Unit 5
Matrices

RANK OF A MATRIX

A matrix of order $m \times n$ is said to be of rank r, if

(*i*) it has atleast one non-zero minor of order r,

(*ii*) all other minors of order greater than r (if any) are zero.

The rank of matrix A is denoted by $\rho(A)$

Remarks : (1) The rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

(2) If A is a square matrix of order n and $|A| \neq 0$ then $\rho(A) = n$.

(3) If A is a matrix of order $m \times n$, then $\rho(A) \leq \min(m, n)$.

(4) The rank of a unit matrix I of order n is n *i.e.* $\rho(I_n) = n$.

(5) Elementary transformation does not change the rank of a matrix.

(6) For any matrix A, $\rho(A) = \rho(A^T)$.

(7) Rank of every non-zero matrix is ≥ 1.

ELEMENTARY OPERATIONS OR ELEMENTARY TRANSFORMATION OF A MATRIX

An elementary transformation or an E-transformation is an operation of any one of the following three types :

(*i*) The interchange of any two columns or rows.

(*ii*) The multiplication of the elements of any column or row by any non-zero number.

(*iii*) The addition of the elements of any column or row to the corresponding elements of any other column or row multiplied by any number.

NORMAL FORM OF A MATRIX

Every $m \times n$ matrix of rank r (non-zero positive integer) can be reduced to one of the following forms by a finite chain of elementary transformations where I_r is the unit matrix of order r.

$$[I_r],\ [I_r \quad 0],\ \begin{bmatrix} I_r \\ 0 \end{bmatrix};\ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

The above forms are called the first canonical forms or normal forms of a matrix.

Remark : The rank of an $m \times n$ type matrix A is r if and only if it can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary operations.

SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS

Consider the following system of m linear simultaneous equations in n unknowns :

$$a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2,$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_m.$$

The above system of linear equations can be written as

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} \end{bmatrix}_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$

A, X and B are known as coefficient matrix, column matrix of unknowns and column matrix of constants, respectively.

AUGMENTED MATRIX

If a matrix A of type $m \times n$ is combined with a matrix B of type $m \times 1$, then we get a matrix $[A : B]$ called augmented matrix.

$$[A : B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & \dots\dots & a_{2n} & : & b_2 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots & a_{mn} & : & b_m \end{bmatrix}$$

LINEAR NON-HOMOGENEOUS SYSTEM

If atleast one b_i $(i = 1, 2, \dots\dots, m) \neq 0$ then above system is known as linear non-homogeneous system.

CONDITIONS FOR CONSISTENCY OF THE NON-HOMOGENEOUS SYSTEM $AX = B$

(*i*) If $\rho[A : B] \neq \rho(A)$, then the given system is inconsistent or it has no solution.

(*ii*) If $\rho[A : B] = \rho(A) < n$ (number of unknowns), then the system has an infinite number of solutions.

(*iii*) If $\rho[A : B] = \rho(A) = n$ (number of unknowns), then the system has a unique solution.

Remark : Use only row transformations for finding the rank of the augmented matrix.

LINEAR HOMOGENEOUS SYSTEM

If all b_i ($i = 1, 2, \ldots\ldots, m$) are zero in the above system of equations then the system is known as linear homogeneous system.

CONDITIONS FOR CONSISTENCY OF THE HOMOGENEOUS SYSTEM $AX = 0$

(*i*) $X = 0$ is always a solution for this type of system. In this solution, each unknown has the value zero that's why it is called the Null Solution or the Trivial Solution. Hence a homogeneous system $AX = 0$ is always consistent.

(*ii*) If $\rho(A) < n$ (number of unknowns), the system has an infinite number of non-trivial solutions.

(*iii*) If $\rho(A) = n$ (number of unknowns), the system has only the trivial solution.

Thus, a system of homogeneous linear equations has either the trivial solution or an infinite number of solutions.

CHARACTERISTIC EQUATION AND CHARACTERISTIC ROOTS (EIGEN VALUES)

Let A be any square matrix of order n. Then the matrix $(A - \lambda I)$ is said to be characteristic matrix of A, where I is an unit matrix of order n.

The determinant

$$|A - \lambda I| = \left|\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \ldots\ldots & 0 \\ 0 & 1 & \ldots\ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots\ldots & 1 \end{bmatrix}\right|$$

or

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots\ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} - \lambda \end{vmatrix}$$

which gives an ordinary polynomial in λ of degree n, is said to be characteristic polynomial of A.

i.e. $(-1)^n \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_n$ (characteristic polynomial)

The equation $|A - \lambda I| = 0$ is known as characteristic equation of A.

Roots of this equation are called the characteristic roots or latent roots or eigen values or proper values of the matrix A.

CHARECTERISTIC VECTORS (EIGEN VECTORS)

If λ is a characteristic root of a square matrix A of order n, then a non zero vector X such that $AX = \lambda X$ is said to be a characteristic vector or eigen vector of A corresponding to the eigen value λ.

Since $$AX = \lambda X \text{ for } X \neq 0$$

or $$AX - \lambda IX = 0$$

or $$(A - \lambda I)X = 0$$

represents a system of n homogeneous equations in n variables $x_1, x_2 \ldots, x_n$.

> *Remark* : The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

PROPERTIES OF EIGEN VALUES

(i) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are considered as the eigen values of matrix kA.

(ii) If λ is an eigen value of the matrix A, then $k + \lambda$ is considered as eigen value of the matrix $A + kI$.

(iii) Matrices A and A^T have the same eigen values.

(iv) If A is a non-singular matrix, then eigen values of A^{-1} are the reciprocals of the eigen values of A.

(v) If the characteristic roots of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$; then the characteristic roots of A^2 are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$.

(vi) If λ be an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of adj A.

(vii) The sum of eigen values of a matrix A is equal to the sum of the elements of the principal diagonal of A.

(viii) If matrix A is singular, then atleast one of its eigen values will be zero.

CAYLEY-HAMILTON THEOREM

Statement. Every square matrix A satisfies its own characteristic equation *i.e.*, if the characteristic equation of a square matrix A of order n is

$$|A - \lambda I| = (-1)^n \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_n = 0,$$

then $(-1)^n A^n + c_1 A^{n-1} + c_2 A^{n-2} + \ldots + c_n I = 0$

> *Remark* : The inverse of a non-singular matrix A can be obtained from above theorem by multiplying both the sides by A^{-1} as
>
> $$A^{-1} [(-1)^n A^n + c_1 A^{n-1} + c_2 A^{n-2} + \ldots + c_n I] = 0$$
>
> or $$A^{-1} = \frac{-1}{c_n} [(-1)^n A^{n-1} + c_1 A^{n-2} + \ldots + c_{n-1} I]$$

DIAGONALIZABLE MATRIX

A matrix A is said to be diagonalizable, if it is similar to a diagonal matrix.

Thus, a matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1} AP = C$ where C is a diagonal matrix.

The matrix P is then said to be helping matrix for diagonalizing A or transforming A to diagonal form.