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Determining the Intruder's Location in a Given Network: Locating-Dominating Sets in a Graph

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Abstract

The exact location of an intruder (e.g. burglar, fire, etc.) in a given network or graph can be determined using the concept of locating-dominating set in a graph. In this paper the concepts of locating, strictly locating, and locating-dominating sets in a graph will be considered. Corresponding parameters will be discussed and some of their relationships will be given. It is shown that the *L*-domination number $\gamma_L(G)$ of a connected graph G of order $n \geq 2$ is $n - 1$ if and only if $G = K_n$ or $G = K_{1,n-1}$. If G is a connected graph and $\gamma_L(G) = 2$, then $3 \leq |V(G)| \leq 5$. The locating-dominating sets in the joins of graphs are characterized in terms of the other concepts and the associated *L*-domination numbers are determined subsequently.

Keywords: locating, strictly locating, locating dominating, join

Introduction

Let $G = (V(G),E(G))$ be a simple connected graph. The ***neighborhood*** of $v \in V(G)$ is the set $N_G(v) = N(v) = \{ x \in V(G): xv \in E(G)\}$. The ***degree*** of $v \in V(G)$, denoted by $deg(v)$, is equal to the cardinality of $N_G(v)$ and the ***maximum degree*** of G is $\Delta(G) = \max\{deg(x): x \in V(G)\}$. A subset S of $V(G)$ is a ***dominating*** set in G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. It is a ***locating set*** in G if $N_G(u) \cap S \neq N_G(v) \cap S$ *for all* $u,v \in V(G) \setminus S$. Set S is said to be a ***strictly locating set*** if it is locating

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and $N_G(u) \cap S \neq S$ for all $u \in V(G) \setminus S$. The minimum cardinality of a locating set in G, denoted by **ln(*G*)**, is called the ***locating number*** of G. The minimum cardinality of a strictly locating set in G, denoted by **sln(*G*)**, is the ***strictly locating number*** of G. A locating (resp. strictly locating) subset S of $V(G)$ which is also dominating is called a ***locating-dominating*** (resp. ***strictly locating-dominating***) set in a graph G. The minimum cardinality of a locating-dominating (resp. strictly locating-dominating) set in G, denoted by $\gamma_L(G)$ (resp. $\gamma_{SL}(G)$), is called the L-domination (resp. SL-domination) number of G. If $S \subseteq V(G)$ is a locating (resp. strictly locating, locating-dominating, or strictly locating-dominating) set with $|S|$ equal to ln(G) (resp. sln(G), $\gamma_L(G)$, or $\gamma_{SL}(G)$), then it is called a ***minimum locating*** (***minimum strictly locating***, ***minimum locating-dominating***, or ***minimum strictly locating-dominating***) set.

In a given network or graph, a locating set can be viewed as a set of monitoring device which can actually determine the exact location of an intruder (e.g. a burglar, a fire, etc.). By requiring such a set to be dominating implies that every node where there is no monitor in it is connected to at least one monitoring device. Hence, determination of the L-domination number of a graph is equivalent to finding the least number of monitoring device that can do the certain task in a given graph.

Domination in graphs and other types of domination can be found in the book by Haynes et al. [6] and studies in [1] and [2]. The concepts of locating set, locating-dominating set and the associated parameters are studied by Haynes et al. in [5]. Other related concepts are studied in [3], [4], [7], and [8].

Results

It is worth mentioning that a locating (strictly locating) set is a non-empty set and always exists in a connected graph G ($V(G)$ is one such set). From the definitions, the following relationships are immediate.

***Remark* 1**. For any connected graph G of order $n \geq 2$, ln(G) ≤ sln(G), sln(G) ≤ $\gamma_{SL}(G)$, and ln(G) ≤ $\gamma_L(G) \leq \gamma_{SL}(G)$.

The following result gives specific relationships between these parameters.

***Theorem* 1**. Let G be a connected graph of order $n \geq 2$.

(*a*) If ln(G) < sln(G), then 1 + ln(G) = sln(G).
(*b*) If ln(G) < $\gamma_L(G)$, then 1 + ln(G) = $\gamma_L(G)$.
(*c*) If sln(G) < $\gamma_{SL}(G)$, then 1 + sln(G) = $\gamma_{SL}(G)$.
(*d*) If $\gamma_L(G) < \gamma_{SL}(G)$, then $1 + \gamma_L(G) = \gamma_{SL}(G)$.

Proof: (*a*) Let S be a minimum locating set in G. Then S is not strictly locating in G. Hence, there exists a $y \in V(G) \setminus S$ such that $N_G(y) \cap S = S$. Set $S^* = S \cup \{y\}$ and

let
$z \in V(G) \setminus S^*$. Then $z \neq y$. Since S is a locating set and $N_G(y) \cap S = S$, $N_G(z) \cap S \neq S$. This implies that there exists $w \in S$ such that $w \notin N_G(z)$. Since $y \notin S$, $w \neq y$. Thus $N_G(z) \cap S^* \neq S^*$. This implies that S^* is a strictly locating set in G. Hence, $\mathrm{sln}(G) \leq 1 + \mathrm{ln}(G)$. Since $\mathrm{ln}(G) < \mathrm{sln}(G)$, $1 + \mathrm{ln}(G) \leq \mathrm{sln}(G)$. Therefore, $1 + \mathrm{ln}(G) = \mathrm{sln}(G)$.

(*b*) Let S be a minimum locating set in G. Then S is not a dominating set in G. Hence, there exists a $y \in V(G) \setminus S$ such that $xy \notin E(G)$ for all $x \in S$. This implies that $N_G(y) \cap S = \varnothing$. Put $S^* = S \cup \{y\}$ and let $z \in V(G) \setminus S^*$. Then $z \neq y$. Since S is a locating set, $N_G(z) \cap S \neq \varnothing$. This implies that there exists $w \in S$ such that $wz \in E(G)$. This shows that S^* is a dominating set in G. Next, let $a,b \in V(G) \setminus S^*$. Then $a,b \in V(G) \setminus S$; hence $N_G(a) \cap S \neq N_G(b) \cap S$ since S is a locating set in G. Thus, $N_G(a) \cap S^* \neq N_G(b) \cap S^*$ for all $a,b \in V(G) \setminus S^*$. This implies that S^* is a locating set in G. Therefore, $\gamma_L(G) \leq |S^*| = 1 + |S| = 1 + \mathrm{ln}(G)$. Since $\mathrm{ln}(G) < \gamma_L(G)$, $1 + \mathrm{ln}(G) \leq \gamma_L(G)$. This shows that $1 + \mathrm{ln}(G) = \gamma_L(G)$.

(*c*) Let S be a minimum strictly locating set in G. Then S is not a dominating set in G. Hence, there exists a $y \in V(G) \setminus S$ such that $xy \notin E(G)$ for all $x \in S$. This implies that $N_G(y) \cap S = \varnothing$. Set $S^* = S \cup \{y\}$. Then, as in the proof of (*b*), S^* is a locating-dominating set in G. Next, let $a \in V(G) \setminus S^*$. Then $a \in V(G) \setminus S$; hence $N_G(a) \cap S \neq S$ since S is a strictly locating set in G. This implies that there exists $z \in S$ such that $z \notin N_G(a)$. Thus, $z \neq y$ and so $N_G(a) \cap S^* \neq S^*$. Consequently, S^* is a strictly locating-dominating set in G. Therefore, $\gamma_{SL}(G) \leq 1 + \mathrm{sln}(G)$. Since $\mathrm{sln}(G) < \gamma_{SL}(G)$, it follows that $1 + \mathrm{sln}(G) \leq \gamma_{SL}(G)$. This shows that $1 + \mathrm{sln}(G) = \gamma_{SL}(G)$.

(*d*) Let S be a minimum locating-dominating set in G. Then S is not a strictly locating set in G. Hence, there exists a $y \in V(G) \setminus S$ such that $N_G(y) \cap S = S$. Put $S^* = S \cup \{y\}$. Since S is a dominating set in G, S^* is also a dominating set in G. Also, as in the proof of (*a*), S^* is a strictly locating (hence, a strictly locating-dominating) set in G. Therefore, $\gamma_{SL}(G) \leq 1 + \gamma_L(G)$. Since $\gamma_L(G) < \gamma_{SL}(G)$, $1 + \gamma_L(G) \leq \gamma_{SL}(G)$. This establishes the desired equality. □

***Remark* 2.** If G is a connected graph of order $n \geq 2$, then $1 \leq \gamma_L(G) \leq n - 1$.

***Theorem* 2**. Let G be a connected graph of order $n \geq 2$. Then $\gamma_L(G) = n - 1$ if and only if $G = K_n$ or $G = K_{1,n-1}$.

Proof: Suppose that $\gamma_L(G) = n - 1$ and suppose further that $G \neq K_n$. Let $x \in V(G)$ with $\deg(x) = \Delta(G)$ Suppose there exists $y \in V(G)\setminus\{x\}$, such that $xy \notin E(G)$. Since G is connected, y can be chosen so that the $d(x,y) = 2$. Let $z \in N(x)$ such that $yz \in E(G)$. If $yw \in E(G)$ for all $w \in N(x)$, then choose $S = V(G) \setminus \{y,z\}$. Since $\deg(x) \geq \deg(z) \geq 2$, there exists $u \in N(x) \setminus \{z\}$. It follows that uy, $xz \in E(G)$, i.e., S is a dominating set in G. Moreover, since $x \in N(z) \setminus N(y)$, $N(z) \cap S \neq N(y) \cap S$. This implies that S is a locating-dominating set in G. Now, if there exists $w \in N(x)$ such that $yw \notin E(G)$, then choose $S = V(G)\setminus\{x,y\}$. Since $zy, xw \in E(G)$, S is a dominating set in G. Also, since $w \in N(x) \setminus N(y)$, $N(x) \cap S \neq N(y) \cap S$. Thus, S is a locating-dominating set in G. In both cases, we have $\gamma_L(G) = |S| = n - 2$, contrary to our assumption. Therefore, $N(x) = V(G)\setminus\{x\}$.

It remains to show that $uv \notin E(G)$ for every two distinct vertices $u, v \in V(G)\setminus\{x\}$. To this end, suppose there exist distinct vertices u and v in $V(G)\setminus\{x\}$ such that $uv \in E(G)$. Since $G \neq K_n$, there exist distinct vertices a and b of G such that $ab \notin E(G)$. If $va \in E(G)$ or $vb \in E(G)$, say $va \in E(G)$, then consider $S = V(G)\setminus\{b,v\}$. Since $a \in N(v) \setminus N(b)$, it follows that $N(v) \cap S \neq N(b) \cap S$. Moreover, $xb, xv \in E(G)$. Thus S is a locating-dominating set in G. If $va \notin E(G)$ and $vb \notin E(G)$, then choose $S = V(G)\setminus\{a,u\}$. Again, S is a locating-dominating set in G. In either case, $\gamma_L(G) = |S| = n - 2$, contrary to our assumption. Therefore, $uv \notin E(G)$ for every two distinct vertices $u, v \in V(G)\setminus\{x\}$.

Accordingly, $G = K_{1,n-1}$.

The converse is easy. □

***Theorem* 3**. Let G be a connected graph. If $\gamma_L(G) = 2$, then $3 \leq |V(G)| \leq 5$.

Proof. Clearly, $2 \leq |V(G)|$. However, by Theorem 2, $\gamma_L(K_2) = 1$. Therefore, $3 \leq |V(G)|$. Suppose now that $|V(G)| > 5$. Let $S = \{x,y\}$ be a minimum locating-dominating set in G. Let $z_1, z_2, z_3, z_4 \in V(G) \setminus S$. Since S is a dominating set, $N(z_i) \cap S \neq \emptyset$ for all $i \in \{1, 2, 3,,4\}$. So, $N(z_i) \cap S$ can only be $\{x\}$, $\{y\}$, or S, which implies that there exist distinct z_i, z_j, where $i, j \in \{1, 2, 3,,4\}$ such that $N(z_i) \cap S = N(z_j) \cap S$. Thus, S is not a locating set in G, contrary to our assumption on S. Therefore, $|V(G)| \leq 5$. □

***Corollary* 1.** Let G be a connected graph of order $n = 4$. Then $\gamma_L(G) = 2$ if and only if $G \neq K_4$ and $G \neq K_{1,3}$.

Proof. This is immediate from Remark 1, Theorem 2 and Theorem 3. □

***Theorem* 4.** Let G be a connected graph of order $n = 5$. Then $\gamma_L(G) = 2$ if and only if there exist distinct vertices x and y that dominate G such that $|N(x) \cap N(y)| = 1$ and and $|N(x) \setminus\{y\}| = |N(y) \setminus\{x\}| = 2$.

Proof. Suppose $\gamma_L(G) = 2$. Then there exist distinct vertices x and y such that $S = \{x, y\}$ is a minimum locating-dominating set in G. Hence $|N(x) \cap N(y)| \leq 1$. Suppose $|N(x) \cap N(y)| = 0$. Then one of x and y, say x, has at least two neighbors in $V(G)\setminus S$ that are not in $N(y)$. This implies that S is not a locating set in G, contrary to our assumption. Thus, $|N(x) \cap N(y) = 1$. Next, let $a \in V(G)$ such that $ax, ay \in E(G)$. Let $b, c \in V(G) \setminus \{x, y, a\}$. Then $b, c \notin N(x) \cap N(y)$. Since S is a dominating set, $b \in N(x)$ or $b \in N(y)$ Assume $b \in N(x)$. Since S is a locating-dominating set, $c \in N(y) \setminus N(x)$. Thus, $|N(x) \setminus\{y\}| = |N(y) \setminus\{x\}| = 2$.

For the converse, suppose there exist distinct vertices x and y of $V(G)$ satisfying the given properties. Let $S = \{x, y\}$. It is easy to show that S is a locating-dominating set in G. Thus $\gamma_L(G) = |S| = 2$. □

***Theorem* 5**. Let G and H be connected non-trivial graphs. Then $S \subseteq V(G+H)$ is a locating dominating set in $G + H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are locating sets in G and H, respectively, where S_1 or S_2 is a strictly locating set.

Proof: Let $S \subseteq V(G+H)$ be a locating-dominating set in $G+H$. Let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. Suppose $S_1 = \varnothing$. Pick distinct vertices u and v in $V(G)$. Then $N_{G+H}(u) \cap S = S = N_{G+H}(v) \cap S$, contrary to our assumption of S. Thus, $S_1 \neq \varnothing$. Similarly, $S_2 \neq \varnothing$. Suppose now that one of S_1 and S_2 is not a locating set, say S_1 is not a locating set in G. Then there exist distinct vertices a and b of G such that $N_G(a) \cap S_1 = N_G(b) \cap S_1$. Since $S_2 \subseteq N_{G+H}(a)$ and $S_2 \subseteq N_{G+H}(b)$, it follows that

$$N_{G+H}(a) \cap S = (N_G(a) \cap S_1) \cup S_2 = N_{G+H}(b) \cap S.$$

This, again, contradicts our assumption of S. Therefore, S_1 and S_2 are locating sets in G and H, respectively.

Next, suppose that both S_1 and S_2 are not strictly locating sets. Then there exist $z \in V(G) \setminus S_1$ and $w \in V(H) \setminus S_2$ such that $N_G(z) \cap S_1 = S_1$ and $N_H(w) \cap S_2 = S_2$. It follows that $N_{G+H}(z) \cap S = S = N_{G+H}(w) \cap S$, contrary to our assumption. Accordingly, S_1 or S_2 is a strictly locating set.

For the converse, suppose that S_1 and S_2 are locating sets in G and H, respectively, and S_1 or S_2 is strictly locating. Let x and y be distinct vertices in $V(G+H) \setminus S$. If $x, y \in V(G)$, then $N_G(x) \cap S_1 \neq N_G(y) \cap S_1$. It follows that

$$N_{G+H}(x) \cap S = (N_G(x) \cap S_1) \cup S_2 \neq (N_G(y) \cap S_1) \cup S_2 = N_{G+H}(y) \cap S.$$

Similarly, $N_{G+H}(x) \cap S \neq N_{G+H}(y) \cap S$ if $x, y \in V(H)$. Suppose $x \in V(G)$ and $y \in V(H)$. Suppose, without loss of generality, that S_1 is a strictly locating set in G. Then $S_1 \not\subset N_{G+H}(x)$. Since $S_1 \subseteq N_{G+H}(y)$, $N_{G+H}(x) \cap S \neq N_{G+H}(y) \cap S$. Thus, S is a locating set in $G + H$. Clearly, S is a dominating set in $G+H$. □

***Corollary* 2**. Let G and H be connected non-trivial graphs. Then

$$\gamma_L(G+H) = \min\{\mathrm{sln}(H) + \mathrm{ln}(G),\ \mathrm{sln}(G) + \mathrm{ln}(H)\}.$$

Proof: Let S be a minimum locating-dominating set in $G + H$. Let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. By Theorem 5, S_1 and S_2 are locating sets in G and H, respectively, and S_1 or S_2 is a strictly locating set. Assume first that S_1 is strictly locating. Then

$$\mathrm{sln}(G) + \mathrm{ln}(H) \le |S_1| + |S_2| = |S| = \gamma_L(G+H).$$

If S_2 is strictly locating, then

$$\mathrm{sln}(H) + \mathrm{ln}(G) \le |S_1| + |S_2| = |S| = \gamma_L(G+H).$$

Thus, $\gamma_L(G+H) \ge \min\{\mathrm{sln}(H) + \mathrm{ln}(G),\ \mathrm{sln}(G) + \mathrm{ln}(H)\}$.

Now suppose without loss of generality that $\mathrm{sln}(G) + \mathrm{ln}(H) \le \mathrm{sln}(H) + \mathrm{ln}(G)$. Let S_1 be a minimum strict locating set in G and S_2 a minimum locating set in H. Then $S = S_1 \cup S_2$ is a locating-dominating set in $G+H$ by Theorem 5. Hence

$$\gamma_L(G+H) \le |S| = |S_1| + |S_2| = \mathrm{sln}(G) + \mathrm{ln}(H).$$

This proves the desired equality. □

***Corollary* 3**. Let G be a non-trivial connected graph and let K_n be the complete graph with $n \ge 2$. Then $\gamma_L(G + K_n) = \mathrm{sln}(G) + n - 1$.

Proof: Clearly, $\mathrm{ln}(K_n) = n$-1 and $\mathrm{sln}(K_n) = n$. From Corollary 2,

$$\gamma_L(G + K_n) = \min\{\mathrm{sln}(G) + n - 1,\ \mathrm{ln}(G) + n\}.$$

From Remark 1 and Theorem 1(*b*), $\mathrm{sln}(G) - 1 \le \mathrm{ln}(G)$. It follows that $\mathrm{sln}(G) - 1 + n \le \mathrm{ln}(G) + n$. Therefore

$$\gamma_L(G + K_n) = \min\{\mathrm{sln}(G) + n - 1,\ \mathrm{ln}(G) + n\} = \mathrm{sln}(G) + n - 1. \quad \square$$

Theorem 6. Let G be a connected non-trivial graph and $K_1 = \langle v \rangle$. Then $S \subseteq V(G+K_1)$ is a locating-dominating set in K_1+G if and only if either $S = S_1 \cup \{v\}$ where S_1 is a locating set in G, or $v \notin S$ and S a strictly locating-dominating set in G.

Proof: For simplicity, let $H = K_1$. Let $S \subseteq V(G+H)$ be a locating-dominating set in $G+H$. Suppose first that $v \in S$. Let $S_1 = V(G) \cap S$. Then, $S_1 \ne \varnothing$, otherwise $S = \{v\}$ which implies that S is not a locating set in $G + H$. Thus, $S = S_1 \cup \{v\}$. Let $x,y \in V(G) \setminus S_1$. Then

$$(N_G(x) \cap S_1) \cup \{v\} = N_{G+H}(x) \cap S \ne N_{G+H}(y) \cap S = (N_G(y) \cap S_1) \cup \{v\}.$$

It follows that $N_G(x) \cap S_1 \ne N_G(y) \cap S_1$. This shows that S_1 is a locating set in G.

Next, suppose that $v \notin S$. Then $S = S_1 \subseteq V(G)$. If $x,y \in V(G) \setminus S$, then, since S is a locating-dominating set in $G+H$,

$$N_{G+H}(x) \cap S = N_G(x) \cap S \neq N_G(y) \cap S = N_{G+H}(y) \cap S.$$

Therefore, S is a locating set in G. Now, if there exists $z \in V(G) \setminus S$ such that $N_G(z) \cap S = S$, then $N_{G+H}(z) \cap S = S = N_{G+H}(v) \cap S$, contrary to our assumption that S is a locating set in $G + H$. This implies that S is a strictly locating set in G. Moreover, because S is also a dominating set in $G+H$, it is a dominating set in G.

For the converse assume first that $S = S_1 \cup \{v\}$, where S_1 is a locating set in G. Clearly, S is a dominating set in $G+H$. Let $x, y \in V(G+H) \setminus S$. Then $x,y \in V(G) \setminus S_1$ and $N_G(x) \cap S_1 \neq N_G(y) \cap S_1$. Thus,

$$N_{G+H}(x) \cap S = (N_G(x) \cap S_1) \cup \{v\} \neq (N_G(y) \cap S_1) \cup \{v\} = N_{G+H}(y) \cap S.$$

This shows that S is a locating-dominating set in $G+H$.

Finally, suppose $v \notin S$ and S a strictly locating-dominating set in G. Then S is a dominating set in $G+H$. Let $x,y \in V(G+H) \setminus S$. If $x,y \in V(G)$, then $N_{G+H}(x) \cap S = N_G(x) \cap S \neq N_G(y) \cap S = N_{G+H}(y) \cap S$. Suppose $x \in V(G)$ and $y = v$. Then $N_{G+H}(v) \cap S = S$. Since S is a strictly locating set in G, $N_G(x) \cap S \neq S$. Thus, $N_{G+H}(v) \cap S \neq N_G(x) \cap S = N_{G+H}(x) \cap S$. This shows that S is a locating-dominating set in $G+H$. □

The following result is immediate from Theorem 6.

Corollary 4. Let G be a connected non-trivial graph. Then

$$\gamma_L(G + K_1) = \min\{\gamma_{SL}(G), \ln(G) + 1\}.$$

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