On Generalized Multi Poly-Euler Polynomials with Two Parameters a and b

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Abstract

In this paper are established more identities of generalized multi poly-Euler polynomials with two parameters which are expressed in terms of Stirling numbers and some generalized Bernoulli polynomials.[36]

Introduction

Euler numbers and polynomials have strong connections with Bernoulli numbers and polynomials, particularly, in the structures of their properties and generalizations. Several properties and generalizations of Bernoulli numbers and polynomials are analogous to those of Euler numbers and polynomials. For example, Kaneko[37] introduced the poly-Bernoulli numbers, denoted by $B_n^{(k)}$, as follows

$$\frac{\mathrm{L}i_k(1-e^{-x})}{1-e^{-x}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{x^n}{n!}$$

where

$$\mathrm{L}i_k(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^k}$$

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[37] M. Kaneko (1997), " Poly-Bernoulli Numbers," *J. Théor. Nombres Bordeaux*, 9 :221–228.

is the known polylogarithm. On the other hand, the paper by Ohno and Sasaki[38] and that of H. Jolany *et al.*[39] defined Euler-type numbers and polynomials, respectively, as follows

$$\frac{\mathrm{Li}_k(1-e^{-4t})}{4t\cosh t}=\sum_{n=0}^{\infty}E_n^{(k)}\frac{t^n}{n!}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{1+e^t}e^{xt}=\sum_{n=0}^{\infty}E_n^{(k)}(x)\frac{t^n}{n!}.$$

Other generalizations of Bernoulli numbers can be found in the paper of Cenkci and Young[40], L. Jang *et al.*[41] and T. Kim[42]. The papers of Jang *et al.* and Kim make use of the concept of q-analogue.

The multiple polylogarithms are usually defined by

$$\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(z_1,z_2,\ldots,z_r)=\sum_{0<m_1<m_2<\ldots<m_r}\prod_{j=1}^{r}m_j^{-k_j}z_j^{n_j},$$

where k_1, k_2, ..., k_r and z_1, z_2, ..., z_r are complex numbers suitably restricted so that the sum (2) converges. These polynomials are certain generalization of the nested harmonic sums as well as the Riemann zeta function and the ordinary polylogarithm, which preserve many interesting properties. The studies that deal with the multiple polylogarithm of the form

$$\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(z)=\sum_{0<m_1<m_2<\ldots<m_r}\frac{z^{m_r}}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}$$

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[38] Y. Ohno and Y. Sasaki (2012), "On the parity of poly-Euler numbers, *RIMS Kokyuroku Bessatsu*, B32: 271–278.

[39]] H. Jolany, R. B. Corcino and T. Komatsu (2015), "More Properties of Multi Poly-Euler Polynomials," *Bol. Soc. Mat. Mex.*, 21: 149–162.

[40] M. Cenkci and P. T. Young (September 2015), "Generalizations of Poly-Bernoulli and Poly-Cauchy Numbers," *Eur. J. Math.*, Published Online DOI 10.1007/s40879-015-0071-3.

[41] L. Jang, T. Kim, and H. K. Pak (2001), " A note on q-Euler and Genocchi numbers," *Proc. Japan Acad. Ser. A Math. Sci.*, **77** :139–141.

[42] T. Kim (2006), "q-Generalized Euler numbers and polynomials," *Russ. J. Math. Phys.* 13(3): 293–298.

have motivated the idea of extending poly-Bernoulli and poly-Euler numbers and polynomials to multiple parameter case. One concrete example is the work of Bayad.[43]

Imatomi *et al.*[44] have defined a certain generalization of Bernoulli numbers in terms of these multiple logarithms as follows

$$\frac{\mathrm{Li}_{(k_1,k_2,\dots,k_r)}(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k_1,k_2,\dots,k_r)}\frac{t^n}{n!}.$$

while H. Jolany *et al.*[45] have introduced the generalized multi poly-Euler polynomials, denoted by $E_n^{(k_1,k_2,\dots,k_r)}(x;a,b,c)$, as follows

$$\frac{2\mathrm{Li}_{(k_1,k_2,\dots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^{t})^r}c^{rxt}=\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\dots,k_r)}(x;a,b,c)\frac{t^n}{n!}.$$

These polynomials have possessed numerous identities parallel to those of poly-Euler polynomials.

In this paper, we derive some identities of $E_n^{(k_1,k_2,\dots,k_r)}(x;a,b,c)$ related to Stirling numbers of the second kind, rising and falling factorials and some Bernoulli-type and Euler-type polynomials under the case where $c=e$. We use the following notation

$$E_n^{(k_1,k_2,\dots,k_r)}(x;a,b):=E_n^{(k_1,k_2,\dots,k_r)}(x;a,b,e)$$

to denote the generalized multi poly-Euler numbers with two parameters a and b.

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[43] A. Bayad and Y. Hamahata, Arakawa-Kaneko (2011), "L-functions and generalized poly-Bernoulli polynomials," *J. Number Theory*, 131 :1020–1036.

[44] K. Imatomi, M. Kaneko and E. Takeda (2014), Multi-Poly-Bernoulli Numbers and Finite Multiple Zeta Values, *J. Integer Seq.*, 17 :Article 14.4.5.

[45] H. Jolany, M. Aliabadi, R.B. Corcino, M.R. Darafsheh (2012)," A note on multi poly-Euler numbers and Bernoulli polynomials," *Gen. Math.*, 20(2â€"3): 122–134.

2 Generalized Multi Poly-Euler Polynomials, Stirling Numbers and the Rising and Falling Factorials

The rising factorial of x of degree m, denoted by $(x)^{(m)}$, is defined as the product of increasing numbers $x, x+1, \ldots, x+m-1$. **More precisely,**

$$(x)^{(m)} = x(x+1)\ldots(x+m-1).$$

On the other hand, the falling factorial of x of degree m, denoted by $(x)_m$, is defined by

$$(x)_m = x(x-1)\ldots(x-m+1).$$

The classical Stirling numbers of the second kind, denoted by $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$, are defined by James Stirling in 1730 in terms of the following relation associated with the falling factorial

$$x^n = \sum_{m=0}^{n} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (x)_m .$$

Several combinatorial properties of these numbers are established including recurrence relations, explicit formulas, asymptotic formulas and the following exponential generating function

$$\frac{(e^t-1)^m}{m!} = \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{t^n}{n!}.$$

These numbers can be interpreted as the number of ways to partition an n-set into m nonempty subsets. This interpretation has been used to give combinatorial meaning of poly-Bernoulli numbers $B_n^{(-k)}$ (see 4 [46]) through the explicit formula of Arakawa and Kaneko[47]

$$B_n^{(-k)} = \sum_{m\geq 0} m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} m! \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\}.$$

It is worth mentioning that C. Brewbaker [5] has also established a combinatorial interpretation of poly-Bernoulli numbers using the concept of lonesum matrices.

[46] B. Be n′ yi (2014), "Advances in Bijective Combinatorics," *Ph.D. Thesis.*

[47] T. Arakawa, M. Kaneko(1999), "On Poly-Bernoulli Numbers," *Comment Math. Univ. St. Paul* 48(2): 159-167.

Throughout the paper, we use the definition of generalized multi poly-Euler polynomials with two-parameters a and b which is taken from (5) by assigning $c=e$. That is,

$$\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^{t})^{r}}e^{rxt}=\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!}.$$

Now, motivated by the work of I. Mezö[48] on expressing Bernoulli polynomials in terms of some Stirling-type numbers, we obtain the following identity for $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ which is expressed in terms of Stirling numbers of the second kind and the rising factorial.

Theorem 2.1 *The generalized multi poly-Euler polynomials $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ with two parameters a and b satisfy the following relation*

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}r^{l}\left\{{l \atop m}\right\}\binom{n}{l}E_{n-l}^{(k_1,k_2,\ldots,k_r)}(-m;a,b)(x)^{(m)}.$$

Proof. Note that (8) can be written as

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!}=\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^{t})^{r}}(1-(1-e^{-rt}))^{-x}$$

Using Newton's Binomial Theorem and the exponential generating function in (6), we have

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!}=\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^{t})^{r}}\sum_{m=0}^{\infty}\binom{x+m-1}{m}(1-e^{-rt})^{m}$$

$$=\sum_{m=0}^{\infty}(x)^{(m)}\frac{(e^{rt}-1)^m}{m!}\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^{t})^{r}}e^{-mrt}$$

[48] I. Mezö (2010)," A new formula for the Bernoulli polynomials," *Result. Math.* 58(3):329–335.

$$= \sum_{m=0}^{\infty}(x)^{(m)}\left(\sum_{n=0}^{\infty}\left\{ {n \atop m} \right\}\frac{(rt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(-mr;a,b)\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\sum_{l=m}^{n} r^l \left\{ {l \atop m} \right\}\binom{n}{l}E_{n-l}^{(k_1,k_2,\ldots,k_r)}(-mr;a,b)(x)^{(m)}\right\}\frac{t^n}{n!}$$

Comparing coefficients completes the proof of (9).

The following theorem contains an identity for $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ which is expressed in terms of Stirling numbers of the second kind and the falling factorial.

Theorem 2.2 *The generalized multi poly-Euler polynomials* $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ *with two parameters* a *and* b *satisfy the following relation*

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b) = \sum_{m=0}^{\infty}\sum_{l=m}^{n} r^l \left\{ {l \atop m} \right\}\binom{n}{l}E_{n-l}^{(k_1,k_2,\ldots,k_r)}(0;a,b)(x)_m.$$

Proof. Note that (8) can be written as

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}((e^{rt}-1)+1)^x$$

Using Newton's Binomial Theorem and the exponetial generating function in (6), we have

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}\sum_{m=0}^{\infty}\binom{x}{m}(e^{rt}-1)^m$$

$$= \sum_{m=0}^{\infty}(x)_m \frac{(e^{rt}-1)^m}{m!}\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}$$

$$= \sum_{m=0}^{\infty}(x)_m\left(\sum_{n=0}^{\infty}\left\{ {n \atop m} \right\}\frac{(rt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(0;a,b)\frac{t^n}{n!}\right)$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\sum_{l=m}^{n}r^{l}\left\{\begin{matrix} l \\ m \end{matrix}\right\}\binom{n}{l}E_{n-l}^{(k_1,k_2,\ldots,k_r)}(0;a,b)(x)_m\right\}\frac{t^n}{n!}$$

Comparing coefficients completes the proof of (10).

3. Generalized Multi Poly-Euler Polynomials, Bernoulli-type and Euler-type Polynomials

There are several polynomials that can be classified as Bernoulli-type and Euler-type polynomials. Two of these are the following:

$$\left(\frac{t}{e^t-1}\right)^s e^{xt}=\sum_{n=0}^{\infty}\mathrm{B}_n^{(s)}(x)\frac{t^n}{n!}$$

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt}=\sum_{n=0}^{\infty}H_n^{(s)}(x;\lambda)\frac{t^n}{n!}.$$

The first one can be classified as Bernoulli-type polynomial and the second one is Euler-type polynomial. Note that, when $s=1$, (11) yields the generating function for Bernoulli polynomials. That is,

$$B_n(x)=\mathrm{B}_n^{(1)}(x).$$

And when $s=1$ and $\lambda=-1$, (12) gives the generating function for Euler polynomials, that is,

$$E_n(x)=H_n^{(1)}(x;-1).$$

In this section, we derive some identities for generalized multi poly-Euler polynomials with two parameters a and b in terms of the above Bernoulli-type and Euler-type polynomials. The following theorem contains an identity expressing $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ in terms of $\mathrm{B}_n^{(s)}(x)$.

Theorem 3.1 *The generalized multi poly-Euler polynomials* $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ *with two parameters* a *and* b *satisfy the following relation*

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)=\sum_{m=0}^{\infty}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}\left\{\begin{matrix}l+s\\ s\end{matrix}\right\}E_{n-m-l}^{(k_1,k_2,\ldots,k_r)}(0;a,b)\mathrm{B}_m^{(s)}(xr).$$

Proof. Note that (8) can be written as

$$G(t)=\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!}$$

$$=\left(\frac{(e^t-1)^s}{s!}\right)\left(\frac{t^s e^{xrt}}{(e^t-1)^s}\right)\left(\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}\right)\frac{s!}{t^s}$$

$$=\left(\sum_{n=0}^{\infty}\left\{\begin{matrix}n+s\\ s\end{matrix}\right\}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}\mathrm{B}_m^{(s)}(xr)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(0;a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{s!}}\left\{\begin{matrix}l+s\\ s\end{matrix}\right\}E_{n-m-l}^{(k_1,k_2,\ldots,k_r)}(0;a,b)\mathrm{B}_m^{(s)}(xr)\right\}\frac{t^n}{n!}.$$

Comparing coefficients completes the proof of (13).

The next theorem contains another identity expressing the generalized multi poly-Euler polynomials with two parameters a and b in terms of $H_n^{(s)}(x;\lambda)$.

Theorem 3.2 *The generalized multi poly-Euler polynomials* $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ *with two parameters* a *and* b *satisfy the following relation*

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)=\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k_1,k_2,\ldots,k_r)}(j;a,b)H_m^{(s)}(xr;\lambda).$$

Proof. Using (8), we have

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!}=\left(\frac{(1-\lambda)^s}{(e^t-\lambda)^s}e^{xrt}\right)\left(\frac{(e^t-\lambda)^s}{(1-\lambda)^s}\right)\left(\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}\right)$$

$$=\left(\sum_{n=0}^{\infty}H_n^{(s)}(xr;\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{jt}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k_1,k_2,\ldots,k_r)}(j;a,b)H_m^{(s)}(xr;\lambda)\right)\frac{t^n}{n!}.$$

Comparing coefficients completes the proof of (14).

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