Unit 2

Differential Calculus – II

DEFINITION OF PARTIAL DIFFERENTIATION

A partial derivative of a function of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are kept constant.

Consider a function z of two independent variables $z = f(x, y)$.

The partial derivative of z with respect to x is denoted by $\frac{\partial z}{\partial x}$ and is defined as the limit $\boxed{\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}}$ and is also denoted by z_x or $f_x(x, y)$.

Similarly, partial derivative of z w.r.t. y is

$\boxed{\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}}$ and is also denoted by z_y or $f_y(x, y)$.

Remarks: (1) Higher order partial derivatives :

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \\ &\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy} \end{aligned}\right\}$$

Similarly third and higher order partial derivatives can be found.

(2) The order of differentiation is immaterial in partial differentiation if the derivatives involved are continuous.

Thus, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ **or** $f_{xy} = f_{yx}$

HOMOGENEOUS FUNCTION

An expression of the form $f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_{n-1} x y^{n-1} + a_n y^n$ is a polynomial in x, y such that the degree of each of the term is same.

Thus, f is a homogenous function of degree n.

Also,

$$f_{(x,y)} = x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \dots + a_{n-1} \left(\frac{y}{x}\right)^n + a_n \left(\frac{y}{x}\right)^n \right] = x^n \phi\left(\frac{y}{x}\right)$$

Thus, $z = f(x, y)$, which can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \psi\left(\frac{x}{y}\right)$ is a homogeneous function of degree n, where n can be positive, negative or zero real number.

Remark: A function $u = f(x, y, z)$ is a homogeneous function of degree n if

$$u = f(x, y, z) = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$$

EULER'S THEOREM FOR HOMOGENEOUS FUNCTION

Theorem : If $u = f(x, y)$ is a homogeneous function of degree n. Then

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf} \quad \text{OR} \quad \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

Corollary : *If f is a homogeneous function of degree n, then*

$$\boxed{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f}$$

Proof : Differentiating $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ partially w.r.t x and y, we get

$$\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y \partial x} = n\frac{\partial f}{\partial x} \quad \text{and} \quad x\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y}$$

Multiplying the above equations by x and y respectively and adding, we get

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = (n-1)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

$$= n(n-1)f \quad \text{[Using Euler's theorem]}$$

Remark: If $u = f(x, y, z)$ is a homogeneous function of three variables of degree n, then $\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf}$

MAXIMA AND MINIMA OF TWO AND THREE VARIABLES (WITHOUT ANY CONSTRAINT)

INTRODUCTION

Before this, You have studied the maxima and minima of functions of one variable. You have already discussed necessary and sufficient conditions for existence of maxima and minima of function of one variable in previous classes.

Now, we have to discuss maxima and minima of functions of two or more variables by the help of various methods. Study of maxima and minima of functions of several variables is very important because of its application in engineering like in designing multistage rockets, differential geometry etc.

Further, we have to discuss maxima and minima of functions of two or more variables given with one or several conditions in same variables. Lagrange's method of multipliers is recognised as one of the important method for this purpose.

MAXIMA AND MINIMA OF A FUNCTION OF TWO VARIABLES $f(x, y)$

(1) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and solve $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ to find all probable extrema i.e. $(a, b), (c, d), \ldots\ldots$ Let us consider one pair of values of $f(x, y)$ as (a, b) for further investigation.

(2) Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at point (a, b).

(3) If $rt - s^2 > 0$ and $r < 0$, then (a, b) is maxima.
If $rt - s^2 > 0$ and $r > 0$, then (a, b) is minima.

(4) If $rt - s^2 < 0$, then (a, b) is neither maxima nor minima.

(5) If $rt - s^2 = 0$, the test fails.
Hence, we need some other method to solve the problem.

Note : In similar manner, we can discuss the possibility of maxima or minima for all points obtained in step (1)

Remark: If $f(x, y, z)$ is a function of three variables x, y, z which is given with a condition $g(x, y, z) = 0$ in variables x, y, z and it is possible to rewrite $g(x, y, z) = 0$ for any of the variables x, y, z in terms of remaining two variables, then we can reduce $f(x, y, z)$ as a function of two variables by the help of $g(x, y, z)$ and thus we can apply the same procedure mentioned above.

MAXIMA AND MINIMA OF A FUNCTION OF THREE VARIABLES $f(x, y, z)$

Here, a set of necessary conditions for existence of maxima and *minima* of $f(x, y, z)$ is given as $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$

On solving these three equations, we get values of x, y, z as a, b, c which are called *stationary* or turning values of function $f(x, y, z)$.

Now, by using

$$A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial y^2}, C = \frac{\partial^2 f}{\partial z^2},$$

$$F = \frac{\partial^2 f}{\partial y \partial z}, G = \frac{\partial^2 f}{\partial z \partial x}, H = \frac{\partial^2 f}{\partial x \partial y}$$

Following steps describe the nature of point (a, b, c)

(i) (a, b, c) is a maxima

when A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

are alternately negative and positive at (a, b, c)

(ii) (a, b, c) is a minima when the above three values are positive at (a, b, c)

WORKING PROCEDURE FOR FINDING THE EXTREME VALUE OF $u = f(x, y, z)$ SUBJECT TO THE CONDITION $g(x, y, z) = 0$ USING LAGRANGE'S METHOD OF MULTIPLIERS

(1) Using $u = f(x, y, z)$ (function for which we have to find extrema) and given condition $g(x, y, z) = 0$ and an unknown real constant λ (Lagrange's multiplier), describe a new function $F(x, y, z)$ as

$$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$$

(2) Differentiate $F(x, y, z)$ partially w.r.t. x, y, z and obtain $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$

(3) Use $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and $g(x, y, z) = 0$ to find values of x, y, z and λ

(4) Substitute values of x, y, z in $u = f(x, y, z)$ to find its extreme value.

Remarks: (1) We can solve all the problems of extremization of $u = f(x, y, z)$ with the condition $g(x, y, z) = 0$ by Lagrange's method of multipliers as well as the method described in Remark (2) of 5.4, if $g(x, y, z) = 0$ can be expressed a function for any variable in terms of remaining two variables.

(2) If, we have to find extreme value of $u = f(x, y, z)$ subject to the conditions $g(x, y, z) = 0$ and $h(x, y, z) = 0$, then use

$$F(x, y, z) = f(x, y, z) + \lambda_1 g(x, y, z) + \lambda_2 h(x, y, z)$$

where λ_1, λ_2 are Lagrange's multipliers.

(3) Lagrange's method of multipliers is applicable for problems of extremization of $u = f(x_1, x_2, \ldots\ldots, x_n)$ *i.e.* for a function of any number of variables given with $g(x_1, x_2, \ldots\ldots, x_n) = 0$

(4) In step (3) of above working procedure, instead of solving $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and $g(x, y, z) = 0$ for four unknowns x, y, z, λ, we can eliminate x, y, z, λ by these four equations in connection of $u = f(x, y, z)$

(5) This method is not capable of describing nature of extreme value (maximum or minimum). It is the main drawback of this method.