Unit 3
Application of Vector Calculus

GREEN'S THEOREM IN A PLANE

Green's theorem provides a relationship between a double integral over a region R in a plane and the line integral over the closed curve C bounding R.

Statment: If $M(x, y)$ and $N(x, y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy-plane bounded by a closed curve C, then

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

where C is traversed in the counter-clockwise direction.

GAUSS'S DIVERGENCE THEOREM

(Relation between surface and volume integrals)

Statement : If $\vec{F}$ is a vector point function having continuous first order partial derivatives in the region V bounded by a closed surface S, then

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, ds$$

where $\hat{n}$ is the outwards drawn unit normal vector to the surface S.

STOKE'S THEOREM (Relation between line and surface integrals)

Stoke's theorem can be regarded as a higher dimensional version of Green's theorem. Stoke's theorem relates a surface integral over a surface S to a line integral around the boundary of curve.

Statement : If S be an open surface bounded by a closed curve C and $\vec{F} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$ be any vector point function having continuons first order partial derivatives, then $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl}\, \vec{F} \cdot \hat{n}\, dS$

where $\hat{n}$ is a unit normal vector at any point of S

GREEN'S THEOREM AS A SPECIAL CASE OF STOKE'S THEOREM

Let $\vec{F} = (F_1\hat{i} + F_2\hat{j})$ be a vector function which is continuously differentiable in a domain in the xy-plane containing region S bounded by a closed curve C.

$$\text{So, } \oint_C \vec{F}\cdot d\vec{r} = \oint_C (F_1\hat{i} + F_2\hat{j})\cdot(\hat{i}dx + \hat{j}dy) = \oint_C (F_1dx + F_2dy)$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Here, $\hat{n} = \hat{k}$

$$\therefore \quad \text{Curl } \vec{F}\cdot\hat{n} = \text{Curl } \vec{F}\cdot\hat{k}$$

$$= \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cdot\hat{k}$$

$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Hence, by Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\hat{n}\,dS$$

or
$$\oint_C (F_1dx + F_2dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$$

which is Green's theorem in a plane.