Unit 1
Integral Calculus

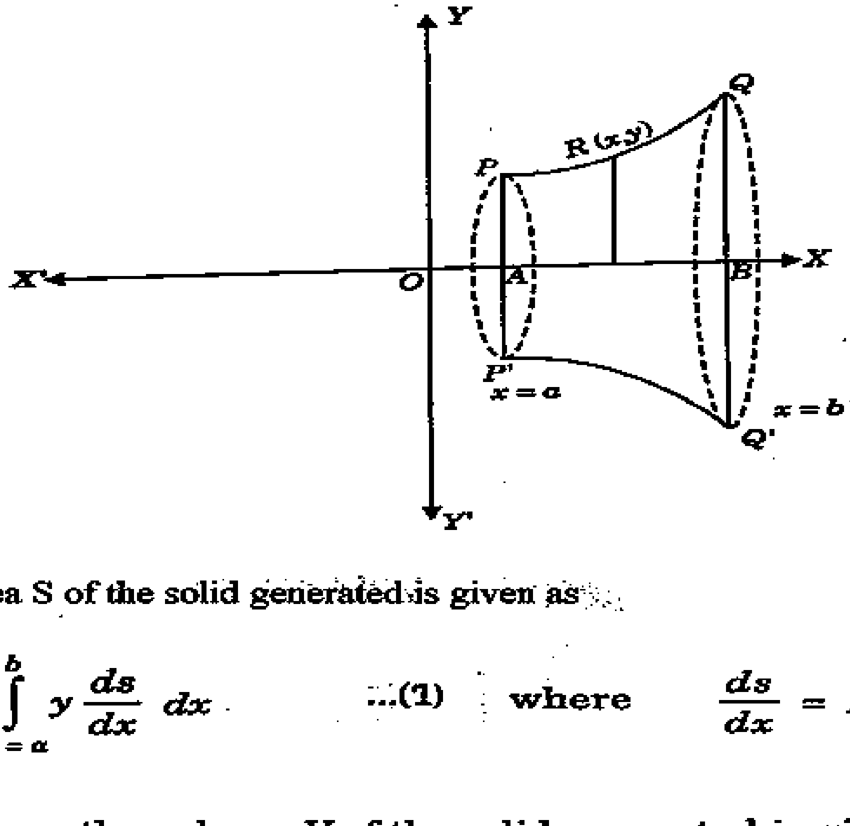
INTRODUCTION

If we revolve a two dimensional plane curve or a part of it about a fixed line (axis of revolution) in its plane, then a solid is generated, which is called solid of revolution e.g. by revolving a semi-circle about its diameter, a sphere is generated, by revolving a straight line about an oblique straight line, a right circular cone is generated, by revolving bounded arc of parabola $y^2 = 4ax$ included between points $(x_1, -y_1)$ and (x_1, y_1) about the tangent at its vertex, a reel is generated etc.

Here, we study the surface area and the volume of these solids of revolution considering co-ordinate axes and lines parallel to them as axis of revolution in general and other lines as axis of revolution in certain problems.

SURFACE AREA AND VOLUME OF SOLID OF REVOLUTION OF CURVE GIVEN IN CARTESIAN FORM $y = f(x)$ OR $x = g(y)$

(I) When axis of revolution is x-axis



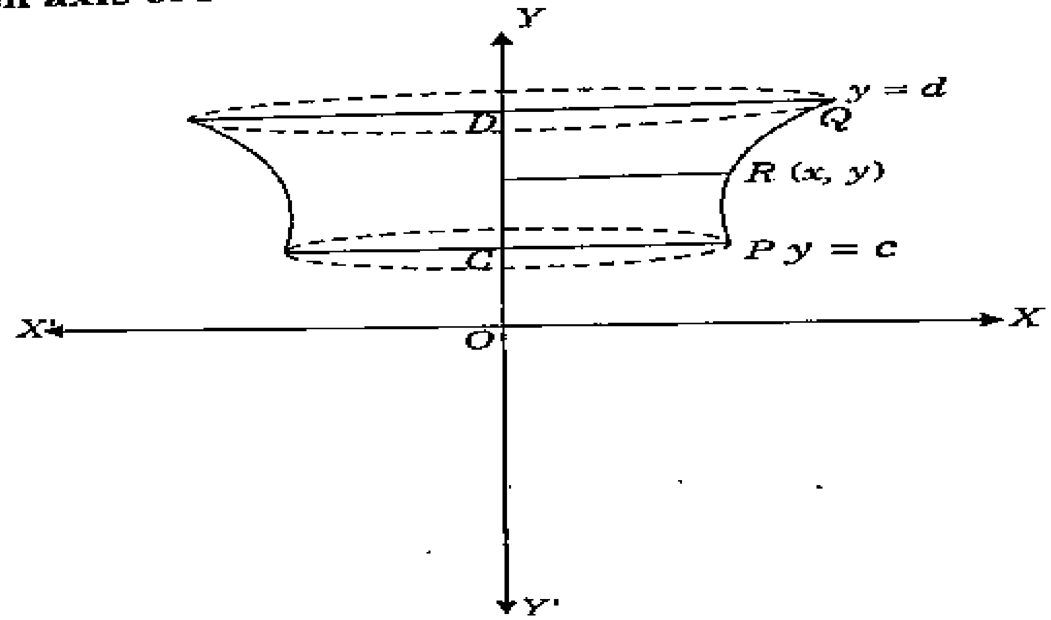
Surface area S of the solid generated is given as

$$S = 2\pi \int_{x=a}^{b} y \frac{ds}{dx}\, dx \quad \ldots(1) \quad \text{where} \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

In this case, the volume V of the solid generated is given by

$$V = \pi \int_{x=a}^{b} y^2 dx \quad \ldots(2)$$

(II) When axis of revolution is y-axis



surface area S of the solid generated is given by

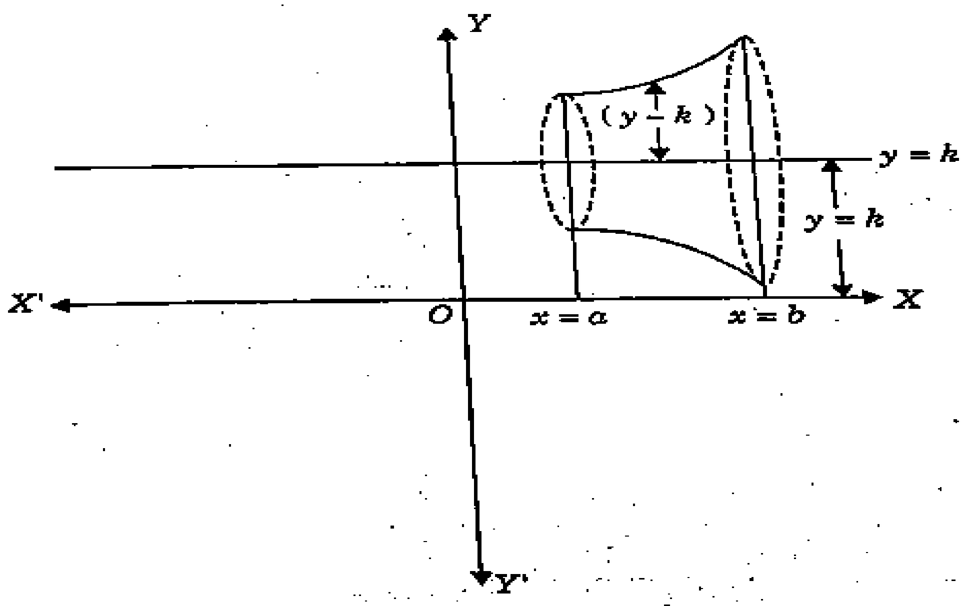
$$S = 2\pi \int_{y=c}^{d} x \frac{ds}{dy}\, dy \qquad \ldots(3) \qquad \text{where} \qquad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

In this case, the volume V of the solid generated is given by

$$V = \pi \int_{y=c}^{d} x^2 dy \qquad \ldots(4)$$

(III) When a line parallel to x-axis *i.e.* line $y = k$ is axis of revolution

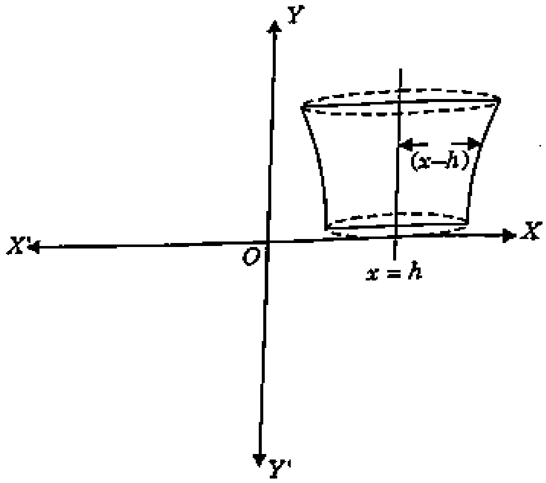
In this case of revolution, the surface area S and the volume V of the solid generated are given by



$$S = 2\pi \int_{x=a}^{b} (y-k) \frac{ds}{dx}\, dx \qquad V = \pi \int_{x=a}^{b} (y-k)^2\, dx$$

(IV) When a line parallel to y-axis i.e. line $x = h$ is axis of revolution

In this case of revolution, the surface area S and the volume V of the solid generated are given by



$$S = 2\pi \int_{y=c}^{d} (x-h) \frac{ds}{dy}\, dy \qquad V = \pi \int_{y=c}^{d} (x-h)^2\, dy$$

Remarks : (1) In any case, revolving part of the curve does not crosses the axis of revolution.

(2) In above article, ds can also be replaced by $\frac{ds}{dy} \cdot dy$ in equations (1), (5) and $\frac{ds}{dx} \cdot dx$ in equations (3), (7) (If required).

(3) If the revolving part is symmetric about one or more lines, then find the surface area or the volume of one symmetric part and multiply that by the number of symmetrical parts.

SURFACE AREA AND VOLUME OF SOLID OF REVOLUTION OF CURVE GIVEN IN PARAMETRIC FORM $x = f(t)$, $y = g(t)$

(I) When axis of=·evolution is x-axis

In this case, the surface area S and the volume V of the solid generated are given by

$$S = 2\pi \int_{t=t_1}^{t_2} y \frac{ds}{dt}\, dt$$

or

$$S = 2\pi \int_{t=t_1}^{t_2} g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left[\because \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right]$$

and

$$V = \pi \int_{t=t_1}^{t_2} y^2 \frac{dx}{dt} \cdot dt$$

or

$$V = \pi \int_{t=t_1}^{t_2} [g(t)]^2 \cdot \frac{d}{dt} [f(t)]\, dt$$

(II) When axis of revolution is y-axis

In this case, the surface area S and the volume V of the solid generated are given by

$$S = 2\pi \int_{t=t_1}^{t_2} x \frac{ds}{dt}\, dt$$

or
$$S = 2\pi \int_{t=t_1}^{t_2} f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left[\because \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\right]$$

and
$$V = \pi \int_{t=t_1}^{t_2} x^2 \frac{dy}{dt}\, dt$$

or
$$V = \pi \int_{t=t_1}^{t_2} [f(t)]^2 \frac{d}{dt} [g(t)]\, dt$$

SURFACE AREA AND VOLUME OF SOLID OF REVOLUTION OF CURVE GIVEN IN POLAR FORM $r = f(\theta)$

(I) When initial line (line $\theta = 0$) is axis of revolution

In this case, the surface area S and the volume V of the solid generated are given by

$$S = 2\pi \int_{\theta=\alpha}^{\beta} y \frac{ds}{d\theta}\, d\theta$$

or
$$S = 2\pi \int_{\theta=\alpha}^{\beta} r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\left[\text{using } y = r\sin\theta \text{ and } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\right]$$

and
$$V = \pi \int_{\theta=\alpha}^{\beta} y^2 \frac{dx}{d\theta}\, d\theta$$

or
$$V = \pi \int_{\theta=\alpha}^{\beta} r^2 \sin^2\theta \frac{d}{d\theta}(r\cos\theta)\, d\theta$$

Using Pappus theorem, volume V in this case can also be obtained by

$$\boxed{V = \frac{2\pi}{3} \int_{\theta=\alpha}^{\beta} r^3 \sin\theta\, d\theta}$$

(II) When line $\theta = \frac{\pi}{2}$ is axis of revolution

In this case, the surface area S and the volume V of the solid generated are given by

$$S = 2\pi \int_{\theta=\alpha}^{\beta} x \frac{ds}{d\theta}\, d\theta$$

or
$$S = 2\pi \int_{\theta=\alpha}^{\beta} r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\left[\text{using } x = r\cos\theta \text{ and } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\right]$$

and
$$V = \pi \int_{\theta=\alpha}^{\beta} x^2 \frac{dy}{d\theta}\, d\theta$$

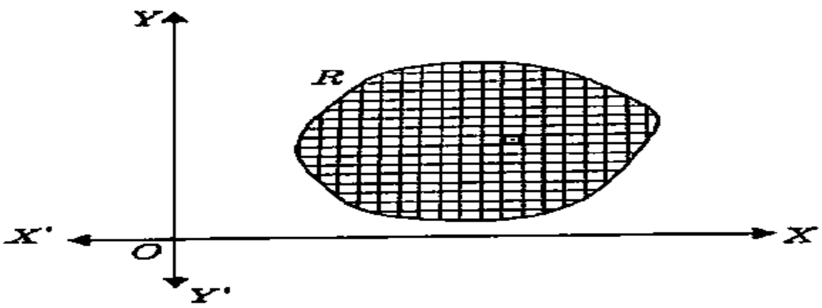
or
$$V = \pi \int_{\theta=\alpha}^{\beta} r^2 \cos^2\theta \frac{d}{d\theta}(r\sin\theta)\, d\theta$$

Using Pappus theorem, volume V in this case can also be obtained by

$$\boxed{V = \frac{2\pi}{3} \int_{\theta=\alpha}^{\beta} r^3 \cos\theta\, d\theta}$$

INTRODUCTION

While describing a double integral over a given region, we have to divide the region into infinite subregions as shown in the figure below:



Double integral of function $f(x, y)$ over the region R is denoted by $\iint_R f(x, y)\, dx\, dy$

PROPERTIES OF DOUBLE INTEGRALS

Like single integral, we can describe several fundamental properties for a double integral.

(1) $$\iint_A f(x, y)\, dxdy = \iint_{A_1} f(x, y)\, dxdy + \iint_{A_2} f(x, y)\, dxdy + \dots\dots\dots + \iint_{A_n} f(x, y)\, dxdy$$

where the area A is divided into n parts, say $A_1, A_2, \dots\dots, A_n$

(2) $$\iint_A [f_1(x, y) + f_2(x, y) + \dots + f_n(x, y)]\, dxdy$$

$$= \iint_A f_1(x, y)\, dxdy + \iint_A f_2(x, y)\, dxdy + \dots + \iint_A f_n(x, y)\, dxdy$$

(3) $$\iint_A k f(x, y)\, dxdy = k \iint_A f(x, y)\, dxdy$$

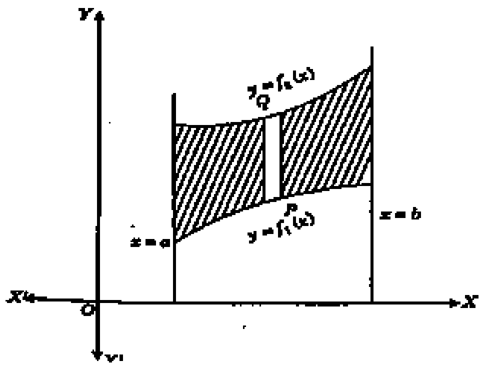
where k is a non-zero real constant.

EVALUATION OF DOUBLE INTEGRALS FOR CARTESIAN FUNCTIONS

The methods of evaluation of double integrals depend upon the nature of curves which bound the region R as stated below

(I) When region R of area A is bounded by two lines $x = a$, $x = b$ (lines parallel to y-axis) and two curves $y = f_1(x)$, $y = f_2(x)$

Here, the bounded region is divided into vertical strips and taking a vertical strip PQ, we can define double integral $\iint_A f(x, y)\, dxdy$ as



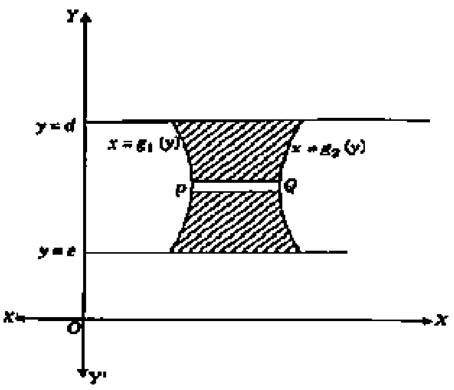
$$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x, y)\, dxdy = \int_{x=a}^{b} \left[\int_{\substack{y = f_1(x) \\ (x \to \text{constant})}}^{f_2(x)} f(x, y)\, dy \right] dx$$

Thus, the above double integral is evaluated first with respect to y (treating x as constant) and then resulting expression which is function of x is integrated with respect to x between the limits $x = a$ and $x = b$

Geometrically, in double integral shown above, internal limits show that all vertical strips exist in between $y = f_1(x)$ and $y = f_2(x)$ and external limits show that all vertical strips lie on horizontal axis in between $x = a$ and $x = b$

(II) When region R of area A is bounded by two lines $y = c$, $y = d$ (lines parallel to x-axis) and two curves $x = g_1(y)$, $x = g_2(y)$

Here, the bounded region is divided into horizontal strips and taking a horizontal strip PQ, we can define double integral $\iint_A f(x, y)\, dxdy$ as



$$\int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} f(x, y)\, dxdy = \int_{y=c}^{d} \left[\int_{\substack{x = g_1(y) \\ (y \to \text{constant})}}^{g_2(y)} f(x, y)\, dx \right] dy$$

Thus, the above double integral is evaluated first with respect to x (treating y as constant) and then resulting expression which is a function of y is integrated with respect to y between the limits $y = c$ and $y = d$

Geometrically, in double integral shown by eqn. (8.6), internal limits show that all horizontal strips exist in between $x = g_1(y)$ and $x = g_2(y)$ and external limits show that all horizontal strips lie on vertical axis in between $y = c$ and $y = d$

(III) When region R of area A is bounded by four lines $x = a, x = b$ (lines parallel to y-axis) and $y = c, y = d$ (lines parallel to x-axis).

Here, the bounded region is a rectangle. Thus, in general it is immaterial whether we integrate first along the horizontal strips and then count them vertically or first along the vertical strips and then count them horizontally.

Thus, we can define double integral $\iint_A f(x, y)\, dxdy$ as

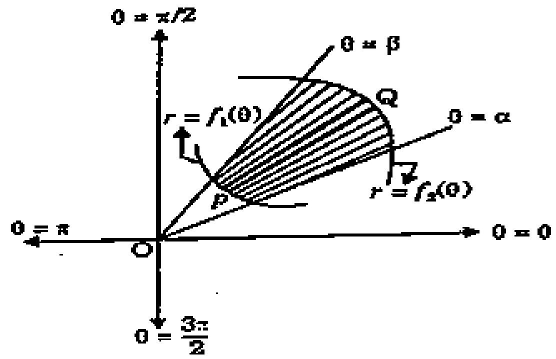
$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\, dxdy = \int_{x=a}^{b}\left[\int_{\substack{y=c \\ (x \to \text{constant})}}^{d} f(x,y)\, dy\right] dx$$

$$\text{or} \quad \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\, dxdy = \int_{y=c}^{d}\left[\int_{\substack{x=a \\ (y \to \text{constant})}}^{x=b} f(x,y)\, dx\right] dy$$

<u>**Note:**</u> If there exist several discontinuities for $f(x, y)$ within or on the boundary of the region of integration, then the change of order of integration does not resuts into the same integrals as shown below

EVALUATION OF DOUBLE INTEGRALS FOR POLAR FUNCTIONS

When region R of area A is bounded by two lines $\theta = \alpha$, $\theta = \beta$ and two curves $r = f_1(\theta)$, $r = f_2(\theta)$



Here, the bounded region is divided into polar strips and taking a polar strip PQ, we can define double integral $\iint_A f(r, \theta)\, dr d\theta$ as

$$\int_{\theta=\alpha}^{\beta}\int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, dr\, d\theta = \int_{\theta=\alpha}^{\beta}\left[\int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, dr\right] d\theta$$

Thus, the above double integral is evaluated first with respect to r (treating θ as constant) and then resulting expression which is a function of θ is integrated with respect to θ between the limits $\theta = \alpha$ and $\theta = \beta$.

Geometrically, in double integral shown by eqn. (8.9), internal limits show that out of all polar strips, a particular polar strip exist in between $r = f_1(\theta)$ and $r = f_2(\theta)$ and external limits show that all polar strips lie between $\theta = \alpha$ and $\theta = \beta$.

Note: Generally, $r = 0$ is taken in place of the curve $r = f_1(\theta)$ in most of the problems discussed in this section.

CHANGE OF DOUBLE INTEGRAL FROM CARTESIAN FORM TO POLAR FORM

To change any cartesian double integral $\iint_A f(x, y)\, dxdy$ into a polar double integral, use the transformations $x = r\cos\theta$ and $y = r\sin\theta$ as well as replace the elementary area $dxdy$ (in cartesian form) by elementary area $rd\theta dr$ (in polar form).

$$\text{Thus,} \quad \iint_A f(x, y)\, dxdy = \iint_A f(r\cos\theta, r\sin\theta)\, rd\theta\, dr$$

Now, the limits of polar double integral expressed in above equation are obtained by dividing the given area of integration *i.e.* area A in polar strips, where the area A is obtained by tracing all the curves associated with the internal and external limits of given cartesian double integral and by considering its geometrical significance either with vertical or with horizontal strips.

CHANGE OF ORDER OF INTEGRATION IN ANY CARTESIAN DOUBLE INTEGRAL

If the limits of integration in a double integral are constant, then the order of integration is immaterial *i.e.* we can change it directly, provided, that the limits of integration are change accordingly.

$$\text{Thus,} \quad \int_{x=a}^{b}\int_{y=c}^{d} f(x, y)\, dxdy = \int_{y=c}^{d}\int_{x=a}^{b} f(x, y)\, dxdy$$

But, if internal limits of a double integral are functions of a variable (x or y), then the change in order of integration will result in the change in the limits of integration as given below

(I) Change of order of integration in $\displaystyle\int_{x=a}^{b}\int_{f_1(x)}^{f_2(x)} f(x, y)\, dxdy$

Plot the curves $x = a$, $x = b$, $y = f_1(x)$ and $y = f_2(x)$ as well as use geometrical significance of given double integral with vertical strips to find area of integration and then divide that area in horizontal strips to change the order of integration in given double integral as

$$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x, y)\, dxdy = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x, y)\, dxdy$$

(Vertical strips) (Horizontal strips)

(II) Change the order of integration in $\int\limits_{y=c}^{d} \int\limits_{x=g_1(y)}^{g_2(y)} f(x, y)\, dxdy$

Plot the curves $y = c, y = d, x = g_1(y)$ and $x = g_2(y)$ as well as use geometrical significance of given double integral with horizontal strips to find area of integration and then divide that area in vertical strips to change the order of integration in given double integral as

$$\int\limits_{y=c}^{d} \int\limits_{x=g_1(y)}^{g_2(y)} f(x, y)\, dxdy = \int\limits_{x=a}^{b} \int\limits_{f_1(x)}^{f_2(x)} f(x, y)\, dxdy$$

(Horizontal strips) (Vertical strips)

Note: (1) While changing order of integration in any double integral, we may find a sum of several double integrals (See Q. 1.)

(2) There exist many double integrals which can be solved conveniently by changing order of integration, infact there are many double integrals which can be solved only by changing order of integration (See Q. 3.)

AREA BY DOUBLE INTEGRAL

(I) Cartesian Form. If we substitute $f(x, y) = 1$ in any cartesian double integral $\iint\limits_A f(x, y)\, dxdy$, then it represents the sum of areas of the elementary rectangles into which the area of integration is divided. Thus, area

$$A = \iint\limits_A dxdy$$

So, if it is required to find area bounded by several cartesian curves $f_1(x, y) = c_1,\ f_2(x, y) = c_2,\ \ldots$ etc., then select the common bounded region after tracing all these curves and then divide the bounded region either in horizontal or vertical strips to find the required area using double integral expressed in above eqn.

(II) Polar Form. In similar fashion, we can state then in polar form

area $$A = \iint\limits_A r d\theta\, dr$$

So, if it is required to find area bounded by several polar curves $f_1(r, \theta) = c_1$, $f_2(r, \theta) = c_2$, ... etc., then select the common bounded region after tracing all these curves and then divide the bounded region in polar strips to find the required area using double integral expressed in above eqn.

VOLUME BY DOUBLE INTEGRAL

If we come across the geometrical significance of double integral

$\iint\limits_A f(x, y)\, dxdy$; $f(x, y) \neq 1$, then it shows volume under the surface $z = f(x, y)$ and over the base of area A in xy-plane where surface $z = f(x, y)$ lies between $z = z_1$ and $z = z_2$ i.e. $z_1 < z < z_2$

So, volume $$= \iint\limits_A (z_2 - z_1)\, dxdy$$

Similar expressions for volume with area in yz-plane or zx-plane are given as

$$\text{Volume} = \iint\limits_A (x_2 - x_1)\, dydz$$

and $$\text{volume} = \iint\limits_A (y_2 - y_1)\, dzdx$$

BETA FUNCTION

Beta function $B(m,n)$ defined by

$$\boxed{B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$

$B(m,n)$ is convergent for $m>0, n>0$

Note: Beta function is also known as Euler's integral of the first kind

ALTERNATIVE FORMS OF BETA FUNCTION

(1) In terms of trigonometric functions:

$$\boxed{B(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta}$$

Proof: Substituting $x=\sin^2\theta$ in the definition of beta function, we get

$$B(m,n)=\int_0^{\pi/2}(\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}(\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

(2) In terms of improper integral:

$$\boxed{B(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx}$$

Proof: Substituting $x=\dfrac{y}{1+y}$ in the definition of beta function,

we get

$$B(m,n)=\int_0^{\infty}\frac{y^{m-1}}{(1+y)^{m-1}}\,\frac{1}{(1+y)^{n-1}}\,\frac{1}{(1+y)^2}\,dy$$

$$=\int_0^{\infty}\frac{y^{m-1}}{(1+y)^{m+n}}\,dy=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

PROPERTIES OF BETA FUNCTION

(1) $\boxed{B(m,n)=B(n,m)}$

Proof: Substituting $x=1-y$ in the definition of beta function, we get

$$B(m,n)=\int_1^0(1-y)^{m-1}\,y^{n-1}(-dy)$$

$$=\int_0^1(1-y)^{m-1}\,y^{n-1}dy$$

$$=\int_0^1 x^{n-1}(1-x)^{m-1}\,dx=B(n,m)$$

(2) $\boxed{B(m,n) = B(m+1,n) + B(m,n+1)}$

Proof: Using $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, we get

$$B(m+1,n) = \int_0^1 x^m (1-x)^{n-1}\,dx$$

and
$$B(m,n+1) = \int_0^1 x^{m-1}(1-x)^n\,dx$$

So, $B(m+1,n) + B(m,n+1)$

$$= \int_0^1 [x^m(1-x)^{n-1} + x^{m-1}(1-x)^n]\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n)$$

Hence, $B(m,n) = B(m+1,n) + B(m,n+1)$

GAMMA FUNCTION

Gamma function Γn defined by

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx \quad ; n>0}$$

Note: (1) Gamma function is also known as Euler's integral of the second kind

(2) $\Gamma 0, \Gamma -1, \Gamma -2, \Gamma -3, \ldots = \infty$

ALTERNATIVE FORMS OF GAMMA FUNCTION

(1) $\boxed{\Gamma n = a^n \int_0^\infty e^{-ax} x^{n-1}\,dx \quad ; n>0}$

[Substituting $x = ay$ in the definition of gamma gunction]

(2) $\boxed{\Gamma n = \frac{1}{n}\int_0^\infty e^{-\left(x^{\frac{1}{n}}\right)}\,dx \quad ; n>0}$

[Substituting $x^n = y$ in the definition of gamma gunction]

(3) $$\boxed{\overline{|n} = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx \quad ; n > 0}$$

[Substituting $x = -\log y$ in the definition of gamma gunction]

(4) $$\boxed{\overline{|n} = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx \quad ; n > 0}$$

[Substituting $x = y^2$ in the definition of gamma gunction]

PROPERTIES OF GAMMA FUNCTION

(1) $$\boxed{\overline{|n} = \underline{|n-1} \quad ; n \in N}$$

e.g. $\overline{|1} = \underline{|0} = 1,\ \overline{|2} = \underline{|1} = 1,\ \overline{|3} = \underline{|2} = 2$ e.t.c.

(2) $$\boxed{\overline{|n} = (n-1)\overline{|n-1} \quad ; n > 1}$$

$$e.g. \overline{\left|\frac{3}{2}\right.} = \frac{1}{2}\overline{\left|\frac{1}{2}\right.},\quad \overline{\left|\frac{5}{2}\right.} = \frac{3}{2}\overline{\left|\frac{3}{2}\right.} = \frac{3}{2}\frac{1}{2}\overline{\left|\frac{1}{2}\right.},$$

$$\overline{\left|\frac{9}{2}\right.} = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\overline{\left|\frac{1}{2}\right.},\quad \overline{\left|\frac{5}{4}\right.} = \frac{1}{4}\overline{\left|\frac{1}{4}\right.} \quad \text{e.t.c.}$$

(3) $$\boxed{\overline{|n}\,\overline{|1-n} = \frac{\pi}{\sin n\pi} \quad ; 0 < n < 1}$$

$$e.g. \overline{\left|\frac{1}{4}\right.}\,\overline{\left|\frac{3}{4}\right.} = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}\pi,\quad \overline{\left|\frac{1}{3}\right.}\,\overline{\left|\frac{2}{3}\right.} = \frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{2\pi}{\sqrt{3}} \quad \text{e.t.c.}$$

Note: $$\overline{\left|\frac{1}{2}\right.}\,\overline{\left|\frac{1}{2}\right.} = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi \Rightarrow \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$$

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(4) $$\boxed{B(m,n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}} \quad ; m,n > 0}$$ **(Relation between beta and gamma function)**

$$e.g.\ B(2,3) = \frac{\overline{|2}\,\overline{|3}}{\overline{|5}} = \frac{\underline{|1}\,\underline{|2}}{\underline{|4}} = \frac{1}{12},$$

$$B\left(5,\frac{1}{2}\right) = \frac{\overline{|5}\,\overline{\left|\frac{1}{2}\right.}}{\overline{\left|\frac{11}{2}\right.}} = \frac{\underline{|4}\sqrt{\pi}}{\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}} = \frac{256}{315} \quad \text{e.t.c.}$$

(5) $$\boxed{\overline{|n}\,\overline{\left|n+\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{2^{2n-1}}\overline{|2n} \quad ; n > 0}$$ **(Legendre's duplication formula)**

$$\text{e.g. } \overline{\left|\frac{1}{3}\right.}\,\overline{\left|\frac{5}{6}\right.} = \frac{\sqrt{\pi}}{2^{-1/3}}\overline{\left|\frac{2}{3}\right.}$$

$$\overline{\left|\frac{1}{5}\right.}\,\overline{\left|\frac{7}{10}\right.} = \frac{\sqrt{\pi}}{2^{-3/5}}\overline{\left|\frac{2}{5}\right.} \quad \text{e.t.c.}$$

$$\text{(6)}\quad \boxed{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\cdots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}\quad ; n\in N}$$

$$\text{e.g.}\quad \Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\cdots\cdots\Gamma\left(\frac{4}{5}\right)=\frac{(2\pi)^{2}}{\sqrt{5}}$$

$$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{6}\right)\cdots\cdots\Gamma\left(\frac{5}{6}\right)=\frac{(2\pi)^{5/2}}{\sqrt{6}}\quad \text{e.t.c.}$$

STANDARD INTEGRALS DERIVED FROM GAMMA FUNCTION

$$\text{(1)}\quad \boxed{\int_0^\infty e^{-ax}\sin bx\ x^{n-1}\,dx=\frac{\Gamma(n)}{\left(a^2+b^2\right)^{n/2}}\sin n\theta\ ;\ \theta=\tan^{-1}(b/a),\ n>0}$$

$$\text{(2)}\quad \boxed{\int_0^\infty e^{-ax}\cos bx\ x^{n-1}\,dx=\frac{\Gamma(n)}{\left(a^2+b^2\right)^{n/2}}\cos n\theta\ ;\ \theta=\tan^{-1}(b/a),\ n>0}$$

Note: **(1)** If $a=0$, then $\int_0^\infty \sin bx\ x^{n-1}\,dx=\frac{\Gamma(n)}{b^n}\sin(n\pi/2)\ ;\ n>0$

(2) If $a=0$, then $\int_0^\infty \cos bx\ x^{n-1}\,dx=\frac{\Gamma(n)}{b^n}\cos(n\pi/2)\ ;\ n>0$