Unit 3

Differential Equation - I

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

DIFFERENTIAL EQUATION : DEFINITION

An equation containing derivaives is called a differential equation.

Examples:

(1) $\frac{dy}{dx} + 2xy = \sin x$

(2) $\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}$

(3) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^x$

(4) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\, \partial y} = 0$

Remark: In above examples, first 3 are ordinary differential equations and the fourth example contains a partial differential equation.

ORDER OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest order derivative appearing in the equation.

Example : In the differential equation $\frac{d^3y}{dx^3} + 3\frac{dy}{dx} + 2y = e^{2x}$,

the order of highest order derivative is 3.

So, it is a differential equation of order 3.

Remark: The order of a differential equation is always a positive integer.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.

Example : Consider the differential equation

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/2} = k\left(\frac{d^2y}{dx^2}\right)$$

when expressed as a polynomial in derivatives it becomes

$$k^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^5 = 0$$

clearly, the power of the highest order differential coefficient is 2.

CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS (LINEAR / NON-LINEAR)

Linear Differential Equation : **A differential equation, in which the dependent variable and all its derivatives occur in the first degree only and there is no product of dependent variable and derivatives occur is called a linear differential equation.**

Non-linear Differential Equation : **A differential equation, which is not linear is called a non-linear differential equation.**

S. No.	Ordinary Differential Equation	Order	Degree	Linear / Non - Linear
1	$\frac{d^2y}{dx^2}+2y=0$	2	1	Linear
2	$\frac{d^4y}{dx^4}=\left[k+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$	4	2	Non-Linear
3	$\frac{d^4y}{dx^4}=\cos\left(\frac{d^3y}{dx^3}\right)=0$	4	Undefined	Non-Linear
4	$\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$	2	Undefined	Non-Linear
5	$\frac{d^3y}{dx^3}+3\frac{dy}{dx}+y=e^{-x}$	3	1	Linear

LINEAR DIFFERENTIAL EQUATION

In general, a linear differential equation of first order and first degree is written as

$$\frac{dy}{dx}+Py=Q\ ,$$

where P & Q are functions of x or may be constants.

Examples: (i) $\frac{dy}{dx}+xy=x^2$ (ii) $\frac{dy}{dx}+y\sin x=\cos x$

Following algorithm may be used to solve above linear differential equation:

Step 1. Write the given differential equation in the form $\frac{dy}{dx}+Py=Q$ and obtain the values of P & Q.

Step 2. Find the integrating factor (I.F.) $= e^{\int P\,dx}$

Step 3. Solution of the given differential equation is obtained by simplifying the equation $y(\text{I.F.})=\int Q\,(\text{I.F.})\,dx+c$

Remark: If we consider as dependent variable and as independent variable, then the above linear differentia equation is written as $\frac{dx}{dy}+Rx=S$,

where $R \& S$ are functions of y or may be constants.

Then the solution of the given differential equation is obtained by simplifying the equation $x\,(\text{I.F.})=\int S\,(\text{I.F.})dy+c$, where $(\text{I.F.}) = e^{\int R dy}$

DIFFERENTIAL EQUATIONS REDUCIBLE TO LINEAR FORM

In general, a differential equation can be reduced in linear form, if it can be written as

$$f'(y)\,\frac{dy}{dx} + P\,f(y) = Q\ ,$$

where $P \,\&\, Q$ are functions of x or may be constants.

OR

A differential equation can be reduced in linear form, if it can be written as

$$f'(x)\frac{dx}{dy}+R\,f(x)=S,$$

where $R \,\&\, S$ are functions of y or may be constants.

EXACT DIFFERENTIAL EQUATION

A differential equation is said to be exact, if it can be directly derived from its primitive (general solution) by differentiation , without any further operation of elimination or reduction.

NECESSARY AND SUFFICIENT CONDITION OF EXACTNESS FOR AN ODE OF FIRST ORDER AND FIRST DEGREE

The differential equation $Mdx + Ndy = 0$, **where** M, N **are functions of** x **and** y, **is said to be exact, if** $\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$

SOLUTION OF AN EXACT DIFFERENTIAL EQUATION

If $Mdx + Ndy = 0$ **is an exact differential equation, then its solution is given as**

$$\boxed{\int_{y=\text{constant}} M\,dx+\int(\text{Terms in } N \text{ not containing } x)\,dy=c}$$

DIFFERENTIAL EQUATIONS REDUCIBLE TO EXACT FORM

Integrating Factor (I.F.)

Some of the differential equations, which are not exact can be made exact after multiplying them by some suitable function of x and y . Such a function is called an integrating factor.

For example, consider the differential equation

$$ydx - xdy = 0 \quad ...(1)$$

Here $M = y$ and $N = -x \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Therefore the differential equation is not exact.

$\Rightarrow$

(1) Multiplying by $\frac{1}{y^2}$, it becomes

$$\frac{ydx - xdy}{y^2} = 0, \text{ which is exact.}$$

(2) Multiplying by $\frac{1}{x^2}$, it becomes

$$\frac{ydx - xdy}{x^2} = 0, \text{ which is exact.}$$

(3) Multiplying by $\frac{1}{xy}$, it becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0, \text{ which is exact.}$$

Therefore $\frac{1}{y^2}, \frac{1}{x^2}$ and $\frac{1}{xy}$ are integrating factors of (1).

Remarks: (1) We have to use any one integrating factor for reducing a non - exact differential equation into exact form.

(2) For finding an integrating factor, different specific cases are to be discussed.

Rules for finding integrating factors

Rule - I: If the differential equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$,

then integrating factor $= \dfrac{1}{Mx + Ny}$

Rule - II: If the differential equation $Mdx + Ndy = 0$ can be presented in the form

$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ and $Mx - Ny \neq 0$,

then integrating factor $= \dfrac{1}{Mx - Ny}$

Rule - III: If the differential equation $Mdx + Ndy = 0$ provides the values of M and N

such that we obtain $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(x)$ or a constant k,

then integrating factor $= e^{\int f(x)dx}$ or $e^{\int k\,dx}$

Rule - IV: If the differential equation $Mdx + Ndy = 0$ provides the values of M and N

such that we obtain $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) = f(y)$ or a constant k,

then integrating factor $= e^{\int f(y)dy}$ or $e^{\int k\,dy}$

Rule - V: If the differential equation $Mdx+Ndy=0$ can be presented in the form

$$x^a y^b\left(mydx+nxdy\right)+x^c y^d\left(pydx+qxdy\right)=0\ ;\ a,b,c,d,m,n,p,q\in R,$$

then we assume $x^h y^k$; $h,k\in R$ as an integrating factor. In order to find the values of h and k, we use the condition of exactness on the given differential equation after multiplying it by the assumed integrating factor $x^h y^k$

Rule - VI: In many problems, integrating factor can be obtained by the method of inspection using various formulae, out of which several are given as:

(i) $\dfrac{xdy-ydx}{x^2}=d\left(\dfrac{y}{x}\right)$ (ii) $\dfrac{ydx-xdx}{y^2}=d\left(\dfrac{x}{y}\right)$

(iii) $\dfrac{xdy-ydx}{xy}=d\left[\log\left(\dfrac{y}{x}\right)\right]$ (iv) $\dfrac{ydx-xdy}{xy}=d\left[\log\left(\dfrac{x}{y}\right)\right]$

(v) $\dfrac{xdy-ydx}{x^2+y^2}=d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]$ (vi) $\dfrac{ydx-xdy}{x^2+y^2}=d\left[\tan^{-1}\left(\dfrac{x}{y}\right)\right]$

(vii) $xdy+ydx=d(xy)$ (viii) $\dfrac{xdy+ydx}{xy}=d\left[\log(xy)\right]$

(ix) $xdx+ydy=d\left[\dfrac{1}{2}(x^2+y^2)\right]$ (x) $\dfrac{xdx+ydy}{x^2+y^2}=d\left[\dfrac{1}{2}\log(x^2+y^2)\right]$ etc.

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

INTRODUCTION

Mathematically, a linear differential equation of order n with constant coefficients is represented as

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+a_n y=X \qquad \ldots(1)$$

where a_0, a_1, ..., a_n are constants and X is either a constant or function of independent variable x only.

If we take D to represent $\dfrac{d}{dx}$, equation (1) becomes

$(a_0D^n+a_1D^{n-1}+\ldots\ldots+a_n)\,y=X$

or $\quad f(D)\,y=X \qquad \ldots(2)$

where $f(D)=a_0D^n+a_1D^{n-1}+\ldots\ldots+a_n$

i.e. $f(D)$ is a polynomial of degree n in D.

COMPLETE SOLUTION OF DIFFERENTIAL EQUATION $f(D)y = X$

Complete solution of equation (2) is given by

$$\boxed{y = \text{C.F.} + \text{P.I.}}$$

Remark : If $X = 0$ occurs in equation (2), then its complete solution is given by

$$\boxed{y = \text{C.F.}}$$

AUXILIARY EQUATION (A.E.) FOR FINDING C.F.

In order to find C.F. for equation (2), consider

$$a_0 m^n + a_1 m^{n-1} + \ldots\ldots + a_n = 0 \qquad \ldots(3)$$

Equation (3), whose left hand side is obtained by substituting D by m in $f(D)$ is called auxilliary equation (A.E.) for equation (2)

RULES FOR FINDING C.F.

Case I : If all the roots of auxiliary equation are real and distinct

i.e. $m = m_1, m_2, \ldots\ldots, m_n,$

then $\boxed{\text{C.F.} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots.. + c_n e^{m_n x}}$

Case II : If two roots of auxiliary equation are real and equal and remaining $(n-2)$ roots are real and distinct

i.e. $m = m_1, m_1, m_3, \ldots\ldots, m_n,$

then $\boxed{\text{C.F.} = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots.. + c_n e^{m_n x}}$

Also, if three roots of auxiliary equation are real and equal and remaining $(n-3)$ roots are real and different i.e. $m = m_1, m_1, m_1, m_4, \ldots.. m_n,$ then proceeding in similar fashion as discussed above

$$\boxed{\text{C.F.} = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots.. + c_n e^{m_n x}}$$

Remark: Above concept can be generalised upto any number of real and equal roots of auxiliary equation.

Case III : If two roots of auxiliary equation are real and occur as pairs in form of surds and remaining $(n-2)$ roots are real and distinct

i.e. $$m = m_1 \pm \sqrt{m_2}\,,\ m_3, m_4, \ldots\ldots, m_n,$$

then $$\boxed{\text{C.F.} = e^{m_1 x}\left[c_1 \cosh \sqrt{m_2}\,x + c_2 \sinh \sqrt{m_2}\,x\right] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}}$$

Remark : Above concept can be generalised upto any number of real and surd roots of auxiliary equation.

Case IV : If two roots of auxiliary equation are imaginary and remaining $(n-2)$ roots are real and distinct

i.e. $$m = m_1 \pm i m_2, m_3, \ldots\ldots, m_n, \text{ (As such roots occur in pairs)}$$

then $$\boxed{\text{C.F.} = e^{m_1 x}(c_1 \cos m_2 x + c_2 \sin m_2 x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}}$$

Remark : Above concept can be generalised upto any number of imaginary roots of auxiliary equation.

Case V : If two pairs (4 roots) of imaginary roots of auxiliary equation are equal and remaining $(n-4)$ roots are real and distinct

i.e. $$m = m_1 \pm i m_2, m_1 \pm i m_2, m_5, \ldots, m_n,$$

then $$\boxed{\text{C.F.} = e^{m_1 x}\left[(c_1 + c_2 x)\cos m_2 x + (c_3 + c_4 x)\sin m_2 x\right] + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}}$$

VARIOUS CASES FOR FINDING PARTICULAR INTEGRAL (P.I.)

RULES FOR FINDING PARTICULAR INTEGRAL

Considering the equation $f(D)\,y = X$

P.I. is given as $\boxed{\text{P.I.} = \dfrac{1}{f(D)}\, X}$

where $\dfrac{1}{f(D)}$ is an inverse operator of $f(D)$.

Now, we will find P.I. with different situations given for X as

(i) e^{ax}

(ii) $\sin ax$ or $\cos ax$

(iii) x^m ; $m \in N$

(iv) $e^{ax}V$; $V = \sin ax, \cos ax$ or x^m

(v) xV ; $V = \sin ax$ or $\cos ax$

(vi) *Any function of* x *(General case)*

P.I. WHEN $X = e^{ax}$

For differential equation with $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)}\, e^{ax}$$

where $f(D) = a_0D^n + a_1D^{n-1} + \dots + a_n$

$$\boxed{\text{P.I.} = \frac{1}{f(D)}\, e^{ax} = \frac{1}{f(a)}\, e^{ax} \; ; f(a) \neq 0}$$

Remark : If $f(a) = 0$ occurs in above P.I., which is possible only if $(D-a)^r$ exists as a factor of $f(D)$ *i.e.*

if $f(D) = (D-a)^r\, \psi(D)$,

then $\text{P.I.} = \dfrac{1}{f(D)}\, e^{ax} = \dfrac{1}{(D-a)^r\, \psi(D)}\, e^{ax}$

or $\text{P.I.} = \dfrac{x^r}{\lfloor r} \cdot \dfrac{e^{ax}}{\psi(a)}$

P.I. WHEN $X = \sin ax$ OR $\cos ax$

For differential equation with $X = \sin ax$ or $\cos ax$

$$\text{P.I.} = \frac{1}{f(D)}\sin ax \text{ or } \frac{1}{f(D)}\cos ax$$

$$\text{P.I.} = \frac{1}{f(D)}\sin ax$$

$$= \frac{1}{f_1(D^2) + Df_2(D^2)}\sin ax$$

$$= \frac{1}{f_1(-a^2) + Df_2(-a^2)}\sin ax$$

$$= \frac{1}{A + DB}\sin ax$$

$$= \frac{(A - DB)}{(A^2 - D^2B^2)}\sin ax$$

$$= \frac{(A - DB)}{A^2 - (-a^2)B^2}\sin ax$$

$$= \frac{(A \sin ax - aB\cos ax)}{(A^2 + a^2B^2)}$$

Remarks : (1) If in above P.I., we obtain $f(-a^2) = 0$, which is possible only if $(D^2 + a^2)$ or its any higher power exists as a factor of $f(D)$ *i.e.*

if $f(D) = (D^2 + a^2)^r$,

$$\frac{1}{(D^2+a^2)^r} \sin ax = \left(-\frac{x}{2a}\right)^r \frac{1}{\lfloor r} \sin\left(ax + \frac{r\pi}{2}\right)$$

and $$\frac{1}{(D^2+a^2)^r} \cos ax = \left(-\frac{x}{2a}\right)^r \frac{1}{\lfloor r} \cos\left(ax + \frac{r\pi}{2}\right)$$

and if $f(D) = (D^2 + a^2)$,

$$\frac{1}{(D^2+a^2)} \sin ax = -\frac{x}{2a} \cos ax$$

and $$\frac{1}{(D^2+a^2)} \cos ax = \frac{x}{2a} \sin ax$$

(2) Above P.I. remains unchanged if we replace $\sin ax$ or $\cos ax$ by $\sin (ax + b)$ or $\cos (ax + b)$ respectively.

P.I. WHEN $X = x^m$; $m \in N$

$$\text{P.I.} = \frac{1}{f(D)} x^m = \frac{1}{(a_0 D^n + a_1 D^{n-1} + \ldots + a_n)} x^m$$

$$= \frac{1}{a_n [(1 + \psi(D))]} x^m$$

$$= \frac{1}{a_n} [1 + \psi(D)]^{-1} x^m$$

Now, expand $[1 + \psi(D)]^{-1}$ in ascending powers of D upto the term containing D^m only as $D^{m+1}(x^m) = D^{m+2}(x^m) = \ldots\ldots = 0$ and finally operate x^m by the various terms of expansion containing different powers of D.

Remark : If $f(D) = (D - \alpha_1)(D - \alpha_2) \ldots (D - \alpha_n)$,

then
$$\text{P.I.} = \frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)} x^m$$

$$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \ldots + \frac{A_n}{D-\alpha_n}\right) x^m$$

[By factorizing partially]

and then we can use the process similar to above article to find $\frac{A_1}{D-\alpha_1} x^m$, $\frac{A_2}{D-\alpha_2} x^m, \ldots, \frac{A_n}{D-\alpha_n} x^m$ separately.

P.I. WHEN $X = e^{ax} V$; $V = \sin ax, \cos ax$ or x^m

$$\text{P.I.} = \frac{1}{f(D)} (e^{ax} V)$$

$$= e^{ax} \frac{1}{f(D+a)} V$$

P.I. WHEN $X = xV$; $V = \sin ax$ or $\cos ax$

$$\text{P.I.} = \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{f(D)^2}V$$

where $\frac{1}{f(D)}V$, $\frac{f'(D)}{[f(D)]^2}V$ can be evaluated by previously discussed methods, if V is given in form of $\sin ax$ or $\cos ax$

Remark : To find $\frac{1}{f(D)}(x^m V)$; $m \in N$ and $m \geq 2$ with $V = \sin ax$ or $\cos ax$, we first obtain $\frac{1}{f(D)}(x^m e^{iax})$ by $\frac{1}{f(D)}(e^{iax}x^m) = e^{iax}\frac{1}{f(D+ia)}x^m$ and then by considering real or imaginary parts, we obtain $\frac{1}{f(D)}(x^m \cos ax)$ or $\frac{1}{f(D)}(x^m \sin ax)$ respectively.

P.I. WHEN X IS ANY FUNCTION OF x

To find $\frac{1}{f(D)}X$, we resolve $f(D)$ into linear factors (whether real or complex)

i.e.
$$\frac{1}{f(D)}X = \frac{1}{(D-\alpha_1)(D-\alpha_2)\dots(D-\alpha_n)}X$$
$$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \dots + \frac{A_n}{D-\alpha_n}\right)X$$
$$= A_1\frac{1}{D-\alpha_1}X + A_2\frac{1}{D-\alpha_2}X + \dots + A_n\frac{1}{D-\alpha_n}X$$

Now, $\frac{1}{D-\alpha_r}X = e^{\alpha_r x}\int Xe^{-\alpha_r x}\,dx$

or
$$\frac{1}{(D-\alpha_1)(D-\alpha_2)\dots(D-\alpha_r)}X = A_1e^{\alpha_1 x}\int Xe^{-\alpha_1 x}\,dx + A_2e^{\alpha_2 x}\int Xe^{-\alpha_2 x}\,dx$$
$$+ \dots\dots\dots\dots + A_n e^{\alpha_n x}\int Xe^{-\alpha_n x}\,dx$$