}^2} x_{4k(n-1)+i}.$$

Iterating the left side of the above equation gives us

$$x_{4kn+i} = \frac{x_i}{(1-x_i^2 y_{i+k}^2)^n},$$

for all $i \in I_1$ and $n \in \mathbb{N}_0$. Using the same idea used above, we turn our attention to the form of y_{4kn+i}, for $i \in I_2$. Now, note that

$$y_{4kn+i} = \frac{y_{4kn-2k+i}}{1-y_{4kn-2k+i}x_{4kn-k+i}} = \left(\frac{1}{1-y_{-2k+i}x_{-k+i}}\right) y_{4kn-2k+i}.$$

But,

$$y_{4kn-2k+i} = \frac{y_{4k(n-1)+i}}{1-y_i x_{i+k}} \quad \text{and} \qquad x_{i+k} = \frac{x_{i-k}}{-1+x_{i-k}y_i},$$

so

$$\begin{aligned} y_{4kn+i} &= \left(\frac{1}{1-y_{-2k+i}x_{-k+i}}\right)\left(\frac{1}{1-y_i x_{i+k}}\right) y_{4k(n-1)+i} \\ &= \frac{y_i}{y_{i-2k}}\left(\frac{1}{1-y_i x_{i+k}}\right) y_{4k(n-1)+i} = \left(\frac{1-y_i x_{i-k}}{1-y_i x_{i+k}}\right) y_{4k(n-1)+i} \\ &= \left(1-y_i^2 x_{i-k}^2\right) y_{4k(n-1)+i}. \end{aligned}$$

Iterating the right side of the equation, we eventually get

$$y_{4kn+i} = y_i\left(1-y_i^2 x_{i-k}^2\right)^n,$$

for all $i \in I_2$ and $n \in \mathbb{N}_0$. Now, we proceed on finding the form for y_{4kn+i} for $i \in I_1$. The idea is similar, but, instead of replacing x_{i+k}, we replace x_{i-k} with an equivalent expression. In particular, we use the identity $x_{i-k} = -x_{i+k}/(1+x_{i-k}y_i)$ to obtain

$$y_{4kn+i} = \left(\frac{1-y_i x_{i-k}}{1-y_i x_{i+k}}\right) y_{4k(n-1)+i} = \left(\frac{1+2y_i x_{i+k}}{(1-y_i x_{i+k})^n}\right) y_{4k(n-1)+i}.$$

Hence, iterating the left-hand side yields the formula

$$y_{4kn+i} = \frac{y_i(1+2y_i x_{i+k})^n}{(1-y_i x_{i+k})^{2n}},$$

for all $i \in I_1$and $n \in \mathbb{N}_0$. Applying the same process to obtain the expression for x_{4kn+i} for $i \in I_2$leads to the following form

$$x_{4kn+i} = \frac{x_i(1-x_i y_{i-k})^{2n}}{(1-2x_i y_{i-k})^n},$$

for all $i \in I_2$ and $n \in \mathbb{N}_0$. For the remaining cases, one can still apply the same ideas presented above to obtain the desired forms for the phase variables $x_{4kn+2k+i}$ and $y_{4kn+2k+i}$ for $i \in I_1 \cup I_2$, for all $n \in \mathbb{N}_0$. To summarize, and to be precise, we give the explicit forms of these quantities together with those obtained previously in the following result (cf. [4, Case (v), Theorem 2.1]).

Theorem 3. *Every solution of (S3) has the following closed forms:*

$$x_{4kn+i+A}=\begin{cases}\dfrac{x_i}{\left(1-x_i^2y_{i+k}^2\right)^n}, for A=0 and i\in I_1,\\ \dfrac{x_i(1-x_iy_{i-k})^{2n}}{(1-2x_iy_{i-k})^n}, for A=0 and i\in I_2,\\ \dfrac{x_i}{(x_iy_{i+k}-1)\left(1-x_i^2y_{i+k}^2\right)^n}, for A=2k and i\in I_1,\\ \dfrac{x_i(x_iy_{i-k}-1)^{2n+1}}{(1-2x_iy_{i-k})^{n+1}}, for A=2k and i\in I_2,\end{cases}$$

and

$$y_{4kn+i+A}=\begin{cases}\dfrac{y_i(1-2y_ix_{i+k})^n}{(1-y_ix_{i+k})^{2n}}, for A=0 and i\in I_1,\\ y_i\left(1-y_i^2x_{i-k}^2\right)^n, for A=0 and i\in I_2,\\ \dfrac{y_i(1-2y_ix_{i+k})^n}{(y_ix_{i+k}-1)^{2n+1}}, for A=2k and i\in I_1\\ y_i(1-y_ix_{i-k})\left(1-y_i^2x_{i-k}^2\right)^n, for A=2k and i\in I_2\end{cases}$$

where $\{x_n\}_{n=-(2k-1)}^{0},\{y_n\}_{n=-(2k-1)}^{0}\subset\mathbb{R}$ *are the initial sets of initial conditions such that* $x_iy_{i+k}\neq 1$ *for all* $i\in I_1$ *and* $x_iy_{i-k}\neq 1$ *for all* $i\in I_2$.

We end our discussion with a concluding remark.

CONCLUDING REMARK

We have successfully carried out the derivation of the closed-form solution of system (2) in a theoretical manner. In particular, through an analytical approach, we were able to obtain the closed-form solutions of system (2), for the cases (i)$(\varepsilon,\delta)=(\rho,\sigma)=(-1,1)$, (ii) $(\varepsilon,\delta)=(\rho,\sigma)=(1,1)$, and (iii) $(\varepsilon,\delta)=(-1,1)$ and $(\rho,\sigma)=(1,-1)$. This was done using appropriate substitutions on the phase variables reducing the system into various difference equations of linear type. Then, employing the method of differences or telescoping (for sums and products), the solution sequences of the resulting linear difference equations were shown to be periodic. The results were then used to obtain the desired forms for the solutions of the given systems for the said three cases. The results presented here coincide with those presented in [4], particularly for the cases (i), (ii) and (v) of Theorem 2.1 in [4]. The other remaining cases were omitted but can be established without any difficulty because same arguments may be used; hence, we leave the proofs to the interested readers.

The key idea to the success of this work relies on the appropriate substitution on the phase variables and the use of the method of differences. This is an alternative approach in dealing with the closed-form solutions of the system of nonlinear difference equations that we considered. Certainly, it is possible that the technique used here can be applied to other classes of difference equations whose closed-form solutions are, in structure, similar to the ones obtained here.

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