Unit 2
Vector Calculus

VECTOR DIFFERENTIAL OPERATOR (DEL)

The vector differential operator del is denoted by $\vec{\nabla}$ and it is defined as

$$\boxed{\vec{\nabla} = \hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}}$$

del is also called nebla.

GRADIENT

If $f(x, y, z)$ is a scalar point function, then gradient of f is defined as $\vec{\nabla} f$ and is written as grad f.

$$\text{grad } f = \vec{\nabla} f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f$$

$$= \hat{i}\ \frac{\partial f}{\partial x} + \hat{j}\ \frac{\partial f}{\partial y} + \hat{k}\ \frac{\partial f}{\partial z}$$

Clearly, grad f is a vector quantity.

Again, $$\vec{\nabla} \cdot \vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

or $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

∇^2 is called Laplacian operator.

SOME IMPORTANT POINTS TO REMEMBER

1. $\vec{\nabla} f$ is a vector normal to the surface $f = c$ and its magnitude is equal to the rate of change in f along this normal.
2. Unit normal vector to the surface $f = c$ is given by $\dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$
3. Directional derivative of f at a point P in the direction of unit vector $\hat{a}$ is equal to

   $\vec{\nabla} f \cdot \hat{a}$

4. Maximum directional derivative of f is along the normal (*i.e.* in the direction of $\vec{\nabla} f$) and its value is $|\vec{\nabla} f|$.

DIVERGENCE OF A VECTOR POINT FUNCTION

Divergence of a vector point function $\vec{f}$ is denoted by divergence $\vec{f}$ and is defined as

$$\text{div } \vec{f} = \vec{\nabla} \cdot \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{f}$$

$$= \hat{i} \cdot \frac{\partial \vec{f}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{f}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{f}}{\partial z}$$

If $\qquad \vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k},$

then $\qquad \text{div } \vec{f} = \vec{\nabla} \cdot f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Clearly, divergence of a vector point function is a scalar point function.

Note: $\vec{\nabla} \cdot \vec{f} \neq \vec{f} \cdot \vec{\nabla}$ because $\vec{f} \cdot \vec{\nabla}$ is an operator and not a vector.

CURL OF A VECTOR POINT FUNCTION

The curl (or rotation) of a vector point function $\vec{f}$ is denoted by curl $\vec{f}$ and is defined as

$$\text{curl } \vec{f} = \vec{\nabla} \times \vec{f}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{f}$$

$$= \hat{i} \times \frac{\partial \vec{f}}{\partial x} + \hat{j} \times \frac{\partial \vec{f}}{\partial y} + \hat{k} \times \frac{\partial \vec{f}}{\partial z}$$

If $\qquad \vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k},$

then $\qquad \text{curl } \vec{f} = \vec{\nabla} \times f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Clearly, curl of a vector point function is a vector point function.

Note: (1) While expanding the determinant mentioned above, operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are to be written always before the functions f_1, f_2, f_3.

(2) The curl is connected with rotation of the vector field, that is why the name rotation is used for curl. If Curl $f = 0$, then f is called irrotational vector.

PROPERTIES AND IDENTITIES OF GRADIENT, DIVERGENCE AND CURL

SOME USEFUL PROPERTIES OF GRAD, DIV AND CURL

1. div grad $\phi = \nabla^2\phi$, where ϕ is a scalar point function.

Proof : div (grad ϕ) $= \vec{\nabla} \cdot (\vec{\nabla}\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Note: $\nabla^2\phi$ can become zero only if $\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial z^2} = 0$

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2. Curl grad $\phi = \vec{0}$

Proof : Curl grad $\phi = \vec{\nabla} \times (\vec{\nabla}\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \vec{0}$$

3. div curl $f = 0$, where f is a vector point function.

Proof : Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Now, curl $\vec{f} = \vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$

$\therefore$ div curl $\vec{f} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{f})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} + \frac{\partial^2 f_1}{\partial y\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} = 0$$

4. Curl curl $\vec{f}$ = grad div $\vec{f} - \sum \frac{\partial^2 \vec{f}}{\partial x^2}$

i.e. $\vec{\nabla} \times (\vec{\nabla} \times \vec{f}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \nabla^2 \vec{f}$

Proof : Let $\vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$, then $\vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\therefore \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{f}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) & \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) & \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \end{vmatrix} = \sum \left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right] \hat{i}$$

$$= \sum \left[\frac{\partial^2 f_2}{\partial y \partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z \partial x}\right] \hat{i} = \sum \left[\left(\frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial x \partial z}\right) - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right] \hat{i}$$

$$= \sum \left[\left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial x \partial z}\right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right] \hat{i}$$

$$= \sum \left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_1\right] \hat{i}$$

$$= \sum \left[\frac{\partial}{\partial x}(\vec{\nabla} \cdot \vec{f}) - \nabla^2 f_1\right] \hat{i} = \sum \hat{i} \frac{\partial}{\partial x}(\vec{\nabla} \cdot \vec{f}) - \nabla^2 \left(\sum f_1 \hat{i}\right)$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \nabla^2 \vec{f}$$

SOME USEFUL VECTOR IDENTITIES

If $\vec{a}$ and $\vec{b}$ are differentiable vector point functions and u is a differentiable scalar point function, then we have

I. div $(u\vec{a}) = u$ (div $\vec{a}$) + $\vec{a}$. grad u or $\vec{\nabla} \cdot (u\vec{a}) = u(\vec{\nabla} \cdot \vec{a}) + \vec{a} \cdot (\vec{\nabla} u)$

Proof : div $(u\vec{a}) = \vec{\nabla} \cdot (u\vec{a}) = \hat{i} \cdot \frac{\partial}{\partial x}(u\vec{a}) + \hat{j} \cdot \frac{\partial}{\partial y}(u\vec{a}) + \hat{k} \cdot \frac{\partial}{\partial z}(u\vec{a})$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x}(u\vec{a}) = \sum \hat{i} \cdot \left[\frac{\partial u}{\partial x}\vec{a} + u\frac{\partial \vec{a}}{\partial x}\right]$$

$$= \sum \left[\hat{i} \cdot \left(\frac{\partial u}{\partial x}\vec{a}\right)\right] + \sum \left[\hat{i} \cdot \left(u\frac{\partial \vec{a}}{\partial x}\right)\right] = \sum \left[\left(\frac{\partial u}{\partial x}\hat{i}\right) \cdot \vec{a}\right] + \sum \left[u\left(\hat{i} \cdot \frac{\partial \vec{a}}{\partial x}\right)\right]$$

$[\because \vec{a} \cdot (k\vec{b}) = (k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})]$

$$= \left[\sum \frac{\partial u}{\partial x}\hat{i}\right] \cdot \vec{a} + u \sum \left(\hat{i} \cdot \frac{\partial \vec{a}}{\partial x}\right) = (\vec{\nabla} u) \cdot \vec{a} + u(\vec{\nabla} \cdot \vec{a})$$

$$= u(\vec{\nabla} \cdot \vec{a}) + \vec{a} \cdot (\vec{\nabla} u)$$

II. $\text{Curl}(u\vec{a}) = (\text{grad}\, u) \times \vec{a} + u \,\text{curl}\, \vec{a}$ or $\vec{\nabla} \times (u\vec{a}) = (\vec{\nabla} u) \times \vec{a} + u(\vec{\nabla} \times \vec{a})$

Proof: $\text{Curl}(u\vec{a}) = \vec{\nabla} \times (u\vec{a}) = \hat{i} \times \frac{\partial}{\partial x}(u\vec{a}) + \hat{j} \times \frac{\partial}{\partial y}(u\vec{a}) + \hat{k} \times \frac{\partial}{\partial z}(u\vec{a})$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(u\vec{a}) = \sum \hat{i} \times \left[\frac{\partial u}{\partial x}\vec{a} + u\frac{\partial \vec{a}}{\partial x}\right]$$

$$= \sum \left[\hat{i} \times \left(\frac{\partial u}{\partial x}\vec{a}\right)\right] + \sum \left[\hat{i} \times \left(u\frac{\partial \vec{a}}{\partial x}\right)\right] = \sum \left[\left(\frac{\partial u}{\partial x}\hat{i}\right) \times \vec{a}\right] + \sum u\left(\hat{i} \times \frac{\partial \vec{a}}{\partial x}\right)$$

$[\because \ \vec{a} \times k\vec{b} = k\vec{a} \times \vec{b} = k(\vec{a} \times \vec{b})]$

$$= \left[\sum \frac{\partial u}{\partial x}\hat{i}\right] \times \vec{a} + u\sum\left(\hat{i} \times \frac{\partial \vec{a}}{\partial x}\right) = (\vec{\nabla} u) \times \vec{a} + u(\vec{\nabla} \times \vec{a})$$

III. $\text{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{curl}\, \vec{a} - \vec{a} \cdot \text{curl}\, \vec{b}$ or $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$

Proof: $\text{div}(\vec{a} \times \vec{b}) = \vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \hat{i} \cdot \frac{\partial}{\partial x}(\vec{a} \times b) + \hat{j} \cdot \frac{\partial}{\partial y}(\vec{a} \times \vec{b}) + \hat{k} \cdot \frac{\partial}{\partial z}(\vec{a} \times \vec{b})$

$$= \sum \left[\hat{i} \cdot \frac{\partial}{\partial x}(\vec{a} \times \vec{b})\right] = \sum \left[\hat{i} \cdot \left(\frac{\partial \vec{a}}{\partial x} \times \vec{b} + \vec{a} \times \frac{\partial \vec{b}}{\partial x}\right)\right]$$

$$= \sum \left[\hat{i} \cdot \left(\frac{\partial \vec{a}}{\partial x}\vec{b}\right)\right] + \sum \left[\hat{i} \cdot \left(\vec{a} \times \frac{\partial \vec{b}}{\partial x}\right)\right] = \sum \left[\left(\hat{i} \times \frac{\partial \vec{a}}{\partial x}\right) \cdot \vec{b}\right] - \sum \left[\hat{i} \cdot \left(\frac{\partial \vec{b}}{\partial x} \times \vec{a}\right)\right]$$

$[\because \ \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}]$

$$= \sum \left[\left(\hat{i} \times \frac{\partial \vec{a}}{\partial x}\right) \cdot \vec{b}\right] - \sum \left[\left(\hat{i} \times \frac{\partial \vec{b}}{\partial x}\right) \cdot \vec{a}\right] \quad [\because \ \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}]$$

$$= \left[\sum \left(\hat{i} \times \frac{\partial \vec{a}}{\partial x}\right)\right] \cdot \vec{b} - \left[\sum \left(\hat{i} \times \frac{\partial \vec{b}}{\partial x}\right)\right] \cdot \vec{a} = (\vec{\nabla} \times \vec{a}) \cdot \vec{b} - (\vec{\nabla} \times \vec{b}) \cdot \vec{a}$$

$$= \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$$

IV. Curl $(\vec{a} \times \vec{b}) = (\vec{b} \cdot \vec{\nabla})\vec{a} - \vec{b}(\vec{\nabla} \cdot \vec{a}) - (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{a}(\vec{\nabla} \cdot \vec{b})$

Proof: Curl $(\vec{a} \times \vec{b}) = \vec{\nabla} \times (\vec{a} \times \vec{b}) = (\vec{\nabla} \cdot \vec{b})\vec{a} - (\vec{\nabla} \cdot \vec{a})\vec{b}$

$$= \hat{i} \times \frac{\partial}{\partial x}(\vec{a} \times \vec{b}) + \hat{j} \times \frac{\partial}{\partial y}(\vec{a} \times \vec{b}) + \hat{k} \times \frac{\partial}{\partial z}(\vec{a} \times \vec{b})$$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(\vec{a} \times \vec{b})$$

$$= \sum \left[\hat{i} \times \left(\vec{a} \times \frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x} \times \vec{b}\right)\right] = \sum \left[\hat{i} \times \left(\vec{a} \times \frac{\partial \vec{b}}{\partial x}\right)\right] + \sum \left[\hat{i} \times \left(\frac{\partial \vec{a}}{\partial x} \times \vec{b}\right)\right]$$

$$= \sum \left[\left(\hat{i} \cdot \frac{\partial \vec{b}}{\partial x}\right)\vec{a} - (\hat{i} \cdot \vec{a})\frac{\partial \vec{b}}{\partial x}\right] + \sum \left[(\hat{i} \cdot \vec{b})\frac{\partial \vec{a}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{a}}{\partial x}\right)\vec{b}\right]$$

$$= \sum \left[\left(\hat{i} \cdot \frac{\partial \vec{b}}{\partial x}\right)\vec{a}\right] - \sum \left[(\vec{a} \cdot \hat{i})\frac{\partial \vec{b}}{\partial x}\right] + \sum \left[(\vec{b} \cdot \hat{i})\frac{\partial \vec{a}}{\partial x}\right] - \sum \left[\left(\hat{i} \cdot \frac{\partial \vec{a}}{\partial x}\right)\vec{b}\right]$$

$$= \left[\sum \left(\hat{i} \cdot \frac{\partial \vec{b}}{\partial x}\right)\right]\vec{a} - \left[\vec{a} \cdot \sum \hat{i}\frac{\partial}{\partial x}\right]\vec{b} + \left[\vec{b} \cdot \sum \hat{i}\frac{\partial}{\partial x}\right]\vec{a} - \left[\sum \left(\hat{i} \cdot \frac{\partial \vec{a}}{\partial x}\right)\right]\vec{b}$$

$$= (\vec{\nabla} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{\nabla})\vec{b} + (\vec{b} \cdot \vec{\nabla})\vec{a} - (\vec{\nabla} \cdot \vec{a})\vec{b}$$

$$= (\vec{b} \cdot \vec{\nabla})\vec{a} - \vec{b}(\vec{\nabla} \cdot \vec{a}) - (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{a}(\vec{\nabla} \cdot \vec{b})$$

V. $\vec{\nabla}(\vec{a} \cdot \vec{b})$ or grad $(\vec{a} \cdot \vec{b}) = (\vec{b} \cdot \vec{\nabla})\vec{a} + (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{b} \times (\vec{\nabla} \times \vec{a}) + \vec{a} \times (\vec{\nabla} \times \vec{b})$

Proof: grad $(\vec{a} \cdot \vec{b}) = \vec{\nabla}(\vec{a} \cdot \vec{b}) = \hat{i}\frac{\partial}{\partial x}(\vec{a} \cdot \vec{b}) + \hat{j}\frac{\partial}{\partial y}(\vec{a} \cdot \vec{b}) + \hat{k}\frac{\partial}{\partial z}(\vec{a} \cdot \vec{b})$

$$= \sum \hat{i}\frac{\partial}{\partial x}(\vec{a} \cdot \vec{b}) = \sum \hat{i}\left(\vec{a} \cdot \frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x} \cdot \vec{b}\right) = \sum \left(\vec{a} \cdot \frac{\partial \vec{b}}{\partial x}\right)\hat{i} + \sum \left(\frac{\partial \vec{a}}{\partial x} \cdot \vec{b}\right)\hat{i} \quad ...(i)$$

By the definition of vector triple product

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \Rightarrow (\vec{a} \cdot \vec{b})\vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - \vec{a} \times (\vec{b} \times \vec{c})$$

$$\Rightarrow \left(\vec{a} \cdot \frac{\partial \vec{b}}{\partial x}\right)\hat{i} = (\vec{a} \cdot \hat{i})\frac{\partial \vec{b}}{\partial x} - \vec{a} \times \left(\frac{\partial \vec{b}}{\partial x} \times \hat{i}\right) = (\vec{a} \cdot \hat{i})\frac{\partial \vec{b}}{\partial x} + \vec{a} \times \left(\hat{i} \times \frac{\partial \vec{b}}{\partial x}\right)$$

$$\Rightarrow \sum \left[\left(\vec{a} \cdot \frac{\partial \vec{b}}{\partial x}\right)\hat{i}\right] = \sum \left[(\vec{a} \cdot \hat{i})\frac{\partial \vec{b}}{\partial x}\right] + \sum \left[\vec{a} \times \left(\hat{i} \times \frac{\partial \vec{b}}{\partial x}\right)\right] = \left[\vec{a} \cdot \sum \hat{i}\frac{\partial}{\partial x}\right]\vec{b} + \vec{a} \times \sum \left(\hat{i} \times \frac{\partial \vec{b}}{\partial x}\right)$$

$$= (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{a} \times (\vec{\nabla} \times \vec{b}) \quad ...(ii)$$

Similarly, $\sum \left(\frac{\partial \vec{a}}{\partial x} \cdot \vec{b}\right)\hat{i} = \sum \left(\vec{b} \cdot \frac{\partial \vec{a}}{\partial x}\right)\hat{i} = (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{b} \times (\vec{\nabla} \times \vec{a}) \quad ...(iii)$

From *(i)*, *(ii)* and *(iii)*

$$\text{grad}(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{a} \times (\vec{\nabla} \times \vec{b}) + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{b} \times (\vec{\nabla} \times \vec{a})$$

$$= (\vec{a} \cdot \vec{\nabla})\vec{b} + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{a} \times (\vec{\nabla} \times \vec{b}) + \vec{b} \times (\vec{\nabla} \times \vec{a})$$

LINE INTEGRAL

Any integral which is to be evaluated along a curve is called a line integral.

CARTESIAN FORM OF LINE INTEGRAL

Let $\bar{F} = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$

Since $\bar{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

$\therefore$ $d\bar{r} = (dx\hat{i} + dy\hat{j} + dz\hat{k})$

Therefore

$$\int_C \bar{F}.d\bar{r} = \int_C (F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}) \,.\, (dx\hat{i} + dy\hat{j} + d_3\hat{k})$$

$$= \int_C (F_1 dx + F_2 dy + F_3 dz)$$

PARAMETRIC FORM OF LINE NTEGRAL

If the equation of curve C is given in parametric form like

$$x = x(t);\ y = y(t);\ z = z(t)$$

then

$$\int_C \bar{F}.d\bar{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_C (F_1 dx + F_2\, dy + F_3\, dz)$$

$$= \int_C \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

WORK DONE BY A FORCE

Let $\bar{F}$ represent the force acting on a particle moving along an arc AB. The work done during a small displacement $\delta\bar{r}$ is $\vec{F}.\delta\bar{r}$.

Hence, the total work done by force $\bar{F}$ during displacement from A to B is given by $\int_A^B \bar{F}.d\bar{r}$

Note: $\int_C \bar{F} \times d\bar{r}$ and $\int_C f\, d\bar{r}$ for any scalar f are other forms of line integral.

SURFACE INTEGRAL

Any integral which is to be evaluated over a surface is said to be a surface integral.

The surface integral over S is defined by

$$\iint_S \bar{F}.d\bar{s} = \iint_S \bar{F}.\hat{n}ds,$$

$$\text{where} \quad ds = \frac{dxdy}{|\hat{k}\cdot\hat{n}|} = \frac{dydz}{|\hat{i}\cdot\hat{n}|} = \frac{dzdx}{|\hat{j}\cdot\hat{n}|}$$

Note: Two other types of surface integrals are $\iint_S \bar{F}\times d\bar{s}$ and $\iint_S \phi\, d\bar{s}$ which are both vectors.

VOLUME INTEGRAL

Any integal which is to be evaluated over a volume is called a volume integral.

If V is a volume bounded by a surface S, then $\iiint_V \phi\, dV$ or $\iiint_V \vec{F}\, dV$ are called volume integrals.

If the volume V is to be subdivided into small cuboids by draving planes parallel to the coordinate planes, then $dV = dxdydz$.

$$\iiint_V \phi\, dV = \iiint_V \phi\, dxdydz$$

If $\quad \vec{F} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$, then

$$\iiint_V \vec{F}\, dV = \iiint_V (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\, dV$$

$$= i\iiint_V F_1\, dxdydz + \hat{j}\iiint_V F_2\, dxdydz + \hat{k}\iiint_V F_3\, dxdydz$$