Unit 1
Differential Calculus – I
ASYMPTOTE

Definition: In other words, a straight line is said to be an asymptote of a curve $y = f(x)$, if the perpendicular distance of the point $P(x,y)$ on the curve from the line tends to zero when x or y or both tend to infinity.

OR

Asymptote can be defined as a pseudo tangent at infinity.

ASYMPTOTES PARALLEL TO THE COORDINATE AXES

Asymptotes parallel to the axes can be obtained by the usual method. Here we shall establish a direct method for finding such asymptotes :

(i) **Asymptotes parallel to x-axis :** The asymptotes parallel to x-axis can be obtained by equating to zero, the coefficients of the highest power of x in the equation of the curve, provided that it is not a constant term.

(i) **Asymptotes parallel to y-axis :** The asymptotes parallel to y-axis can be obtained by equating to zero, the coefficients of the highest power of y in the equation of the curve, provided that it is not a constant term.

PROCEDURE OF FINDING ALL ASYMPTOTES OF A POLYNOMIAL CURVE IN CARTESIAN FORM

Step - 1: Find all asymptotes parallel to coordinate axes using the process defined earlier.

Step - 2: For finding all oblique asymptotes of a curve of degree n, we put $x = 1$ and $y = m$ in terms of degree n and n - 1 to find $\phi_n(m)$ and $\phi_{n-1}(m)$.

Step - 3: We put $\phi_n(m) = 0$ to find n values of m, If these n values of m are real and different i.e. $m = m_1, m_2, \ldots, m_n$, then we calculate c using the following formula:

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

Then by substituting $m = m_1, m_2, \ldots, m_n$ in above expression, we get $c = c_1, c_2, \ldots, c_n$. Combining every set of values of m and c, asymptotes are given as $\boxed{y = m_i x + c_i}$

Step - 4: If n values of m are real such that two values are identical and remaining n - 2 values are different i.e. $m = m_1, m_1, m_3, \ldots, m_n$, then we calculate c for $m = m_1, m_1$ using the following formula:

$$\frac{c^2}{\underline{\lvert 2}}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0,$$

where $\phi_{n-2}(m)$ can be obtained by substituting x = 1 and y = m in terms of degree n - 2. Also, for finding asymptotes for remaining different values, use step - 2.

Step - 5: If three values of m are identical, then c is calculated as per the formula

$$\frac{c^3}{\underline{\lvert 3}}\phi_n'''(m) + \frac{c^2}{\underline{\lvert 2}}\phi_{n-1}'''(m)\,c\,\phi_{n-2}'(m) + \phi_{n-3}(m) = 0 \quad \text{etc.}$$

METHOD OF INSPECTION

If the equation of the curve is expressible in the form $\boxed{F_n + P = 0,}$ where F_n contains product of n linear and different factors of the type $(ax + by + c)$ and P contains terms of degree $(n-2)$ and lower, then all asymptotes of the given curve are obtained by

$$\boxed{F_n = 0}$$

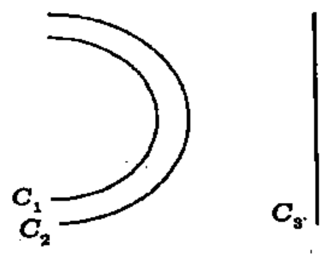
INTERSECTION OF THE CURVE AND ITS ASYMPTOTES

If $S_n = 0$ is equation of given curve of degree 'n' and $A_n = 0$ is combined equation of its all asymptotes, then curve which contains $n(n-2)$ points of intersection of given curve and its all asymptotes is given as :

$$\boxed{S_n + \lambda A_n = 0\ ;\ \lambda \text{ is any real number.}}$$

CONCEPT OF THE CURVATURE

Let us look at the following curves.



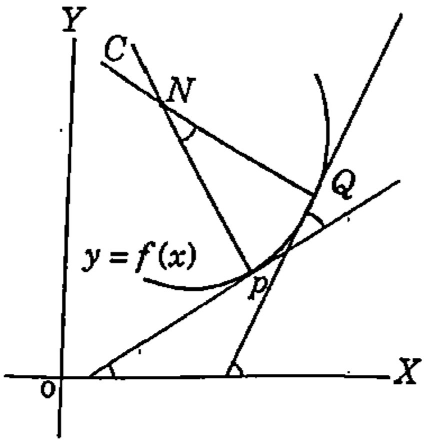
The curve C_1 bends more sharply than C_2 and bending tendency is zero in the straight line C_3.

CURVATURE

Definition: 'Curvature' is used for the quantitative measure of the bending of the curve at a point or in other words, curvature is a mathematical tool which measures the bending of the curve at every point.

GEOMETRY OF CURVATURE

Let the normals at P and Q intersect at N. As Q approaches P, the point N takes up a definite position C, on the normal at P. The length CP is called the radius of curvature, and the point C is called centre of curvature at the point P, whereas the circle with centre C and radius CP is called the circle of curvature which touches the curve $y = f(x)$ at P.



The radius of curvature is donated by ρ and curvature is donated by $\boxed{\kappa = \frac{1}{\rho}}$

CARTESIAN FORMULA FOR RADIUS OF CURVATURE

Let the equation of the curve be $y = f(x)$.

Then $$\boxed{\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

Remark: The above formula does not hold good if the tangent is parallel to the axis of y *i.e.* when $\frac{dy}{dx}$ is infinite. In such cases we interchange the axis of x and y because the value of ρ depends only on the curve and not on the axes.

Hence, we have $$\boxed{\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}}$$

FORMULA FOR RADIUS OF CURVATURE FOR CURVE IN PARAMETRIC FORM

If $x = f(t)$, $y = g(t)$ is the parametric form of the given curve, then radius of curvature is given as:

$$\boxed{\rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - x''y')}}$$

where x', y', x'', y'' are respective first and second order derivatives of x and y with respect to 't'

RADIUS OF CURVATURE AT ORIGIN

Step - 1 Equate the lowest degree terms of the given curve with zero to find all tangents of the curve at origin.

Step - 2 For any tangent except x - axis and y - axis, find the slope and assume that as p

Step - 3 Substitute $y = \left(px + \frac{qx^2}{\lfloor 2} + \ldots \right)$ in the equation of the curve and by equating coefficient of like power of x, find the value of q

Step - 4 For finding value of radius of curvature concerned with that particular tangent at origin, use the formula

$$\boxed{\rho = \frac{(1+p^2)^{3/2}}{q}}$$ [Maclaurine's expansion method]

Step - 5 If for the given curve, x - axis (line $y = 0$) is obtained as tangent at origin, then for finding the concerned radius of curvature, use the formula

$$\boxed{(\rho)_{(0,0)} = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^2}{2y} \right)}$$ [Newton's Method]

Step - 6 If for the given curve, y - axis (line $x = 0$) is obtained as tangent at origin, then for finding the concerned radius of curvature, use the formula

$$\boxed{(\rho)_{(0,0)} = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{y^2}{2x} \right)}$$ [Newton's Method]

CENTRE AND CIRCLE OF CURVATURE

For a curve $f(x, y) = c$, centre of curvature $(\bar{x}, \bar{y})$ is given as:

$$\boxed{\bar{x} = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}}$$ and $$\boxed{\bar{y} = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}}$$

Using $(\bar{x}, \bar{y})$ as centre of curvature and ρ as radius of curvature, the equation of the circle of curvature is given as:

$$\boxed{(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2}$$

CHORDS OF CURVATURE

Length of Chord of Curvature Parallel to x-axis

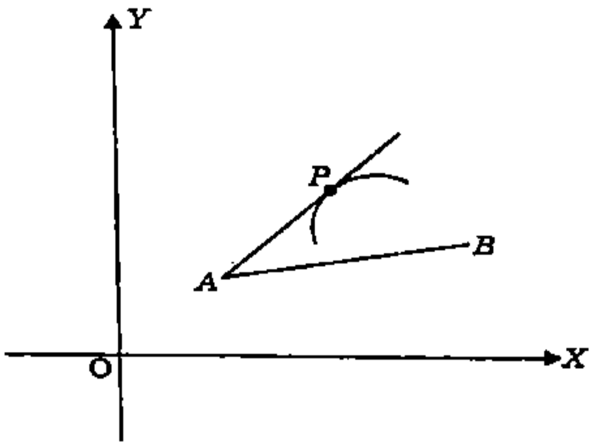
$$= 2\,\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]\Big/\frac{d^2y}{dx^2}$$

Length of Chord of Curvature Parallel to y-axis

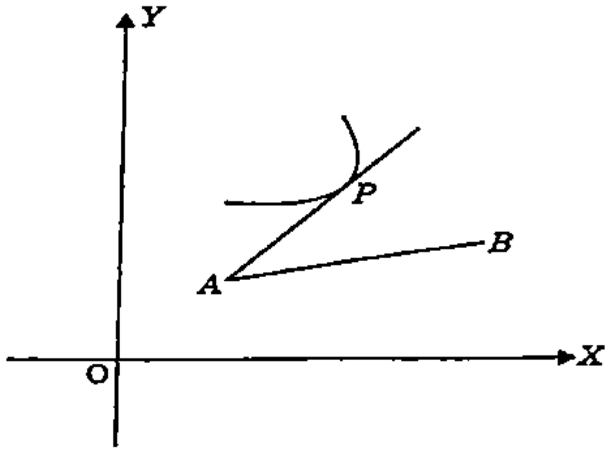
$$= \frac{2\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

CONCAVITY AND CONVEXITY OF A CURVE WITH RESPECT TO A GIVEN LINE

Let P be a given point on a curve and let AB be a given fixed straight line not passing through P. The curve is said to be concave or convex at P with respect to the line AB according as a sufficiently small arc containing P lies entirely within or without the acute angle formed by the tangent at P and the given line AB.



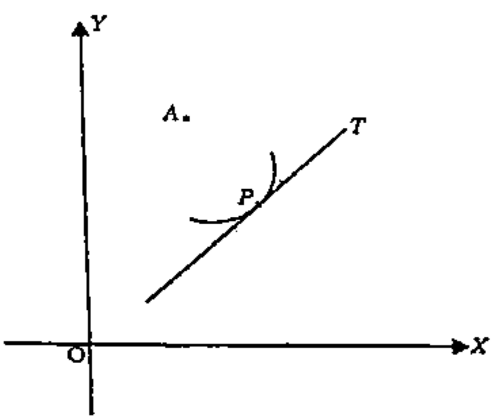
(a) Curve at P is concave w.r.t. AB



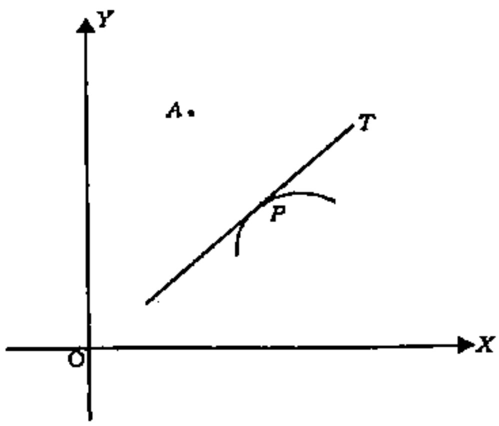
(b) Curve at P is convex w.r.t. AB

CONCAVITY AND CONVEXITY OF A CURVE WITH RESPECT TO A POINT

Let P be a given point on a curve. The curve is said to be concave or convex at P with respect to a given fixed point A, according as the curve in the immediate neighbourhood of P and the point A lie on the same side or opposite sides of the tangent PT to the curve at P.



(a) Curve at P is concave w.r.t A.



(b) Curve at P is convex w.r.t. A.

A TEST FOR CONCAVITY AND CONVEXITY

1. A curve is convex or concave at point P wrt x-axis according as $y \dfrac{d^2y}{dx^2}$ is positive or negative at point P.

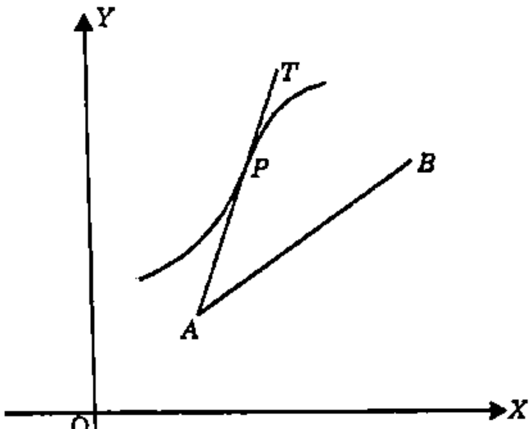
2. A curve is convex or concave at point P wrt y-axis according as $x \dfrac{d^2x}{dy^2}$ is positive or negative at point P.

3. A curve is convex or concave at point P wrt the foot of the ordinate at point P according as $y \dfrac{d^2y}{dx^2}$ is positive or negative at point P.

4. A curve is convex in interval $[a, b]$, if $\dfrac{d^2y}{dx^2} > 0,\ \forall\ x \in [a, b]$ and concave in interval $[a, b]$, if $\dfrac{d^2y}{dx^2} < 0\ \forall\ x \in [a, b]$.

POINT OF INFLEXION

A point on a curve, at which the curve changes from cancavity to convexity or from convexity to concavity, is called a point of inflexion *e.g.* for the following curve, point P is that point.



P is point of inflexion.

A TEST FOR POINT OF INFLEXION

For any curve $f(x, y) = 0$, all points where we obtain

$$\frac{d^2y}{dx^2} = 0 \text{ but } \frac{d^3y}{dx^3} \neq 0,$$

are called points of inflexion.

CURVE TRACING IN CARTESIAN COORDINATE SYSTEM

INTRODUCTION

The main objective to study curve tracing is to find the approximate shape of a curve with the knowledge of asymptotes, tangents, points of inflexion and multiple points etc. without plotting a large number of points.

IMPORTANT DEFINITIONS AND CONCEPTS INVOLVED IN CURVE TRACING

(*a*) **Singular Point.** A point on a curve at which the curve shows an extra-ordinary or abnormal behaviour is called singular point. Point of inflexion, maxima, minima and multiple points are the examples of singular points.

(*b*) **Multiple Point.** A point on a curve 'through which more than one branch(es) of the curve pass' is called a multiple point.

(*c*) **Double Point.** A point on a curve is called a double point if two branches of the curve pass through it.

(*d*) **Triple Point.** A point on a curve is called a triple point if three branches of the curve pass through it.

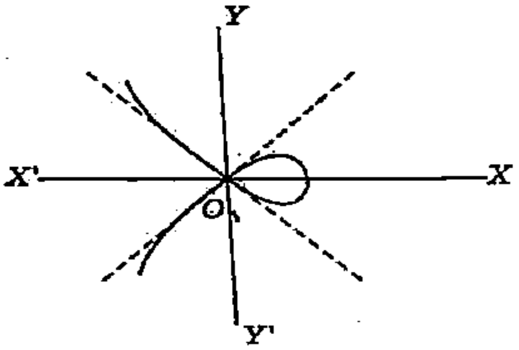
(*e*) **Tangents at the Origin/other Point.** To investigate the nature of a multiple point, the equation(s) of the tangent(s) at the point is/are required.

(*f*) **Classification of the Double Point.** There are three categories of double points.

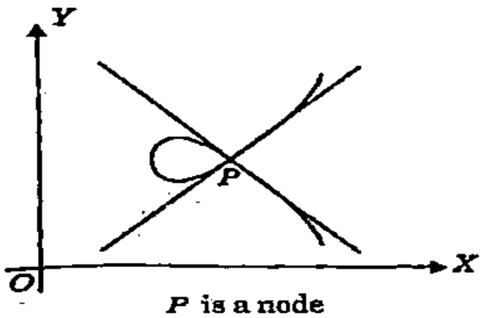
(*i*) **Node.** If the two branches of the curve passing through a point P are real and the tangents to them are also real and different, then P is called a node.

e.g. for $y^2(a+x) = x^2(3a-x)$,

$y = \pm x\sqrt{3}$ are real and different tangents at the origin

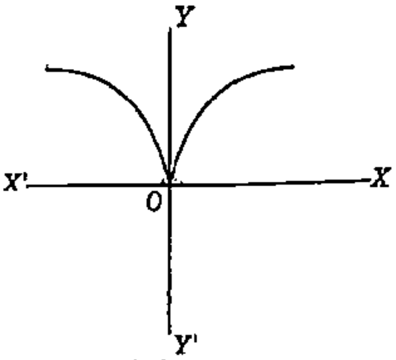


Origin is a node

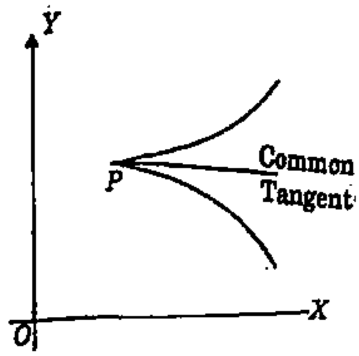


P is a node

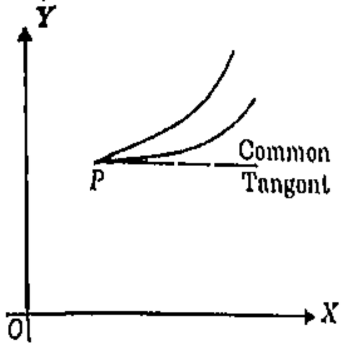
(*ii*) **Cusp.** If the two branches of the curve passing through a point P are real and the tangents to them are also real and coincident, then P is called a cusp *e.g.* for $x^2 = ay^3$, $x = 0$, $x = 0$ are real and coincident tangents at the origin.



Origin is a cusp

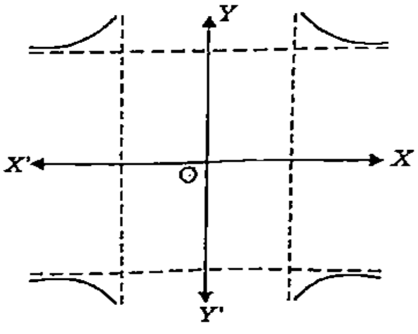


P is a cusp



P is a cusp

(*iii*) **Conjugate or Isolated Point.** If the two branches of a curve passing through a point P be imaginary *i.e.* if there are no real points of the curve in the neighbourhood of P, then P is called a conjugate or an isolated point of the curve. The co-ordinates of this point satisfy the equation of the curve and generally at such point there exist two imaginary tangents.

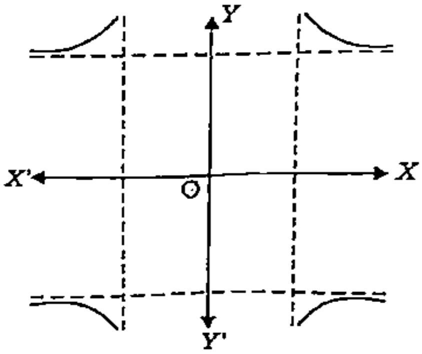


Origin is an isolated point

e.g. for $(x^2-a^2)(y^2-b^2)=a^2b^2$,

$y=\pm\dfrac{ibx}{a}$ are two imaginary tangents at the origin.

(*iii*) **Conjugate or Isolated Point.** If the two branches of a curve passing through a point P be imaginary *i.e.* if there are no real points of the curve in the neighbourhood of P, then P is called a conjugate or an isolated point of the curve. The co-ordinates of this point satisfy the equation of the curve and generally at such point there exist two imaginary tangents.



Origin is an isolated point

e.g. for $(x^2-a^2)(y^2-b^2)=a^2b^2$,

$y=\pm\dfrac{ibx}{a}$ are two imaginary tangents at the origin.

(*iv*) If $f(x,y)=f(-x,-y)$ *i.e.* if x and y are replaced by $-x$ and $-y$ respectively and the equation of the curve remains unaltered without fulfilling the condition of even powers of x and y, then the curve is symmetrical in opposite quadrants *e.g.* $y=x(x^2-1)$

(*v*) If $f(x,y)=f(y,x)$ *i.e.* if x and y are replaced by y and x respectively and the equation of the curve remains unaltered, then the curve is symmetrical about the line $y=x$, *e.g.* $x^3+y^3=3axy$

(*vi*) If $f(x,y)=f(-y,-x)$ *i.e.* if x and y are replaced by $-y$ and $-x$ respectively and equation of the curve remains unaltered, then the curve is symmetrical about the line $y=-x$, *e.g.* $x^3-y^3=3axy$

2. Nature of Origin. Now, we have to find out whether the curve passes through the origin or not. If the equation of the curve does not contain any constant term, then the curve clearly passes through the origin or if on putting $x=0$ in the equation, y also becomes zero, then the curve passes through the origin. *e.g.* $y^2=4ax$

If the curve passes through the origin, then find the equation(s) of the tangent(s) of the curve at the origin by equating to zero the lowest degree terms in the equation of the curve. In case origin is a double point, find whether it is a node, a cusp or a conjugate point.

3. Intersection with the Co-ordinate Axes. To find the points on co-ordinate axes, we first put $x = 0$ and calculate the values of y, then put $y = 0$ and calculate the values of x. After finding the points of intersection with co-ordinate axes, we find the tangents at those points by shifting the origin to those points. For example, if the origin is shifted to a point $(a, 0)$, then the given curve is transformed from $f(x, y) = 0$ to $f(x + a, y) = 0$

4. Asymptotes. Determine all the real asymptotes of the curve *i.e.*

(*i*) Asymptotes parallel to the co-ordinate axes.

(*ii*) Oblique asymptotes with the help of the methods already discussed.

5. Region of Existence. Determine the region in which the curve lies. To do this, solve the given equation of the curve for y (or x) and determine the values of x (or y) for which the values of y (or x) are imaginary. If y is seen to be imaginary in $a < x < b$, the curve does not exist in the region bounded by the lines $x = a$ and $x = b$. Above process can be simplified as :

(*i*) If any of x (or y) is given as pure quadratic variable, then solve given equation for x (or y) in terms of y (or x) by virtue of a square root to decide the certain region *e.g.* In $y^2(a - x) = x^2(a + x)$, y is a pure quadratic variable.

Thus, $$y = \pm x\sqrt{\left(\frac{a+x}{a-x}\right)}$$

$\Rightarrow$ $$x \in [-a, a)$$

So, curve exist only if $-a \le x < a$

(*ii*) If neither x nor y is given as pure quadratic variable, then check the existence of the curve in all four quadrants one by one using sign combinations *i.e.* in first quadrant (+, +), in second quadrant (–, +), in third quadrant (–, –), in fourth quadrant (+, –) *e.g.* curve $x^3 + y^3 = 3axy$ doesn't exist in third quadrant because in third quadrant, $x^3 + y^3$ is negative and $3axy$ is positive. (Here, we consider 'a' as a positive constant).

6. Points of Maxima and Minima.

To find these points, use equation of curve to obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. By substituting $\frac{dy}{dx} = 0$, we obtain points as probable maxima or minima and then by evaluating $\frac{d^2y}{dx^2}$ at these points, we can state as

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ point is maxima, $\frac{d^2y}{dx^2} < 0 \Rightarrow$ point is minima,

$\frac{d^2y}{dx^2} = 0 \Rightarrow$ calculate higher order derivatives at that point

and if $$\frac{d^3y}{dx^3} = \frac{d^4y}{dx^4} = \ldots\ldots = \frac{d^{n-1}y}{dx^{n-1}} = 0\ ;\ \frac{d^ny}{dx^n} \neq 0,$$

then there exist a maxima $\left(\text{if } n \text{ is even and } \frac{d^ny}{dx^n} < 0\right)$

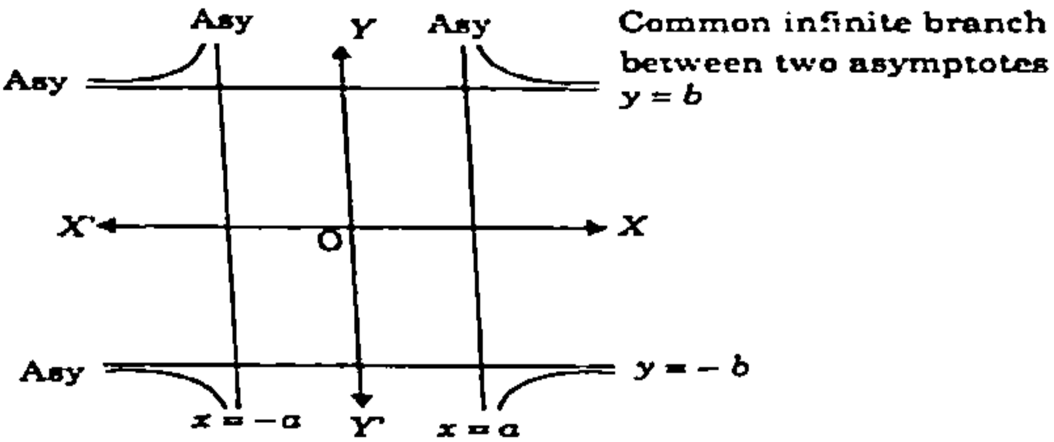
and minima $\left(\text{if } n \text{ is even and } \frac{d^ny}{dx^n} > 0\right)$

and there does not exist any maxima or minima (if n is odd).

7. Plotting of Points. For the correct shape of the curve, we can plot some points on the curve. Notice also how y varies as x varies continuously from $-\infty$ to $+\infty$.

Remarks : (1) For tracing a curve, we have to draw asymptotes, points of intersection of curve with the co-ordinate axes and tangents and at last apply symmetricity of the curve with the region of existence.

(2) For any curve if two asymptotes intersect, then there exist a common infinite branch of the curve *e.g.* For the curve $(x^2 - a^2)(y^2 - b^2) = a^2b^2$,
$x = \pm a$ and $y = \pm b$ are the asymptotes and its tracing is given as



(3) For any curve, if origin or any other point exist as a node, then there exists a loop at that node in absence of asymptote.

PROCEDURE FOR TRACING POLAR CURVES

Apply following procedure for tracing of a curve whose equation is given in the polar form $f(r, \theta) = 0$

(1) Symmetry. (*a*) If $f(r, \theta) = f(r, -\theta)$ *i.e.* if θ is replaced by $-\theta$ and the equation of the curve remains unaltered, then the curve is symmetrical about the initial line $\theta = 0$ (*i.e.* x-axis).

e.g. $r = a(1 + \cos\theta)$

(*b*) If $f(r, \theta) = f(r, \pi - \theta)$ *i.e.* if θ is replaced by $(\pi - \theta)$ and the equation of the curve remains unaltered, then the curve is symmetrical about the line $\theta = \dfrac{\pi}{2}$ (*i.e.* y-axis).

e.g. $r = a(1 + \sin\theta)$

(c) If $f(r, \theta) = f\left(r, \frac{\pi}{2} - \theta\right)$ *i.e.* if θ is replaced by $\left(\frac{\pi}{2} - \theta\right)$ and the equation of the curve remains unaltered, then the curve is symmetrical about the line $\theta = \frac{\pi}{4}$ (*i.e.* line $y = x$).

e.g. $$r = \frac{3a \sin\theta \cos\theta}{(\cos^3\theta + \sin^3\theta)}$$

(*d*) If $f(r, \theta) = f\left(r, \frac{3\pi}{2} - \theta\right)$ *i.e.* if θ is replaced by $\left(\frac{3\pi}{2} - \theta\right)$ and the equation of the curve remains unaltered, then the curve is symmetrical about the line $\theta = \frac{3\pi}{4}$ (*i.e.* line $y = -x$).

e.g. $$r = \frac{3a \sin\theta \cos\theta}{(\cos^3\theta - \sin^3\theta)}$$

(*e*) If $f(r, \theta) = f(-r, \theta)$ *i.e.* if r is replaced by $-r$ and the equation of the curve remains unaltered, then the curve is symmetrical about the pole.

e.g. $$r^2 = a^2 \cos 2\theta.$$

(2) Nature of Pole. Now, we have to find out whether the curve passes through the pole or not. If for some finite value of θ (say) $\theta = \alpha$, $r = 0$ then curve passes through the pole and line $\theta = \alpha$ is considered as a tangent of the curve at the pole.

(3) Intersection with the Line $\theta = 0$ and $\theta = \frac{\pi}{2}$. If the curve intersects with the initial line $\theta = 0$, then the points of intersection can be obtained by putting $\theta = 0$ in the eqn. of the curve and if the curve intersects with the line $\theta = \frac{\pi}{2}$, then the points of intersection can be obtained by putting $\theta = \frac{\pi}{2}$ in the equation of the curve.

(4) Asymptotes. For a polar curve, generally asymptote exists if $r \to \infty$, when $\theta \to a$ finite angle, which can be obtained by the following process :

(*i*) Represent the given curve as $\frac{1}{r} = g(\theta)$

(*ii*) Solve the equation $g(\theta) = 0$

(*iii*) If $g(\theta) = 0$ provides a finite solution *i.e.* $\theta = \beta$, then calculate

$$g'(\beta) = [g'(\theta)]_{\theta=\beta}$$

(*iv*) Asymptote is given by $r \sin(\theta - \beta) = \frac{1}{g'(\beta)}$

Using above process, we can conclude that for any polar curve, asymptote doesn't exist in any of three conditions :

(*i*) If curve is not expressible as $\frac{1}{r} = g(\theta)$

(*ii*) If $g(\theta) = 0$ does not provide a finite solution

(*iii*) If $g'(\beta) = 0$

(5) Region of Existence. (*i*) If r^2 is given in the equation of the curve, then solve the equation for r in terms of θ by virtue of a square root to decide the certain region. *e.g.* In $r^2 = a^2 \cos 2\theta$,

$$r = \pm a\sqrt{\cos 2\theta} \quad \Rightarrow \quad \cos 2\theta \geq 0$$

$\Rightarrow$ Curve exist only if $\cos 2\theta \geq 0$

(*ii*) If $\sin\theta$ or $\cos\theta$ is given in the equation of the curve, then use $-1 \leq \sin\theta \leq 1$ or $-1 \leq \cos\theta \leq 1$ to select the region of existence.

(6) Periodicity. In polar equations generally periodic functions like $\sin\theta$ or $\cos\theta$ occur and so values of θ are considered from 0 to 2π. The remaining vlaues of θ will not provide any new branch of the curve.

(7) Table. Solve the given equation for r and see how r varies as θ increases from 0 to $+\infty$ and θ decreases from 0 to $-\infty$. Thus, construct a table between corresponding values of θ and r as

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	......
r	r_1	r_2	r_3	r_4	r_5	......

(8) Transformation to Cartesian Co-ordinate System. Sometimes it is useful to transform $f(r, \theta) = 0$ into $F(x, y) = 0$ by using the relations

$$r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Remarks : (1) While observing nature of pole for a particular polar curve, we can obtain $r = 0$ for several values of θ, which can create a confusion about pole for that curve as a multiple point. But it is not true and it happens because of periodicity of the given curve. *e.g.* for the curve $r = a(1 + \cos\theta)$, $r = 0$ for $\theta = \pm\pi, \pm 3\pi, \ldots$, but we consider pole as a single point and one of the tangents as $\theta = \pi$.

(2) For most of the curves, while calculating the values of θ and r, periodicity helps to consider range of θ from 0 to 2π and using symmetricity this range can be further minimized.

e.g. For curve $r = a(1 - \cos\theta)$, using periodicity, range of θ is considered from 0 to 2π and it is minimized from 0 to π only because of symmetricity of the curve with respect to the line $\theta = 0$
