Question 1.

EE5121

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EE19B115

a) minimize $\sum_{i=1}^{m} \left((x_c - x_i)^2 + (y_c - y_i)^2 - r^2 \right)^2$

$= \sum_{i=1}^{m} \left(x_i^2 + x_c^2 - 2x_c x_i + y_c^2 + y_i^2 - 2y_c y_i - r^2 \right)^2$

$= \sum_{i=1}^{m} \left(x_i^2 + y_i^2 + x_c(-2x_i) + y_c(-2y_i) + x_c^2 + y_c^2 - r^2 \right)^2$

if $A = \begin{bmatrix} 2x_i & 2y_i & -1 \\ \vdots & \vdots & -1 \\ & & \vdots \end{bmatrix}$, $b = \begin{bmatrix} x_1^2 & y_1^2 \\ x_2^2 & y_2^2 \\ \vdots & \vdots \end{bmatrix}$

above problem can be transformed to

minimize $\| Az - b \|$

$z(1) = x_c$, $z(2) = y_c$

$z(3) = x_c^2 + y_c^2 - r^2$

1) by solving, we get

x_c = -2.567

y_c = 6.468

r = 1.321

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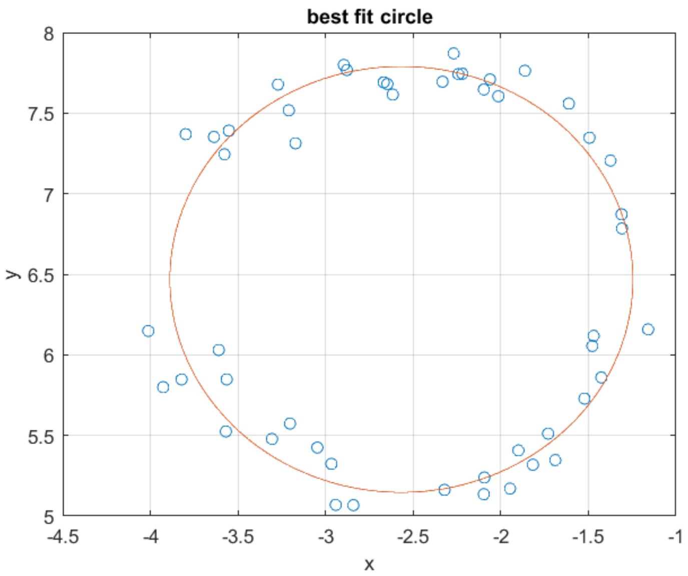

Screenshot of Command window

```
--------------------------------------------------------------------
 number of iterations   =  7
 primal objective value = -2.08222236e+00
 dual   objective value = -2.08222238e+00
 gap := trace(XZ)       = 2.16e-08
 relative gap           = 4.18e-09
 actual relative gap    = 3.81e-09
 rel. primal infeas (scaled problem)   = 7.96e-12
 rel. dual     "        "       "      = 4.76e-12
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.4e+00, 4.7e+01, 2.9e+00
 norm(A), norm(b), norm(C) = 1.0e+02, 2.0e+00, 3.7e+02
 Total CPU time (secs)  = 0.37
 CPU time per iteration = 0.05
 termination code       =  0
 DIMACS: 8.0e-12  0.0e+00  2.5e-11  0.0e+00  3.8e-09  4.2e-09
--------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +2.08222
```

Question 2

minimize $f(x) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$

a) Gradient descent ($\alpha = 0.1$)

algo : $x_{new} = x - \alpha \nabla f(x)$

no: of iterations = 20

optimal value of f is 2.559274

optimal x is $\begin{bmatrix} -0.3489 \\ 0 \end{bmatrix}$

b) Gradient descent with backtracking line-search, $\alpha = 0.1$, $\beta = 0.5$

algo:

while $f(x+t\Delta x) > f(x) + \alpha t \nabla f(x)^T \Delta x$,

$t = \beta t$

else

$x_{new} = x - t \nabla f(x)$

no: of iterations = 16

optimal value of f is 2.559273

optimal x is $\begin{bmatrix} -0.3487 \\ 0 \end{bmatrix}$

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(c) Newton's method with backtracking line search, $\alpha = 0.1$, $\beta = 0.5$

Algo:

given a starting point $x \in \text{dom} f$

1) $\Delta x = -\nabla^2 f(x)^{-1} \nabla f(x) = v$

2) if $f(x + t^* v') > f(x) + \alpha t \nabla f v$

$t = \beta t$

else

$x = x + tv$

3) repeat

no: of iterations $= 4$

optimal value of f is 2.559267

optimal x is $\begin{bmatrix} -0.346 \\ 0 \end{bmatrix}$

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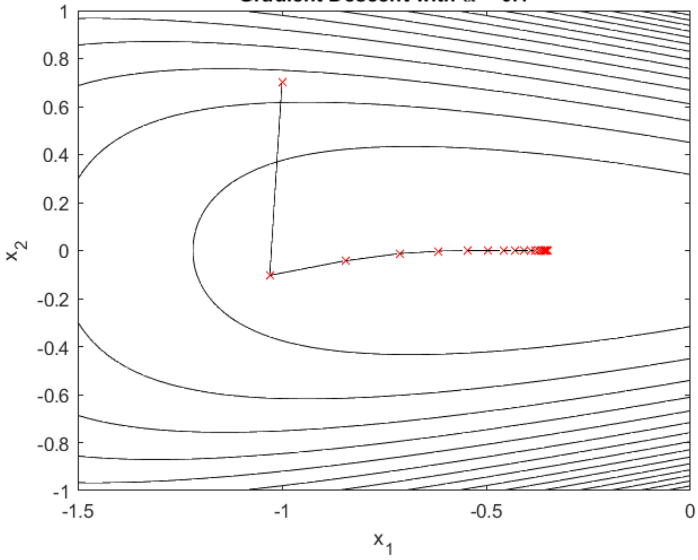
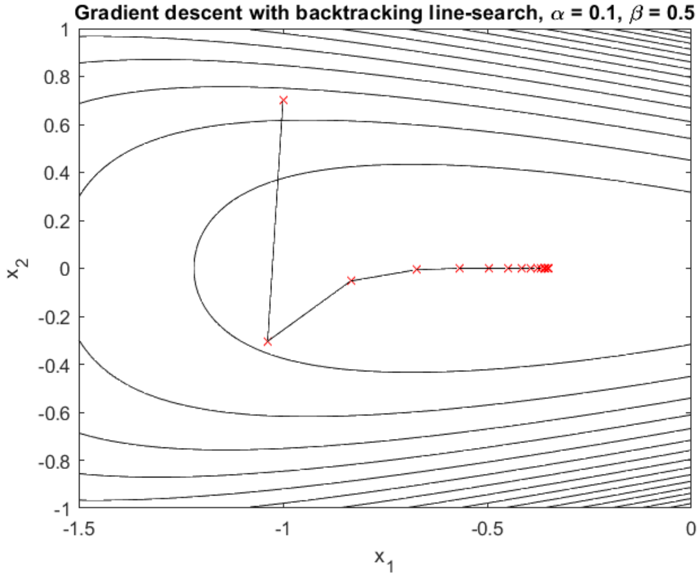
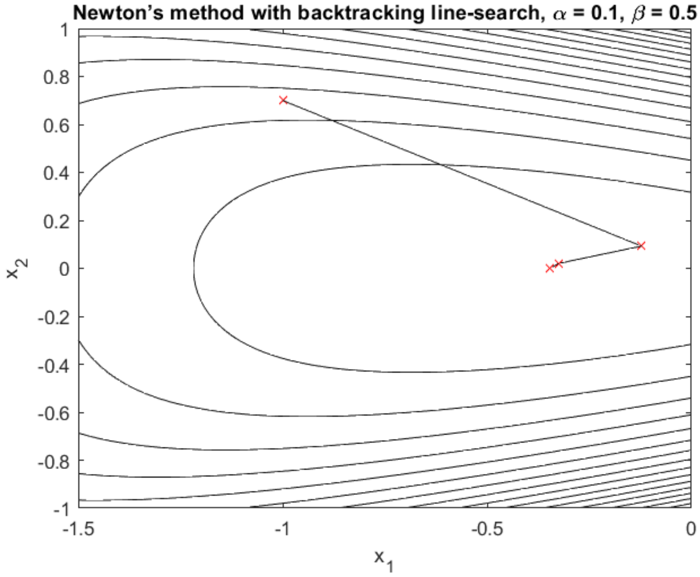
Gradient Descent with α = 0.1



Gradient descent with backtracking line-search, α = 0.1, β = 0.5



Newton's method with backtracking line-search, α = 0.1, β = 0.5



Question 3

minimize $c^T x$

Subject to $Ax \leq b$

$x_i \in \{0,1\}, \quad i = 1 \dots n$

a) as $x_i \in \{0,1\} \Rightarrow x_i(1-x_i) = 0$

$x_i(1-x_i) = 0 \Rightarrow x_i = \{0,1\}$

$\Longleftrightarrow$

So, both are equivalent

b) The problem is not convex as equality constraint isn't affine.

c) Dual:

consider $\lambda \in \mathbb{R}_m, \mu \in \mathbb{R}_n$

$L(x,\lambda,\mu) = c^T x + \lambda^T(Ax-b) + \mu^T(x(1-x))$

$= c^T x + \lambda^T(Ax-b) + \sum \mu_i x_i - \sum \mu_i x_i^2$

$v = \text{diag}(\mu_1 \mu_2 \dots)$,

$\nabla_x L(\bar{x},\lambda,\mu) = 0 \Rightarrow c + A^T\lambda - 2v\bar{x} + \mu$

$\bar{x} = \frac{1}{2} v^{-1} [c + A^T\lambda + \mu]$

$$g(\lambda,\mu) = \inf_x L(x,\lambda,\mu)$$

$$= \inf L(\bar{x},\lambda,\mu)$$

$$= \frac{1}{2} c^T V^{-1} [c + A^T\lambda + \mu] +$$

$$+ \frac{1}{2} \lambda^T A V^{-1} [c + A^T\lambda + \mu] - \lambda^T b +$$

$$\frac{1}{2} \mu^T V^{-1} (c + A^T\lambda + \mu) -$$

$$\frac{1}{4} V^{-1} (c + A^T\lambda + \mu)^T V V^{-1} [c + A^T\lambda + \mu]$$

many things cancel, we end up with

$$= -b^T\lambda - \frac{1}{4} \cdot (c + A^T\lambda + \mu)^T V^{-1} (c + A^T\lambda + \mu)$$

$$= - \quad V^{-1} = \mathrm{diag}\left(\frac{1}{\mu_1}, \ldots \frac{1}{\mu_n}\right)$$

$$\text{maximize } -b^T\lambda - \frac{1}{4} \sum_{i=1}^{n} \frac{(c_i + a_i^T\lambda + \mu_i)^2}{\mu_i}$$

subject to $\lambda \geq 0$

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maximize $-\frac{(c_i + a_i^T\lambda + \mu_i)^2}{\mu_i}$

minimize: $\frac{(c_i + a_i^T\lambda)^2}{\mu_i} + \mu_i + 2(c_i + a_i^T\lambda)$

~~maximized~~ minimized when $(c_i + a_i^T\lambda)^2 = \mu_i^2$

~~Mice~~.

If $c_i + a_i^T\lambda \geq 0$, ~~max~~ min possible is 0

when $\mu_i = -(c_i + a_i^T\lambda)$

If $c_i + a_i^T\lambda \leq 0$, ~~max~~ min possible is

$4(c_i + a_i^T\lambda)$

So finally it becomes

$$\underset{\lambda}{\text{maximize}} \quad -b^T x + \sum_{i=1}^{n} \min\{0, c_i + a_i^T x\}$$

subject to $\lambda \geq 0$

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d) - positive combination of affine functions and pointwise infimum of them is concave.

maximizing concave function is a convex problem

e) optimum value of objective function is found out to be -3

f) $d^* = p^*$

```
-----------------------------------------------------------------
 number of iterations   =  7
 primal objective value =  3.00000005e+00
 dual   objective value =  2.99999999e+00
 gap := trace(XZ)       = 5.94e-08
 relative gap           = 8.48e-09
 actual relative gap    = 8.47e-09
 rel. primal infeas (scaled problem)   = 3.87e-13
 rel. dual     "        "       "      = 7.72e-12
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 2.2e+00, 1.4e+00, 4.2e+00
 norm(A), norm(b), norm(C) = 4.9e+00, 3.2e+00, 6.1e+00
 Total CPU time (secs)  = 1.66
 CPU time per iteration = 0.24
 termination code       =  0
 DIMACS: 4.2e-13  0.0e+00  9.5e-12  0.0e+00  8.5e-09  8.5e-09
-----------------------------------------------------------------

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Status: Solved
Optimal value (cvx_optval): -3
```

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Question 4

$$x(t+1) = Ax(t) + Bu(t) + \underbrace{w(t)}_{\text{disturbance}}$$

(a) to find A, B,

our objective function should be

$$\text{minimize } \sum \| x(t+1) - Ax(t) - Bu(t) \|_2^2$$

finally formulation:

```
cvx_begin
    variables A(10,10) B(10,4)
    minimize ||X' - AX - BU||_F
cvx_end
```

here $X' = X[2:100]$

(b) - cardinality of a matrix is not a convex function, so it isn't a convex problem.

- we need to relax the problem to 1-norm of matrix.

$$\underset{A,B}{\text{minimize}} \; \|X' - AX - BU\|_F + \lambda \left(\|A\|_1 + \|B\|_1 \right)$$

- I varied λ from 0-100 in steps of 2 each, graph looked like this.

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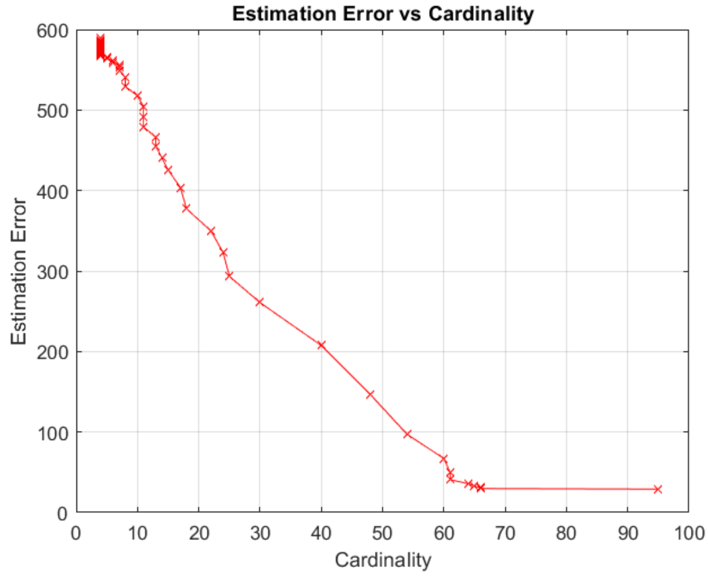


```
------------------------------------------------------------------
 number of iterations   =  7
 primal objective value = -2.90664650e+01
 dual   objective value = -2.90664651e+01
 gap := trace(XZ)       = 3.98e-08
 relative gap           = 6.73e-10
 actual relative gap    = 6.30e-10
 rel. primal infeas (scaled problem)   = 1.48e-14
 rel. dual     "        "       "      = 2.77e-12
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.4e+00, 2.9e+01, 4.1e+01
 norm(A), norm(b), norm(C) = 3.0e+03, 2.0e+00, 9.1e+02
 Total CPU time (secs)  = 8.15
 CPU time per iteration = 1.16
 termination code       =  0
 DIMACS: 1.5e-14  0.0e+00  1.5e-11  0.0e+00  6.3e-10  6.7e-10
------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +29.0665
```

for lambda = 0

```
------------------------------------------------------------------
 number of iterations   = 15
 primal objective value = -3.25316383e+02
 dual   objective value = -3.25316385e+02
 gap := trace(XZ)       = 1.82e-06
 relative gap           = 2.80e-09
 actual relative gap    = 2.59e-09
 rel. primal infeas (scaled problem)   = 8.02e-12
 rel. dual     "        "       "      = 1.04e-12
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.8e+02, 5.0e+01, 7.1e+01
 norm(A), norm(b), norm(C) = 3.0e+03, 1.4e+02, 9.1e+02
 Total CPU time (secs)  = 0.13
 CPU time per iteration = 0.01
 termination code       =  0
 DIMACS: 8.8e-11  0.0e+00  5.6e-12  0.0e+00  2.6e-09  2.8e-09
------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +325.316
```

for lambda = 12

In these kind of errors trade offs, the graph looks like elbow and elbow point should be taken.

$\lambda = 12$, cardinality = ~~64~~ 61

est_error = 40.93

```
lambda = 0, cardinality is 95, est_error is 29.066465
lambda = 2, cardinality is 66, est_error is 29.755367
lambda = 4, cardinality is 66, est_error is 30.923479
lambda = 6, cardinality is 65, est_error is 32.741305
lambda = 8, cardinality is 64, est_error is 35.769196
lambda = 10, cardinality is 61, est_error is 40.933947
lambda = 12, cardinality is 61, est_error is 49.680926
lambda = 14, cardinality is 60, est_error is 67.182094
lambda = 16, cardinality is 54, est_error is 97.050043
lambda = 18, cardinality is 48, est_error is 146.633256
lambda = 20, cardinality is 40, est_error is 207.503971
lambda = 22, cardinality is 30, est_error is 260.779194
lambda = 24, cardinality is 25, est_error is 293.318123
lambda = 26, cardinality is 24, est_error is 322.607559
lambda = 28, cardinality is 22, est_error is 349.695581
lambda = 30, cardinality is 18, est_error is 377.300417
lambda = 32, cardinality is 17, est_error is 403.727601
lambda = 34, cardinality is 15, est_error is 425.551737
lambda = 36, cardinality is 14, est_error is 440.518124
lambda = 38, cardinality is 13, est_error is 454.727427
lambda = 40, cardinality is 13, est_error is 466.136786
lambda = 42, cardinality is 11, est_error is 478.806873
lambda = 44, cardinality is 11, est_error is 490.759117
lambda = 46, cardinality is 11, est_error is 503.774730
lambda = 48, cardinality is 10, est_error is 517.688546
lambda = 50, cardinality is 8, est_error is 529.294196
lambda = 52, cardinality is 8, est_error is 540.429041
lambda = 54, cardinality is 7, est_error is 549.490619
lambda = 56, cardinality is 7, est_error is 552.707211
lambda = 58, cardinality is 7, est_error is 556.096387
lambda = 60, cardinality is 6, est_error is 559.119186
lambda = 62, cardinality is 6, est_error is 561.395430
lambda = 64, cardinality is 5, est_error is 563.598802
lambda = 66, cardinality is 5, est_error is 564.866997
lambda = 68, cardinality is 5, est_error is 566.183140
lambda = 70, cardinality is 4, est_error is 567.548214
lambda = 72, cardinality is 4, est_error is 568.858261
lambda = 74, cardinality is 4, est_error is 570.025526
lambda = 76, cardinality is 4, est_error is 571.232277
lambda = 78, cardinality is 4, est_error is 572.479227
```

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Question 5

we have to approximate $\sin x$ between $x \in [-\pi, \pi]$ as a cubic polynomial

$N = 20, K = 3$

we can think that the equation is

$f(x) = y_0 + y_1 x + y_2 x^2 + y_3 x^3$

1-Norm minimization :-

$$\underset{y \in \mathbb{R}^4}{\text{minimize}} \; \|y^T X - b\|_1$$

it is not an LP problem, but it's a convex optimization problem.

consider $|y^T x - b|_i \le z_i$

now, its minimizing Σz_i

subject to

$(y^T x - b) <= z$

$-y^T X + b <= z$

in standard form LP, $z_i \ge 0$, also, X should be written as difference of non-negative values, so finally

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formulation becomes

```
cvx_begin
    variable a(4) nonnegative;
    variable c(4) nonnegative;
    variable z(20) nonnegative;

    minimize sum(z(:))
    subject to
        x*(a-c) - b <= z
        -x*(a-c) + b <= z

cvx-end
```

final f(x)

$$f(x) = 0.88x - 0.09x^3$$

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Norm- Minimization

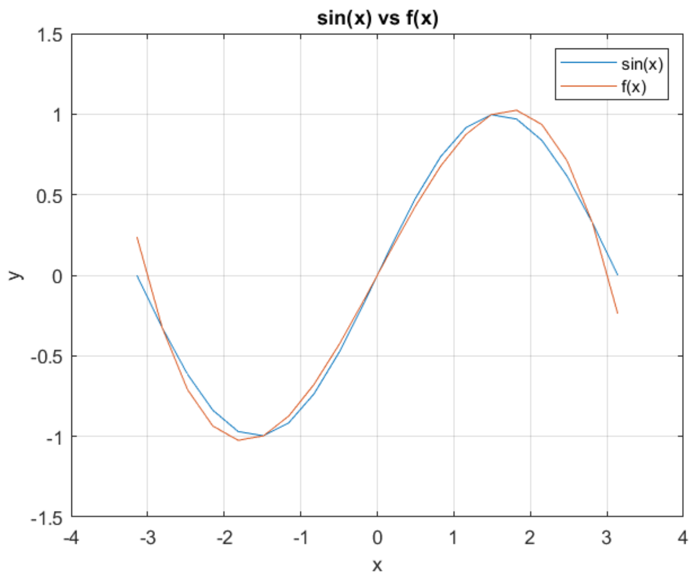
```
--------------------------------------------------------------------
 number of iterations    =  9
 primal objective value =  1.31208546e+00
 dual   objective value =  1.31208542e+00
 gap := trace(XZ)       = 4.36e-08
 relative gap           = 1.20e-08
 actual relative gap    = 1.09e-08
 rel. primal infeas (scaled problem)   = 1.81e-13
 rel. dual     "        "       "      = 1.99e-09
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.1e+00, 4.2e+00, 6.1e+00
 norm(A), norm(b), norm(C) = 9.3e+01, 4.1e+00, 5.5e+00
 Total CPU time (secs)  = 0.46
 CPU time per iteration = 0.05
 termination code       =  0
 DIMACS: 3.7e-13  0.0e+00  5.5e-09  0.0e+00  1.1e-08  1.2e-08
--------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +1.31209
```

LP

```
--------------------------------------------------------------------
 number of iterations   = 10
 primal objective value = -1.31208541e+00
 dual   objective value = -1.31208544e+00
 gap := trace(XZ)       = 3.67e-08
 relative gap           = 1.01e-08
 actual relative gap    = 8.86e-09
 rel. primal infeas (scaled problem)   = 2.33e-12
 rel. dual     "        "       "      = 4.87e-11
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 4.3e+00, 4.8e+02, 4.8e+02
 norm(A), norm(b), norm(C) = 1.3e+02, 5.5e+00, 5.4e+00
 Total CPU time (secs)  = 0.13
 CPU time per iteration = 0.01
 termination code       =  0
 DIMACS: 6.4e-12  0.0e+00  1.3e-10  0.0e+00  8.9e-09  1.0e-08
--------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +1.31209
```



Question 6

minimize $\|C - \hat{C}\|_F$

constraining C is semidefinite,

all diagonal entries to be 1

C is symmetric

final formulation:

```
cvx_begin
    variable X(4,4) symmetric semidefinite;
    minimize norm(Ct - X, 'fro');
    subject to
    X(1,1) == 1
    X(2,2) == 1
    X(3,3) == 1
    X(4,4) == 1
cvx_end
```

final C

$$\begin{bmatrix} 1 & -0.64 & 0 & -0.78 \\ -0.64 & 1 & 0.13 & 0.16 \\ 0 & 0.13 & 1 & 0.34 \\ -0.78 & 0.16 & 0.34 & 1 \end{bmatrix}$$

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Command window Screenshot

```
------------------------------------------------------------------
 number of iterations   =  9
 primal objective value = -3.58138386e-01
 dual   objective value = -3.58138387e-01
 gap := trace(XZ)       = 1.93e-09
 relative gap           = 1.12e-09
 actual relative gap    = 1.11e-09
 rel. primal infeas (scaled problem)   = 1.02e-12
 rel. dual     "        "       "      = 7.39e-12
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.9e+00, 5.1e-01, 2.6e+00
 norm(A), norm(b), norm(C) = 4.6e+00, 2.0e+00, 3.7e+00
 Total CPU time (secs)  = 1.19
 CPU time per iteration = 0.13
 termination code       =  0
 DIMACS: 1.0e-12  0.0e+00  1.4e-11  0.0e+00  1.1e-09  1.1e-09
------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +0.358138
```