The Amalgamation Number of Graphs Involving Wheels and Fans*

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Abstract

Let $G = \big(V(G), E(G)\big)$ and $H = \big(V(H), E(H)\big)$ be finite, connected, simple graphs, such that $V(G)$ and $V(H)$ are disjoint sets. A ***k-amalgamation*** of G and H, denoted by $\boldsymbol{G \star^{k} H}$, is the graph obtained by identifying k distinct vertices of G with k distinct vertices of H such that no adjacent vertices of one graph are identified with adjacent vertices of the other. The **amalgamation number** of G and H, denoted by ***a*(*G,H*)**, is the maximum k such that a k-amalgamation $G \star^{k} H$ is possible. Let a wheel and a fan be denoted by $W_m = K_1 + C_m$ and $F_n = K_1 + P_n$, respectively, where $m \geq 3$ and $n \geq 2$. Results on the amalgamation number of wheels and fans and of two wheels will be presented.

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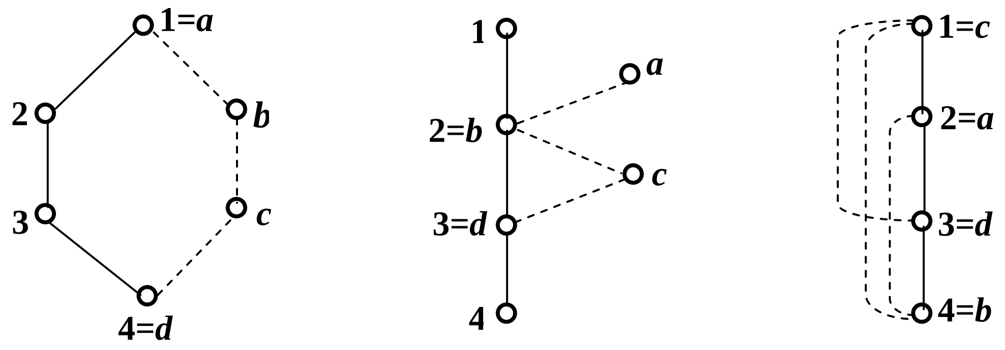
1. Introduction

This study includes only graphs which are simple, connected and finite. Graph-theoretic terms which are used but not explicitly defined here are adopted from [1] and [4].

Definition 1.1. ([5]) Let $G=(V(G),E(G))$ and $H=(V(H),E(H))$ be finite, connected, simple graphs such that $V(G)$ and $V(H)$ are disjoint, and let $k \le \min\{|V(G)|, |V(H)|\}$. A ***k-amalgamation*** of G and H, denoted by $\boldsymbol{G\star^k H}$, is the graph obtained by identifying k distinct vertices of G with k distinct vertices of H such that no adjacent vertices of one graph are identified with adjacent vertices of the other. When $\boldsymbol{G\star^k H}$ results through the identification of the vertices in the ordered set $U=\{u_1,u_2,u_3,\cdots,u_k\}$, where $U\subseteq V(G)$, with the vertices in the ordered set $W=\{w_1,w_2,w_3,\cdots,w_k\}$, where $W\subseteq V(H)$, and $u_i=w_i$ for each $i=1,2,3,\cdots,k$, the k-amalgamation is indicated by $\star^k = (U,W)$. In the case when $U = \{u\}$ and $W = \{w\}$, the 1-amalgamation is denoted by $\star = (u,w)$. A 1-amalgamation of G and H will also be referred to as an **amalgamation** of G and H.

Example 1.1. Let G and H be two paths of order 4, where $G=P_4=(1,,2,3,4)$ and $H=P_4=(a,b,c,d)$. Figure 1.1(a) and Figure 1.1(b) show two non-isomorphic 2-amalgamations, and Figure 1.1(c) shows a 4-amalgamation of P_4 with itself.



(a) $\star^2 = (\{1,4\},\{a,d\})$ *(b)* $\star^2 = (\{2,3\},\{b,d\})$ *(c)* $\star^4 = (\{2,4,1,3\},\{a,b,c,d\})$

Figure 1.1 Some non-isomorphic k-amalgamations of P_4 with itself

Definition 1.2. The ***amalgamation number*** of G and H, denoted by $a(G,H)$, is the maximum k such that a k-amalgamation $G \star^{k} H$ is possible.

Example 1.2. Figure 1.1 shows that $a(P_4, P_4) = 4$.

Remark 1. If $G \star^{k} H$ exists with $\star^{k} = (U, W)$, then $H \star^{k} G$ exists with $\star^{k} = (W, U)$; thus, $G \star^{k} H = H \star^{k} G$.

Remark 2. Let $|V(G)|$ and $|V(H)|$ be the order (*i.e.* the number of vertices) of the graphs G and H, respectively. Then, the amalgamation number is bounded by the orders of G and H, that is

$$1 \le a(G,H) \le \min\{|V(G)|, |V(H)|\}.$$

Let G be a connected graph. The ***distance*** between the vertices v and w in G, denoted by $d(v,w)$ or $d(v,w;G)$, is the length of a shortest path in G between v and w.

Theorem 1.1. Let G and H be connected finite simple graphs. Suppose there are vertices v in G and w in H such that $d(x,v;G)=1$ for any vertex $x \ne v$ in G and $d(y,w;H)=1$ for any vertex $y \ne w$ in H. If $G \star^{k} H$ is a k-amalgamation which is obtained through $\star^{k} = (U, W)$, where $v \in U$, $w \in W$, and $v = w$, then $k = 1$.

Proof. Suppose $v \in U$ and $w \in W$, and $G \star^{k} H$ is a k-amalgamation which is obtained through $\star^{k} = (U, W)$, with $v = w$. Any other vertex x in G cannot be identified with another vertex y in H since $d(x,v;G)=1$ and $d(y,w;H)=1$. Therefore, $x \notin U$ and $y \notin V$ for any vertex $x \ne v$ in G and for any vertex $y \ne w$ in H. Thus, $|U| = |V| = 1$, and $k = 1$. ▓

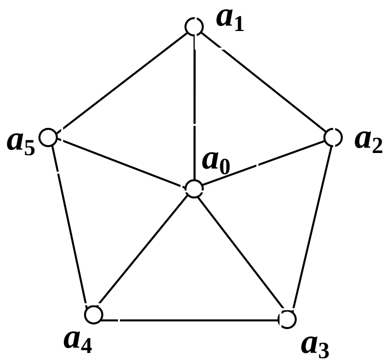
Corollary 1.2. If G and H are complete graphs, then $a(G,H)=1$.

Definition 1.3. Let G and H be two graphs such that $V(G)$ and $V(H)$ are disjoint sets. The **sum** or **join** of G and H, denoted by $G + H$, consists of the graphs G and H together with the edges between every vertex of G and every vertex of H.

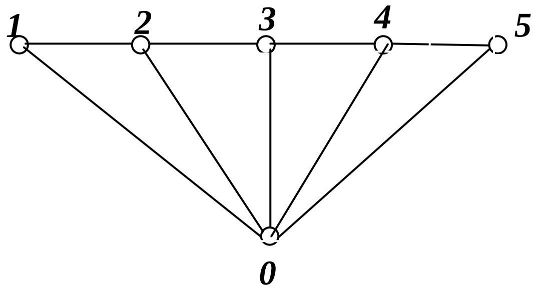
Definition 1.4. A ***wheel*** of order $n+1$ is the graph $W_n = K_1 + C_n$, $n \ge 3$, where C_n is a cycle of order n.

Definition 1.5. A ***fan*** of order $n+1$ is the graph $F_n = K_1 + P_n$, $n \ge 2$, where P_n is a path of order n.

Example 1.3. A wheel $W_5 = K_1 + C_5$ and a fan $F_5 = K_1 + P_5$ are shown in Figure 1.2(a) and Figure 1.2(b), respectively.



(a) W_5



(b) F_5

Figure 1.2 A wheel W_5 and a fan F_5

This study establishes the amalgamation numbers $a(W_m, F_n)$, for $m \geq 3$ and $n \geq 2$, and $a(W_m, W_n)$, for $n \geq m \geq 3$. The investigation can be extended to the study of the amalgamation of other special graphs. The amalgamation of graphs involving paths and cycles was studied in [2] and [3].

2. The Amalgamation Number of a Wheel and a Fan

Let $W_m = K_1 + C_m$ be a wheel, with $K_1 = \{a_0\}$ and $C_m = [a_1, a_2, \cdots, a_m]$, and edges $[a_0, a_i]$, $i = 1, 2, 3, \cdots, m$, $[a_1, a_m]$, and $[a_i, a_{i+1}]$, $i = 1, 2, 3, \cdots, m-1$. Let $F_n = K_1 + P_n$ be a fan, with $K_1 = \{0\}$ and $P_n = (1, 2, 3, \cdots, n)$, and edges $[0, i]$, for $i = 1, 2, 3, \cdots, n$, and $[i, i+1]$, for $i = 1, 2, 3, \cdots, n-1$.

The following result is a corollary of Theorem 1.1.

Lemma 2.1. If $W_m \star^k F_n$ is obtained through $\star^k = (U, W)$, with $a_0 \in U$, $0 \in W$, and $a_0 = 0$, then $k = 1$.

Theorem 2.2. For $n \geq 7$, $a(W_3, F_n) = 4$.

Proof. Let $n \geq 7$. As shown in Figure 2.1, we obtain a 4-amalgamation of W_3 and F_n through $\star^4 = (U, W)$, with $U = \{a_1, a_2, a_3, a_0\}$ and $W = \{1, 3, 5, 7\}$. Thus, $a(W_3, F_n) \geq 4$. By Remark 2, $a(W_3, F_n) \leq \min\{|V(W_3)|, |V(F_n)|\} = \min\{4, n+1\} = 4$. Therefore, $a(W_3, F_n) = 4$. ▓

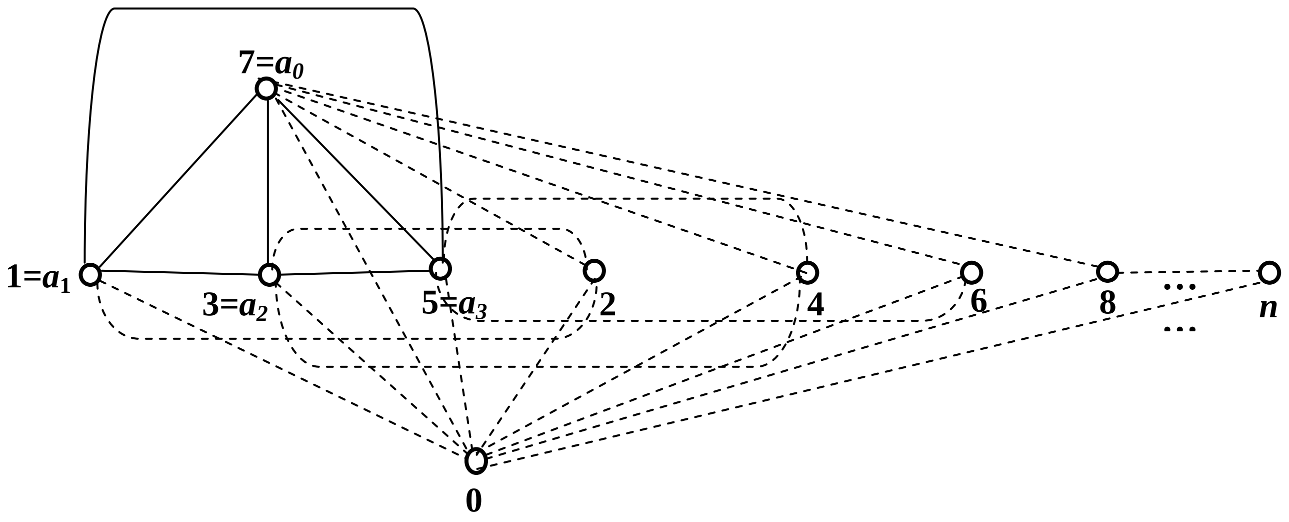


Figure 2.1 $\star^4 = (U, W)$ in $W_3 \star^4 F_n$, $n \geq 7$

Since W_3 is a complete graph, Theorem 2.2 holds when W_3 is replaced with a connected graph of order 4 with fewer edges. The following result is a corollary of Theorem 2.2.

Corollary 2.3. If $n \geq 7$ and G is a connected graph of order 4, then $a(G, F_n) = 4$.

Theorem 2.4.

i. $a(W_3, F_2) = 1$
ii. $a(W_3, F_n) = 2$ if $n = 3$ or 4
iii. $a(W_3, F_n) = 3$ if $n = 5$ or 6

Proof. Since W_3 is the complete graph of order 4, we may assume without loss of generality that $a_1 \in U$ for any k-amalgamation $W_3 \star^k F_n$ with $\star^k = (U, W)$.

(i) Since W_3 and F_2 are both complete graphs, so $a(W_3, F_2) = 1$.

(ii) Let n = 3 or 4. A 2-amalgamation of W_3 and F_n is isomorphic to one of the following graphs: G_1 which is obtained through $\star^2 = (U, W)$ with $U = \{a_1, a_2\}$ and $W = \{1,3\}$; G_2 which is obtained through $\star^2 = (U, W)$, with $U = \{a_1, a_2\}$ and $W = \{1,4\}$; or G_3 which is obtained through $\star^2 = (U, W)$ with $U = \{a_2, a_3\}$ and $W = \{1,3\}$. Thus, $a(W_3,F_n) \geq 2$. For each of G_1, G_2, and G_3, $d(a_k, j) \leq 2$ for all $a_k \in W_3$ and $j \in F_n$, hence the number of amalgamation vertices cannot be increased. Therefore, $a(W_3,F_n) = 2$ if n = 3 or 4.

(iii) Let n = 5 or 6. A 3-amalgamation of W_3 and F_n is isomorphic to G which is obtained through $\star^3 = (U, W)$, with $U = \{a_1, a_2, a_3\}$ and $W = \{1,3,5\}$ or $W = \{1,3,6\}$. Thus, $a(W_3, F_n) \geq 3$. By a similar argument in (ii), $a(W_3, F_n) = 3$ if n = 5 or 6. ▩

Theorem 2.5. For $n \geq 7$, $a(W_4, F_n) = 5$.

Proof. Let $n \geq 7$. As shown in Figure 2.2, we obtain a 5-amalgamation of W_4 and F_n through $\star^5 = (U, W)$, with $U = \{a_1, a_3, a_2, a_4, a_0\}$ and $W = \{1, 2, 4, 5, 7\}$. By Remark 2, $a(W_4, F_n) \leq \min\{|V(W_4)|, |V(F_n)|\} = \min\{5, n+1\} = 5$. Thus, $a(W_4, F_n) = 5$ if $n \geq 7$. ▓

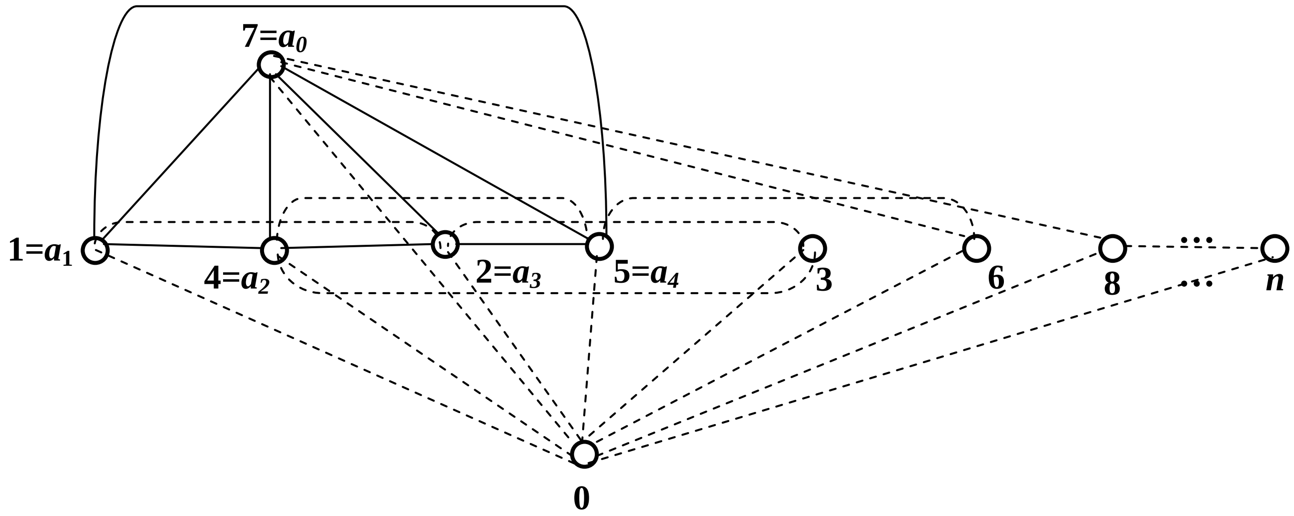


Figure 2.2 $\star^5 = (U,W)$ in $W_4 \star^5 F_n$, $n \geq 7$

Theorem 2.6.

i. $a(W_4, F_n) = 2$ if n = 2 or 3
ii. $a(W_4, F_4) = 3$
iii. $a(W_4, F_n) = 4$ if n = 5 or 6

Proof. (i) F_2 is the complete graph of order 3. Without loss of generality, we may assume that $a_1 \in U$ for any k-amalgamation $W_4 \star^k \boldsymbol{F}_2$ with $\star^k = (U,W)$. A 2-amalgamation of W_4 and F_2 is isomorphic to $G = W_4 \star^2 \boldsymbol{F}_2$ which is obtained through $\star^2 = (U,W)$, with $U = \{a_1, a_3\}$ and $W = \{1,2\}$. Thus, $a(W_4, F_2) \geq 2$. Since $d(a_k, j; G) \leq 2$ for all $a_k \in W_4$ and $j \in F_2$, so the number of amalgamation vertices cannot be increased. Therefore, $a(W_4, F_2) = 2$.

A 2-amalgamation of W_4 and $\boldsymbol{F}_3$ is isomorphic to $G = W_4 \star^2 \boldsymbol{F}_3$ which is obtained through $\star^2 = (U,W)$, with $U = \{a_1, a_3\}$ and $W = \{1,2\}$, $W = \{1,3\}$ or $W = \{0,1\}$, $U = \{a_1, a_2\}$ and $W = \{1,3\}$, $U = \{a_2, a_4\}$ and $W = \{1,2\}$ or $W = \{0,1\}$, or $U = \{a_1, a_4\}$ and $W = \{1,3\}$. Since $d(a_k, j; G) \leq 2$ for all $a_k \in W_4$ and $j \in F_3$, so the number of amalgamation vertices cannot be increased. Thus, $a(W_4, F_3) = 2$.

(ii) A 3-amalgamation of W_4 and $\boldsymbol{F}_4$ is isomorphic to $G = W_4 \star^3 \boldsymbol{F}_4$ which is obtained through $\star^3 = (U,W)$, with $U = \{a_1, a_3, a_2\}$ and $W = \{1,2,4\}$, or $U = \{a_1, a_2, a_4\}$ and $W = \{1,3,4\}$. Since $d(a_k, j; G) \le 2$ for all $a_k \in W_4$ and $j \in F_4$, so the number of amalgamation vertices cannot be increased. Thus, $a(W_4, F_4) = 3$.

(iii) Let n = 5 or 6. A 4-amalgamation of W_4 and $\boldsymbol{F}_n$ is isomorphic to $G = W_4 \star^4 \boldsymbol{F}_n$ which is obtained through $\star^4 = (U,W)$, with $U = \{a_1, a_3, a_2, a_4\}$ and $W = \{1,2,4,5\}$, or $U = \{a_1, a_3, a_4, a_2\}$ and $W = \{1,2,4,5\}$. Since $d(a_k, j; G) \le 2$ for all $a_k \in W_4$ and $j \in F_n$, so the number of amalgamation vertices cannot be increased. Thus, $a(W_4, F_n) = 4$. ▓

Theorem 2.7. If $m \ge 5$ and $n \ge m+2$, then $a(W_m, F_n) = m+1$.

Proof. Let $m \ge 5$ and $n \ge m + 2$. By Lemma 2.1 we may not identify the vertices a_0 and 0. We consider two cases (see Figure 2.3 for m even, and Figure 2.4 for m odd). When m is even, we obtain an $(m+1)$-amalgamation of W_m and F_n with $\star^{m+1} = (U,W)$, $U = \{a_1, a_3, a_5, \cdots, a_{m-1}, a_2, a_4, \cdots, a_m, a_0\}$ and $W = \{1,2,3,4,\cdots,m,m+2\}$. When m is odd, we obtain an $(m+1)$-amalgamation of W_m and F_n with $\star^{m+1} = (U,W)$, $U = \{a_1, a_3, a_5, \cdots, a_m, a_2, a_4, \cdots, a_{m-1}, a_0\}$ and $W = \{1,2,3,4,\cdots,m,m+2\}$. In both cases, $|U| = |V(W_m)| = m+1$; hence, $a(W_m, F_n) \ge m+1$. By Remark 2,

$$a(W_m, F_n) \le \min\{|V(W_m)|, |V(F_n)|\} = \min\{m+1, n+1\} = m+1.$$

Thus, $a(W_m, F_n) = m+1$ if $m \ge 5$ and $n \ge m+2$. ▓

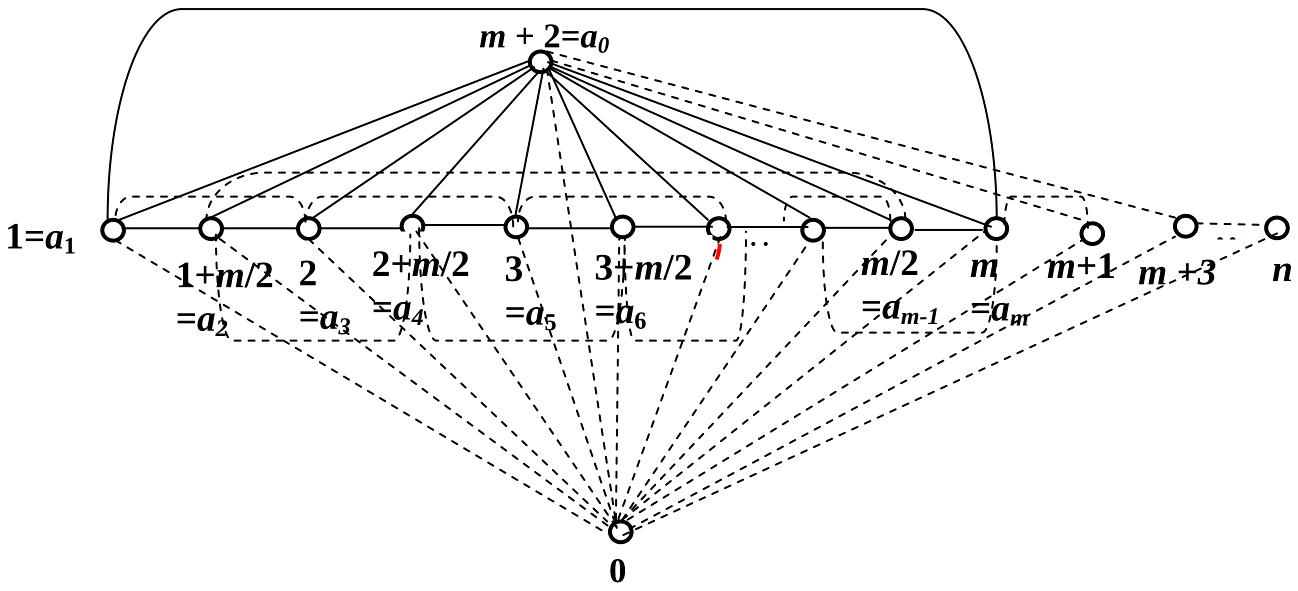


Figure 2.3 $\star^{m+1} = (U,W)$ in $W_m \star^{m+1} F_n$ for m even, $m \ge 5$ and $n \ge m+2$

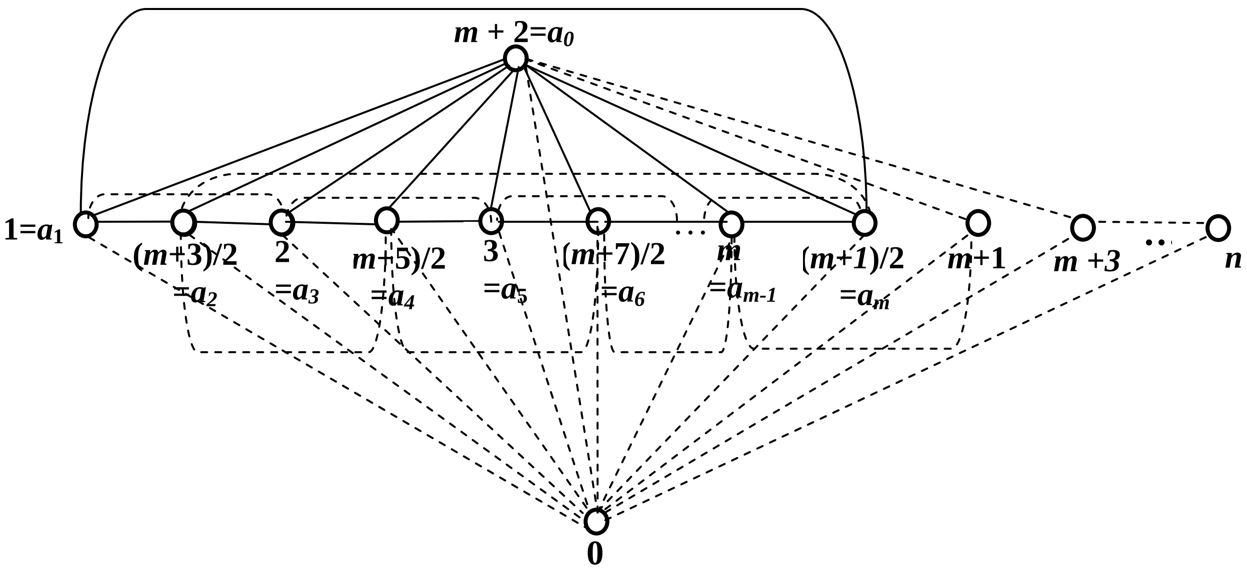


Figure 2.4 $\star^{m+1} = (U, W)$ in $W_m \star^{m+1} F_n$ for m odd, $m \geq 5$ and $n \geq m+2$

Theorem 2.8. If $m \geq 5$ and $m \leq n \leq m+1$, then $a(W_m, F_n) = m$.

Proof. Let $m \geq 5$ and $m \leq n \leq m+1$. By Lemma 2.1 identifying the vertices $a_0 \in W_m$ and $0 \in F_n$ will not yield a maximum number of amalgamation vertices, thus, $a_0 \neq 0$. When m is even, we obtain an m-amalgamation of W_m and F_n, with $\star^m = (U,W)$, $U = \{a_1, a_3, a_5, \cdots, a_{m-1}, a_2, a_4, \cdots, a_m\}$ and $W = \{1, 2, 3, 4, \cdots, m\}$. When m is odd, we obtain an m-amalgamation of W_m and F_n, with $\star^m = (U,W)$, $U = \{a_1, a_3, a_5, \cdots, a_m, a_2, a_4, \cdots, a_{m-1}\}$ and $W = \{1, 2, 3, 4, \cdots, m\}$. Thus, $a(W_m, F_n) \geq m$. By Remark 2, $m \leq a(W_m, F_n) \leq m+1$. Suppose that $W_m \star^{m+1} F_n$ exists, with $\star^{m+1} = (U,W)$ and $|U| = |V(W_m)| = m+1$. Since $a_0 \neq 0$, so a_0 is identified with a vertex w in $\{1, 2, 3, \cdots, n\} = V(F_n) - \{0\}$. Thus, $n \geq m+1$. Therefore, $W_m \star^{m+1} F_n$ is not possible if $n = m$; hence, $a(W_m, F_m) = m$. Let $n = m+1$. Without loss of generality, let $a_0 = w$, where $w \in \{1, 2, 3, \cdots, m+1\}$. Note that the vertices $a_1, a_2, a_3, a_4, \cdots, a_m$ are identified with the vertices in $\{1, 2, 3, \cdots, m+1\} - \{w\}$. Thus, there are vertices a_k and a_j, $k \neq 0$ and $j \neq 0$, which are identified with $w-1$ and $w+1$, that is, $a_k = w-1$ and $a_j = w+1$. However, $(a_0 = w, a_k = w-1)$ and $(a_0 = w, a_j = w+1)$ are edges in W_m, and $(a_0 = w, w-1)$ and $(a_0 = w, w+1)$ are edges in F_{m+1}; thus, multiple edges occur. Therefore, $W_m \star^{m+1} F_n$ is not possible if $n = m+1$; hence, $a(W_m, F_{m+1}) = m$. ▓

3. The Amalgamation Number of Two Wheels

Let $W_m = K_1 + C_m$ be a wheel with $K_1 = \{a_0\}$ and $C_m = [a_1, a_2, a_3, \cdots, a_m]$ and edges $[a_o, a_i]$, $i = 1, 2, 3, \cdots, m$, and $[a_1, a_m]$, $[a_i, a_{i+1}]$, $i = 1, 2, 3, \cdots, m-1$. Let $W_n = K_1 + C_n$ be a second wheel with $K_1 = \{0\}$ and the cycle $C_n = [1, 2, 3, \ldots, n]$ and edges $[0,i]$, $i = 1, 2, 3, \cdots, n$, and $[1, n]$, $[i, i+1]$, $i = 1, 2, 3, \cdots, n-1$. By Remark 1, $G \star^n H = H \star^n G$, thus we will consider only $a(W_m, W_n)$ with $m \le n$.

The following result is a corollary of Theorem 1.1.

Lemma 3.1. If $W_m \star^k W_n$ is obtained through $\star^k = (U, W)$, with $a_0 \in U$, $0 \in W$, and $a_0 = 0$, then $k = 1$.

Theorem 3.2. For $n \ge 8$, $a(W_3, W_n) = 4$.

Proof. Refer to Figure 3.1. We obtain a 4-amalgamation of W_3 and W_n through $\star^4 = (U, W)$, with $U = \{a_1, a_2, a_3, a_0\}$ and $W = \{1, 3, 5, 7\}$. Therefore, $a(W_3, W_n) \ge 4$. By Remark 2, $a(W_3, W_n) \le \min\{|V(W_3)|, |V(W_n)|\} = \min\{4, n+1\} = 4$. Thus, $a(W_3, W_n) = 4$ if $n \ge 8$. ▓

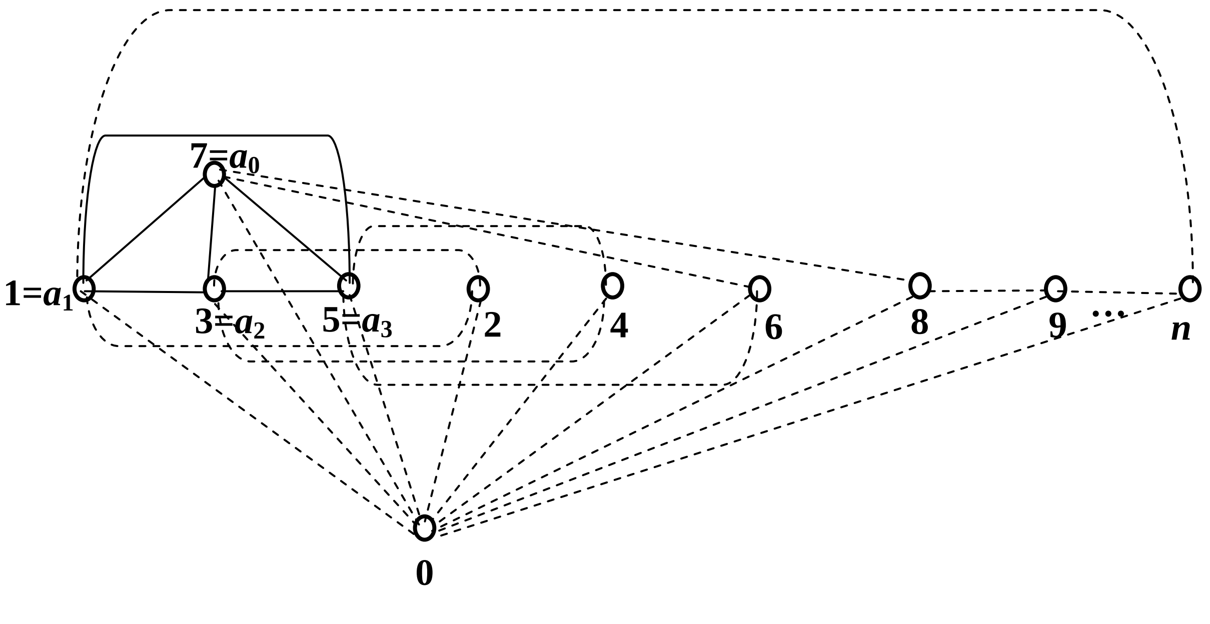


Figure 3.1 $\star^4 = (U, W)$ in $W_3 \star^4 W_n$ *for* $n \ge 8$

Theorem 3.3.

i. $a(W_3, W_3) = 1$

ii. $a(W_3, W_n) = 2$ if $n = 4$ or 5

iii. $a(W_3,W_n)=3$ if n = 6 or 7

Proof. Note that W_3 is the complete graph of order 4. Without loss of generality, we may assume that $a_1 \in U$ for any k-amalgamation $W_3 \star^k W_n$ with $\star^k = (U,W)$.

(i) By Corollary 1.2, $a(W_3,W_3)=1$.

(ii) Let n = 4 or 5. A 2-amalgamation of W_3 and either a W_4 or a W_5 is isomorphic to $G = W_3 \star^2 \mathbf{W}_n$ which is obtained through $\star^2 = (U,W)$ where $U=\{a_1,a_2\}$ and $W=\{1,3\}$ when $n=4$, and $U=\{a_1,a_2\}$ and $W=\{1,3\}$ or $W=\{1,4\}$ when $n=5$. Thus, $a(W_3,W_n)\geq 2$. Since $d(a_k,j;G)\leq 2$ for all $a_k \in W_3$ and $j \in W_n$, so the number of amalgamation vertices cannot be increased. Therefore, $a(W_3,W_n)=2$.

(iii) Let n = 6 or 7. A 3-amalgamation of W_3 and W_{n} is isomorphic to $G = W_3 \star^3 \mathbf{W}_{\mathrm{n}}$ which is obtained through $\star^3 = (U,W)$ where $U=\{a_1,a_2,a_3\}$ and $W=\{1,3,5\}$ when $n=6$, and $U=\{a_1,a_2,a_3\}$ and $W=\{1,3,5\}$, $W=\{1,4,6\}$ or $W=\{1,3,6\}$ when $n=7$. Thus, $a(W_3,W_n)\geq 3$. Since $d(a_k,j;G)\leq 2$ for all $a_k \in W_3$ and $j \in W_n$, so the number of amalgamation vertices cannot be increased. Therefore, $a(W_3,W_n)=3$. ▓

Theorem 3.4. For $n \geq 8$, $a(W_4,W_n)=5$.

Proof. Refer to Figure 3.2. For $n \geq 8$, we obtain a 5-amalgamation of W_4 and W_n through $\star^5 = (U,W)$, with $U=\{a_1,a_3,a_2,a_4,a_0\}$ and $W=\{1,2,4,5,7\}$. By Remark 2, $a(W_4,W_n)\leq \min\{|V(W_4)|,|V(W_n)|\}=\min\{5,n+1\}=5$. Thus, $a(W_4,W_n)=5$ if $n \geq 8$. ▓

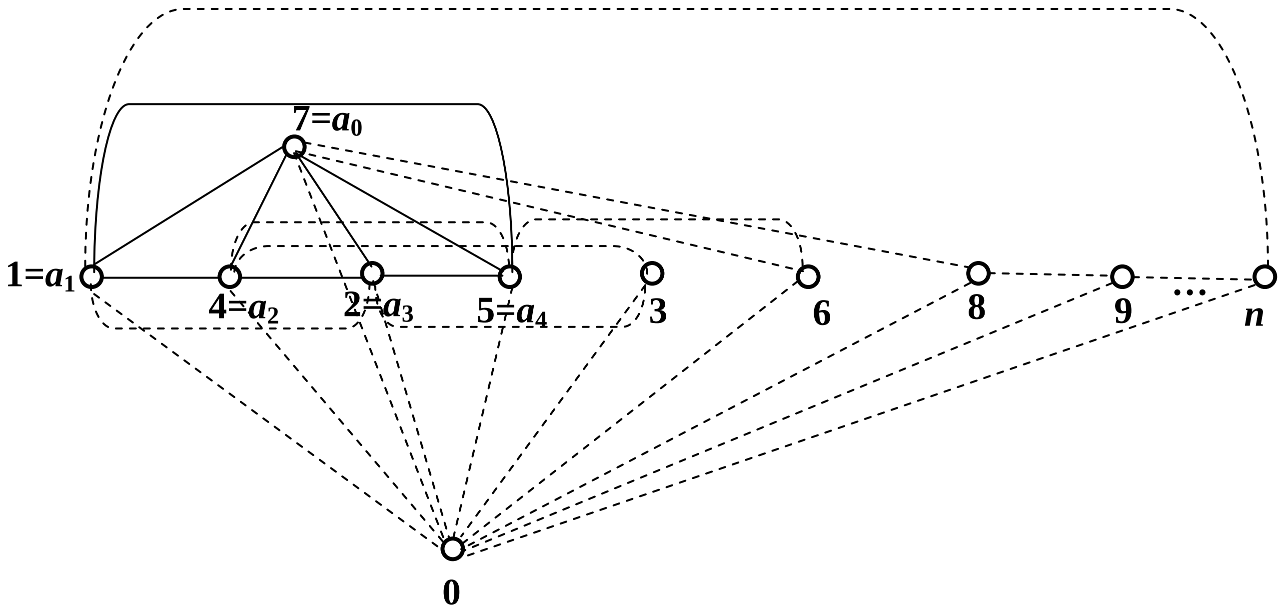


Figure 3.2 $\bigstar^5 = (U,W)$ in $W_4 \bigstar^5 W_n$, $n \geq 8$

Theorem 3.5.

i. $a(W_4, W_4) = 2$
ii. $a(W_4, W_5) = 3$
iii. $a(W_4, W_n) = 4$ if $n = 6$ or 7

Proof. (i) A 2-amalgamation of W_4 and W_4 is isomorphic to $G = W_4 \bigstar^2 W_4$ which is obtained through $\bigstar^2 = (U,W)$, with $U = \{a_1, a_3\}$ and $W = \{1,2\}$, $W = \{1,3\}$ or $W = \{1,4\}$, or $U = \{a_1, a_2\}$ and $W = \{1,3\}$. Thus, $a(W_4, W_4) \geq 2$. Since $d(a_k, j; G) \leq 2$ for all $a_k \in W_4$ and $j \in W_4$, so the number of amalgamation vertices cannot be increased. Therefore, $a(W_4, W_4) = 2$.

(ii) A 3-amalgamation of W_4 and W_5 is isomorphic to $G = W_4 \bigstar^2 W_5$ which is obtained through $\bigstar^3 = (U,W)$, with $U = \{a_1, a_2, a_3\}$ and $W = \{1,3,5\}$, $U = \{a_1, a_2, a_4\}$ and $W = \{1,3,4\}$, $U = \{a_1, a_3, a_2\}$ and $W = \{1,2,4\}$, $U = \{a_1, a_3, a_4\}$ and $W = \{1,2,4\}$, or $U = \{a_2, a_4, a_3\}$ and $W = \{1,2,4\}$. (Refer to Figure 3.3 for $W_4 \bigstar^3 W_5$ with $U = \{a_1, a_3, a_2\}$ and $W = \{1,2,4\}$.) Thus, $a(W_4, W_5) \geq 3$. Since $d(a_k, j; G) \leq 2$ for all $a_k \in W_4$ and $j \in W_5$, so the number of amalgamation vertices cannot be increased. Therefore, $a(W_4, W_5) = 3$.

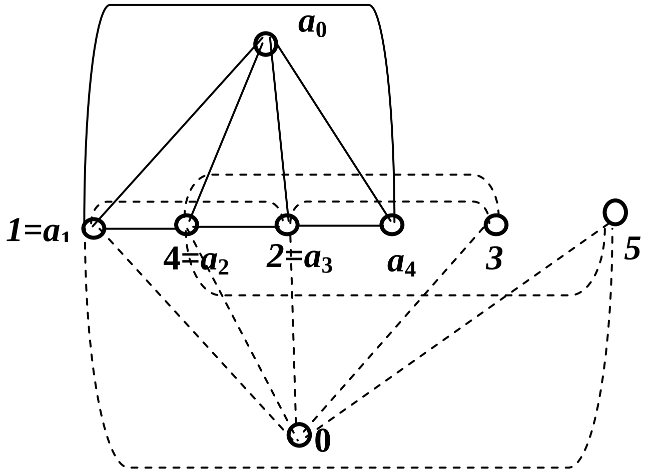


Figure 3.3 $W_4 \bigstar^3 W_5$, with $\bigstar^3 = (U,W)$, $U = \{a_1, a_3, a_2\}$ and $W = \{1,2,4\}$

(iii) Let $n = 6$ or 7. A 4-amalgamation of W_4 and either a W_6 or a W_7 is isomorphic to $G = W_4 \bigstar^2 W_n$ which is obtained through $\bigstar^4 = (U,W)$ with $U = \{a_1, a_3, a_4, a_2\}$ and $W = \{1,2,4,5\}$, $U = \{a_1, a_3, a_2, a_4\}$ and $W = \{1,2,4,5\}$, or $U = \{a_1, a_2, a_4, a_3\}$ and $W = \{1,3,4,6\}$. Thus, $a(W_4, W_n) \geq 4$. Since $d(a_k, j; G) \leq 2$ for all

$a_k \in W_4$ and $j \in W_n$, so the number of amalgamation vertices cannot be increased. Therefore, $a(W_4, W_n) = 4$ for $n = 6$ or 7.. ∎

Theorem 3.6. If $m \geq 5$ and $n \geq m+3$, then $a(W_m, W_n) = m+1$.

Proof. Let $m \geq 5$ and $n \geq m+3$. By Lemma 3.1 we may not identify the vertices a_0 and 0. We consider two cases (see Figure 3.4 when m is even, and Figure 3.5 when m is odd). When m is even, we obtain an $(m+1)$-amalgamation of W_m and W_n through $\star^{m+1}$ = (U,W), with $U = \{a_1, a_3, a_5, \cdots, a_{m-1}, a_2, a_4, a_6, \cdots, a_m, a_0\}$ and $W = \{1, 2, 3, \cdots, m, m+2\}$. When m is odd, we obtain an $(m+1)$-amalgamation of W_m and W_n through $\star^{m+1}$ = (U,W), with $U = \{a_1, a_3, a_5, \cdots, a_m, a_2, a_4, a_6, \cdots, a_{m-1}, a_0\}$and $W = \{1, 2, 3, \cdots, m, m+2\}$. In both cases, $|U| = |V(W_m)| = m+1$ and $|W| = m+1$. Hence, $a(W_m, W_n) \geq m+1$. By Remark 2, $a(W_m, W_n) \leq \min\{|V(W_m)|, |V(W_n)|\} = \min\{m+1, n+1\} = m+1$. Thus, $a(W_m, W_n) = m+1$ if $m \geq 5$ and $n \geq m+3$. ∎

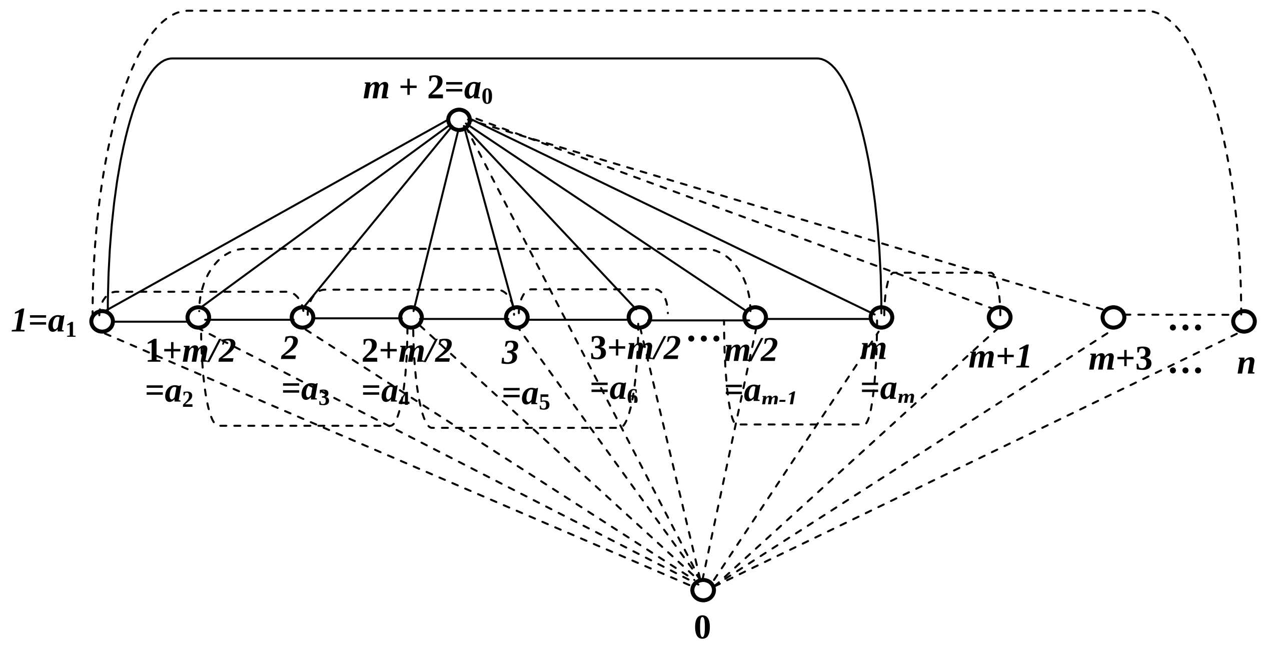


Figure 3.4 $\star^{m+1} = (U,W)$ in $W_m \star^{m+1} W_n$, m even, $m \geq 5$ and $n \geq m+3$

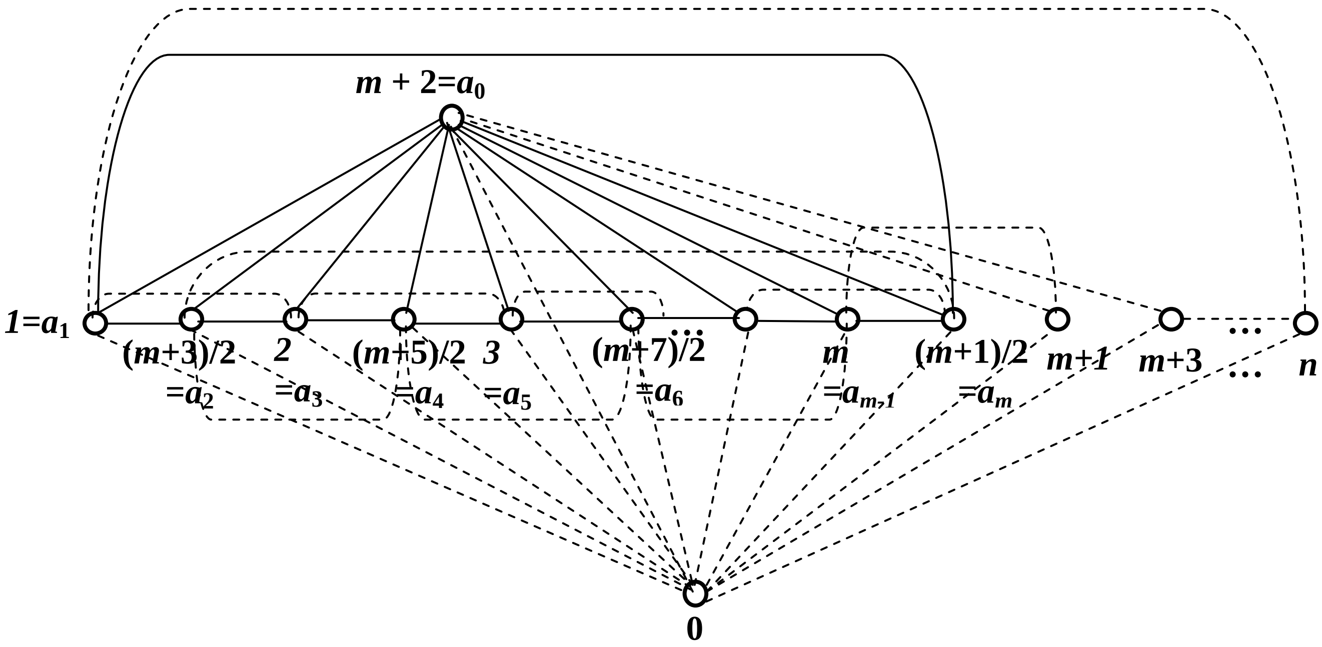


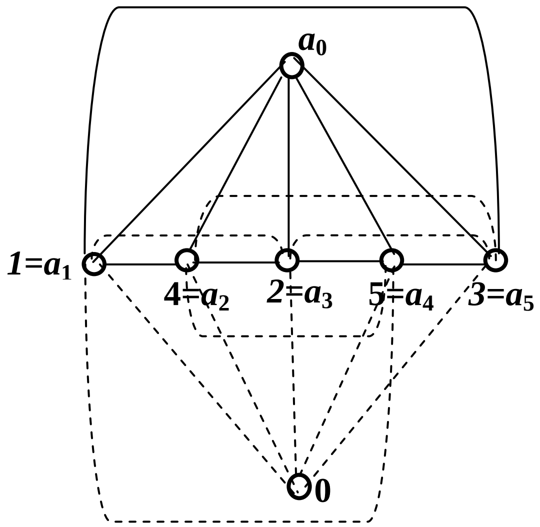
Figure 3.5 $\star^{m+1} = (U,W)$ in $W_m \star^{m+1} W_n$, m odd, $m \geq 5$ and $n \geq m+3$

Theorem 3.7. If $m \geq 5$ and $m \leq n \leq m+2$, then $a(W_m, W_n) = m$.

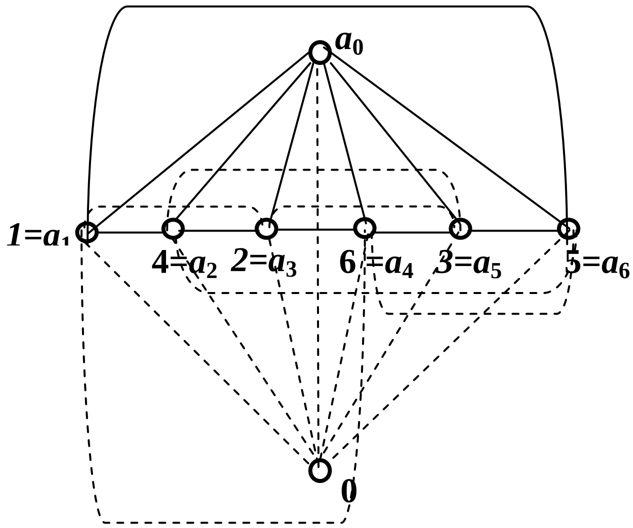
Proof. Let $m \geq 5$ and $m \leq n \leq m+2$. By Lemma 3.1 we may not identify the vertices $a_0 \in W_m$ and $0 \in W_n$ in order to maximize the number of vertices of amalgamation , that is, $a_0 \neq 0$. When m is even, we obtain an m-amalgamation of W_m and W_n through $\star^m$ = (U,W), with $U = \{a_1, a_3, a_5, \cdots, a_{m-1}, a_2, a_m, a_4, a_6, \cdots, a_{m-2}\}$ and $W = \{1, 2, 3, \cdots, m\}$. When m is odd, we obtain an m-amalgamation of W_m and W_n through $\star^m = (U, W)$, with $U = \{a_1, a_3, a_5, \cdots, a_m, a_2, a_4, a_6, \cdots, a_{m-1}\}$ and $W = \{1, 2, 3, \cdots, n\}$. Hence, $a(W_m, W_n) \geq m$. By Remark 2, $a(W_m, W_n) \leq m+1$. Suppose that a $W_m \star^{m+1} W_n$ exists, with $\star^{m+1}$ = (U,W). Then, $U = V(W_m)$. Since $a_0 \neq 0$, so a_0 is identified with a vertex $w \in \{1, 2, 3, \cdots, n\}$. Therefore, $n \geq m+1$. Hence, $W_m \star^{m+1} W_n$ does not exist if $n = m$, and therefore, $a(W_m, W_m) = m$. Let $n = m+1$ or $m+2$. Without loss of generality, let $a_0 = w$, where $w \in \{1, 2, 3, \cdots, m, m+1, m+2\}$. The m vertices $a_1, a_2, a_3, a_4, \cdots, a_m$ of W_m are identified with m vertices from $\{1, 2, 3, \cdots, m, m+1, m+2\} = V(W_n) - \{0\}$. We consider two cases. Case (1). Let $w \in \{1, 2, 3, \cdots, m+1\}$. Then, there is a vertex a_k, $k \neq 0$, which is identified with $w-1$ or $w+1$, that is, $a_k = w-1$ or $a_k = w+1$. Thus, either $(a_0, a_k = w-1)$ or $(a_0, a_k = w+1)$, or both, are edges in W_m, and $(a_0 = w, w-1)$ and

$(a_0 = w, w+1)$ are edges in W_n. In this case, multiple edges occur. Case (2). Let $w = n = m+2$. Then, either there is a vertex a_k, $k \neq 0$, which is identified with $w-1 = m+1$, or not. If $a_k = w-1$, then $(a_0, a_k = w-1)$ is an edge in W_m and $(a_0 = w, w-1)$ is an edge in W_n; hence, multiple edges occur. If $a_k \neq w-1$, then $1 = a_k$ for some $k \neq 0$. Then, $(a_0 = m+2,\ a_k = 1)$ is an edge in W_m and $(a_0 = m+2,\ a_k = 1)$ is an edge in W_n; hence, multiple edges occur. Thus, a $W_m \star^{m+1} W_n$ does not exist, and $a(W_m, W_{m+2}) = m$ if $n = m+1$ or $m+2$. This completes the proof of the theorem. ▓

Example 3.1. A 5-amalgamation $W_5 \star^5 W_5$ which is obtained through $\star^5 = (U, W)$, with $U = \{a_1, a_3, a_5, a_2, a_4\}$ and $W = \{1,2,3,4,5\}$, and a 6-amalgamation $W_6 \star^6 W_6$ which is obtained through $\star^6 = (U, W)$, with $U = \{a_1, a_3, a_5, a_2, a_6, a_4\}$ and $W = \{1,2,3,4,5,6\}$ are shown in Figure 3.6. By Theorem 3.7, $a(W_5, W_5) = 5$ and $a(W_6, W_6) = 6$.



(a) $W_5 \star^5 W_5$



(b) $W_6 \star^6 W_6$

Figure 3.6 Some m-amalgamations $W_m \star^m W_m$, $m \geq 5$

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