Unit 5
Partial Differential Equations

FORMATION OF PARTIAL DIFFERENTIAL EQUATION BY THE ELIMINATION OF ARBITRARY CONSTANTS

Consider an equation

$$f(x, y, z, a, b) = 0 \qquad \ldots(1)$$

where a and b are arbitrary constants.

Let z be a function of two independent variables x and y. Differentiating equation (1) partially with respect to x and y, we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \qquad \ldots(2)$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \qquad \ldots(3)$$

Eliminating two constants a and b from these equations (1), (2) and (3), we obtain a partial differential equation of the form

$$g(x, y, z, p, q) = 0 \qquad \ldots(4)$$

FORMATION OF PARTIAL DIFFERENTIAL EQUATION BY THE ELIMINATION OF ARBITRARY FUNCTION

Let u and v are two functions of x, y and z which are connected by the relation

$$\phi(u, v) = 0$$

or

$$u = \phi(v) \qquad \ldots(5)$$

Differentiating equation (5) partially with respect to x and y, we obtain

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(6)$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(7)$$

Here, z is considered as a function of x and y.

Now, eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between equations (6) and (7) by the determinant method, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

or

$$p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right) + q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$$

or

$$\frac{\partial(u, v)}{\partial(y, z)}\, p + \frac{\partial(u, v)}{\partial(z, x)}\, q = \frac{\partial(u, v)}{\partial(x, y)} \qquad \ldots(8)$$

where $\frac{\partial(u, v)}{\partial(x, y)} = \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$

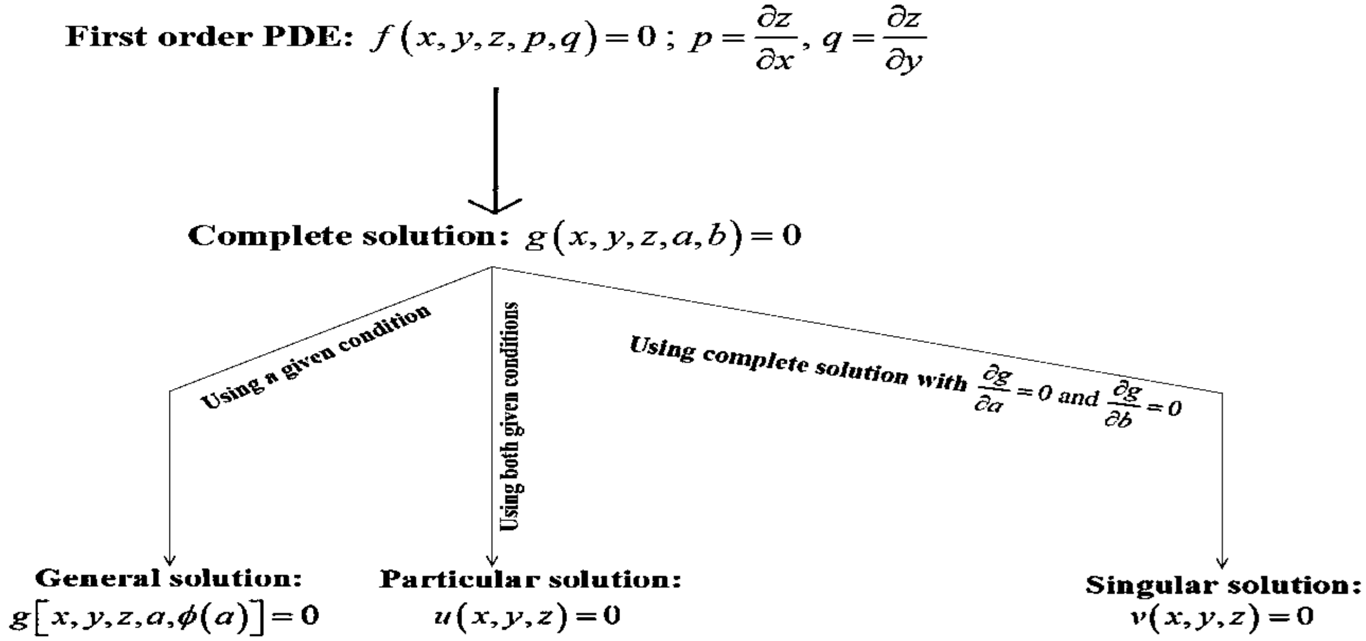
Above equation (8) is generally denoted by

$$Pp + Qq = R \qquad \ldots(9)$$

where P, Q and R are functions of x, y and z.

Above PDE (9) is known as Lagrange's PDE

Different types of solutions of a first order PDE



WORKING PROCEDURE FOR SOLVING $Pp + Qq = R$

(1) Put the given linear partial differential equation of the first order in the standard form $Pp + Qq = R$.

(2) Form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(3) Solve above auxiliary equations by the method of multipliers or by the method of grouping or both to get two independent solutions $u = a$ and $v = b$, where a and b are arbitrary constants.

(4) The general solution of equation written in (1) is then written in one of the following three equivalent forms :

$$\phi(u, v) = 0 \quad \text{or } u = \phi(v) \quad \text{or } v = \phi(u)$$

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER

General form of any non-linear partial differential equation of first order can be given as

$$f(x, y, z, p, q) = 0 \qquad ...(1)$$

Obviously it is a partial differential equation which contains p and q with degree higher than one and terms involving product of p and q.

As there are two independent variables (usually x and y) is partial differential equation (1), its complete solution involves two arbitrary constants.

There exist a general method of solution for partial differential equation (1). Before discussing that, we discuss some standard forms of equation (1) with different combinations of variables x, y, z, p and q which can be solved by specific methods.

STANDARD FORM I

There are some partial differential equations in which x, y and z do not occur, so that those can be written as

$$\boxed{f(p, q) = 0} \qquad \ldots(1)$$

Complete integral of equation (1) is given by

$$z = ax + by + c \qquad \ldots(2)$$

where a, b and c are arbitrary constants and a, b are connected by the relation

$$f(a, b) = 0 \qquad \ldots(3)$$

If $f(a, b) = 0$ reduces to $b = \phi(a)$, the complete integral of the given partial differential equation will be of the form

$$z = ax + \phi(a)y + c \qquad \ldots(4)$$

Remark : Differentiating equation (4) partially with respect to a and c, we get

$$0 = x + \phi'(a)y$$

and $\qquad 0 = 1$

As $0 = 1$ is absurd, hence singular solution does not exist for standard form I.

STANDARD FORM II

There are some partial differential equations in which x, y, z, p and q are combined specifically as

$$\boxed{z = px + qy + f(p, q)} \qquad \ldots(1)$$

Complete integral of equation (1) is given by

$$z = ax + by + f(a, b) \qquad \ldots(2)$$

where a and b are arbitrary constants.

Remark: For above standard form II, singular solution is to be provided in any problem.

STANDARD FORM III

There are some differential equations which do not contain x and y, so that those can be written as

$$\boxed{f(z, p, q) = 0} \qquad \ldots(1)$$

Remark : Singular solution may exist for equation (1)

WORKING PROCEDURE FOR SOLVING $f(z, p, q) = 0$

1. We have to put the values of p and q in the partial differential equation as $\dfrac{dz}{dX}$ and $a\dfrac{dz}{dX}$ respectively.
2. Solve the ordinary differential equation of first order in z and X.
3. Substitute $X = (x + ay)$ in the solution of above ordinary differential equation to find the complete integral of given partial differential equation.

STANDARD FORM IV

Equations of the form $\boxed{f_1(x,p) = f_2(y,q)}$

i.e. partial differential equations in which the variable z does not appear as well as the terms containing p and x are separated from those containing q and y.

To solve such partial differential equations, we put each side equal to an arbitrary constant a

i.e. $\quad f_1(x,p) = f_2(y,q) = a$

Thus, we get $\quad p = F_1(x,a)$ and $q = F_2(y,a)$

Since $\quad dz = p\,dx + q\,dy$

We have, $\quad dz = F_1(x,a)\,dx + F_2(y,a)\,dy$

Integrating it, we get

$$z = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + b$$

which is the required complete integral of the given partial differential equation.

Remarks : (1) Partial differential equations in which functions of x, p and y, q are separated as well as z is contained but distributed in the same ratio with p and q are also solvable by the above method.

e.g. $\quad z^2(p+q) = x+y,\ x^2y^3p^2q = z^3$ etc.

As $\quad z^2(p+q) = x+y$ can be written as

$$pz^2 - x = y - qz^2$$

i.e. z is multiplied with p as well as with q

and $\quad x^2y^3p^2q = z^3$ can be written as

$$\frac{x^2p^2}{z^2} = \frac{z}{y^3q}$$

i.e., $\frac{1}{z^2}$ is multiplied with p^2 as well as $\frac{1}{z}$ is multiplied with q.

(2) While solving a partial differential equation like

$$yzp^2 = q \quad \text{or } z^{2/3}p^2 = \frac{q}{yz^{1/3}},$$

z can not be distributed in the same ratio with p and q

i.e. $z^{2/3}$ is multiplied with p^2 but $z^{1/3}$ is not multilied with q.

(3) Partial differential equations of the form

$$y - p = f(x-q)$$

or $\quad x - q = \quad (y-p)$

can also be solved by the above method.

Assuming $\quad x - q = a, \quad y - p = k \quad$ and $\ k = f(a)$

i.e. $\quad p = y - k \quad$ and $q = x - a$

and $\quad dz = p\,dx + q\,dy$ gives

$$dz = (y-k)\,dx + (x-a)\,dy$$
$$= (y\,dx + x\,dy) - k\,dx - a\,dy$$

or $\quad \int dz = \int d(xy) - k\int dx - a\int dy$

or $\quad z = xy - kx - ay + b$

or $\quad z = xy - f(a)\,x - ay + b \qquad [\because k = f(a)]$

(4) Singular solution may exist for above standard form.

GENERAL METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE BUT OF ANY DEGREE (CHARPIT'S METHOD)

This is a general method for solving partial differential equations of order one with two independent variables. Let the given equation be

$$\boxed{f(x, y, z, p, q) = 0} \qquad \ldots(1)$$

Now, if we are able to find another relation

$$g(x, y, z, p, q) = 0 \qquad \ldots(2)$$

containing x, y, z, p and q, then we can solve equations (1) and (2) for p and q and substitute in

$$dz = pdx + qdy \qquad \ldots(3)$$

Solution of equation (3), (if it exists) is the complete solution of given partial differential equation (1).

Remarks : (1) Since the solution by this method is generally more complicated, this method is applied to solve equations which can not be reduced to any of the standard forms as discussed before.

(2) This method is also applicable to solve Lagrange's linear partial differential equations of first order.

WORKING PROCEDURE WHILE USING CHARPIT'S METHOD

(1) Shift all the terms of the given partial differential equation to L.H.S. and consider the given partial differential equation as

$$f(x, y, z, p, q) = 0$$

(2) Write down the Charpit's auxilliary equation

$$\frac{dp}{\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)} = \frac{dz}{\left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

(3) Substitute the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}$ and $\frac{\partial f}{\partial q}$ in the auxilliary equation written in step (2).

(4) Select two proper fractions from the auxilliary equation so that the resulting integral may come out to be the simplest relation containing atleast one of p and q.

(5) Solve the relation obtained in step (4) with given partial differential equation to find values of p and q in terms of $x, y,$ and z.

(6) Substitute values of p and q in

$$dz = pdx + qdy$$

which on integration yields the complete solution of given partial differential equation.