(2) Phase : The phase of oscillations is given by

$$\tan \phi = \frac{Rq}{k - mq^2}$$

Or

$$\tan \phi = \frac{Rq}{m(\omega^2 - q^2)}$$

Thus, the phase depends on $(\omega^2 - q^2)$ and R. Some important cases are tabulated below.

Condition	**tan ϕ**	**Angle ϕ**
$q < \omega$	Positive	$0 < \phi < \pi/2$
$q = \omega$	Infinity	$\pi/2$
$q > \omega$	Negative	$\pi/2 < \phi < \pi$

(3) Frequency : The angular frequency of forced oscillations is the same as that of the forcing angular frequency 'q'. Thus, the frequency of the forced oscillator is $q/2\pi$. Forced oscillations are executed as long as external periodic force acts on the body. If the external force is withdrawn, damped oscillations with diminishing amplitude occur.

3.3 Resonance

- We shall now discuss the remarkable phenomenon of great importance throughout physics that occur when an oscillator is subjected to an external periodic force by an external agency. This phenomenon is called as 'resonance'. The most striking feature of an oscillator is the way in which periodic force of fixed size produces very different results depending on its frequency. In particular, if the forced frequency is made very close to the natural frequency of an oscillator, then the amplitude of the oscillation can be made very large by application of small periodic force. This is the phenomenon of resonance. Thus, when frequency of the applied force is equal to the natural frequency of an oscillator, it oscillates with maximum amplitude.
- As the natural frequency of the oscillator differs more and more from the forcing frequency, the response becomes less, so that the amplitude and energy of oscillation become less.

Resonance phenomena occurs widely in natural and in technological applications :

Emission and absorption of light
Lasers
Tuning of radio and television sets
Mobile phones
Microwave communications
Machine, building and bridge design
Musical instruments
Medicine - nuclear magnetic resonance
X-rays
Hearing

- **M**agnetic **R**esonance **I**maging (MRI) is a type of scan which uses magnetism to build up a picture of the inside of the body. Fig. 3.2 is an example of an MRI scan of the head.

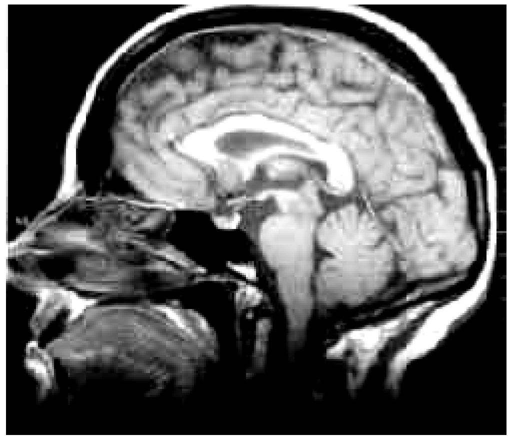


Fig. 3.2 : MRI Scan of head

There are number of applications of resonance. We will study here few of them.

(1) Mechanical resonance : A child's swing has a natural frequency of its own. If the swing is subjected to a pushing force of frequency equal to the natural frequency of the swing, it is set into resonant oscillations of large amplitude with little effort.

(2) Electrical resonance : The radio tuning circuit contains LC circuit. The frequency of circuit depends on the values of L and C. The radio circuit is tuned by changing the value of the capacitor. If the frequency of circuit becomes equal to the frequency of radio waves transmitted by the transmitter from the radio station, the response of the circuit is more and the circuit accepts that signal, so that radio program is heard.

(3) Optical resonance : Light from a sodium vapour lamp may cause sodium atoms in the lamp at low pressure to glow with its characteristic yellow colour.

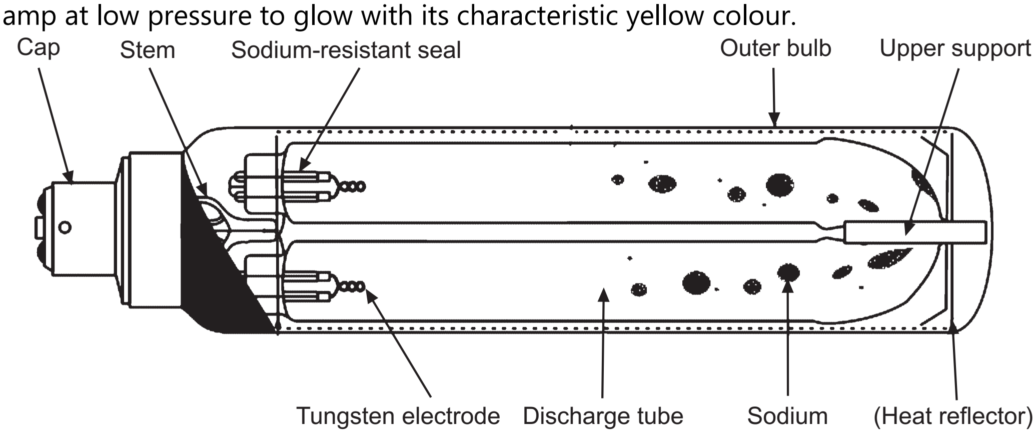


Fig. 3.3 : Structure of sodium vapour lamp

- The sodium vapour lamp consists of a **U** shaped inner glass tube filled with neon gas at a pressure of 10 mm. It also contains a small quantity of sodium and argon gas. The initial ionization voltage is reduced, as the ionization potential of argon is low. Two oxide

3.4 Types of Resonance

The mechanical resonance has two types : (1) Amplitude resonance, (2) Velocity resonance.

3.4.1 Amplitude Resonance

- The amplitude of forced oscillations is given by

$$A = \frac{f_o}{m\sqrt{(\omega^2 - q^2)^2 + \frac{R^2 q^2}{m^2}}}$$

- The amplitude is maximum when denominator of the above equation is minimum. The denominator depends on the driving frequency q. The condition for minimum denominator is that

$$\frac{d}{dq}\left[(\omega^2 - q^2)^2 + \frac{R^2 q^2}{m^2}\right] = 0$$

$$\therefore \qquad 2(\omega^2 - q^2)(-2q) + \frac{2R^2 q}{m^2} = 0$$

Or

$$q^2 = \omega^2 - \frac{R^2}{2m^2}$$

$$\therefore \qquad q = \sqrt{\omega^2 - \frac{R^2}{2m^2}} \qquad \ldots (3.9)$$

- This is the angular frequency of forced oscillations at which the amplitude is maximum. This frequency is called *resonant* frequency.
- When the damping is very small, we can neglect the term $\frac{R}{2m}$ and we get

$$q = \omega$$

- Thus, for low damping, the amplitude of the oscillations is maximum when the angular frequency of applied force is equal to the natural angular frequency of the body. This phenomenon is called *amplitude resonance*.
- Fig. 3.5 shows the variation of the amplitude with the driving frequency q. When the frequency of applied force increases from smaller value, the difference $(\omega^2 - q^2)$ decrease and the amplitude increases. At $q = \omega$, the amplitude is maximum and decreases when q increases beyond ω.

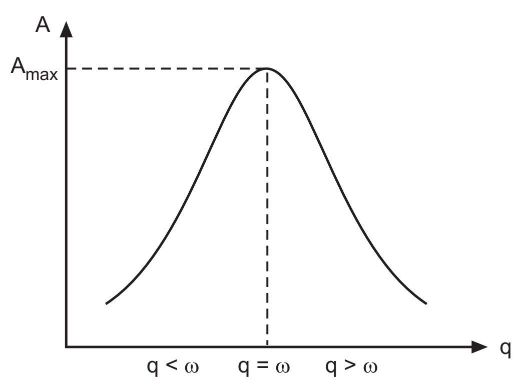


Fig. 3.5 : Variation of amplitude with driving frequency

When $q = \omega$, the amplitude is maximum and is given by

$$A_{max} = \frac{f_o}{m\sqrt{(\omega^2 - \omega^2)^2 + \frac{R^2\,\omega^2}{m^2}}}$$

Or
$$A_{max} = \frac{f_o}{\omega R} \qquad \text{... (3.10)}$$

- Thus, A_{max} is inversely proportional to the damping coefficient R.
- When R = 0, $A_{max} = \infty$. This is the ideal case and in this case the sharpness of resonance is maximum.
- Thus, when the driving frequency approaches a natural frequency of vibration, the resulting oscillations dramatically increase in amplitude. Resonance occurs when the driving frequency matches the natural frequency and the amplitude of the oscillation reaches a maximum value.
- At resonance, most of the energy is added to the mechanical energy of the vibrating system, very little energy is returned to the driving source. The smaller the damping, the greater is the amplitude of vibration.

3.4.2 Velocity Resonance

- The displacement of the body in forced oscillations is given by

$$x = A \sin(qt - \phi)$$

- The velocity of the body in forced oscillations is given by

$$v = \frac{dx}{dt}$$

$\therefore$
$$v = Aq \cos(qt - \phi)$$

Or
$$v = v_o \cos(qt - \phi)$$

where $v_o = Aq$, called velocity amplitude.

$\therefore$
$$v_o = \frac{f_o\,q}{m\sqrt{(\omega^2 - q^2)^2 + \frac{R^2\,q^2}{m^2}}}$$

Or
$$v_o = \frac{f_o}{\frac{m}{q}\sqrt{(\omega^2 - q^2)^2 + \frac{R^2\,q^2}{m^2}}}$$

Or
$$v_o = \frac{f_o}{m\sqrt{\frac{(\omega^2 - q^2)^2}{q^2} + \frac{R^2}{m^2}}}$$

$\therefore$
$$v_o = \frac{f_o}{m\sqrt{\left(\frac{\omega^2 - q^2}{q}\right)^2 + \frac{R^2}{m^2}}}$$

The velocity amplitude depends upon $\left(\frac{\omega^2 - q^2}{q}\right)$. The velocity amplitude is maximum when this term is minimum. The condition for minimum denominator is

$$\frac{d}{dq}\left[\left(\frac{\omega^2 - q^2}{q}\right)^2 + \frac{R^2}{m^2}\right] = 0$$

Or $$\frac{d}{dq}\left[\left(\frac{\omega^2 - q^2}{q}\right)^2\right] = 0$$

Or $$\frac{d}{dq}\left[\left(\frac{\omega^2}{q} - q\right)^2\right] = 0$$

$\therefore$ $$2\left(\frac{\omega^2}{q} - q\right)\left(-\frac{\omega^2}{q^2} - 1\right) = 0$$

Or $$-2q\left(\frac{\omega^2}{q^2} - 1\right)\left(\frac{\omega^2}{q^2} + 1\right) = 0$$

$\therefore$ $$-2q\left(\frac{\omega^4}{q^4} - 1\right) = 0$$

Or $$\left(\frac{\omega^4}{q^4} - 1\right) = 0$$

which gives $$\omega^4 = q^4$$

Or $$q = \omega$$

- Thus, *the velocity amplitude is maximum when the driving angular frequency is equal to the natural angular frequency of the oscillating body.* This phenomenon is called the velocity resonance.
- The variation of velocity amplitude with the driving frequency q is shown in Fig. 3.6.

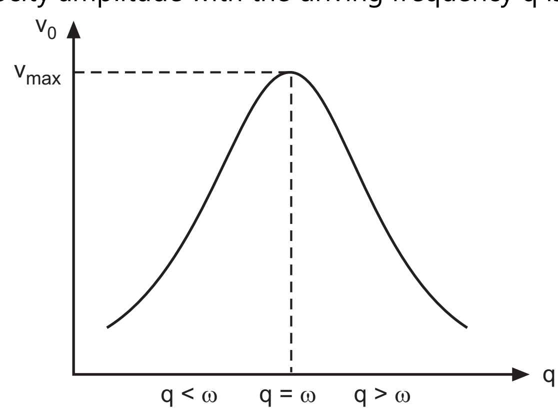


Fig. 3.6 : Variation of velocity amplitude with frequency

- When frequency of applied force increases, the difference ($\omega^2 - q^2$) decreases and the velocity amplitude increases. At $q = \omega$, the velocity amplitude is maximum and decreases when q increases beyond ω.

- The maximum value of velocity amplitude is

$$v_{max} = \frac{f_o}{m\sqrt{\left(\frac{\omega^2 - \omega^2}{\omega}\right)^2 + \frac{R^2}{m^2}}}$$

Or
$$v_{max} = \frac{f_o}{R} \qquad \text{... (3.11)}$$

- Thus, v_{max} is inversely proportional to the damping coefficient R. At velocity resonance, v is large when R is small. In the absence of damping, v_{max} would be infinity. However, this is an ideal case and never possible in actual practice.

  At velocity resonance,

$$\tan\phi = \frac{Rq}{m(\omega^2 - q^2)} = \infty$$

Or
$$\phi = \frac{\pi}{2}$$

  i.e. at resonance the displacement and applied force have the phase difference $\pi/2$ or 90^{o}.

3.5 Sharpness of Resonance

- When the frequency of external periodic force differs from the value required for resonance, the response of the oscillator diminishes. Sharpness of resonance expresses the fall of the amplitude with the change of driving frequency on each side of the maximum amplitude.
- The amplitude in forced oscillations is given by

$$A = \frac{f_o}{m\sqrt{(\omega^2 - q^2)^2 + \frac{R^2 q^2}{m^2}}}$$

- The variation of amplitude with driving angular frequency q is shown in Fig. 3.7.

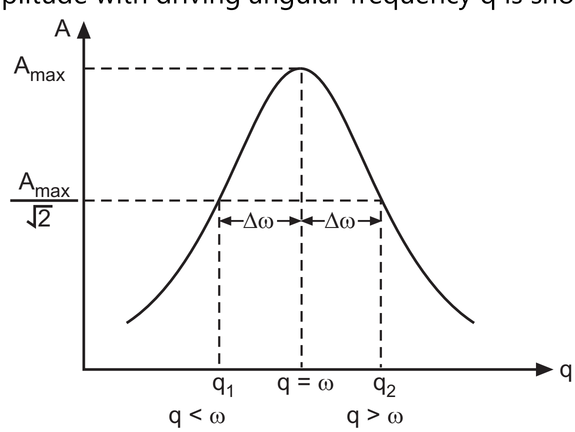


Fig. 3.7 : Resonance curve

- At $q = \omega$, the amplitude is maximum. Amplitude A decreases when $|\omega - q|$ increases. The difference $|\omega - q|$, for which the amplitude falls from A_{max} to $\frac{A_{max}}{\sqrt{2}}$, is called half width of the resonance curve. Sharpness of resonance is characterized by the half width. When the half width of the resonance curve is small, the resonance is said to be *sharp*. If the half width is large, the resonance is said to be *flat*. Let us, therefore, obtain the expression of half width.

  We can write, $\omega^2 - q^2 = (\omega + q)(\omega - q)$

  If $\omega \approx q$, then $\omega + q = 2\omega$ and $\omega - q = \Delta\omega$

  $$\therefore \quad \omega^2 - q^2 = 2\omega\,\Delta\omega$$

  $$\therefore \quad A = \frac{f_o}{m\sqrt{4\omega^2(\Delta\omega)^2 + \frac{R^2\omega^2}{m^2}}}$$

  From equation (3.10), we have $A_{max} = \frac{f_o}{\omega R}$

  At half width, $$A = \frac{A_{max}}{\sqrt{2}}$$

  $$\therefore \quad \frac{f_o}{m\sqrt{4\omega^2(\Delta\omega)^2 + \frac{R^2\omega^2}{m^2}}} = \frac{f_o}{\sqrt{2}\,\omega R}$$

  Or $$m\sqrt{4\omega^2(\Delta\omega)^2 + \frac{R^2\omega^2}{m^2}} = \sqrt{2}\,\omega R$$

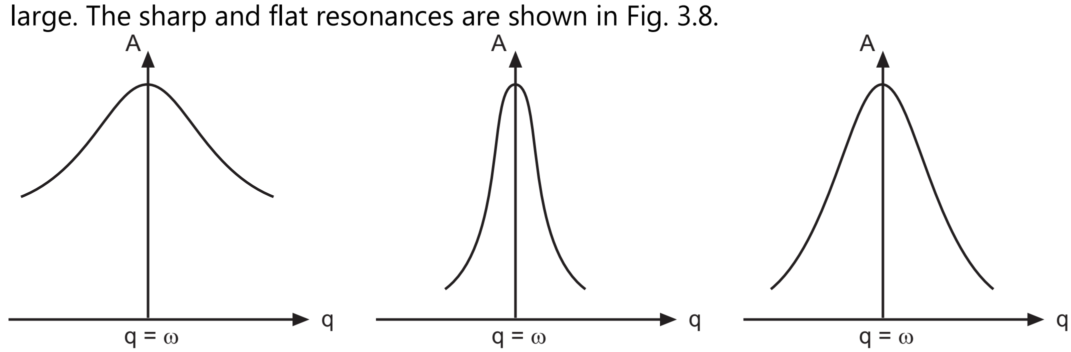
  Squaring on both sides, we get

  $$m^2\,4\omega^2(\Delta\omega)^2 + R^2\omega^2 = 2\omega^2 R^2$$

  Or $$m^2\,4\omega^2(\Delta\omega)^2 = \omega^2 R^2$$

  $$\therefore \quad \Delta\omega = \frac{R}{2m} \qquad \ldots (3.12)$$

  $\Delta\omega$ is called the half width. Thus, the half width depends upon mass m and damping coefficient R. In order to make the resonance sharp, R should be made small and m be large. The sharp and flat resonances are shown in Fig. 3.8.



(a) Flat **(b) Sharp** **(c) Medium**

Fig. 3.8

- Sharpness of resonance is very important in radio receiver. By tuning the radio, natural frequency of the electrical circuit in the receiver is adjusted to the same value as the frequency of the radio waves transmitted by the radio station. In good radio set, selectivity is high so that the response ceases for small mistuning. In such cases, the damping is due to electrical resonance.

3.6 Energy Equation of Forced Oscillations

The equation of forced oscillations is given by

$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + kx = f_o \sin qt$$

Multiplying above equation by $\frac{dx}{dt}$, we get

$$m\left(\frac{d^2x}{dt^2}\right)\frac{dx}{dt} + R\left(\frac{dx}{dt}\right)^2 + kx\frac{dx}{dt} = f_o \sin qt\frac{dx}{dt}$$

$$\therefore \qquad m\left(\frac{d^2x}{dt^2}\right)\frac{dx}{dt} + kx\frac{dx}{dt} = f_o \sin qt\frac{dx}{dt} - R\left(\frac{dx}{dt}\right)^2$$

Or

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2\right] = f_o \sin qt\frac{dx}{dt} - R\left(\frac{dx}{dt}\right)^2$$

$$\frac{d}{dt}[\text{K.E.} + \text{P.E.}] = f_o \sin qt\frac{dx}{dt} - R\left(\frac{dx}{dt}\right)^2$$

Or

$$\frac{dE}{dt} = f_o \sin qt\frac{dx}{dt} - R\left(\frac{dx}{dt}\right)^2$$

where E is the total energy and it is sum of kinetic energy (K.E.) and potential energy (P.E.).

$$\text{K.E.} = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

and

$$\text{P.E.} = \frac{1}{2}kx^2$$

Taking average energy over the period, we get

$$\overline{\frac{dE}{dt}} = \overline{f_o \sin qt\frac{dx}{dt}} - \overline{R\left(\frac{dx}{dt}\right)^2} \qquad \text{... (3.13)}$$

The displacement in forced oscillations is given by

$$x = A \sin (qt - \phi)$$

$$\therefore \qquad \frac{dx}{dt} = Aq \cos (qt - \phi)$$

$\therefore \qquad f_o \sin qt \dfrac{dx}{dt} = f_o Aq \sin qt \cos(qt - \phi)$

Or $\qquad f_o \sin qt \dfrac{dx}{dt} = f_o Aq \sin qt [\cos qt \cos \phi + \sin qt \sin \phi]$

$$= f_o Aq \sin qt \cos qt \cos \phi + f_o Aq \sin^2 qt \sin \phi$$

$$\overline{f_o \sin qt \frac{dx}{dt}} = f_o Aq \cos \phi \, \overline{\sin qt \cos qt} + f_o Aq \sin \phi \, \overline{\sin^2 qt}$$

Over one complete cycle of oscillation, we have

$$\overline{\sin qt \cos qt} = 0$$

and $\qquad \overline{\sin^2 qt} = \dfrac{1}{2}$

$\therefore \qquad \overline{f_o \sin qt \dfrac{dx}{dt}} = f_o Aq \sin \phi \cdot \dfrac{1}{2}$

From equation (3.4), we have $RAq = f_o \sin \phi$

$\therefore \qquad \overline{f_o \sin qt \dfrac{dx}{dt}} = \dfrac{RA^2 q^2}{2} \qquad \text{... (3.14)}$

- Equation (3.14) gives the average rate of absorption of energy by the oscillating body from the applied force.

We have, $\qquad R\left(\dfrac{dx}{dt}\right)^2 = RA^2 q^2 \cos^2 (qt - \phi)$

$\therefore \qquad \overline{R\left(\dfrac{dx}{dt}\right)^2} = RA^2 q^2 \, \overline{\cos^2 (qt - \phi)}$

Or $\qquad \overline{R\left(\dfrac{dx}{dt}\right)^2} = \dfrac{RA^2 q^2}{2} \qquad \text{... (3.15)}$

- Equation (3.15) gives the average rate of dissipation of energy i.e. power dissipation in overcoming the external damping force.
- Using equations (3.14) and (3.15) in (3.13), we get

$$\frac{\overline{dE}}{dt} = \frac{RA^2q^2}{2} - \frac{RA^2q^2}{2}$$

Or $\qquad \dfrac{\overline{dE}}{dt} = 0$

$\therefore \qquad \overline{E} = \text{Constant}$

- Thus, the average energy over a cycle of oscillation is constant. In other words, the average rate of dissipation of energy is equal to the average rate of absorption of energy.

3.7 Quality Factor of Forced Oscillations

- The quality factor Q of an oscillator represents its efficiency. This is the quantity in which the damping effect is expressed. The quality factor Q is defined as 2π times the ratio of the energy stored to the dissipation of energy in one oscillation.

$$\therefore \qquad Q = 2\pi \frac{\text{Energy stored}}{\text{Dissipation of energy in one oscillation}}$$

- The average energy stored in the oscillator is

$$\bar{E} = \overline{\frac{1}{2} m \left(\frac{dx}{dt}\right)^2} + \overline{\frac{1}{2} kx^2}$$

We have,
$$x = A \sin (qt - \phi)$$
$$\frac{dx}{dt} = Aq \cos (qt - \phi)$$

$$\therefore \qquad \bar{E} = \frac{1}{2} mA^2q^2 \, \overline{\cos^2 (qt - \phi)} + \frac{1}{2} kA^2 \, \overline{\sin^2 (qt - \phi)}$$

We have
$$\overline{\cos^2 (qt - \phi)} = \overline{\sin^2 (qt - \phi)} = \frac{1}{2} \quad \text{and} \quad k = m\omega^2$$

$$\therefore \qquad \bar{E} = \frac{1}{4} mA^2q^2 + \frac{1}{4} m\omega^2A^2$$

Or
$$\bar{E} = \frac{mA^2}{4} (q^2 + \omega^2)$$

- The average rate of dissipation of energy is

$$\overline{R\left(\frac{dx}{dt}\right)^2} = \frac{RA^2q^2}{2}$$

- The dissipation of energy over one cycle is

$$T\,\overline{R\left(\frac{dx}{dt}\right)^2} = T \frac{RA^2q^2}{2}$$

- Thus, the quality factor is

$$Q = 2\pi \frac{\frac{mA^2}{4} (q^2 + \omega^2)}{T \frac{RA^2q^2}{2}}$$

Or
$$Q = \frac{2\pi}{T} \cdot \frac{m}{2R} \left(\frac{\omega^2}{q^2} + 1\right)$$

Since $q = \frac{2\pi}{T}$, therefore,

$$Q = \frac{mq}{2R}\left(\frac{\omega^2}{q^2} + 1\right) \qquad \ldots (3.16)$$

As a special case, at resonance ($q \approx \omega$), we have

$$Q = \frac{m\omega}{R}$$

3.8 Power and Band Width

- During the forced oscillations the energy is continuously supplied by the external periodic force. The rate of supply of energy is called power. The power is given by

$$P = \text{Force} \times \text{Velocity}$$

$$P = f_o \sin qt \cdot \frac{dx}{dt}$$

The average power absorbed by the oscillator is

$$\bar{P} = \overline{f_o \sin qt \cdot \frac{dx}{dt}} \qquad \ldots (3.17)$$

From equation (3.14),

$$\overline{f_o \sin qt \cdot \frac{dx}{dt}} = \frac{RA^2q^2}{2}$$

$$\therefore \qquad \bar{P} = \frac{RA^2q^2}{2}$$

We have,

$$A = \frac{f_o}{m\sqrt{(\omega^2 - q^2)^2 + \frac{R^2q^2}{m^2}}}$$

$$\therefore \qquad \bar{P} = \frac{Rq^2f_o^2}{2m^2\left[(\omega^2 - q^2)^2 + \frac{R^2q^2}{m^2}\right]}$$

Or

$$\bar{P} = \frac{Rf_o^2}{2m^2\left[\left(\frac{\omega^2 - q^2}{q}\right)^2 + \frac{R^2}{m^2}\right]} \qquad \ldots (3.18)$$

- Thus, $\bar{P}$ is a function of q. As the difference $|\omega - q|$ decreases, the power increases. The power $\bar{P}$ is maximum at $q = \omega$. $\bar{P}$ decreases symmetrically as $|\omega - q|$ increases. The variation is shown in Fig. 3.9. The power is maximum at $q = \omega$ and from equation (3.18), we get

$$\bar{P}_{max} = \frac{f_o^2}{2R} \qquad \ldots (3.19)$$

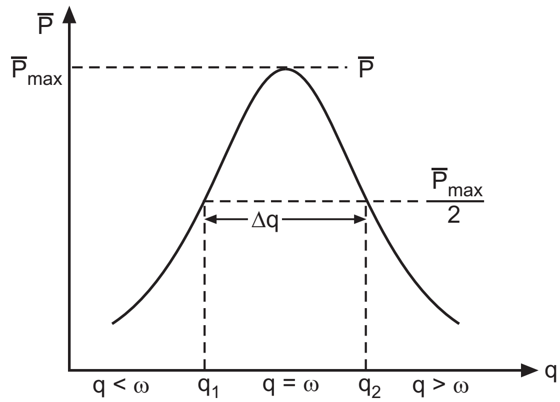


Fig. 3.9 : Power resonance curve

- Band width Δq is defined as the difference in angular frequencies of applied force for which power falls from $\overline{P}_{max}$ to $\frac{\overline{P}_{max}}{2}$. Let q_1 and q_2 be the corresponding frequencies as shown in Fig. 3.9. Then the band width is

$$\Delta q = q_2 - q_1$$

We can write, $$\Delta q = (q_2 - \omega) + (\omega - q_1)$$

Since the curve is symmetric about $q = \omega$, we have $(q_2 - \omega) = (\omega - q_1) = \frac{\Delta q}{2}$

If $\omega = \frac{q_1 + q_2}{2}$, we get

$$\Delta q = 2(\omega - q_1)$$

Or $$(\omega - q_1) = \frac{\Delta q}{2}$$

For sharp resonance, $q_1 \approx \omega$ and $(\omega + q_1) = 2q_1$

$\therefore$ $$(\omega^2 - q_1^2) = (\omega - q_1)(\omega + q_1)$$

Or $$(\omega^2 - q_1^2) = q_1 \Delta q \qquad \text{... (3.20)}$$

The power or average power at $q = q_1$ is

$$\overline{P} = \frac{Rf_o^2}{2m^2\left[\left(\frac{\omega^2 - q_1^2}{q_1}\right)^2 + \frac{R^2}{m^2}\right]}$$

Or $$\overline{P} = \frac{Rf_o^2}{2m^2\left[(\Delta q)^2 + \frac{R^2}{m^2}\right]} \qquad \text{... (3.21)}$$

But at $q = q_1$, $\bar{P} = \frac{\bar{P}_{max}}{2}$, therefore,

$$\frac{\bar{P}_{max}}{2} = \frac{Rf_o^2}{2m^2\left[(\Delta q)^2 + \frac{R^2}{m^2}\right]}$$

Using equation (3.19) in the above equation, we get

$$\frac{f_o^2}{4R} = \frac{Rf_o^2}{2m^2\left[(\Delta q)^2 + \frac{R^2}{m^2}\right]} \quad \text{Or} \quad \frac{f_o^2}{4R^2} = \frac{f_o^2}{2m^2\left[(\Delta q)^2 + \frac{R^2}{m^2}\right]}$$

$$\therefore \quad 4R^2 = 2m^2\left[(\Delta q)^2 + \frac{R^2}{m^2}\right]$$

This equation gives $\Delta q = \frac{R}{m}$ ... (3.22)

- Thus, the band width is directly proportional to the damping coefficient and inversely to the mass of oscillator.

 The quality factor is given by

$$Q = \frac{m\omega}{R}$$

$$\therefore \quad Q = \frac{\omega}{\Delta q} \qquad \text{... (3.23)}$$

Thus, Band width = $\frac{\text{Resonant angular frequency}}{\text{Quality factor}}$

3.9 Application of Forced Oscillations in LCR Series Circuit

- Fig. 3.10 shows an electrical circuit containing a capacitor of capacitance C, inductor of inductance L, and resistor of resistance R in series with an alternating e.m.f. $E = E_o \sin \omega' t$, where E_o is peak value of e.m.f. and ω' is its angular frequency. Let ω be the natural angular frequency of the circuit.

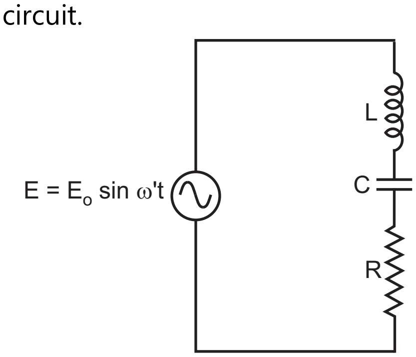


Fig. 3.10 : LCR series circuit

- Let I be the instantaneous current flowing through the circuit and 'q' be the charge on the capacitor at that instant.

3.11 Problems

Problem 1 : *The equation of forced oscillations is given by*

$$2\left(\frac{d^2x}{dt^2}\right) + 3\left(\frac{dx}{dt}\right) + 16\,x = 30 \sin 2t$$

All quantities are expressed in CGS units.

Find the velocity amplitude and the maximum kinetic energy.

Solution : From given equation, we get,

$m = 2$ g, $R = 3$ dyne/cm-s^{-1}, $k = 16$ dyne/cm, $f_o = 30$ dyne and $q = 2$ rad/s

The displacement in forced oscillations is given as

$$x = A \sin (qt - \phi)$$

and velocity is $$v = \frac{dx}{dt} = Aq \cos (qt - \phi)$$

The velocity amplitude is $$v_o = Aq = \frac{f_o\, q}{\sqrt{(k - mq^2)^2 + R^2q^2}}$$

$$= \frac{30 \times 2}{\sqrt{(16 - 2 \times 2^2)^2 + 3^2 \times 2^2}} = \frac{60}{\sqrt{(16 - 8)^2 + 9 \times 4}}$$

$$= \frac{60}{\sqrt{8^2 + 36}} = \frac{60}{\sqrt{100}} = \frac{60}{10}$$

$$= \mathbf{6\ cm/s}$$

The maximum kinetic energy is

$$K.E. = \frac{1}{2} m v_o^2 = \frac{1}{2} \times 2 \times 6^2$$

$$= \mathbf{36\ ergs}$$

Problem 2 : *In the case of forced oscillator, the amplitude of oscillations increases from 1 mm at very low frequencies to 100 mm at the amplitude resonant angular frequency. What is the quality factor of the forced oscillator ?*

Solution : The amplitude of forced oscillations is given by

$$A = \frac{f_o}{\sqrt{(k - mq^2)^2 + R^2q^2}}$$

At very low frequency, $q = 0$ and $A = A_0$

$$\therefore \qquad A_0 = \frac{f_o}{k} = \frac{f_o}{m\omega^2}$$

At resonance, the amplitude is maximum and it is given as,

$$A_{max} = \frac{f_o}{\omega R}$$

$$\therefore \qquad \frac{A_{max}}{A_0} = \frac{f_o / \omega R}{f_o / m\omega^2} = \frac{m\omega}{R}$$

We have quality factor, $Q = \frac{m\omega}{R}$

$\therefore$ $Q = \frac{A_{max}}{A_0} = \frac{100}{1} = \mathbf{100}$

Problem 3 : *The equation of forced oscillations of an oscillator is given as*

$$4\left(\frac{d^2x}{dt^2}\right) + 3\left(\frac{dx}{dt}\right) + 36\,x = 2.7 \sin 3t$$

where all quantities are expressed in SI units.

Determine the amplitude and phase difference between the periodic force.

Solution : From the given equation, we get

$m = 4$ kg, $R = 3$ N/m-s^2, $k = 36$ N/m, $f_o = 2.7$ N and $q = 3$ rad/s

The amplitude of forced oscillations is given by

$$A = \frac{f_o}{\sqrt{(k - mq^2)^2 + R^2q^2}}$$

$$\therefore \quad A = \frac{2.7}{\sqrt{(36 - 4 \times 3^2)^2 + 3^2 \times 3^2}} = \frac{2.7}{\sqrt{(36 - 36)^2 + 9 \times 9}}$$

$$= \frac{2.7}{9}$$

$$\therefore \quad A = \mathbf{0.3\ m}$$

The phase difference is given by

$$\tan\phi = \frac{qR}{k - mq^2} = \frac{3 \times 3}{36 - 4 \times 3^2} = \frac{9}{0} = \infty$$

$$\therefore \quad \phi = \boldsymbol{\frac{\pi}{2}}$$

Problem 4 : *The equation of forced oscillations of an oscillator is given as*

$$4\left(\frac{d^2x}{dt^2}\right) + 2\left(\frac{dx}{dt}\right) + 144\,x = 25 \sin qt$$

Determine the resonant frequency at which velocity resonance takes place. Also determine quality factor at resonance and half width.

Solution : From the given equation,

$m = 4$ units, $R = 2$ units, $k = 144$ units, $f_o = 25$ units

(i) The velocity resonance takes place when forcing frequency equals the natural frequency i.e. $q = \omega$.

We have, $\omega = \sqrt{\frac{k}{m}}$

$$\therefore \quad q = \sqrt{\frac{k}{m}} = \sqrt{\frac{144}{4}} = \sqrt{36} = \mathbf{6\ units}$$

(ii) The quality factor at resonance is

$$Q = \frac{m\omega}{R} = \frac{4 \times 6}{2} = \mathbf{12} \qquad (\because \omega = q = 6 \text{ units})$$

(iii) Half width is given by

$$\Delta\omega = \frac{R}{2m} = \frac{2}{2 \times 4} = \mathbf{0.25\ units}$$

Problem 5 : *An alternating e.m.f. of peak value 200 volt is applied across the series combination of an inductor of inductance 20 mH, a capacitor of capacitance 2 μF and resistance of 50 Ω. Determine resonant frequency, quality factor and band width.*

Solution : Given : $L = 20$ mH $= 20 \times 10^{-3}$ H, $C = 2\ \mu$F $= 2 \times 10^{-6}$ F, $R = 50\ \Omega$

(i) The resonant angular frequency is given by

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{20 \times 10^{-3} \times 2 \times 10^{-6}}}$$

$$= \frac{1}{\sqrt{4 \times 10^{-8}}}$$

$$= \frac{1}{2 \times 10^{-4}} = 0.5 \times 10^{4} \text{ rad/sec}$$

Or $\omega_o = \mathbf{5000\ rad/sec}$

The resonant frequency is

$$f_r = \frac{\omega_o}{2\pi} = \frac{5000}{2 \times 3.14} = \mathbf{796.18\ Hz}$$

(ii) The quality factor is given as

$$Q = \frac{\omega_o L}{R} = \frac{5000 \times 20 \times 10^{-3}}{50} = \mathbf{2}$$

(iii) $$\text{Band width} = \frac{\text{Resonant angular frequency}}{\text{Quality factor}}$$

$$= \frac{5000}{2} = \mathbf{2500\ rad/sec}$$

Problem 6 : *The equation of forced oscillations of an oscillator is given as*

$$4\left(\frac{d^2x}{dt^2}\right) + 3\left(\frac{dx}{dt}\right) + 36\,x = 2.7 \sin 3t$$

where all quantities are expressed in SI units.

Determine the average power absorbed by the oscillator.

Solution : From the given equation, we get

$m = 4$ kg, $R = 3$ N/m-s^{-1}, $k = 36$ N/m, $f_o = 2.7$ N and $q = 3$ rad/sec

The average power absorbed by the oscillator is given as

$$\bar{P} = \frac{RA^2 q^2}{2}$$

where $$A = \frac{f_o}{\sqrt{(k - mq^2)^2 + R^2 q^2}}$$

$$\therefore \quad A = \frac{2.7}{\sqrt{(36 - 4 \times 3^2)^2 + 3^2 \times 3^2}}$$

$$= \frac{2.7}{\sqrt{(36 - 36)^2 + 9 \times 9}} = \frac{2.7}{9} = 0.3 \text{ m}$$

$$\therefore \quad \bar{P} = \frac{RA^2 q^2}{2} = \frac{3 \times (0.3)^2 \times 3^2}{2} = \mathbf{1.215\ J/sec}$$

Summary

1. If some external periodic force is constantly applied to a damped harmonic oscillator, it oscillates with constant amplitude. These oscillations are called forced oscillations.
2. The amplitude of forced oscillations is given as

$$A = \frac{f_o}{m\sqrt{(\omega^2 - q^2)^2 + \frac{R^2 q^2}{m^2}}}$$

   where ω is the natural angular frequency and q is the forcing angular frequency.
3. When angular frequency of applied force is equal to natural angular frequency of oscillator, the oscillator oscillates with maximum amplitude. This phenomenon is called **resonance**.
4. At resonance, the phase difference between displacement and applied force is $\pi/2$.
5. Response of the oscillator to the applied periodic force is maximum at resonance.
6. The half width of the resonance is given by

$$\Delta\omega = \frac{R}{2m}$$

   The sharpness of resonance is decided by the half width. When half width is small, resonance is sharp and when half width is large, resonance is flat.
7. In forced oscillations, the average energy over one cycle of oscillation is constant i.e. the rate of absorption of energy is equal to rate of dissipation of energy.
8. The bandwidth of power resonance is the difference in angular frequencies of applied force for which power falls from $\bar{P}_{max}$ to $\frac{\bar{P}_{max}}{2}$. It is given as

$$\Delta q = \frac{R}{m}$$