# Examining a Derivation of Hankel's Loop Integral Using Integration

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## Abstract

*This paper presents a derivation of Hankel's Loop integral using integration through a branch cut. It justifies the derivation of Hankel's loop integral mentioned in references where the Hankel contour is collapsed into the negative real axis. Moreover, it discusses an application to $r$-Whitney numbers with complex parameters.*

***Keywords:*** *Analysis, Gamma Function, Hankel contour, Hankel Identity, Hankel's loop integral*

## Introduction

In the study of the asymptotics of $r$-Whitney numbers[v] (C.B.Corcino, R.B. Corcino, and N.G. Acala 2014), the Hankel's loop integral plays an important role. It is through this integral that computations in the case where the parameters are integers, are extended to the case where real parameters are considered. Temme (2014) used the Hankel's loop integral to obtain an asymptotic expansion of the integral

$$G_\lambda(z) = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} s^{\lambda-1}\, q(s)e^{zs}ds, \tag{1}$$

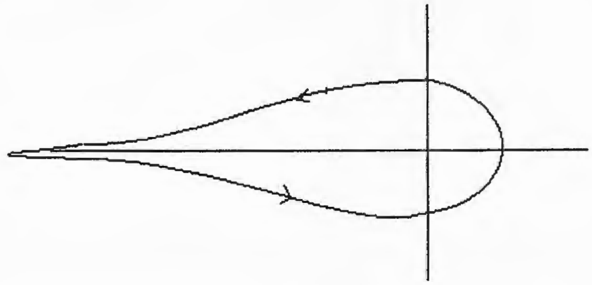

**Figure 1. Hankel Contour**

where $\lambda$ is an unrestricted real or complex constant and the path is the Hankel contour shown in Figure 1.

The Hankel's Loop Integrals are alternative integral representations of the Gamma Function. There are two forms of Hankel's loop Integrals. The First Form which we shall refer to as the Hankel's Loop Integral [6] is given by

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_H e^t\, t^{-z} dt, \tag{2}$$

where the contour $H$ is a contour that starts at $-\infty$ below the negative real axis with $arg(t) = -\pi$, encircles the origin in the positive direction and goes back to $-\infty$ above the real axis with $arg(t) = +\pi$. $H$ is called the Hankel contour (See Figure 1).

The Second Form will not be discussed here. What follows are facts about the Gamma function and concepts that will be used in the derivation of Hankel's Loop Integral.

**The Gamma Function**

The Gamma function was discovered by the famous mathematician L. Euler in an attempt to find an analytic continuation of the factorial function $f(n) = n!$. The classical Gamma function, denoted by Γ, is defined as

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt \tag{3}$$

with $Re\ z > 0$ and the path of integration is the real axis. The many-valued function $t^z$ has its principal value for $t > 0$, where $\arg(t) = 0$.

That the function $\Gamma(z)$ is analytic in the domain $Re\ z > 0$ follows from the following theorem. Proof of this theorem can be found in Copson (1967).

**Theorem 1.1.** Let $t$ be a real variable ranging over a finite or infinite interval $(a, b)$ and $z$ a complex variable ranging over a domain $\boldsymbol{D}$ in $\mathbb{C}$. Assume that the function $f: (\boldsymbol{D} \times (a, b)) \to \mathbb{C}$ satisfies the following conditions:

1. $f$ is a continuous function of both variables.
2. For each fixed value of $t$, $f(z, t)$ is an analytic function in $\boldsymbol{D}$ of the first variable.
3. The integral

$$F(z) = \int_a^b f\,(z, t)dt, \qquad z \in D$$

4. converges uniformly at both limits in any compact set in $D$.

Then $F(z)$ is analytic in $\boldsymbol{D}$, and its derivatives of all orders may be found by differentiating under the sign of integration.

The uniform convergence of the Gamma function follows from the Weierstrass M-test which is stated below.

**Theorem 1.2.** (The Weierstrass M-Test) Suppose $f(x, t)$ is Riemann integrable over $[a, c]$ for all $c \geq a$ and all $x \in [\alpha, \beta]$. Suppose that there exists a positive function $M(t)$ defined for $t \geq a$ such that $|f(x, t)| \leq M(t)$ for $t \geq a$ and $x \in [\alpha, \beta]$, and such that

$$\int_a^\infty M\,(t)dt$$

exists. Then for each $x \in [\alpha, \beta]$,

$$\int_a^\infty f\,(x, t)dt$$

is absolutely convergent and the convergence is uniform on $[\alpha, \beta]$.

Since

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1,$$

for positive integer $n$, successive application of integration by parts give

$$\Gamma(n) = (n-1)(n-2)\dots(2)\int_0^{\infty} e^{-t}\,dt = (n-1)! \tag{4}$$

This shows that the Gamma function is a natural extension of the factorial function.

Another important property of the gamma function is that it has simple poles at the points $z = -n$, $(n = 0,1,2,\dots)$. This follows from the fact that

$$\Gamma(z) = \int_0^1 e^{-t}\,t^{z-1}dt + \int_1^{\infty} e^{-t}\,t^{z-1}dt, \tag{5}$$

from which we have (see [7])

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{1}{k+z} + entirefunction. \tag{6}$$

Hence $\Gamma(z)$ is a meromorphic function with simple poles at $z = 0,-1,-2,\dots$. Definition of a meromorphic function is given below.

**Definition 1.3.** Meromorphic Function

In the mathematical field of complex analysis, a meromorphic function on an open subset $D$ of the complex plane is a function that is holomorphic on all $D$ except a set of isolated points (the poles of the function), at each of which the function must have a Laurent series. (The terminology comes from the Ancient Greek meros, meaning part, as opposed to holos, meaning whole.)

The concept of analytic continuation will be used to show that (Churchill and Brown) holds for all $z$. A definition of analytic continuation is as follows:

**Definition 1.4.** Analytic Continuation

Suppose $f$ is an analytic function defined on a non-empty open subset $\boldsymbol{U}$ of the complex plane $\mathbb{C}$. If $\boldsymbol{V}$ is a larger open subset of $\mathbb{C}$, containing $\boldsymbol{U}$, and $F$ is an analytic function defined on $\boldsymbol{V}$ such that $F(z) = f(z)$ for all $z \in \boldsymbol{U}$, then $F(z) = f(z)$ for all $z \in \boldsymbol{V}$ and $F$ is called an analytic continuation of $f$. (Whittaker and Watson 1927)

## 2. Proof of the Hankel's Loop Integral

Let

$$I(z) = \int_H e^t\,t^{-z}dt, \tag{6}$$

where the branch of $t^{-z}$ takes its principal value at the points (or points) where the contour crosses the positive real axis.

The technique used here to derive the Hankel's loop Integral is similar to that in [2] making only modifications to obtain the appropriate branches of the multi-valued function $e^t t^{-z}$ and the contours to be used. The proof that follows willjustify the derivation of Hankel's Loop

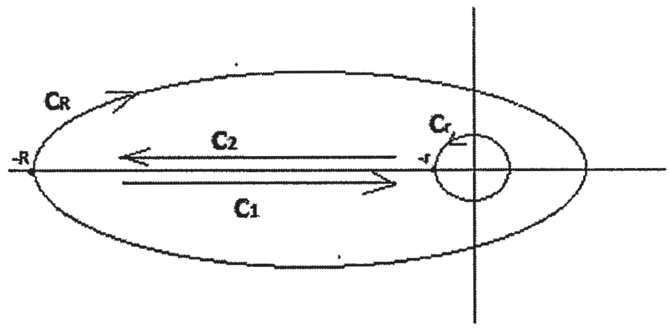


**Figure 2**

Integral where the contour is collapsed into the branch cut (See Oliver 1974).

We begin by letting $C_r$ denote the circle $|z| = r$, where $0 < r < 1$, $C_R$ be the contour enclosing $C_r$ as shown in Figure 2 with the given orientation. Let $C_1$ be the line segment from $-R$ to $-r$ traversed in the direction shown with $arg(t) = -\pi$, while $C_2$ is the line segment from $-r$ to $-R$ traversed in the direction shown with $arg(t) = \pi$.

The branch

$$f(t) = e^t t^{-z} \qquad (|t| > 0, \;\; -\pi < arg(t) < \pi)$$

of the multi-valued function $e^t t^{-z}$, with branch cut $arg(t) = \pi$, is piecewise continuous on $C_r$ and $C_R$. Thus, the integrals

$$\int_{C_r} f(t)dt, \qquad \int_{C_R} f(t)dt.$$

exist.

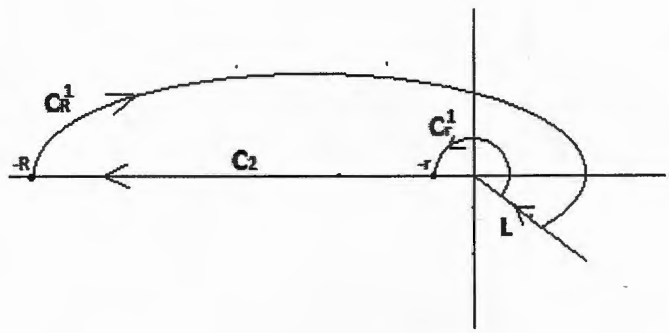


**Figure 3**

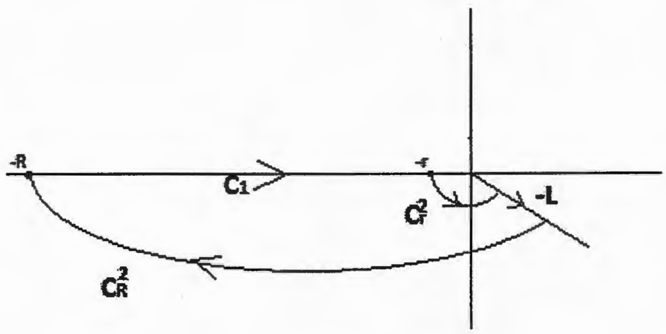


**Figure 4**

We then introduce the two simple closed contours shown in Figure 3 and Figure 4, obtained by cutting the annulus in Figure 2 along $L$.

Note that the branch

$$f_1(t) = e^t t^{-z} \qquad D_1: \left(|t| > 0, -\frac{\pi}{2} < arg(t) < \frac{3\pi}{2}\right)$$

is analytic within and on the closed contour in Figure 3, the branch cut being the lower half of the imaginary axis. Note that on this branch $arg(t) = \pi$ when $t$ is a point on the line segment joining $-r$ and $-R$.

Then, by Cauchy-Goursat Theorem, the integral of $f_1(t)$ along the closed contour in Figure 3 is 0. That is,

$$\int_{C_2} f_1(t)dt + \int_{C_R^1} f_1(t)dt + \int_L f_1(t)dt + \int_{C_r^1} f_1(t)dt = 0. \tag{7}$$

On the other hand, the branch

$$f_2(t)=e^t t^{-z} \qquad D_2:\left(|t|>0,-\frac{3\pi}{2}<arg(t)<\frac{\pi}{2}\right)$$

is analytic everywhere within and on the closed contour in Figure 4. Its branch cut is the upper half of the imaginary axis. Note also that on this branch, the $arg(t)=-\pi$ when $t$ is a point on the line segment joining $-r$ and $-R$ on the negative real axis. Again, by Cauchy-Goursat Theorem, we have

$$\int_{C_1} f_2(t)dt+\int_{C_r^2} f_2(t)dt-\int_L f_2(t)dt+\int_{C_R^2} f_2(t)dt=0. \tag{8}$$

Write $t^{-z}=exp[-z(logt+i\, arg\, t)]$. Then ·

$$f(t)=e^t e^{-z[logt+iarg(t)]}, \qquad (|t|>0,-\pi<arg(t)<\pi).$$

Observe that except for the legs, $C_1$ and $C_2$ along the negative real axis, $f_1(t)=f(t)$ on the contour in Figure 3, and $f_2(t)=f(t)$ on the contour in Figure 4. So (7) becomes

$$\int_{C_2} f_1(t)dt+\int_{C_R^1} f(t)dt+\int_L f(t)dt+\int_{C_r^1} f(t)dt=0. \tag{9}$$

and (8) becomes

$$\int_{C_1} f_2(t)dt+\int_{C_r^2} f(t)dt-\int_L f(t)dt+\int_{C_R^2} f(t)dt=0. \tag{10}$$

Note that $C_r^1+C_r^2=C_r$ and $C_R^1+C_R^2=C_R$. Adding (9) and (10), we have

$$\int_{C_2} f_1(t)dt+\int_{C_1} f_2(t)dt+\int_{C_r} f(t)dt+\int_{C_R} f(t)dt=0, \tag{11}$$

from which we have

$$\int_{C_R} f(t)dt = -\int_{C_2} f_1(t)dt - \int_{C_1} f_2(t)dt - \int_{C_r} f(t)dt. \tag{12}$$

We now prove that the integral along $C_r$ goes to zero as $r \to 0$ provided that $Re\,(1-z) > 0$. Note that

$$\left|\int_{C_r} f(z)dz\right| \leq \int_{C_r} |\, e^t t^{-z} dt \,| \leq e^r 2\pi r^{1-a}, \tag{13}$$

where $a = Re\, z$. As $r \to 0$, the expression at the right hand side goes to 0 when $1-a > 0$, that is, when $Re\,(1-z) > 0$. Thus, the integral also goes to 0 as $r \to 0$.

Now let $r \to 0$ and $R \to \infty$. Then $C_R$ will become the negative of the Hankel contour described in (2), the contribution of the integral on $C_r$ is 0 and (12) becomes

$$-\int_H f(t)dt = -\int_{C_2^+} f_1(t)dt - \int_{C_1^+} f_2(t)dt,$$

or

$$\int_H f(t)dt = \int_{C_1^+} f_2(t)dt + \int_{C_2^+} f_1(t)dt, \tag{14}$$

where $C_1^+$ and $C_2^+$ are $C_1$ and $C_2$, respectively when $r \to 0$ and $R \to \infty$. The legs $C_1^+$ and $C_2^+$ may be considered the two sides of the interval $(-\infty, -r]$ [6].

We now proceed to compute the integrals at the right hand side of (14).

On $C_1^+$, $t = \tau e^{-i\pi}$, $0 < \tau < \infty$. Thus,

$$t^{-z} = e^{-zlogt} = e^{-z(log\tau - i\pi)} = \tau^{-z} e^{i\pi z}.$$

Consequently,

$$\int_{C_1^+} f_2(t)dt = \int_{\infty}^{0} e^{-\tau} \tau^{-z} e^{i\pi z} (-d\tau)$$

$$= -\int_{0}^{\infty} e^{-\tau} \tau^{-z} e^{i\pi z} (-d\tau)$$

$$= e^{i\pi z} \int_{0}^{\infty} e^{-\tau} \tau^{-z} d\tau. \tag{15}$$

On $C_2^+$, $t = \tau e^{i\pi}$, $\quad 0 < \tau < \infty$. Thus,

$$t^{-z} = e^{-z(log\tau + i\pi)} = \tau^{-z} e^{-i\pi z},$$

and

$$\int_{C_2^+} f_1(t)dt = \int_{0}^{\infty} e^{-\tau} \tau^{-z} e^{-i\pi z} (-d\tau)$$

$$= -e^{-i\pi z} \int_{0}^{\infty} e^{-\tau} \tau^{-z} d\tau. \tag{16}$$

Then (14) becomes

$$\int_H f(t)dt = e^{i\pi z} \int_{0}^{\infty} e^{-\tau} \tau^{-z} d\tau - e^{-i\pi z} \int_{0}^{\infty} e^{-\tau} \tau^{-z} d\tau$$

$$= (e^{i\pi z} - e^{-i\pi z}) \int_{0}^{\infty} e^{-\tau} \tau^{-z} d\tau$$

$$= 2i \sin(\pi z) \Gamma(1-z), \tag{17}$$

provided that $Re\,(1-z) > 0$.

An identity of the Gamma function known as the *Euler Reflection Formula* states that, for non-integral $z$,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{sin\pi z}.$$

Hence,

$$\frac{1}{2\pi i}\int_H e^t t^{-z}dt=\frac{1}{\Gamma(z)}, \quad Re\,(1-z)>0. \tag{18}$$

Next we remove the condition $Re\,(1-z)>0$. From the Euler Reflection Formula, it also follows that the Gamma function does not have zeros. Moreover, in the quotient $\frac{1}{\Gamma(z)}$, the singularities of $\Gamma(z)$ which are simple poles at $z=0,-1,-2,\ldots$, are removed. Hence, $\frac{1}{\Gamma(z)}$ is an entire function.

On the other hand, the integrals

$$\int_{C_1^+} e^t t^{-z}dt = e^{i\pi z}\int_0^{+\infty} e^{-\tau}\tau^{-z}d\tau, \tag{19}$$

$$\int_{C_2^+} e^t t^{-z}dt = -e^{-i\pi z}\int_0^{\infty} e^{-\tau}\tau^{-z}d\tau \tag{20}$$

both converge uniformly in any compact set $D$, $z\in D$ (This follows from the Weierstrss M-Test). Thus Theorem 1.1 is satisfied and so $I(z)$ is analytic in $D$. By the principle of analytic continuation (see Definition 1.4), we conclude that

$$\frac{1}{2\pi i}\int_H e^t t^{-z}dt=\frac{1}{\Gamma(z)}$$

holds for all values of $z$.

## 3. Application to Second Kind r-Whitney Numbers

The $r$-Whitney numbers of the second kind have been introduced by Mezo (2010) to obtain a new formula for Bernoulli polynomials. These numbers satisfy

$$\frac{1}{\beta^m (m!)} e^{rz}(e^{\beta z}-1)^m = \sum_{n=m}^{\infty} W_{\beta,r}(n,m)\frac{z^n}{n!}, \tag{21}$$

where $n$ and $m$ are positive integers. These numbers are equivalent to the numbers considered by Rucinski and Voight [8] and the $(r,\beta)$-Stirling numbers [5].

Applying the Cauchy Integral Formula, the following integral representation is obtained:

$$W_{\beta,r}(n,m) = \frac{n!}{2\pi i \beta^m (m!)} \int_C \frac{e^{rz}(e^{\beta z}-1)^m}{z^{n+1}} dz, \tag{22}$$

where $C$ is a circle about the origin.

Following Flajolet and Prodinger (1999), we define

$$W_{\beta,r}(y,x) = \frac{y!}{2\pi i \beta^x x!} \int_H \frac{e^{rz}(e^{\beta z}-1)^x}{z^{y+1}} dz, \tag{23}$$

where $x$ and $y$ are complex numbers, $y!$ and $x!$ are generalized factorials defined via the gamma function and $H$ is the Hankel contour described in Churchill and Brown.

As in Corcino *et al.* (2014), (23) can be written in the form

$$W_{\beta,r}(y,x) = \beta^{y-x} \frac{y!}{x! 2\pi i} \int_H \frac{e^{vu}(e^u-1)^x}{u^{y+1}} du. \tag{24}$$

Consequently,

$$W_{\beta,r}(y,y-x)=\binom{y}{x}\frac{x!\beta^x}{2\pi i}\int_H \frac{e^{vu}(e^u-1)^{y-x}}{u^{y+1}}du \tag{25}$$

$$=\binom{y}{x}\frac{x!\beta^x}{2\pi i}\int_H \frac{T(q,w,v)}{w^{x+1}}dw, \tag{26}$$

where

$$T(q,w,v)=exp\{-w+vqw+\frac{2}{q}log[f(qw)+1]\}, \tag{27}$$

$$q=2/(y-x), \quad f(u)=\frac{e^u-1}{u}-1,$$

and the logarithm is taken to be the principal branch.

Since $log(f(qw)+1$ is analytic on the half plane $Re\, z>-1$, $T(q,w,v)$ as a function of $q$, is analytic on the said half plane. In particular, $T(q,w,v)$ is analytic within the unit circle. Thus, $T(q,w,v)$ has a Maclaurin expansion about $q=0$, given by

$$T(q,w,v)=1+\sum_{k=1}^{\infty}T_k(w,v)q^k. \tag{28}$$

Hence, the computations in [1] are still valid for complex parameters $x$ and $y$. The Hankel's loop integral is used in Corcino *et al.* (2014) to compute the integrals in the following expression for $r$-Whitney numbers, which also holds for complex $x$ and $y$.

$$W_{\beta,r}(y,y-x)=\beta^x\binom{y}{x}\left(\frac{y-x}{2}\right)^x\left[\frac{x!}{2\pi i}\int_H \frac{e^w}{w^{x+1}}dw\right]$$

$$+\beta^x\binom{y}{x}\left(\frac{y-x}{2}\right)^x\left[\frac{x!}{2\pi i}\int_H \sum_{k=1}^{\infty}\frac{T_k(w,v)q^k e^w}{w^{x+1}}dw\right], \tag{29}$$

The first integral on the right hand side of (29) is

$$\frac{1}{2\pi i}\int_H e^w w^{-(x+1)} dw = \frac{1}{\Gamma(x+1)} \tag{30}$$

For the second integral of (29) we compute three terms. The $T_k(w,\nu)$'s for $k=1,2,3$ are given below.

$$T_1(w,\nu) = \nu w + \frac{w^2}{12}, \tag{31}$$

$$T_2(w,\nu) = \frac{1}{2}\left[\nu^2 w^2 + \frac{\nu w^3}{6} + \frac{w^4}{144}\right], \tag{32}$$

$$T_3(w,\nu) = \frac{1}{3!}\left[\nu^3 w^3 + \left(\frac{\nu^2}{4} - \frac{1}{240}\right) w^4 + \frac{\nu w^5}{48} + \frac{w^6}{1728}\right]. \tag{33}$$

Computing the corresponding integrals:

$$\frac{1}{2\pi i}\int_H \left(\nu w + \frac{w^2}{12}\right) q e^w dw = q\left[\frac{\nu}{\Gamma(x)} + \frac{1}{12\Gamma(x-1)}\right]$$

$$= q\left[\frac{\nu}{(x-1)!} + \frac{1}{12(x-2)!}\right]. \tag{34}$$

Similarly,

$$\frac{1}{2\pi i}\int_H \frac{T_2(w,\nu)q^2e^w}{w^{x+1}}dw = \frac{q^2}{2!}\left[\frac{\nu^2}{(x-2)!}+\frac{\nu}{6}\cdot\frac{1}{(x-3)!}\right.$$

$$\left.+\frac{1}{144}\cdot\frac{1}{(x-4)!}\right] \tag{35}$$

$$\frac{1}{2\pi i}\int_H \frac{T_3(w,\nu)q^3e^w}{w^{x+1}}dw = \frac{q^3}{3!}\left[\frac{\nu^3}{(x-3)!}+\left(\frac{\nu^2}{4}-\frac{1}{240}\right)\frac{1}{(x-4)!}\right.$$

$$\left.+\frac{\nu}{48}\cdot\frac{1}{(x-5)!}+\frac{1}{1728}\cdot\frac{1}{(x-6)!}\right]. \tag{36}$$

An Asymptotic Formula for $r$-Whitney numbers of the second kind with complex parameters $x$ and $y$, is given by

$$W_{\beta,r}(y,y-x) = \binom{y}{x}\left(\frac{\beta(y-x)}{2}\right)^x\left[1+\frac{1}{y-x}\left\{2x\nu+\frac{(x)_2}{6}\right\}\right.$$

$$+\frac{1}{(y-x)^2}\left\{2\nu^2(x)_2+\frac{\nu(x)_3}{3}+\frac{(x)_4}{72}\right\}$$

$$+\frac{1}{(y-x)^3}\left\{\frac{4}{3}\nu^3(x)_3+\left[\frac{\nu^2}{3}-\frac{1}{180}\right](x)_4\right.$$

$$\left.\left.+\frac{\nu(x)_5}{36}+\frac{(x)_6}{1296}\right\}+\ldots\right], \tag{37}$$

valid when $|x| = o(|y-x|)$. (See Corcino *et al.* (2014)

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