EE17B115_LAB9

April 16, 2019

1 Assignment 9

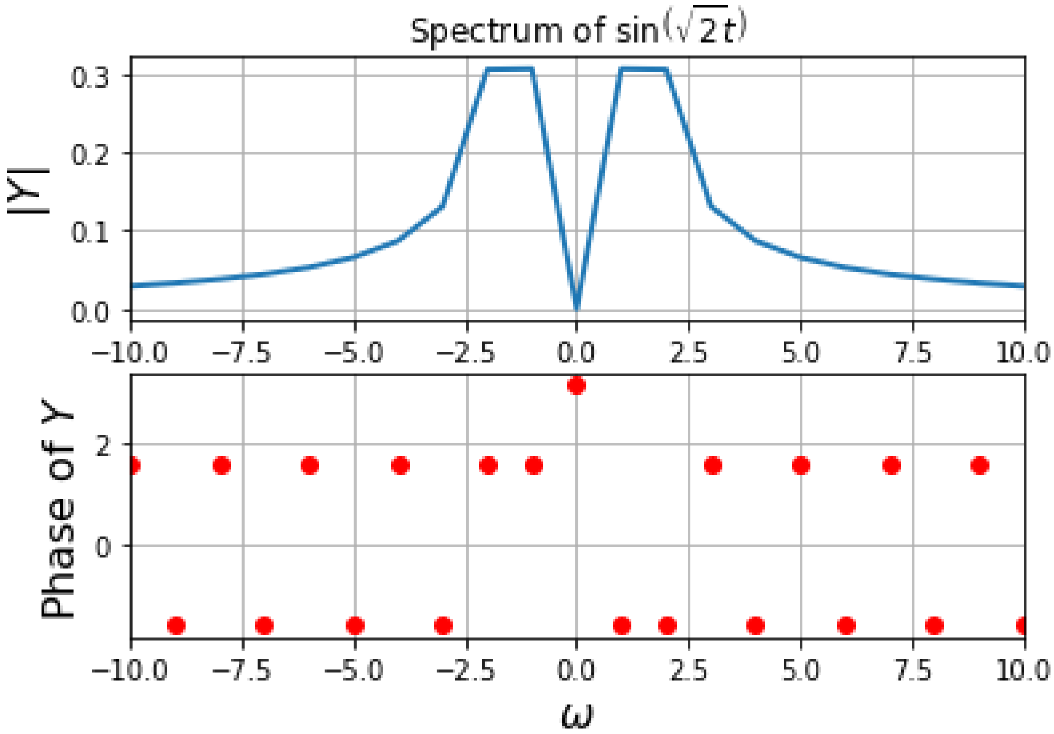
Let us perform FFT on functions which are discontinuous when periodically extended. An example of this is sin($\sqrt{2}$t). The discontiuity causes fourier components in frequencies other than the sinusoids frequency which decay as $1/w$, due to Gibbs phenomenon. We resolve this problem using the process of windowing. We use this windowed transform to analyse signals known to contain a sinusoid of unknown frequencies and extract its phase and frequency. We then perform a sliding DFT on a chirped signal and plot a spectrogram or a time-frequency plot.

```
In [ ]: from pylab import *
```

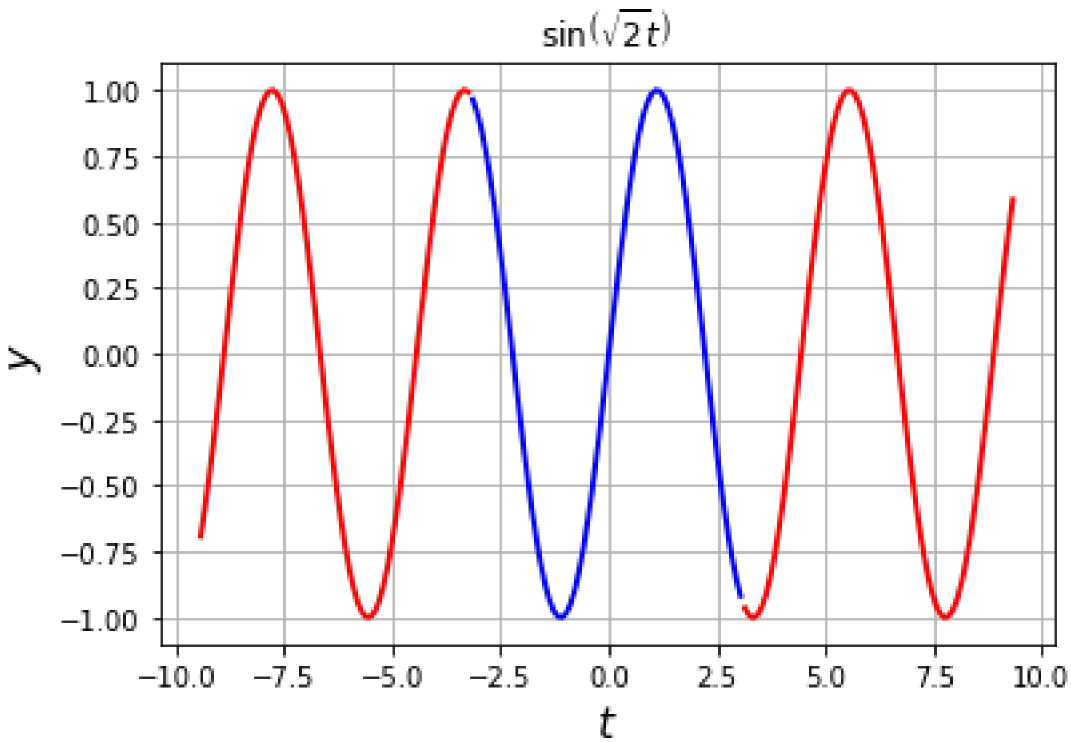
1.1 Question 1

1.1.1 Spectrum of sin($\sqrt{2}$t)

```
In [2]: t=linspace(-pi,pi,65);t=t[:-1]
        dt=t[1]-t[0];fmax=1/dt
        y=sin(sqrt(2)*t)
        y[0]=0 # the sample corresponding to -tmax should be set zeroo
        y=fftshift(y) # make y start with y(t=0)
        Y=fftshift(fft(y))/64.0
        w=linspace(-pi*fmax,pi*fmax,65);w=w[:-1]
        figure()
        subplot(2,1,1)
        plot(w,abs(Y),lw=2)
        xlim([-10,10])
        ylabel(r"$|Y|$",size=16)
        title(r"Spectrum of $\sin\left(\sqrt{2}t\right)$")
        grid(True)
        subplot(2,1,2)
        plot(w,angle(Y),'ro',lw=2)
        xlim([-10,10])
        ylabel(r"Phase of $Y$",size=16)
        xlabel(r"$\omega$",size=16)
        grid(True)
        show()
```

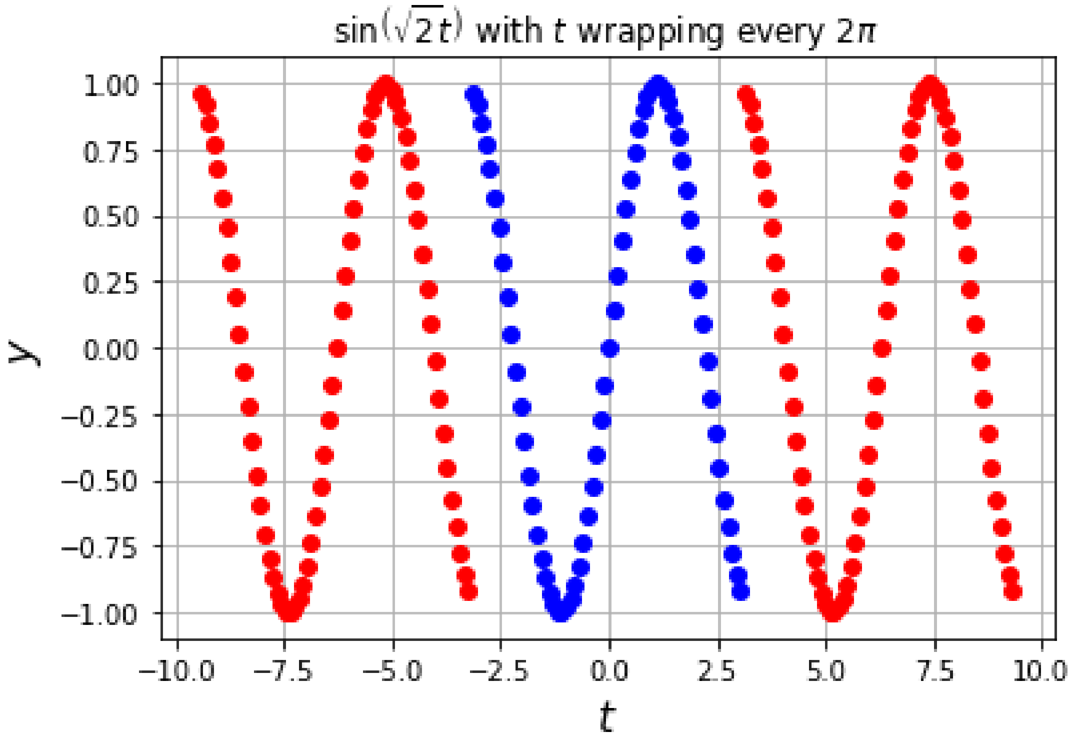



```
In [3]: t1=linspace(-pi,pi,65);t1=t1[:-1]
        t2=linspace(-3*pi,-pi,65);t2=t2[:-1]
        t3=linspace(pi,3*pi,65);t3=t3[:-1]
        # y=sin(sqrt(2)*t)
        figure(2)
        plot(t1,sin(sqrt(2)*t1),'b',lw=2)
        plot(t2,sin(sqrt(2)*t2),'r',lw=2)
        plot(t3,sin(sqrt(2)*t3),'r',lw=2)
        ylabel(r"$y$",size=16)
        xlabel(r"$t$",size=16)
        title(r"$\sin\left(\sqrt{2}t\right)$")
        grid(True)
        savefig("fig10-2.png")
        show()
```



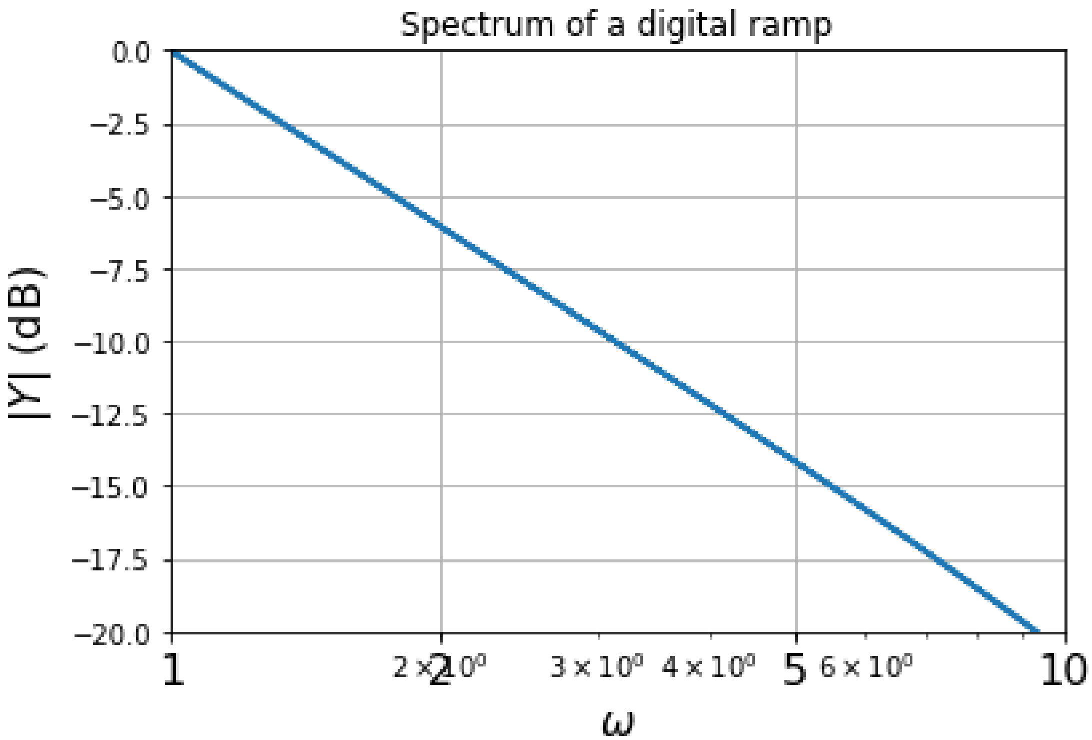
However, when we calculate the DFT by sampling over a finite time window, we end up calculating the DFT of the following periodic signal:

```
In [4]: t1=linspace(-np.pi,np.pi,65);t1=t1[:-1]
        t2=linspace(-3*np.pi,-np.pi,65);t2=t2[:-1]
        t3=linspace(np.pi,3*np.pi,65);t3=t3[:-1]
        t4=linspace(-3*np.pi,3*np.pi,);t4=t4[:-1]
        y=sin(sqrt(2)*t1)
        figure(3)
        plot(t1,y,'bo',lw=2)
        plot(t2,y,'ro',lw=2)
        plot(t3,y,'ro',lw=2)
        ylabel(r"$y$",size=16)
        xlabel(r"$t$",size=16)
        title(r"$\sin\left(\sqrt{2}t\right)$ with $t$ wrapping every $2\pi$ ")
        grid(True)
        show()
```



This results in discontinuites in the signal. These discontinuities lead to spectral components which decay as 1/. To confirm this, we plot the spectrum of the periodic ramp below:

```
In [5]: t=linspace(-pi,pi,65);t=t[:-1]
        dt=t[1]-t[0];fmax=1/dt
        y=t
        y[0]=0 # the sample corresponding to -tmax should be set zeroo
        y=fftshift(y) # make y start with y(t=0)
        Y=fftshift(fft(y))/64.0
        w=linspace(-pi*fmax,pi*fmax,65);w=w[:-1]
        figure()
        semilogx(abs(w),20*log10(abs(Y)),lw=2)
        xlim([1,10])
        ylim([-20,0])
        xticks([1,2,5,10],["1","2","5","10"],size=16)
        ylabel(r"$|Y|$ (dB)",size=16)
        title(r"Spectrum of a digital ramp")
        xlabel(r"$\omega$",size=16)
        grid(True)
        savefig("fig10-4.png")
        show()
```

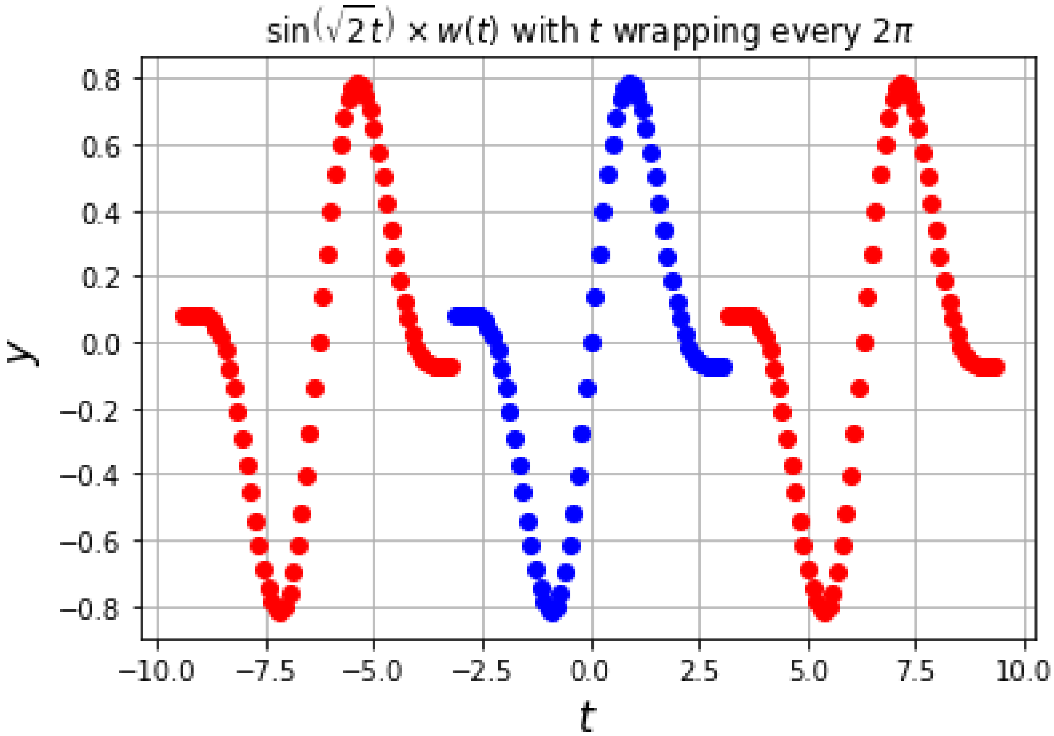


We can observe that the error in previous fft is due to the sudden jumps of input signal.

1.1.2 Hamming Window

We resolve the problem of discontinuities by attenuating the signal near the endpoints of our time window, to reduce the discontinuities caused by periodically extending the signal. This is done by multiplying by a so called windowing function.

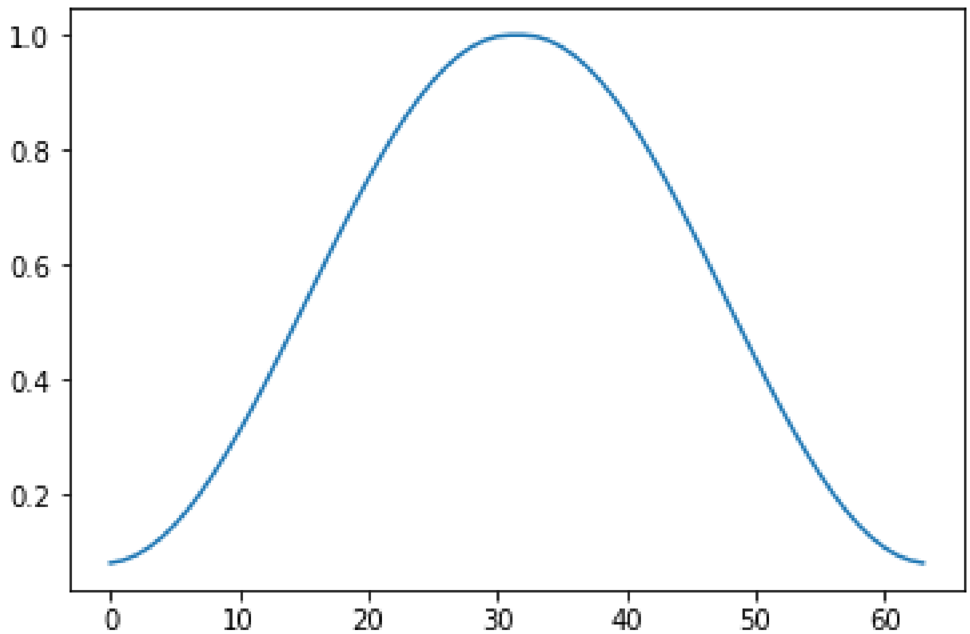
```
In [6]: t1=linspace(-pi,pi,65);t1=t1[:-1]
        t2=linspace(-3*pi,-pi,65);t2=t2[:-1]
        t3=linspace(pi,3*pi,65);t3=t3[:-1]
        n=arange(64)
        wnd=fftshift(0.54+0.46*cos(2*pi*n/63))
        y=sin(sqrt(2)*t1)*wnd
        figure(3)
        plot(t1,y,'bo',lw=2)
        plot(t2,y,'ro',lw=2)
        plot(t3,y,'ro',lw=2)
        ylabel(r"$y$",size=16)
        xlabel(r"$t$",size=16)
        title(r"$\sin\left(\sqrt{2}t\right)\times w(t)$ with $t$ wrapping every $2\pi$ ")
        grid(True)
        show()
```



This is the plot of our windowing function in time domain. We can clearly see that we are trying to attenuate at end points.

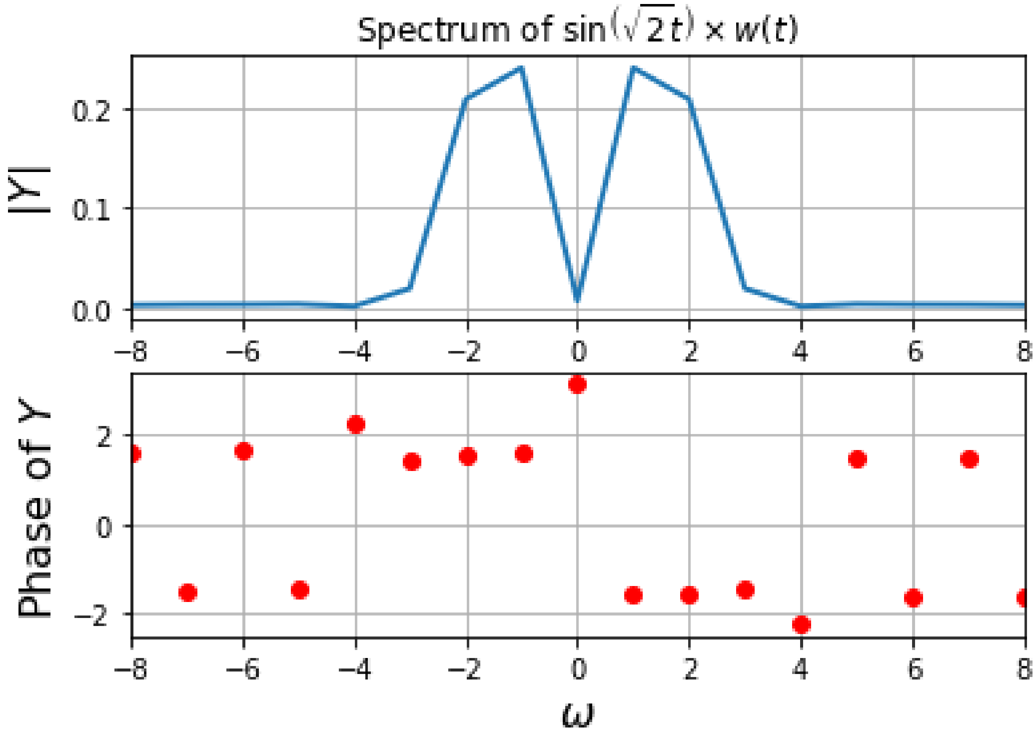
```
In [7]: plot(wnd)
```

```
Out[7]: [<matplotlib.lines.Line2D at 0x7fb04ddf8dd8>]
```



1.1.3 FFT after windowing

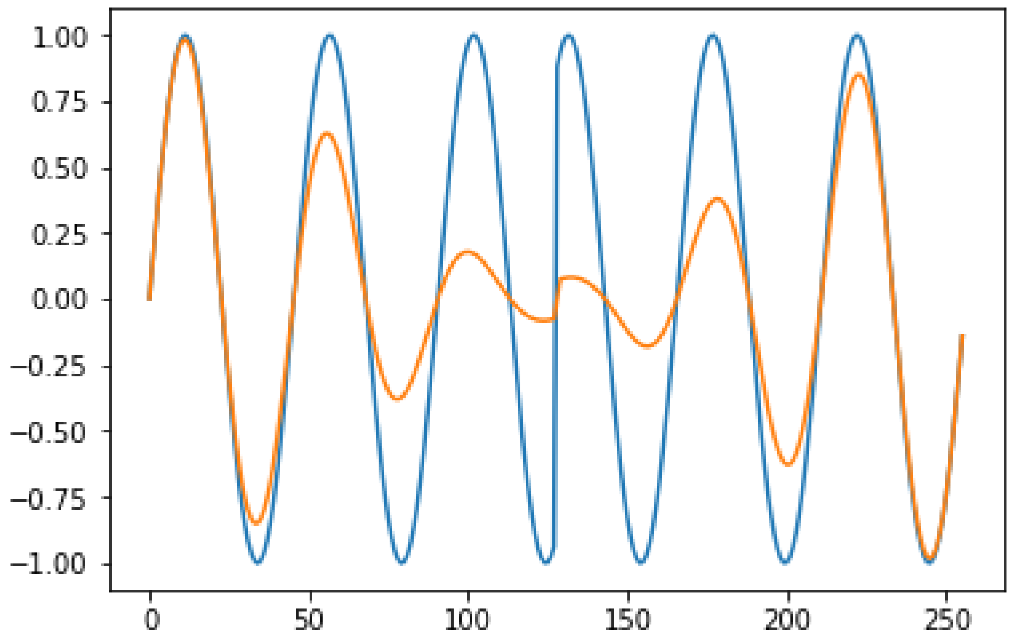
In [8]:
```
t=linspace(-pi,pi,65);t=t[:-1]
dt=t[1]-t[0];fmax=1/dt
n=arange(64)
wnd=fftshift(0.54+0.46*cos(2*pi*n/63))
y=sin(sqrt(2)*t)*wnd
y[0]=0 # the sample corresponding to -tmax should be set zeroo
y=fftshift(y) # make y start with y(t=0)
Y=fftshift(fft(y))/64.0
w=linspace(-pi*fmax,pi*fmax,65);w=w[:-1]
figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-8,8])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\sin\left(\sqrt{2}t\right)\times w(t)$")
grid(True)
subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
xlim([-8,8])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()
```

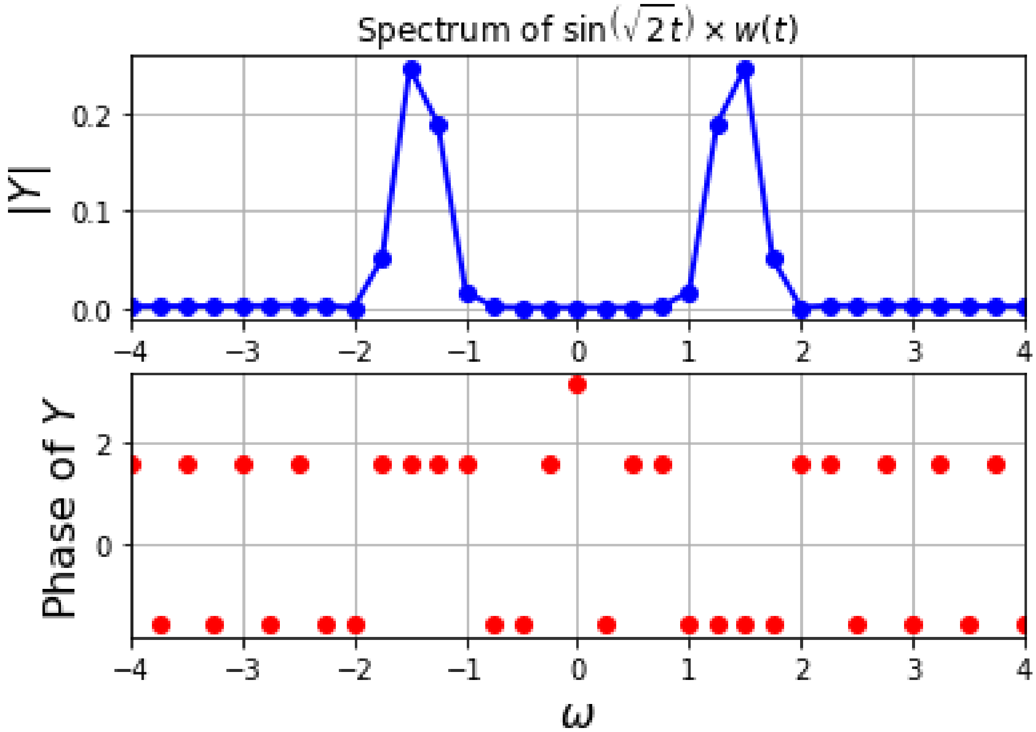


Let us make it crystal clear by increasing window size.

In [9]:
```
t=linspace(-4*pi,4*pi,257);t=t[:-1]
dt=t[1]-t[0];fmax=1/dt
n=arange(256)
wnd=fftshift(0.54+0.46*cos(2*pi*n/256))
y=sin(sqrt(2)*t)
plot(fftshift(y))
# y=sin(1.25*t)
y=y*wnd
y[0]=0 # the sample corresponding to -tmax should be set zeroo
y=fftshift(y) # make y start with y(t=0)
plot(y)
Y=fftshift(fft(y))/256.0
w=linspace(-pi*fmax,pi*fmax,257);w=w[:-1]
figure()
subplot(2,1,1)
plot(w,abs(Y),'b',w,abs(Y),'bo',lw=2)
xlim([-4,4])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\sin\left(\sqrt{2}t\right)\times w(t)$")
grid(True)
subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
```

```
xlim([-4,4])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()
```



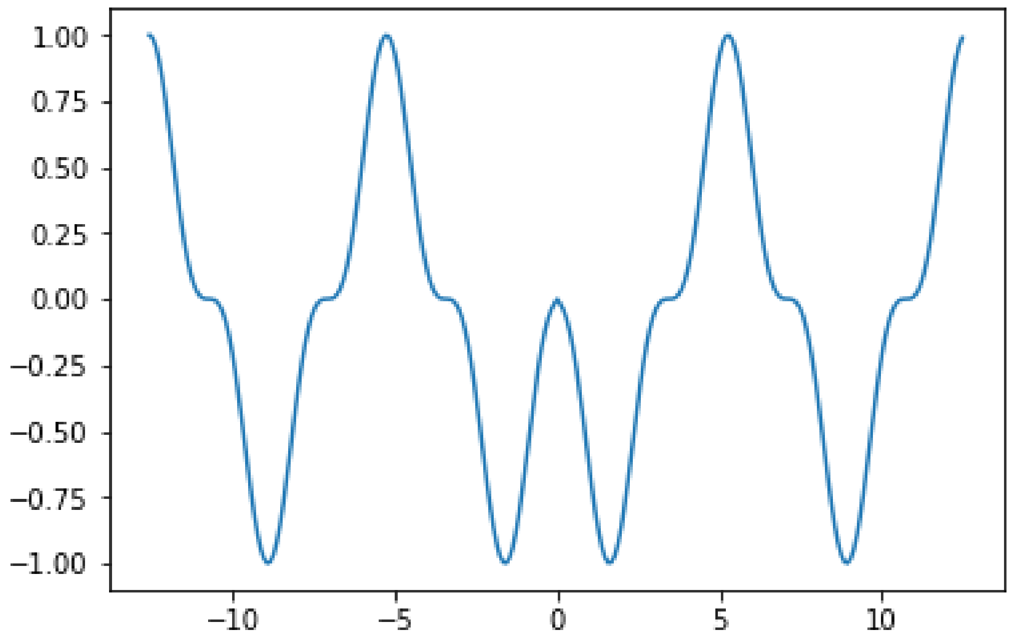


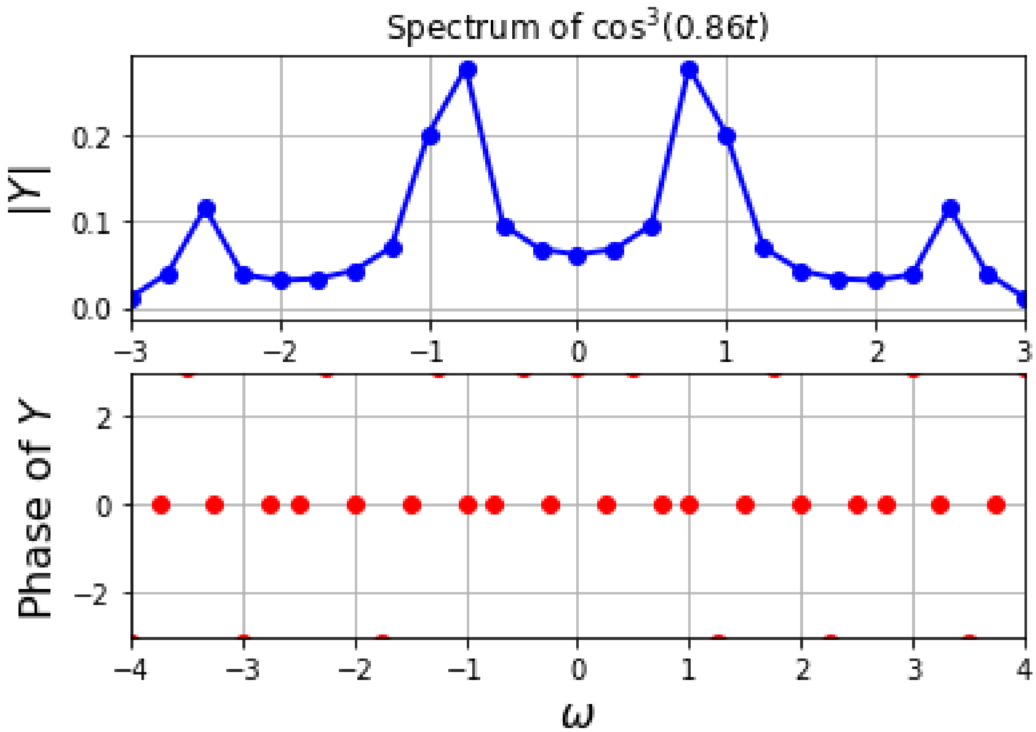
1.2 Question 2

1.2.1 Spectrum of $cos^3(0.86t)$ without windowing

In [10]:
```
t=linspace(-4*pi,4*pi,257);t=t[:-1]
dt=t[1]-t[0];fmax=1/dt
y=(cos(0.86*t))**3
y[0]=0 # the sample corresponding to -tmax should be set zeroo
y=fftshift(y)# make y start with y(t=0)
plot(t,y)
Y=fftshift(fft(y))/256.0
w=linspace(-pi*fmax,pi*fmax,257);w=w[:-1]
figure()
subplot(2,1,1)
plot(w,abs(Y),'b',w,abs(Y),'bo',lw=2)
ylabel(r"$|Y|$",size=16)
xlim([-3,3])
title(r"Spectrum of $\cos^3(0.86t)$")
grid(True)
subplot(2,1,2)
i=where(abs(Y)>1e-3)
plot(w[i],((angle(Y[i]))),'ro',lw=2)
xlim([-4,4])
```

```
ylim([-3,3])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()
```

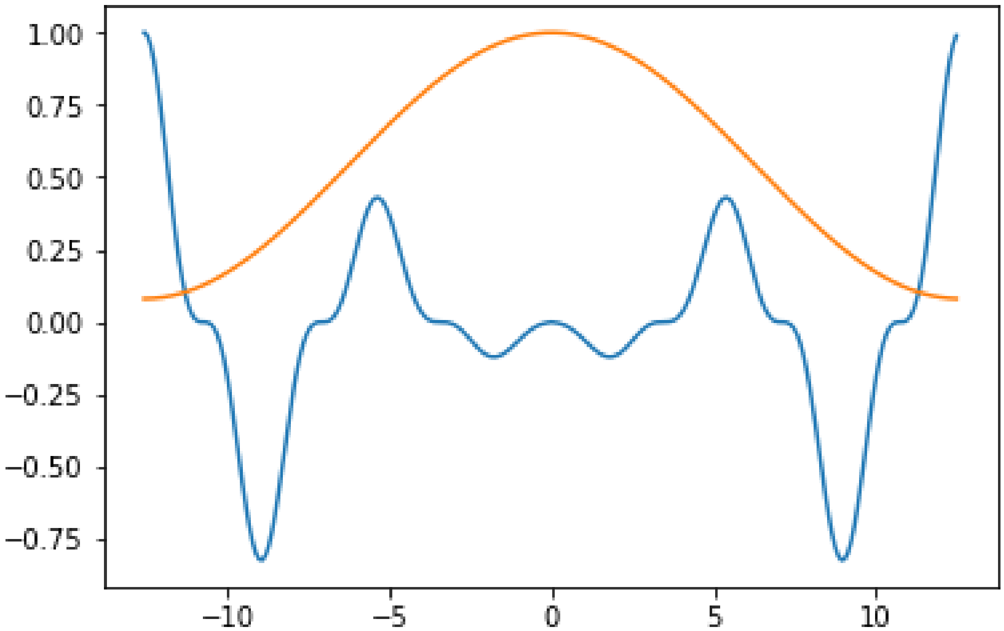


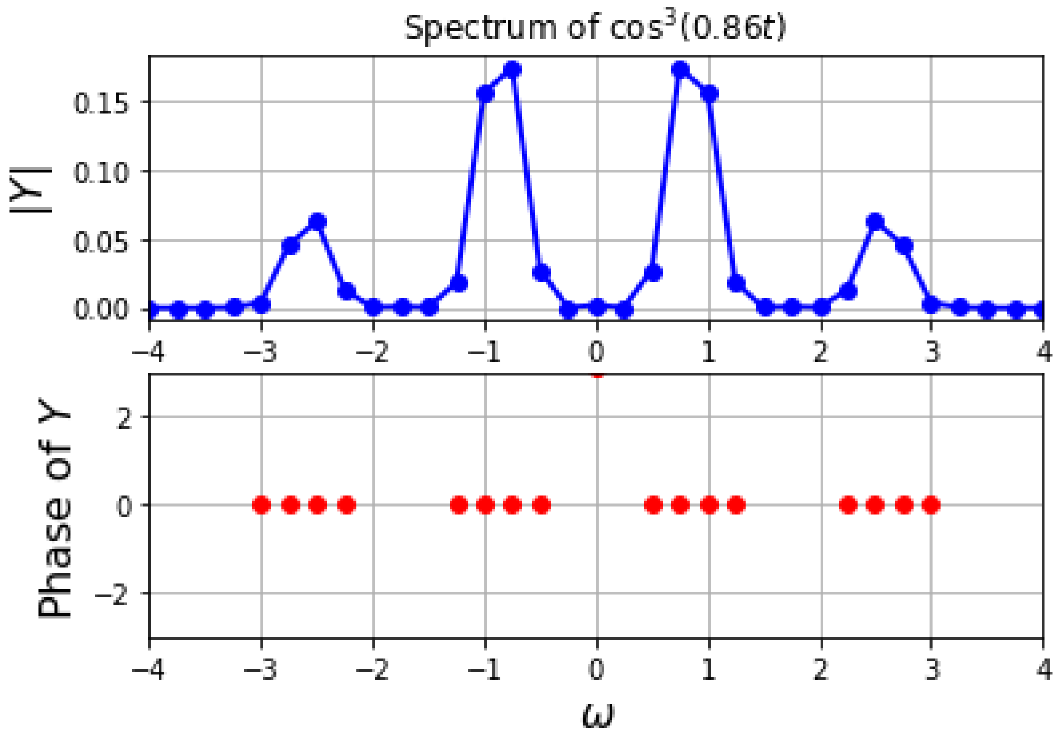


1.2.2 Spectrum of $cos^3(0.86t)$ with windowing

In [11]:
```
t=linspace(-4*pi,4*pi,257);t=t[:-1]
dt=t[1]-t[0];fmax=1/dt
n=arange(256)
wnd=fftshift(0.54+0.46*cos(2*pi*n/256))
y=(cos(0.86*t))**3
y=y*wnd
y[0]=0 # the sample corresponding to -tmax should be set zeroo
y=fftshift(y) # make y start with y(t=0)
plot(t,y,t,wnd)
Y=fftshift(fft(y))/256.0
w=linspace(-pi*fmax,pi*fmax,257);w=w[:-1]
figure()
subplot(2,1,1)
plot(w,abs(Y),'b',w,abs(Y),'bo',lw=2)
xlim([-4,4])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\cos^3(0.86t)$")
grid(True)
subplot(2,1,2)
i=where(abs(Y)>1e-3)
plot(w[i],angle(Y[i]),'ro',lw=2)
```

```
xlim([-4,4])
ylim([-3,3])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()
```



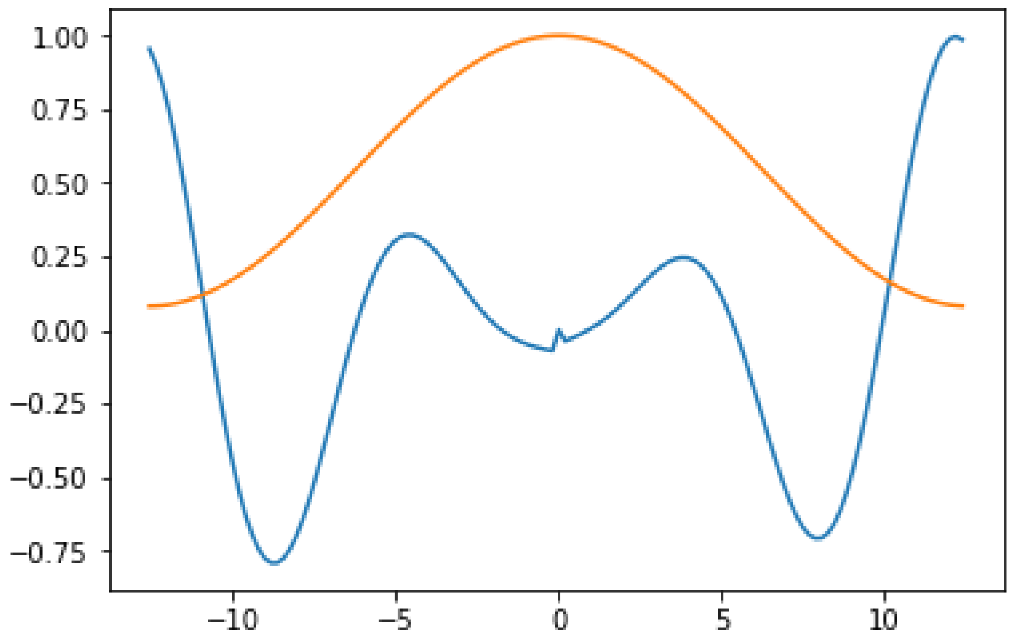


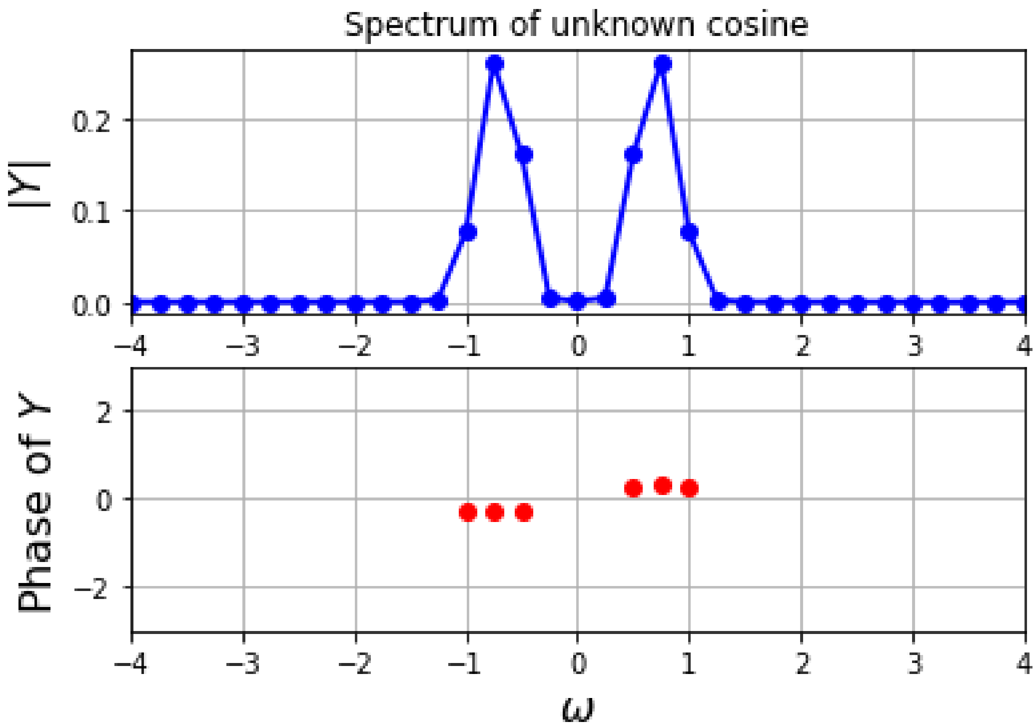
1.3 Question 3

1.3.1 Estimating w and ϕ

```
In [108]: t=linspace(-4*pi,4*pi,129)
          t=t[:-1]
          omega=0.7
          phase=0.3
          y=cos(omega*t+phase)
          dt=t[1]-t[0];fmax=1/dt
          n=arange(128)
          wnd=fftshift(0.54+0.46*cos(2*pi*n/128))
          y=y*wnd
          y[0]=0 # the sample corresponding to -tmax should be set zeroo
          y=fftshift(y) # make y start with y(t=0)
          plot(t,y,t,wnd)
          Y=fftshift(fft(y))/128
          w=linspace(-pi*fmax,pi*fmax,129);w=w[:-1]
          figure()
          subplot(2,1,1)
          plot(w,abs(Y),'b',w,abs(Y),'bo',lw=2)
          xlim([-4,4])
          ylabel(r"$|Y|$",size=16)
```

```
title(r"Spectrum of unknown cosine")
grid(True)
subplot(2,1,2)
i=where(abs(Y)>1e-2)
plot(w[i],angle(Y[i]),'ro',lw=2)
xlim([-4,4])
ylim([-3,3])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()
```





We will estimate phase and angular frequency by taking the average of phase and frequency at points where the considerable magnitude is present.

```
In [1]: ii=i[0]
        print(w[ii])
        ww=w[ii][:len(w[ii])//2]
        abst=abs(Y)[ii][:len(abs(Y)[ii])//2]
        estimated_w=-sum(abst*ww)/sum(abst)
        ang=angle(Y)[ii][:len(angle(Y)[ii])//2]
        estimated_phi=-(sum(ang)/len(ang))
```

```
        ---------------------------------------------------------------------------

        NameError                                 Traceback (most recent call last)

        <ipython-input-1-368817e248eb> in <module>
    ----> 1 ii=i[0]
          2 print(w[ii])
          3 ww=w[ii][:len(w[ii])//2]
          4 abst=abs(Y)[ii][:len(abs(Y)[ii])//2]
          5 estimated_w=-sum(abst*ww)/sum(abst)
```

```
NameError: name 'i' is not defined
```

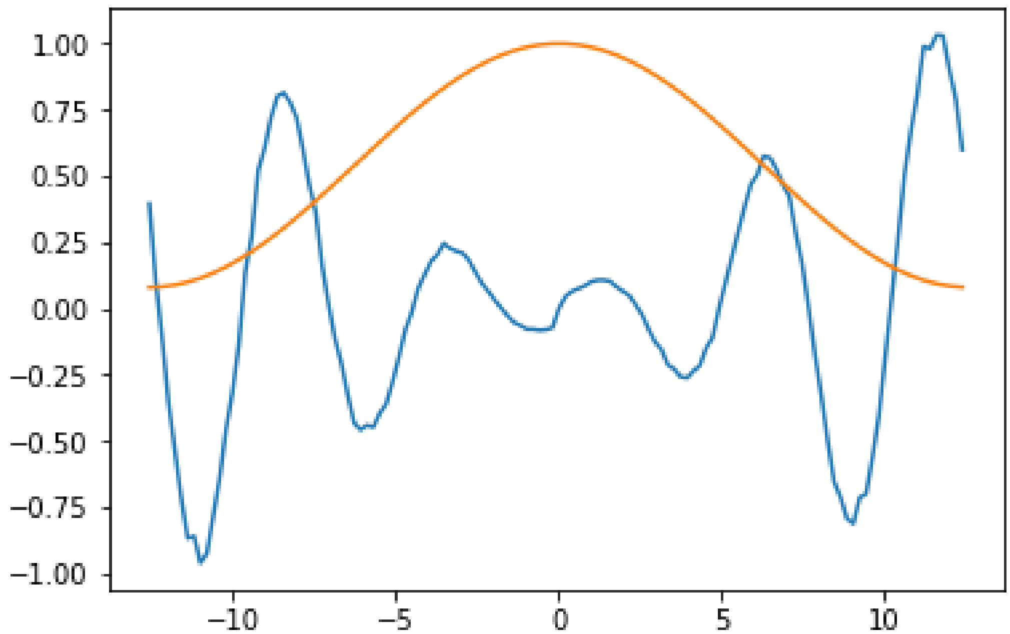
```
In [110]: print(estimated_w)
          print(estimated_phi)
```

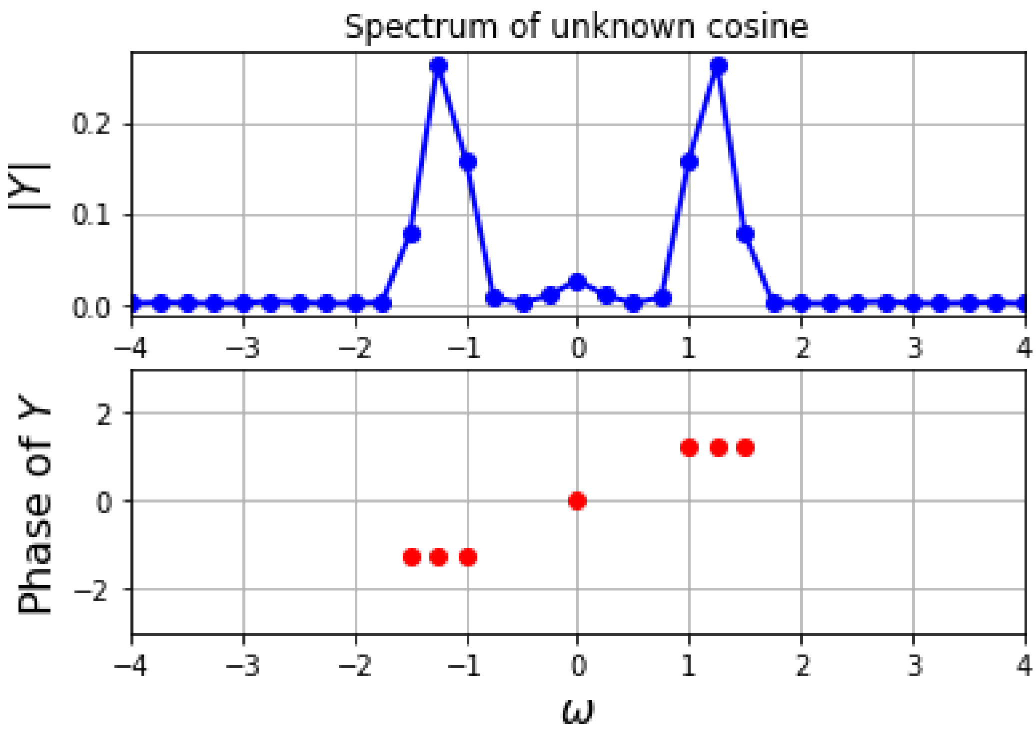
```
0.7079072572004657
0.29634287744684723
```

1.4 Question 4

1.4.1 Estimating w and ϕ (with noise)

```
In [112]: t=linspace(-4*pi,4*pi,129)
          t=t[:-1]
          omega=1.2
          phase=1.25
          y=cos(omega*t+phase)
          y=y+(rand(129)*0.1)[:-1]
          dt=t[1]-t[0];fmax=1/dt
          n=arange(128)
          wnd=fftshift(0.54+0.46*cos(2*pi*n/128))
          y=y*wnd
          y[0]=0 # the sample corresponding to -tmax should be set zeroo
          y=fftshift(y) # make y start with y(t=0)
          plot(t,y,t,wnd)
          Y=fftshift(fft(y))/128
          w=linspace(-pi*fmax,pi*fmax,129);w=w[:-1]
          figure()
          subplot(2,1,1)
          plot(w,abs(Y),'b',w,abs(Y),'bo',lw=2)
          xlim([-4,4])
          ylabel(r"$|Y|$",size=16)
          title(r"Spectrum of unknown cosine")
          grid(True)
          subplot(2,1,2)
          i=where(abs(Y)>1e-2)
          plot(w[i],angle(Y[i]),'ro',lw=2)
          xlim([-4,4])
          ylim([-3,3])
          ylabel(r"Phase of $Y$",size=16)
          xlabel(r"$\omega$",size=16)
          grid(True)
          show()
```






```
In [113]: ii=i[0]
          print(w[ii])
```

```
ww=w[ii][:len(w[ii])//2]
abst=abs(Y)[ii][:len(abs(Y)[ii])//2]
estimated_w=-sum(abst*ww)/sum(abst)
ang=angle(Y)[ii][:len(angle(Y)[ii])//2]
estimated_phi=-(sum(ang)/len(ang))
print(estimated_w)
print(estimated_phi)
```

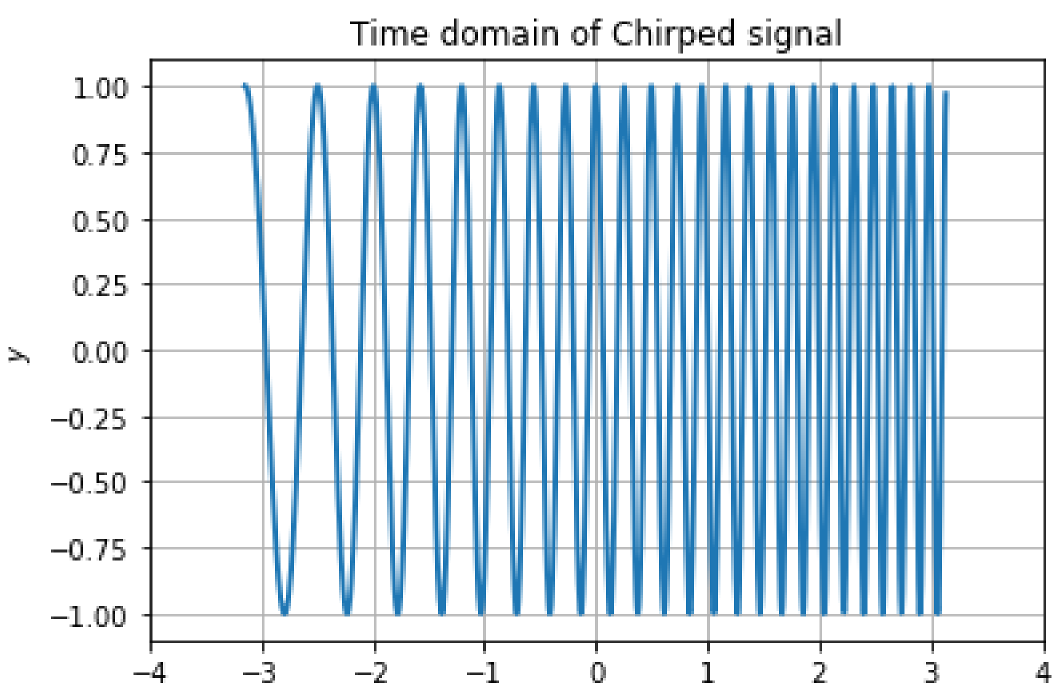
```
[-1.5  -1.25 -1.    0.    1.    1.25  1.5 ]
1.209138153915868
1.2513239769924993
```

We used this windowed DFT to estimate the frequency and phase of an unknown sinusoid from its samples, even if considerable amount of noise is present.

1.5 Question 5

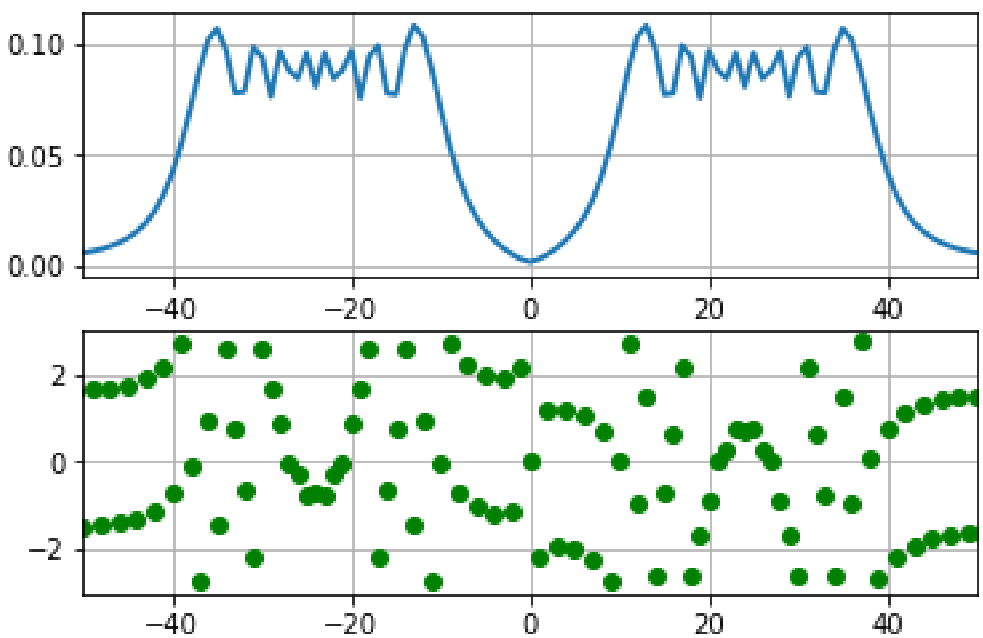
1.5.1 Spectrum of Chirp Signal

```
In [14]: t=linspace(-np.pi,np.pi,1025);t=t[:-1]
         dt=t[1]-t[0]
         fmax=1/dt
         y=cos(16*(1.5+t/(2*np.pi))*t)
         plot(t,y,lw=2)
         xlim([-4,4])
         ylabel(r"$y$")
         title(r"Time domain of Chirped signal")
         grid(True)
```



1.5.2 Without Windowing

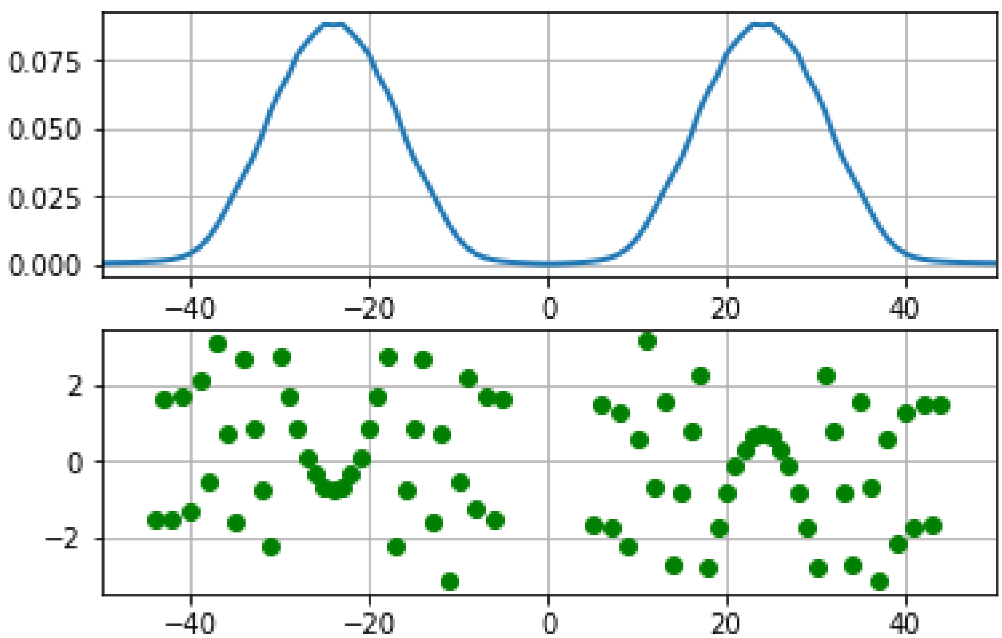
In [15]:
```
y=fftshift(y)
Y=fftshift(fft(y))/1024
w=linspace(-fmax*pi,fmax*pi,1025)
w=w[:-1]
figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-50,50])
grid(True)
subplot(2,1,2)
i=where(abs(Y)>1e-4)
plot(w[i],angle(Y[i]),"go")
xlim([-50,50])
grid(True)
```



1.5.3 With Windowing

In [73]:
```
t=linspace(-pi,pi,1025)
t=t[:-1]
dt=t[1]-t[0]
fmax=1/dt
```

```
y=cos(16*(1.5+t/(2*pi))*t)
n=arange(1024)
wnd=fftshift(0.54+0.46*cos(2*pi*n/1024))
y=y*wnd
y=fftshift(y)
Y=fftshift(fft(y))/1024
w=linspace(-fmax*pi,fmax*pi,1025)
w=w[:-1]
figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-50,50])
grid(True)
subplot(2,1,2)
i=where(abs(Y)>1e-3)
plot(w[i],angle(Y[i]),"go")
xlim([-50,50])
grid(True)
```



We now observe that the frequencies are more confined to the range between 16 and 32, as expected. The extra components due to the discontinuity have been suppressed due to hamming window

1.6 Question 6

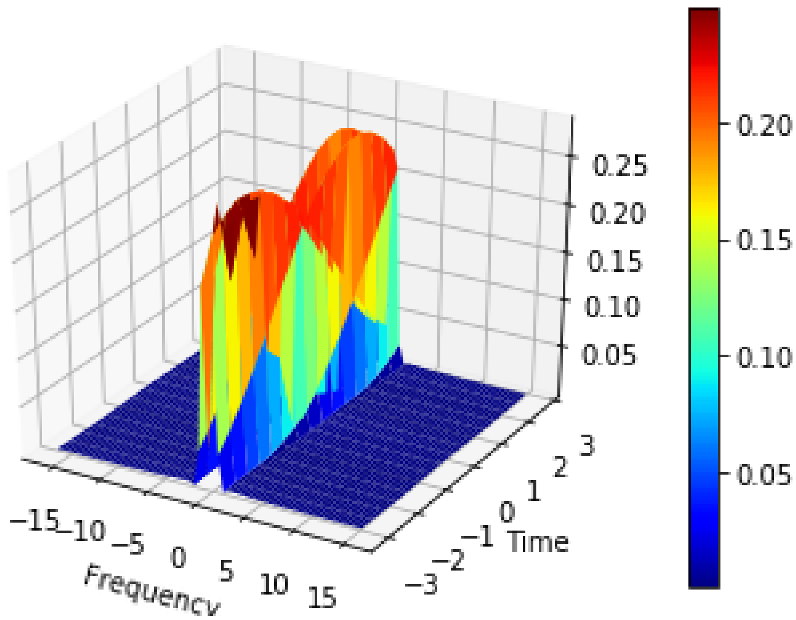
1.6.1 Spectrum vs Time, frequency

To obtain a better picture of what is going in the chirp signal, we take the DFT of a small window of samples around each time instant, and plot a 2D surface of the resulting spectra vs time.

```
In [104]: t=linspace(-pi,pi,1025);t=t[:-1]
          splitted=split(t,16)
```

```
In [105]: Ymag=zeros((16,64))
          Ypha=zeros((16,64))
          for i in range(16):
              tt=splitted[i]
              dt=tt[1]-tt[0]
              fmax=1/dt
              n=arange(64)
              wnd=fftshift(0.54+0.46*cos(2*pi*n/64))
              y=cos(16*(1.5+tt/(2*pi))*tt)
              y=y*wnd
              Y=fftshift(fft(fftshift(y)))/64
              w=linspace(-fmax*np.pi,fmax*np.pi,64+1)
              w=w[:-1]
              Ymag[i,:]=abs(Y)
              Ypha[i,:]=angle(Y)
```

```
In [114]: from mpl_toolkits.mplot3d import Axes3D
          from matplotlib import cm
          fig =figure()
          ax = fig.add_subplot(111, projection='3d')
          t=linspace(-pi,pi,1025);t=t[:-1]
          t=t[::64]
          w=linspace(-fmax*pi,fmax*pi,64+1);w=w[:-1]
          t,w=np.meshgrid(t,w)
          surf=ax.plot_surface(w,t,Ymag.T,rstride=1, cstride=1, cmap=cm.jet)
          fig.colorbar(surf)
          ylabel("Time")
          xlabel("Frequency")
          show()
```



By performing localized DFTs at different time isntants, we obtained a time-frequency plot which allowed us to better analyse signals with varying frequencies in time.

```
In [ ]:
```