Learning Deterministic One-Counter Automata

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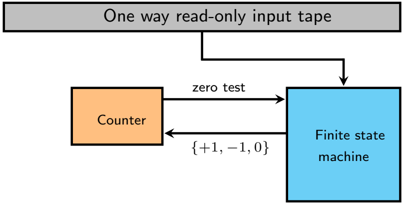
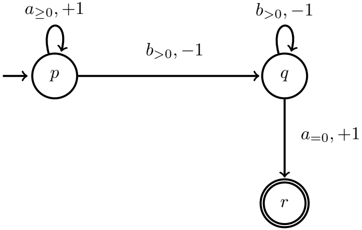
RP 2025

One-counter automata

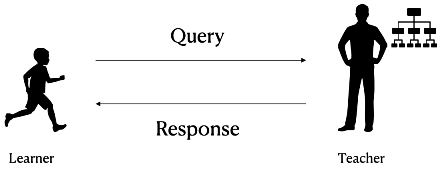


Counter: Can be incremented, decremented or tested for zero.



OCA accepting $\{a^n b^n a \mid n \geq 0\}$.

Active learning framework



- Teacher knows the language of a DOCA $\mathcal{T}$.
- Learner's aim: Learn a DOCA $\mathcal{L}$ equivalent to $\mathcal{T}$.
- Learner can query Teacher.
- Learner uses Teacher's response to learn.

Queries

- The Teacher knows the language of a DOCA $\mathcal{T}$.
- Learner's aim: Learn a DOCA $\mathcal{L}$ equivalent to $\mathcal{T}$.

Membership query

Learner: Is w in the language of $\mathcal{T}$?

Teacher: Yes or No.

Minimal-equivalence query

Learner: Is DOCA $\mathcal{L}$ equivalent to $\mathcal{T}$?

Teacher: Yes or "No and a minimal word w that distinguishes $\mathcal{L}$ and $\mathcal{T}$".

Literature review: Active learning of DOCA

Angluin, 1987	DFA – Membership, Equivalence P
Fahmy & Roos, 1995	DROCA[1] – Membership, Min. Equivalence EXPTIME
Neider & Löding, 2010	VOCA[2] – Membership, Min. Equivalence EXPTIME
Bruyère, Pérez & Staquet, 2022	DROCA[1] – Membership, Min. Equivalence, Counter Value EXPTIME
M., Penelle & Sreejith, 2025	DROCA[1] – Membership, Min. Equivalence, Counter Value EXPTIME, Polynomial queries
M., Penelle & Sreejith, 2025	DOCA – Membership, Min. Equivalence P

[1] Realtime DOCA: strict subclass of DOCA

[2] VOCA: Visibly OCA

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M., Penelle & Sreejith, 2025 → DOCA – Membership, Min. Equivalence
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The OL^* Algorithm

OL^* - Active learning of DOCA

Theorem (OL^* is in P)

- *Let Teacher knows a language recognised by a DOCA $\mathcal{T}$.*
- *The OL^* algorithm learns a DOCA $\mathcal{L}$ that is equivalent to $\mathcal{T}$ in time polynomial in $|\mathcal{T}|$, using membership and minimal-equivalence queries.*

Assumptions on DOCA

- The Teacher knows a language accepted by a DOCA $\mathcal{T}$.
  - We denote by $n = |\mathcal{T}|$, the number of states.
- To make the presentation simpler, we assume the following about $\mathcal{T}$:
  - There are no ε-transitions.
  - In a transition, the counter is incremented or decremented at most by one.
- Learner wants to learn a DOCA $\mathcal{L}$ equivalent to $\mathcal{T}$.

OL^*: Step 1. Learning k-behaviour DFA

- A DFA $\mathcal{A}$ is a k-**behaviour DFA** of $\mathcal{T}$ if $\mathcal{A}$ is k-equivalent to $\mathcal{T}$. That is,

$$w \text{ is accepted by } \mathcal{A} \quad \text{iff} \quad w \text{ is accepted by } \mathcal{T}, \quad \text{for all } |w| \leq k.$$

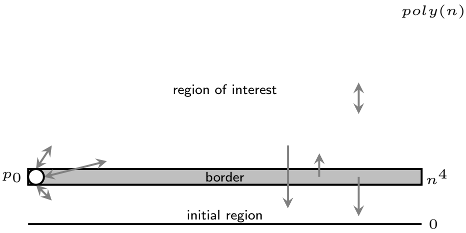
- Angluin's L^* algorithm can learn a k-behaviour DFA in time polynomial in k and n.
  - This is done using membership and minimal-equivalence queries.

- Step 1. of learner is to **learn a** $poly(n)$**-behaviour DFA**.

OL^*: Step 2. Partitioning the behaviour DFA

The DFA $\mathcal{A}$ is partitioned into:

- **Initial region**: States reachable by words of length $< n^4$.
- **Border region**: States reachable by words of length n^4 but not less.
- **Region of interest**: Remaining states.



- What next?
  - Learner constructs a partial OCA for all border states.
  - Combine all these partial OCA to get the final DOCA $\mathcal{A}$.

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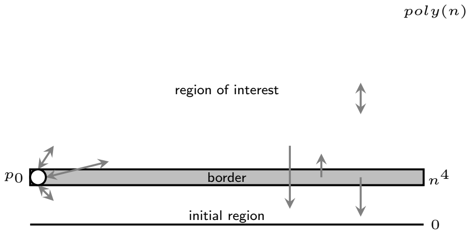

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OL^*: Step 3. Finding a winning sequence

Definition

We say $w_0, w_1, w_2, \ldots, w_K$ is a **winning sequence** for a state p if the run of these words on $\mathcal{T}$ reach configurations

$$(p, i),\ (p, i+d),\ (p, i+2d), \ldots, (p, i+Kd)$$

respectively, for some $d \leq n^2$ and $i > n^3$.

Lemma (Winning sequence lemma)

For any state p_0 in behaviour DFA $\mathcal{A}$, one can enumerate polynomially many sequences of the form $w_0, w_1, w_2, \ldots, w_K$ in polynomial time such that one of them is a winning sequence for p_0.

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OL^*: Step 4. Parallel BFS on $\mathcal{A}$

- Consider a *winning sequence*

$$w_0, w_1, w_2, \quad \ldots, \quad w_K.$$

- Run these words on the behaviour DFA. We reach the state sequence

$$p_0, p_1, p_2, \quad \ldots, \quad p_K.$$

- Run parallel BFS (depth at most n^3) from this sequence.
  - All distinct sequences identified.
  - At most n^3 distinct sequences.
  - These sequences are the states of DOCA $\mathcal{L}_{p_0}$.

Parallel BFS

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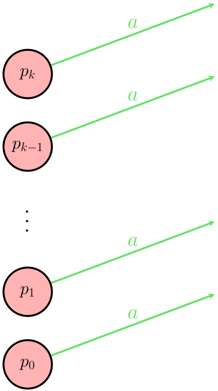

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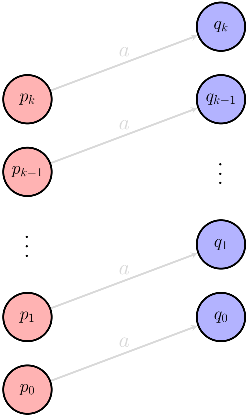

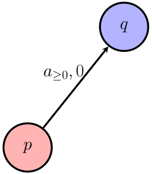

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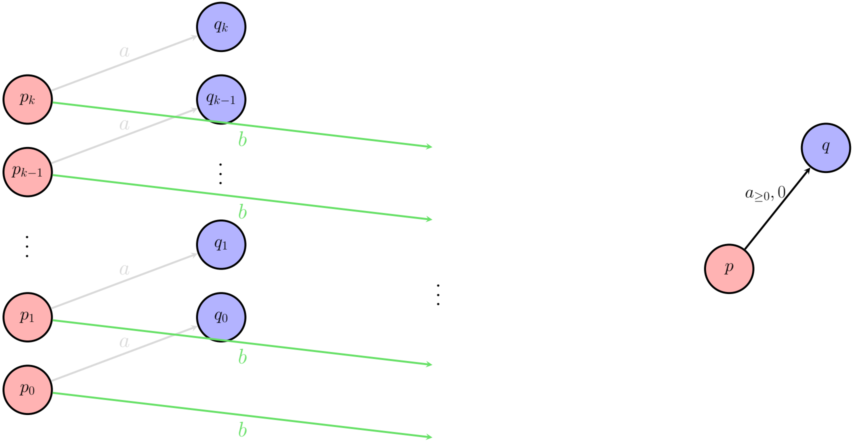

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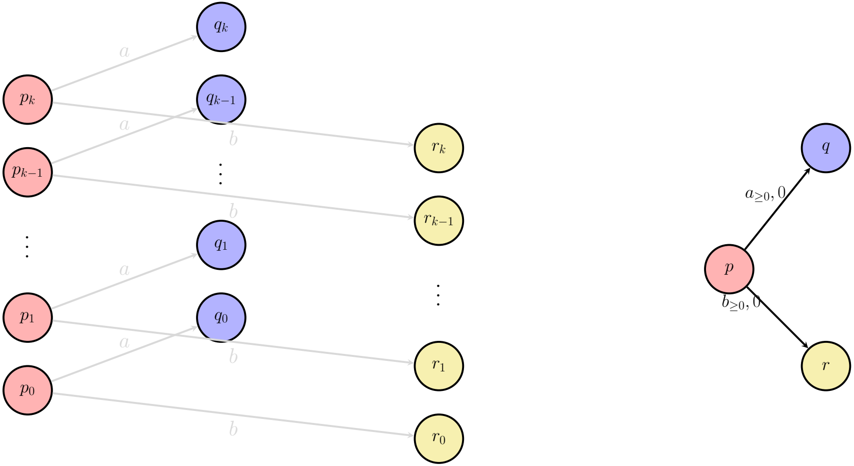

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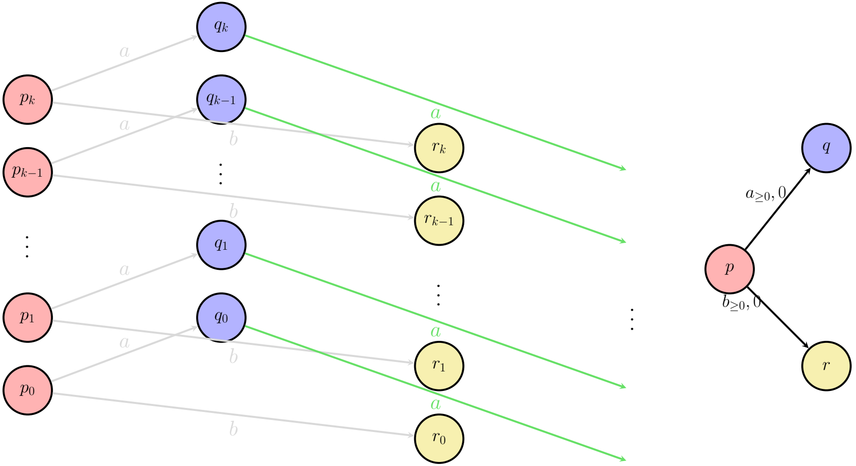

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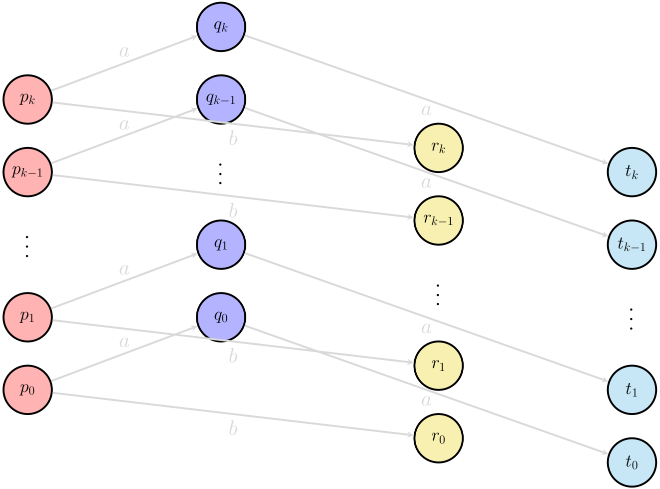

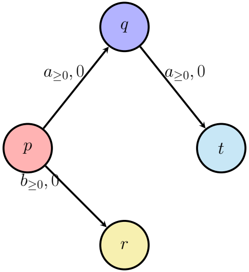

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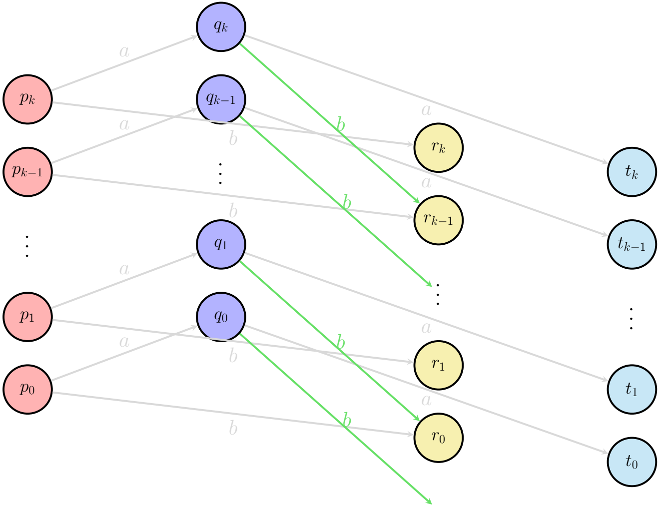

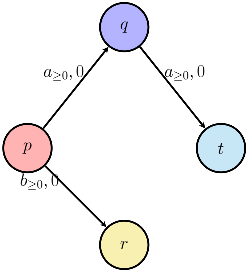

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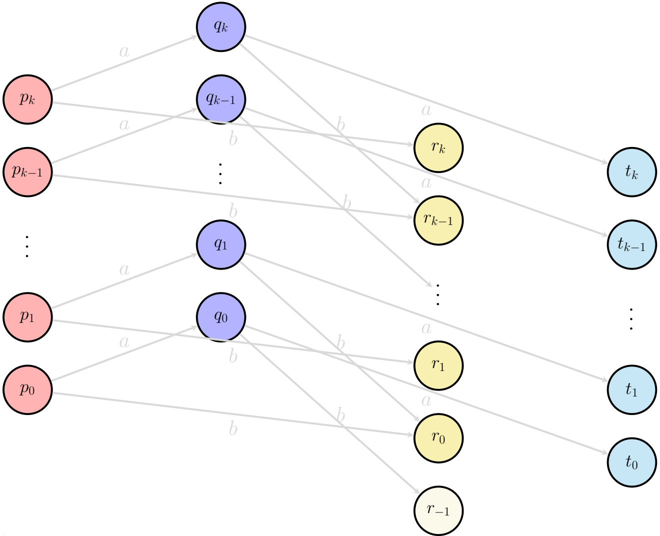

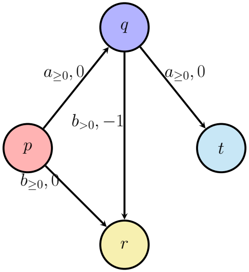

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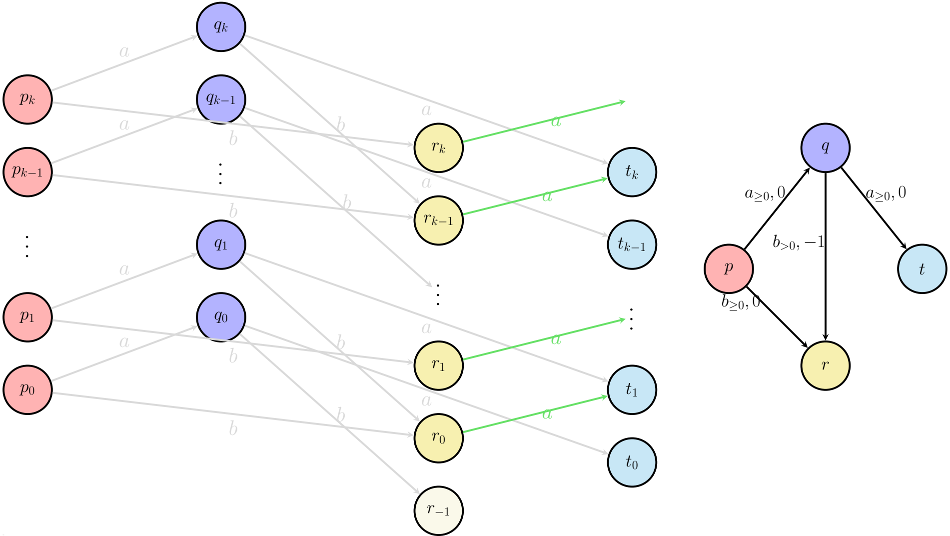

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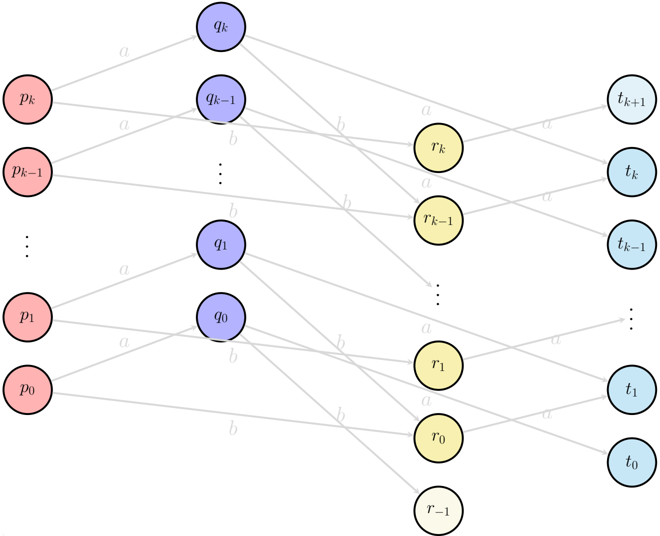

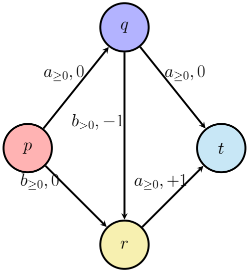

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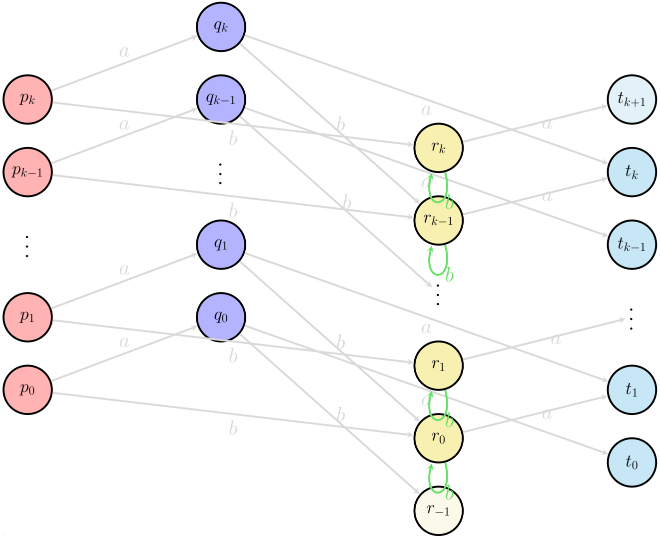

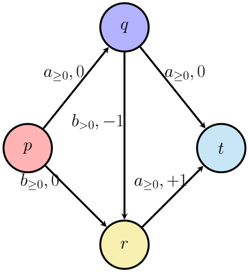

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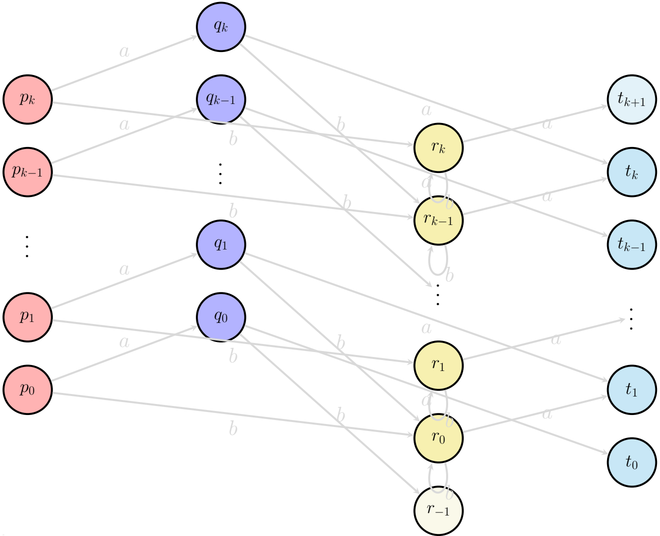

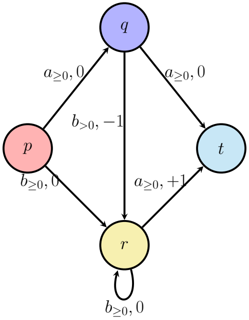

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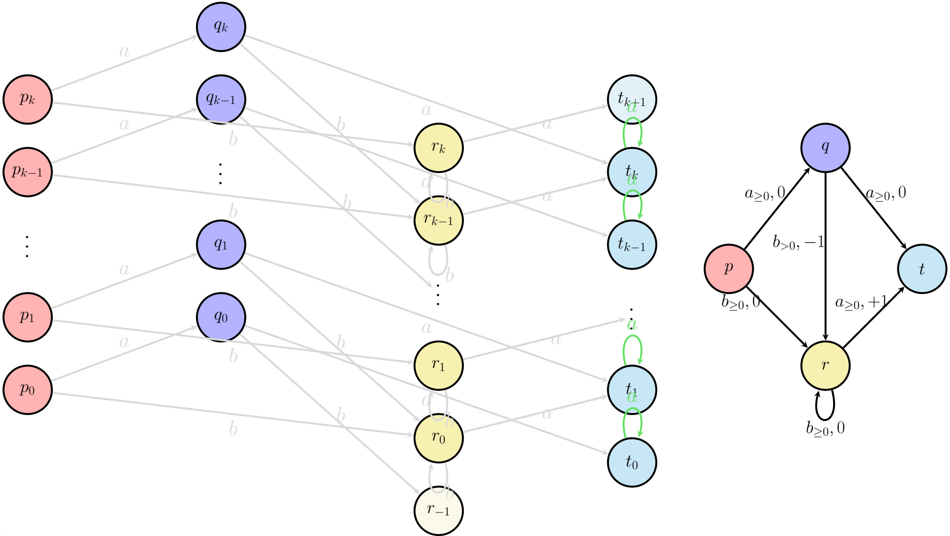

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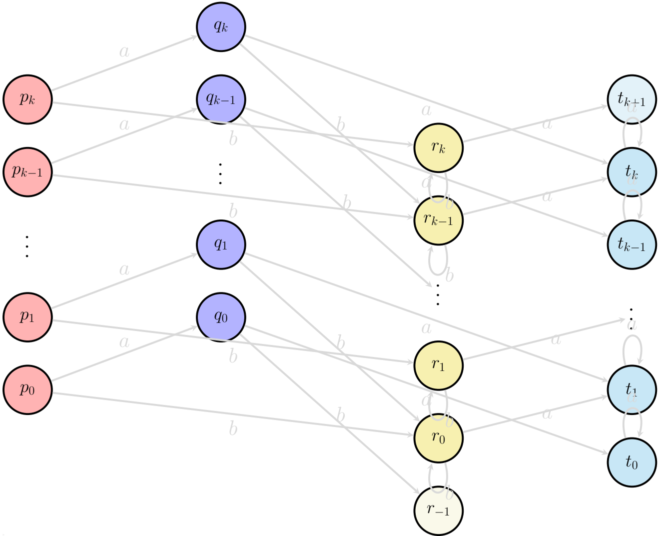

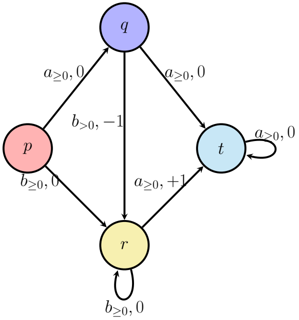

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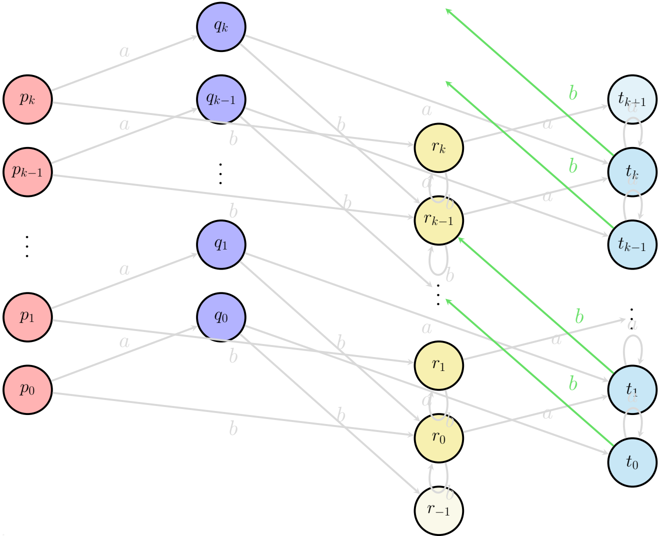

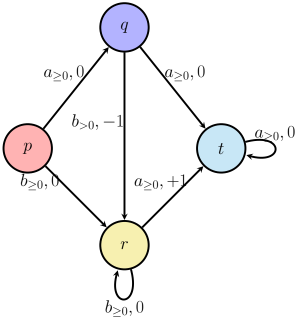

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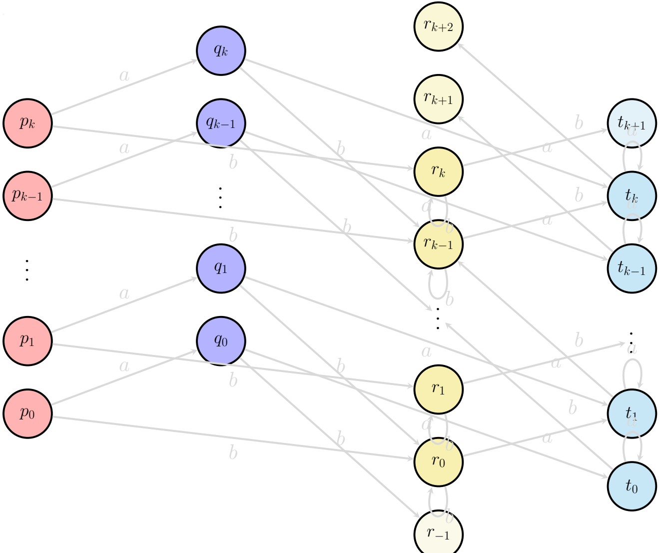

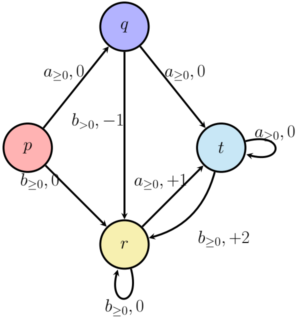

OL*: Step 5. Constructing $\mathcal{L}_{p_0}$

- The parallel BFS colors “most” of the reachable states from border state p.
- However, upto n^3 number of states are not colored.
  - eg. the state r_{-1} in the example, and some states reachable from r_{-1}.
- These are added to the partial OCA with zero counter value.
- The initial region is also added to the partial OCA with zero counter value.

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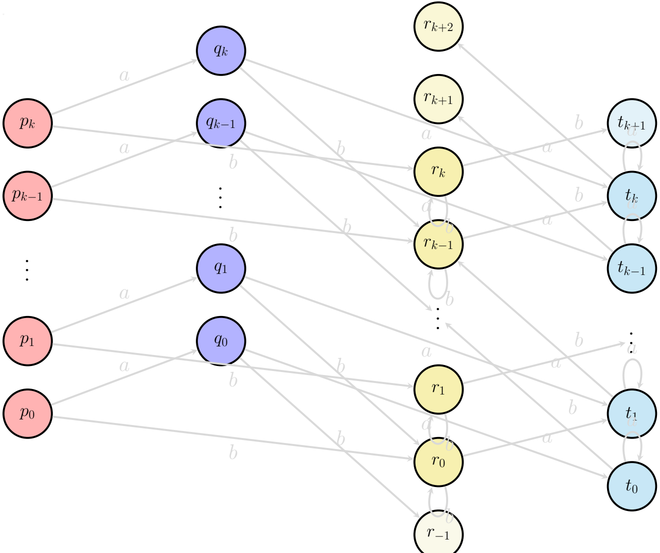

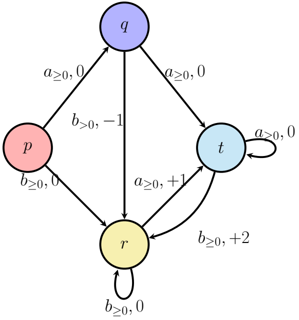

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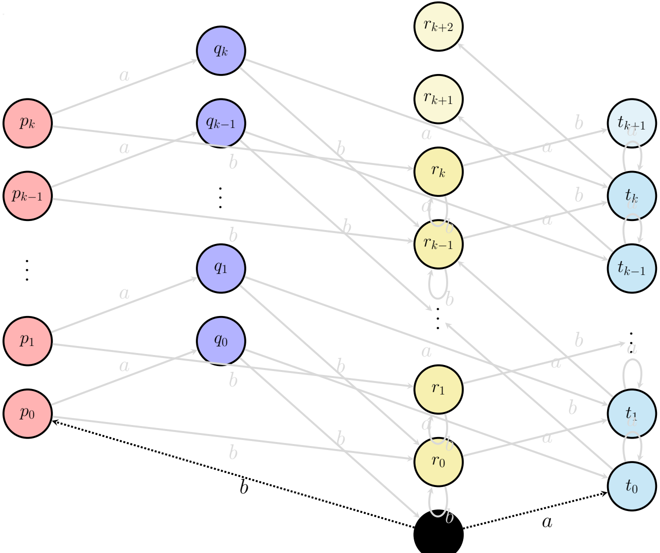

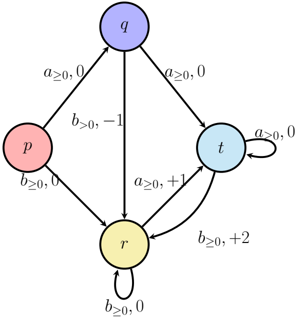

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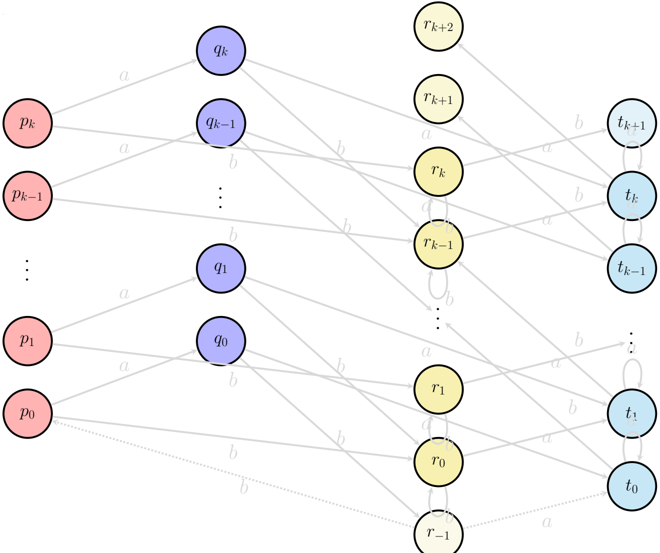

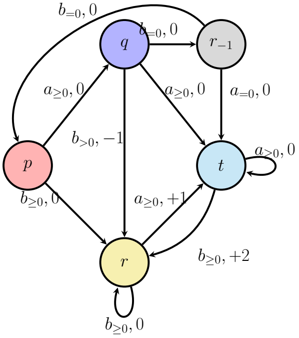

OL^*: Step 6. Constructing $\mathcal{L}$

- The final OCA $\mathcal{L}$ is the union of partial OCA corresponding to all border states and the DFA corresponding to the initial region.
- Size of $\mathcal{L}$ is $\mathcal{O}(n^8)$.
- There exists a $poly(n)$ such that if $\mathcal{L}$ is $poly(n)$ equivalent to $\mathcal{T}$, then $\mathcal{L}$ and $\mathcal{T}$ are equivalent (Böhm, Göller & Jančar, 2013).
- Correctness:

$\mathcal{L}$ is $poly(n)$-equivalent to $\mathcal{A}$	(Step 6. construction)
$\mathcal{A}$ is $poly(n)$-equivalent to $\mathcal{T}$	(Step 1. construction)
$\mathcal{L}$ is $poly(n)$-equivalent to $\mathcal{T}$	(from above)
$\mathcal{L}$ is equivalent to $\mathcal{T}$	(Böhm, Göller & Jančar, 2013)

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Summary of OL^*

- Construct $poly(n)$-behaviour DFA using L^* algorithm.
- Partition the behaviour DFA into *initial region*, *border*, and *region of interest*.
- For each border state:
  - Generate a winning sequence of words: $w_0, w_1, \ldots, w_K$.
  - Run these words on the DFA to get the sequence of states: $p_0, p_1, \ldots, p_K$.
  - Run parallel BFS from this sequence.
  - All reachable sequences of parallel BFS form states of partial OCA.
  - Counter values are incremented / decremented based on the shift in the sequence.
  - Add the missed states and initial region to get partial OCA.
- Construct the final OCA by combining the partial OCA.

Conclusion

Theorem

OL^* learns a DOCA equivalent to the Teacher's DOCA using membership and minimal-equivalence queries, and in time polynomial in the size of a smallest DOCA recognising the language.

Corollary

Polynomial approximation for minimisation of DOCA.

Future work

- Replacing minimal-equivalence with equivalence query.
- Practical OL^* algorithm.
- Improving running time of equivalence.
- Learning weighted models (like weighted visibly OCA).

Thank You!