Unit 5
Electromagnetism

GRADIENT

It is defined for a scalar field/function $f(x, y, z)$ and is given by

$$\boxed{\vec{\nabla} f(x, y, z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}}$$

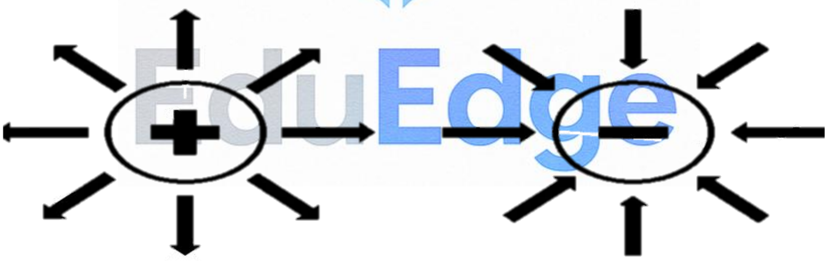
Physical Significance: The Gradient of a scaler field $\vec{\nabla} f(x, y, z)$ represents both the magnitude and direction of the maximum increase of the scaler field in space.
E.g. Grad of T - Direction of heat flow
Grad of V - Direction of electric field

DIVERGENCE

It is defined for a vector field $\vec{A}$ & given by

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Physical Significance: Divergence of $\vec{A}$ is the net outward flux per unit volume over a closed surface.
The divergence of a vector field $\vec{A}$ at a given point is a measure of how much the vector field $\vec{A}$ spread out i.e. Diverges.



If $\vec{V}$ velocity of fluid flow
$\vec{\nabla} \cdot \vec{V} =$ Net rate of flow of fluid out of a small region.

CURL

Curl of a further field is defined as

$$\boxed{\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_y}{\partial y}\right)}$$

Physical Significance: The Curl of a vector field $\vec{A}$ is the measure of the rotation or circulation of the field at a given point.
E.g. Fluid flow of water

Properties VECTOR ALGEBRA

1. Gauss Divergence Theorem $\oint_s \vec{A} \cdot \overrightarrow{dS} = \int_V (\vec{\nabla} \cdot \vec{A})\, dV$
2. Stokes Theorem $\oint_l \vec{A} \cdot \overrightarrow{dl} = \oint_s (\vec{\nabla} \times \vec{A}) \overrightarrow{dS}$
3. Vector Identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$

DIVERGENCE AND CURL OF ELECTRIC FIELD

Divergence

According to this law, the total electric flux passing through a closed surface is equal to the $\frac{1}{\varepsilon_0}$ times of the total charge in closed in the surface.

$$\oint_s \vec{E} \cdot \overrightarrow{dS} = \frac{Q}{\varepsilon_0}$$

$$\boxed{\oint_s \vec{E} \cdot \overrightarrow{dS} = \frac{1}{\varepsilon_0} \int_V \rho\, dV} \quad \text{......... (1)} \qquad \text{[Integral Form]}$$

Using gauss divergence theorem,

$$\oint_s \vec{E} \cdot \overrightarrow{dS} = \int_V (\vec{\nabla} \cdot \vec{E})\, dV \quad \text{......... (2)}$$

By eq 1 and 2,

$$\int_V (\vec{\nabla} \cdot \vec{E})\, dV = \frac{1}{\varepsilon_0} \int_V \rho\, dV$$

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}} \qquad \text{[Differential From]}$$

Curl

We know that, point charge in static Electric Field

$$\boxed{\oint_l \vec{E} \cdot \overrightarrow{dl} = 0} \quad \text{......... (1)} \qquad \text{[Integral Form]}$$

Using stokes theorem,

$$\oint_l \vec{E} \cdot \overrightarrow{dl} = \oint_s (\vec{\nabla} \times \vec{E}) \overrightarrow{dS} \quad \text{......... (2)}$$

By eq 1 and 2,

$$\boxed{\vec{\nabla} \times \vec{E} = 0} \qquad \text{[Differential From]}$$

DIVERGENCE AND CURL OF MAGNETIC FIELD

Divergence

According to this law, the total magnetic field through closed surface is zero.

$$\boxed{\oint_s \vec{B} \cdot \overrightarrow{dS} = 0} \quad \text{......... (1)} \qquad \text{[Integral Form]}$$

Using gauss divergence theorem,

$$\oint_s \vec{B} \cdot \overrightarrow{dS} = \int_V (\vec{\nabla} \cdot \vec{B})\, dV \quad \text{......... (2)}$$

By eq 1 and 2,

$$\int_V (\vec{\nabla} \cdot \vec{B})\, dV = 0$$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0} \qquad \text{[Differential From]}$$

Curl

According to this law, the line integral of the magnetic field induction over a closed path is equal to µ times the total current passing through the loop.

$$\oint_l \vec{B}\cdot\vec{dl} = \mu_0 I_{(total)} \qquad \text{.........} (1)$$

If J is current density, then

$$I_{(total)} = \oint_S \vec{J}\cdot\vec{dS} \qquad \text{.........} (2)$$

From eq 1 and 2,

$$\boxed{\oint_l \vec{B}\cdot\vec{dl} = \mu_0 \oint_S \vec{J}\cdot\vec{dS}} \qquad \text{.........} (3) \qquad \text{[Integral Form]}$$

Using stokes theorem,

$$\oint_l \vec{B}\cdot\vec{dl} = \oint_S (\vec{\nabla}\times\vec{B})\vec{dS} \qquad \text{.........} (4)$$

By eq 3 and 4,

$$\oint_S (\vec{\nabla}\times\vec{B})\vec{dS} = \mu_0 \oint_S \vec{J}\cdot\vec{dS}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}} \qquad \text{[Differential From]}$$

FARADAY LAW

First Law

According to this law, whenever a magnetic flux linked with a closed loop changes and emf is induced in the loop.

Second Law

According to this law, the magnitude of induced emf in a closed loop is directly proportional to the rate of change in magnetic flux linked with the loop.

$$\boxed{\epsilon = -\frac{d\phi}{dt}} \qquad \text{where } [\therefore \phi = \vec{B}\cdot\vec{A}]$$

If the loop has N turns, then

$$\boxed{\epsilon = -N\frac{d\phi}{dt}}$$

EQUATION OF CONTINUITY

An equation that expresses the equality of incoming and outgoing chares V is called continuity equation.

Physical Significance: It implies that in absence of source of charge the rate of increase of charge in a conductor is equals to a rate of flow of charge into the conductor. continuity equation represents the conservation of charge.

We know that,

$$I = -\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial t}\oint \rho\, dV \qquad \text{.........} (1) \qquad [\therefore Q = \oint \rho\, dV]$$

If J is current density, then

$$I = \oint \vec{J}\cdot\vec{dS} \qquad \text{.........} (2) \qquad \left[\therefore J = \frac{I}{A}\right]$$

From eq 1 and 2,

$$\oint \vec{J}\cdot\vec{dS} = -\frac{\partial}{\partial t}\oint \rho\, dV \qquad \text{.........} (3)$$

Using gauss divergence theorem,

$$\oint \vec{J} \cdot \overrightarrow{dS} = \int_V (\vec{\nabla} \cdot \vec{J})\, dV \quad \text{......... (4)}$$

By eq 3 and 4,

$$\int_V (\vec{\nabla} \cdot \vec{J})\, dV = -\frac{\partial}{\partial t} \oint \rho\, dV$$

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

As charge inside a conductor is 0, then $\rho = 0$ and $\vec{\nabla} \cdot \vec{J} = 0$.

DISPLACEMENT CURRENT

Displacement current (I_d)t is the rate of change of an electric displacement field over a time.
According to equation of continuity,

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{............ (1)}$$

If charge inside a conductor is 0, then

$$\vec{\nabla} \cdot \vec{J} = 0$$

Let us consider the total current density is $(\vec{J} + \overrightarrow{J_D})$,

$$\vec{\nabla} \cdot (\vec{J} + \overrightarrow{J_D}) = 0$$

$$\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \overrightarrow{J_D} = 0$$

$$\vec{\nabla} \cdot \overrightarrow{J_D} = -\vec{\nabla} \cdot \vec{J}$$

From eq 1,

$$\vec{\nabla} \cdot \overrightarrow{J_D} = -\left(-\frac{\partial \rho}{\partial t}\right) = \frac{\partial \rho}{\partial t}$$

According to Maxwell's 1st eq,

$$\vec{\nabla} \cdot \overrightarrow{J_D} = \frac{\partial (\vec{\nabla} \cdot \vec{D})}{\partial t} \qquad [\because \rho = \vec{\nabla} \cdot \vec{D}]$$

$$\vec{\nabla} \cdot \overrightarrow{J_D} = \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t} \qquad \text{[Where D = Electric Flux Density]}$$

$$\boxed{\overrightarrow{J_D} = \frac{\partial \vec{D}}{\partial t}} \qquad \text{[Where } J_D \text{ = Displacement Current Density]}$$

According to Ampere's Law,

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \overrightarrow{J_D})$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \qquad \text{[Modify Form of Ampere's Law]}$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}} \qquad [\because i_d = \varepsilon_0 \frac{\partial \phi_E}{\partial t}]$$

MAXWELL EQUATIONS

Maxwell 1 Equation

According to this law, the total electric flux passing through a closed surface is equal to the $\frac{1}{\varepsilon_0}$ times of the total charge in closed in the surface.

$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \frac{Q}{\varepsilon_0}$$

$$\boxed{\oint_S \vec{E} \cdot \overrightarrow{dS} = \frac{1}{\varepsilon_0} \int_V \rho \, dV} \quad .... (1) \qquad \text{[Integral Form]}$$

Using gauss divergence theorem,

$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \int_V (\vec{\nabla} \cdot \vec{E}) \, dV \quad .... (2)$$

By eq 1 and 2,

$$\int_V (\vec{\nabla} \cdot \vec{E}) \, dV = \frac{1}{\varepsilon_0} \int_V \rho \, dV$$

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}} \qquad \text{[Differential From]}$$

$$\vec{\nabla} \cdot (\vec{E}\varepsilon_0) = \rho \qquad [Displacement\ current(D) = \vec{E} \cdot \varepsilon_0]$$

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$

Maxwell 2 Equation

According to this law, the total magnetic field through closed surface is zero.

$$\boxed{\oint_S \vec{B} \cdot \overrightarrow{dS} = 0} \quad .... (1) \qquad \text{[Integral Form]}$$

Using gauss divergence theorem,

$$\oint_S \vec{B} \cdot \overrightarrow{dS} = \int_V (\vec{\nabla} \cdot \vec{B}) \, dV \quad .... (2)$$

By eq 1 and 2,

$$\int_V (\vec{\nabla} \cdot \vec{B}) \, dV = 0$$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0} \qquad \text{[Differential From]}$$

Maxwell 3 Equation

According to faraday law of electromagnetic induction, emf induced in closed loop is equal to the rate of change of magnetic flux enclosed in loop.

$$\varepsilon = -\frac{d\phi_B}{dt} \quad .... (1)$$

Since, the line integral of electric field also represent the emf.

$$\varepsilon = \oint_l \vec{E} \cdot \vec{dl} \quad .... (2)$$

From eq 1 and 2,

$$\oint_l \vec{E} \cdot \vec{dl} = -\frac{d\phi_B}{dt} \quad .... (3)$$

As we know that,

$$\phi_B = \oint_S \vec{B} \cdot \overrightarrow{dS}$$

Put the value in eq 3,

$$\oint_l \vec{E} \cdot \vec{dl} = -\frac{d[\oint_S \vec{B} \cdot \overrightarrow{dS}]}{dt}$$

$$\boxed{\oint_l \vec{E} \cdot \vec{dl} = -\oint_S \frac{d\vec{B}}{dt} \cdot \overrightarrow{dS}} \quad .... (4) \qquad \text{[Integral Form]}$$

Using stokes theorem,

$$\oint_l \vec{E} \cdot \vec{dl} = \oint_s (\vec{\nabla} \times \vec{E}) \vec{dS} \qquad \text{......... (5)}$$

From eq 4 and 5,

$$\oint_s (\vec{\nabla} \times \vec{E}) \vec{dS} = -\oint_s \frac{d\vec{B}}{dt} \cdot \vec{dS}$$

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}} \qquad \text{[Differential From]}$$

Maxwell 4 Equation

According to this law, the line integral of the magnetic field induction over a closed path is equal to μ times the total current passing through the loop.

$$\oint_l \vec{B} \cdot \vec{dl} = \mu_0 I_{(total)} \qquad \text{......... (1)}$$

If J is current density, then

$$I_{(total)} = \oint_s \vec{J} \cdot \vec{dS} \qquad \text{......... (2)}$$

From eq 1 and 2,

$$\oint_l \vec{B} \cdot \vec{dl} = \mu_0 \oint_s \vec{J} \cdot \vec{dS} \qquad \text{......... (3)}$$

According to maxwell total J is due to two types of current are:

(a) Free current / conduction current density ($\vec{J}$), due to motion of free charges.

(b) Displacement current density ($\vec{J_D}$), due to variation of electric field w.r.to time.

From eq 3,

$$\boxed{\oint_l \vec{B} \cdot \vec{dl} = \mu_0 \oint_s (\vec{J} + \vec{J_D}) \cdot \vec{dS}} \qquad \text{......... (4)} \qquad \text{[Integral Form]}$$

Using stokes theorem,

$$\oint_l \vec{B} \cdot \vec{dl} = \oint_s (\vec{\nabla} \times \vec{B}) \vec{dS} \qquad \text{......... (5)}$$

From eq 4 and 5,

$$\oint_s (\vec{\nabla} \times \vec{B}) \vec{dS} = \mu_0 \oint_s (\vec{J} + \vec{J_D}) \cdot \vec{dS}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{J_D}) \qquad \text{......... (6)}$$

Taking divergence both the sides in eq 6,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{J_D})$$

According to vector identity,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$$

$$\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{J_D} = 0$$

$$\vec{\nabla} \cdot \vec{J_D} = -\vec{\nabla} \cdot \vec{J} \qquad \text{......... (7)}$$

According to equation of continuity,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

From eq 7,

$$\vec{\nabla} \cdot \vec{J_D} = \frac{\partial \rho}{\partial t}$$

$$\vec{\nabla}\cdot\overrightarrow{J_D} = \frac{\partial(\vec{\nabla}\cdot\vec{D})}{\partial t}$$

$$\overrightarrow{J_D} = \frac{\partial\vec{D}}{\partial t}$$

Put value of $\overrightarrow{J_D}$ in eq 6, we get

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)$$

$$\vec{\nabla}\times\mu_0\vec{H} = \mu_0\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) \qquad [\because \vec{B} = \mu_0\vec{H}]$$

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}+\frac{\partial\vec{D}}{\partial t}} \text{ Or } \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}} \qquad \text{[Differential From]}$$

ELECTROMAGNETIC WAVE PROPAGATION IN FREE SPACE

Electromagnetic Waves are waves that consist of oscillating electric and magnetic fields, which are propagate through space without requiring a medium. These waves obey maxwell equations and travel at the speed of light in vacuum.

Maxwell's equations describe the behavior of EM Waves.

1. Gauss's Law for Electricity

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

2. Gauss's Law for Magnetism

$$\vec{\nabla}\cdot\vec{B} = 0$$

3. Faraday's Law of Electromagnetic Induction

$$\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$$

4. Ampere's Law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Using these equations, we derive the wave equation for EM Waves in free space.
Maxwell's equations in free space
Since, free space has no charges ($\rho = 0$) and no currents ($\vec{J} = 0$), Maxwell's equations simplify to:

1. $\vec{\nabla}\cdot\vec{E} = 0$

2. $\vec{\nabla}\cdot\vec{B} = 0$

3. $\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$

4. $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

Wave Equation for $\vec{E}$

According to Faraday's Law,

$$\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$$

Now, take the curl on both sides

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{d\vec{B}}{dt} \quad \text{.......... (1)}$$

Using vector identity,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} \quad \text{.......... (2)}$$

From eq 1 and 2,

$$\vec{\nabla}^2 \vec{E} = \vec{\nabla} \times \frac{d\vec{B}}{dt} \quad \text{.......... (3)} \qquad [\because \vec{\nabla} \cdot \vec{E} = 0]$$

From Ampere's Law,

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Take the time derivative on both sides

$$\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Put the vale in eq 3, we get

$$\boxed{\vec{\nabla}^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \text{ OR } \boxed{\vec{\nabla}^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

Wave Equation for $\vec{B}$

According to Ampere's Law,

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Now, take the curl on both sides

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \varepsilon_0 \vec{\nabla} \times \frac{\partial \vec{E}}{\partial t} \quad \text{.......... (1)}$$

Using vector identity,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2 \vec{B} \quad \text{.......... (2)}$$

From eq 1 and 2,

$$-\vec{\nabla}^2 \vec{B} = \mu_0 \varepsilon_0 \vec{\nabla} \times \frac{d\vec{E}}{dt} \quad \text{.......... (3)} \qquad [\because \vec{\nabla} \cdot \vec{B} = 0]$$

From Faraday's Law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Take the time derivative on both sides

$$\vec{\nabla} \times \frac{\partial \vec{E}}{\partial t} = -\frac{\partial^2 \vec{B}}{\partial t^2}$$

Put the vale in eq 3, we get

$$\boxed{\vec{\nabla}^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}} \text{ OR } \boxed{\vec{\nabla}^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0}$$

Compare this equation with Wave Standard Equation

$$\nabla^2 f = \frac{1}{V^2} \frac{\partial^2 f}{\partial t^2}$$

We can identify wave speed as

$$V = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

In free space,

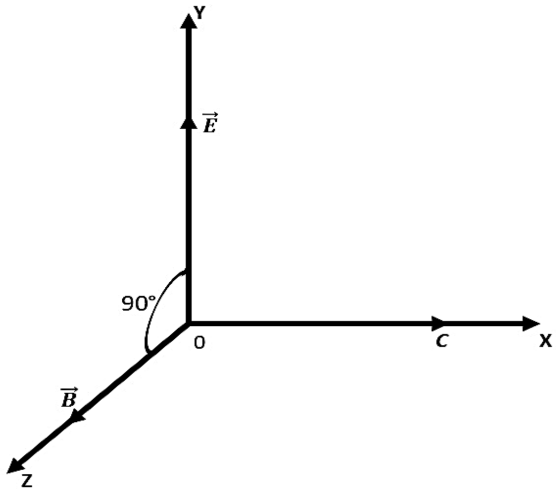
$$\boxed{V = \frac{1}{\sqrt{(4\pi \times 10^{-7})(8.85 \times 10^{-12})}} = 3 \times 10^8\, m/s = c}$$

Speed of light in vacuum.

Characteristics of EM Waves in Free Space

1. *Transverse nature*
   The E and B are perpendicular to each other and to the direction of wave propagation(k).
   If the wave propagates in the x-direction, then

$$E = E_0\hat{y}\, cos(kx - \omega t)$$
$$B = B_0\hat{z}\, cos(kx - \omega t)$$



2. *Speed of Wave*
   The speed of the wave in free space is $V = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8\, m/s = c.$
3. *Relation Between E and B*
   The magnitudes E and B are related as $c = \frac{E_0}{B_0}.$

ENERGY FLOW AND POYNTING VECTOR

The energy carried by EM Waves is characterized by the Poynting vector, which represents the power per unit area transparent by the wave.

$$\boxed{S = E \times B}$$

Where,

S is Poynting vector (W/m^2)
E is Electric Field (V/m)
B is Magnetic Field (T)