Unit 4
Coordinate Geometry of Three Dimensions

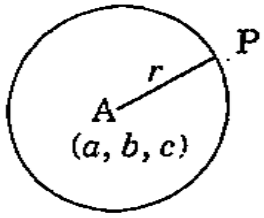
DEFINITION

The locus of a variable point in space, whose distance from a fixed point is constant, is called a sphere.

The fixed point is called the centre and constant distance is called the radius of the sphere.

EQUATION OF A SPHERE IN DIFFERENT FORMS

(a) **Central Form :** Let $A(a, b, c)$ be the coordinates of centre and r be the radius of a given sphere.



Then $\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2}$

is the required equation of the sphere.

Remark : If centre of the sphere is origin (0,0,0) and radius is r, then equation of sphere is given as $(x-0)^2 + (y-0)^2 + (z-0)^2 = r^2$

or $\boxed{x^2 + y^2 + z^2 = r^2}$

which is called the simplest form of a sphere.

(b) **General Form :** $\boxed{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$

By observing above equation, it is obvious to say that

(i) It is a second degree equation in variables x, y, z.

(ii) Coefficients of x^2, y^2 and z^2 are unity (generally equal).

(iii) There is no term containing xy, yz and zx.

To find centre and radius of a general sphere :

centre of the general sphere is the point $(-u, -v, -w)$

and its radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Remarks : (1) Equation (1.4) contains four arbitrary constant u, v, w and d. So, four conditions are required to find the equation of sphere.

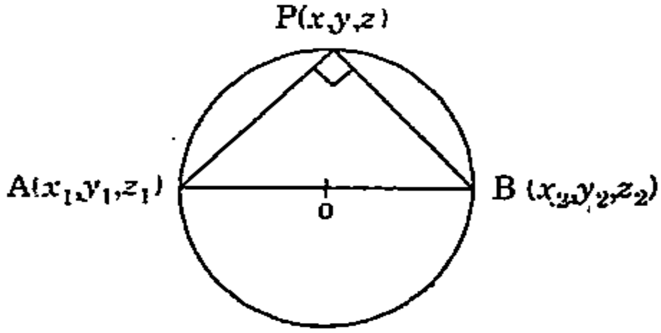
(2) It is to be carefully noted that in equation (1.4), coefficients of x^2, y^2, z^2 should be unity. If they are equal but not unity, then divide by the coefficient of x^2 or y^2 or z^2 to make them unity.

(3) Sphere (1.4) will be real, imaginary or a point sphere according as $u^2 + v^2 + w^2 > d$ or $< d$ or $= d$.

(4) Coordinates of centre for general equation of a shpere are

$$\left(-\frac{1}{2} \text{ coeff. of } x, -\frac{1}{2} \text{ coeff. of } y, -\frac{1}{2} \text{ coeff. of } z\right)$$

(*c*) **Diameter Form :** Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the end points of the diameter of the sphere. Let $P(x, y, z)$ be any point on the surface of the sphere.



Then, $$\boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0}$$

is the required equation of the sphere in diameter form.

(*d*) **Equation of a sphere passing through four given points (x_r, y_r, z_r) ; $r = 1, 2, 3, 4$.**

Let the required equation of the sphere represented by equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

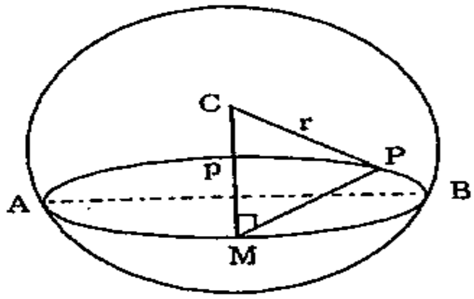
which passes through four points (x_r, y_r, z_r); $r = 1, 2, 3, 4$. Hence, we have

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$
$$x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0$$
$$x_3^2 + y_3^2 + z_3^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0$$
$$x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0$$

Eliminating 1, u, v, w and d from above equations, we get the four point form of the equation of the sphere.

INTERSECTION OF SPHERE WITH PLANE AND INTERSECTION OF TWO SPHERES

PLANE SECTION OF A SPHERE



Let C be the centre of sphere and P be any point on the intersection of the sphere and the plane. If CM is the perpendicular to the plane section APB, then

$$PM = \sqrt{(CP^2 - CM^2)}$$
$$= \text{constant.}$$

as $CP = r$(radius of the sphere)

and the points C and M are also fixed. Therefore the point P moves in the plane such that its distance from a fixed point M is constant. Hence P describes a circle. The centre of the circle is point M (foot of the perpendicular CM) and radius is PM.

Since point P lies on both the plane and the sphere, the two equations, *i.e.*, of plane and the sphere together represent a circle. Thus, the general equations of a circle are

$$\boxed{x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d = 0; \quad Ax + By + Cz + D = 0}$$

GREAT CIRCLE

A circle is said to be great circle if its plane passes through the centre of the sphere and then the centre of the circle coincides with the centre of the sphere and radius with the radius of the sphere.

INTERSECTION OF TWO SPHERES

The curve of intersection of two spheres is a circle. Let the equations of two spheres be

$$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \quad ...(1)$$

and $$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0 \quad ...(2)$$

Now, $S_1 - S_2 = 0$ provides $(x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1)$

$$- (x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2) = 0$$

or $$2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + d_1 - d_2 = 0 \quad ...(3)$$

which being a first degree equation in x, y, z represents a plane. Thus, the curve of intersection of two given spheres is the same as that of the intersection of the plane (3) with any of the given spheres (1) and (2)

Hence, the curve of intersection of two spheres is a circle.

ANY SPHERE THROUGH A GIVEN CIRCLE

Let the circle be given by

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$
$$P = Ax + By + Cz + D = 0$$

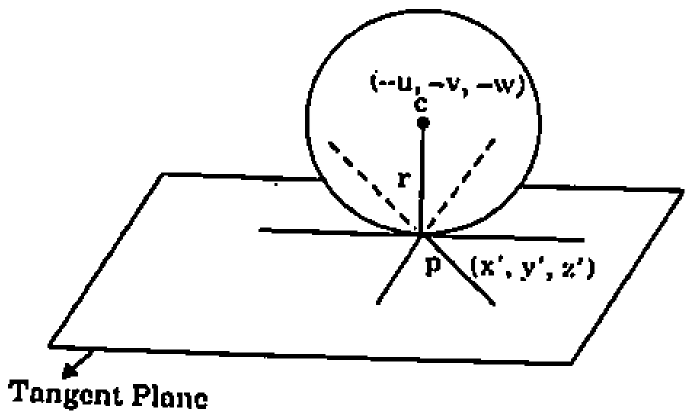
Consider the equation $S + \lambda P = 0$

i.e., $(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) + \lambda (Ax + By + Cz + D) = 0$

Here, $S + \lambda P = 0$ represents a sphere through the circle $S = 0, P = 0$

TANGENT PLANE
AND
ANGLE OF INTERSECTION OF TWO SPHERES

EQUATION OF THE TANGENT PLANE AT A POINT



The equation of the tangent plane of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point (x', y', z') is given as :

$$\boxed{xx' + yy' + zz' + u(x + x') + v(y + y') + w(z + z') + d = 0}$$

Remarks : (1) The equation of tangent plane to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x', y', z') is $xx' + yy' + zz' = a^2$

(2) Before applying above result, students are advised to make the coefficients of x^2, y^2, z^2 unity, if not in the problem.

(3) In the given equation of the sphere, we have to change

$$x^2 \to xx', \quad y^2 \to yy', \quad z^2 \to zz',$$

$$x \to \frac{1}{2}(x + x'), \quad y \to \frac{1}{2}(y + y'), \quad z \to \frac{1}{2}(z + z')$$

to find the equation of tangent plane at point (x', y', z') of the sphere.

CONDITION FOR A PLANE $Ax + By + Cz + D = 0$ TO TOUCH THE SPHERE $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Required condition is given as :

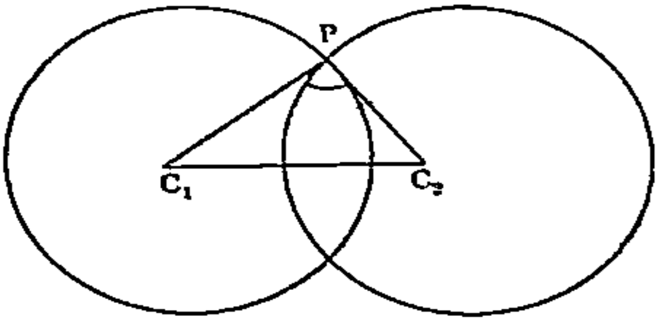
length of the perpendicular from the centre of the sphere on the plane = radius of the sphere

i.e.

$$\boxed{\frac{-Au - Bv - Cw + D}{\sqrt{A^2 + B^2 + C^2}} = \sqrt{u^2 + v^2 + w^2 - d}}$$

ANGLE OF INTERSECTION OF TWO SPHERES

The angle of intersection between two spheres is the angle between the two radii of the sphere through their common point of intersection.



ORTHOGONAL SPHERES

If the angle of intersection between two spheres is a right angle, then the spheres are said to be orthogonal spheres.

CONDITION OF ORTHOGONALITY BETWEEN TWO SPHERES

If two spheres are given as :

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \quad ...(1)$$

$$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0 \quad ...(2)$$

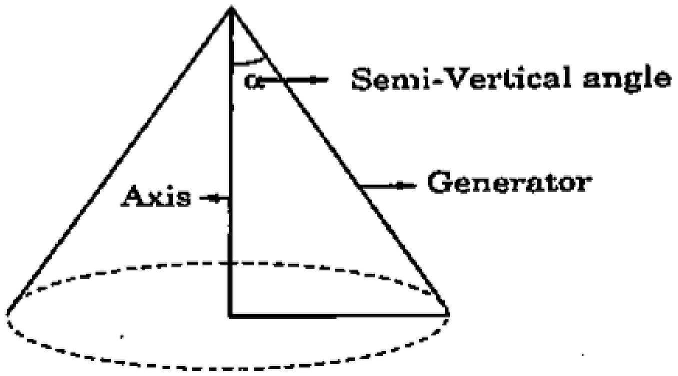
Then the condition of orthogonality between spheres (1) and (2) is given as :

$$\boxed{2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2}$$

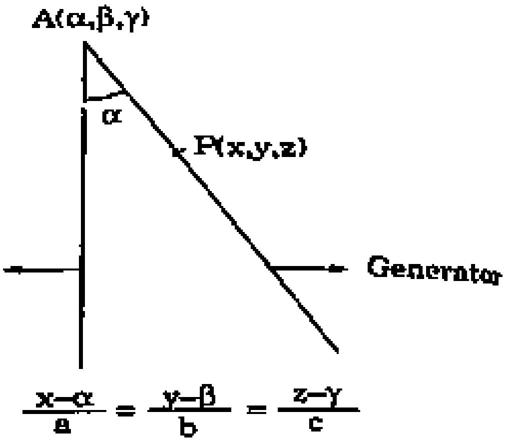
RIGHT CIRCULAR CONE

A right circular cone is a surface generated by a moving line which passes through a fixed point and makes a constant angle with a fixed line through that fixed point.

The fixed point is called the vertex of the cone, the fixed line is called the axis of the cone, the moving line is called the generator of the cone and the constant angle is called the semi-vertical angle of the cone.



VERTEX, AXIS AND SEMI-VERTICAL ANGLE



Let a right circular cone be given with vertex $A(\alpha,\beta,\gamma)$, axis

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$$

and semi-vertical angle α. Now, if we consider $P(x,y,z)$ as any point on the cone, then direction ratios of line AP will be $x-\alpha$, $y-\beta$, $z-\gamma$ and thus angle α between axis of the cone and line AP is obtained as

$$\cos\alpha = \frac{(a)(x-\alpha)+(b)(y-\beta)+(c)(z-\gamma)}{\sqrt{a^2+b^2+c^2}\ \sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

or $$\cos^2\alpha\,(a^2+b^2+c^2)\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2] = [a(x-\alpha)+b(y-\beta)+c(z-\gamma)]^2 \qquad ...(1)$$

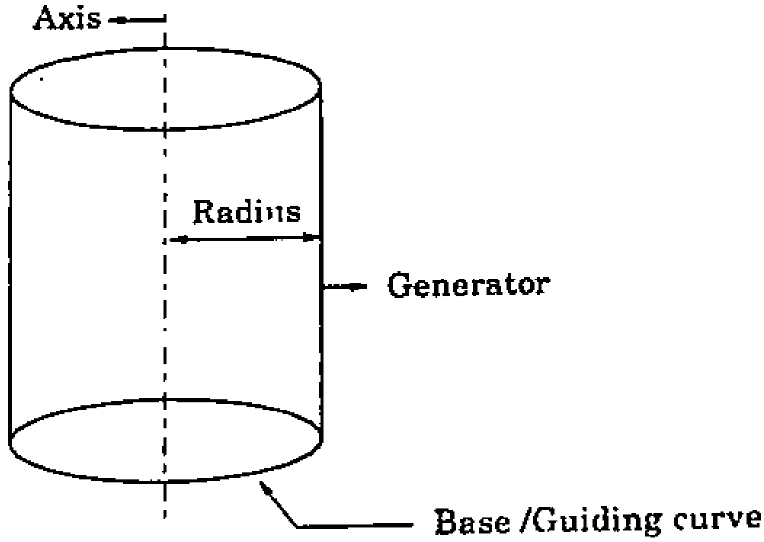
Equation (1) represents the required equation of right circular cone.

Remark : Equation of right circular cone with vertex at the origin, z-axis as axis and semi-vertical angle α is given by

$x^2 + y^2 = z^2\tan^2\alpha$

Similarly, equations with x and y-axis as axis are $y^2 + z^2 = x^2\tan^2\alpha$ and $x^2 + z^2 = y^2\tan^2\alpha$ respectively.

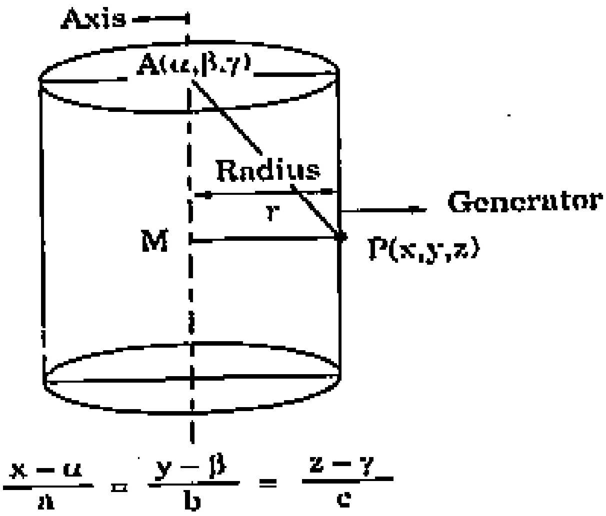
RIGHT CIRCULAR CYLINDER



A right circular cylinder is a surface generated by a moving line which intersects a fixed circle and maintains constant distance from a parallel fixed line.

The fixed line is called the axis of the cylinder, the moving line is called the generator of the cylinder, fixed circle is called the base/guiding curve of the cylinder and the constant distance is called the radius of the cylinder.

EQUATION OF A RIGHT CIRCULAR CYLINDER GIVEN WITH AXIS AND RADIUS



Let a right circular cylinder is given with axis $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ and radius r. Now, if we consider $P(x, y, z)$ as any point on the cylinder and draw a perpendicular PM from P to the axis, then we obtain a right angled triangle APM, in which

$$AP^2 = AM^2 + MP^2 \qquad ...(1)$$

where $MP = r$ (radius), ...(2)

$$AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2} \quad ...(3)$$

and AM = projection of the line joining the points $A(\alpha, \beta, \gamma)$ and $P(x, y, z)$ upon the line AM with a, b, c as direction ratios

So, $$AM = \frac{a(x-\alpha) + b(y-\beta) + c(z-\gamma)}{\sqrt{a^2 + b^2 + c^2}} \qquad ...(4)$$

Now, using equations (2), (3) and (4) in equation (1), we get

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \frac{[a(x-\alpha) + b(y-\beta) + c(z-\gamma)]^2}{(a^2 + b^2 + c^2)} + r^2 \qquad ...(5)$$

Equation (5) represents the required equation of right circular cylinder.