Mathematical Methods in Engineering and Applied Science
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Problem Set 6. Due on Nov. 13 at 23:59.

(1) Given the data:

x_i	0	1	2	3	5
y_i	1	3	3	4	6

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(a) Find the best linear fit by solving the 2×2 normal system by hand $A^T A u = A^T b$, with $A = \begin{bmatrix} x^T & 1 \end{bmatrix}$ and $b = y^T$. Plot the data and the fit.
(b) Calculate the Moore-Penrose pseudo-inverse A^+ of A directly from its definition.
(c) Write down the SVD of A^+.
(d) What is the error vector e of the approximation and its 2-norm?

(2) Find the best plane in $\mathbb{R}^3$, in the least-squares sense, through the data given in the table:

x_i	1	1	2	3	5
y_i	5	3	4	10	7
z_i	2	1	2	5	5

. What is the error vector and its norm?

(3) Determine the dominant modes in the function f of space x and time t:

$$f(x,t) = e^{-x^2} \sin(x + 3t) \cos(x - t),$$

considering the interval $x \in [-5, 5]$, $t \in [0, 10]$.
(a) Plot the singular values in uniform as well as semilog scales.
(b) Plot the solution in the x-t plane over the given interval.
(c) How much "energy" of the solution is contained in mode 1 and in modes 1+2?
(d) Plot the first two columns of U and V in the SVD of matrix F obtained by calculating $f(x,t)$ over a grid with 100 points in x and 50 points in t. Explain their meaning.

(4) To find a root of $f(x) = 0$, Newton's method tells to start with some initial guess x_0 and then to iterate following the scheme: $x_{n+1} = x_n - f(x_n)/f'(x_n)$.
(a) Use this method to find the root $x = 1$ of $f(x) = x^2 - 1$.
(b) What is the range of initial conditions x_0 that give convergence to $x = 1$?
(c) How fast do the iterations converge? Plot the error $e_n = |x_n - 1|$ as a function of n (maybe, in log scale).

(5) Now apply the same Newton iterations as in the previous problem to the equation $f(x) = x^2 + 1 = 0$. Clearly, this equation has no real roots.
(a) The question is: What do the iterations do? Do they converge to anything?
(b) How does the behavior of the iterations depend on the initial point x_0?
(c) What if you start the iterations in the complex plane? Can you get convergence to the actual roots $\pm i$ of the equation? What are the domains of attraction of the roots?

(6) Consider the function $f = 2x^2 + 2xy + y^2 - x - 2y$.
(a) Find its minimum analytically by representing f as $\frac{1}{2}u^T A u - b^T u$. Plot the function together with its contour levels using, for example, **surfc** function in Matlab.
(b) Now find the minimum using the gradient descent. Determine the step τ in the descent method.

(c) Starting with $(x_0, y_0) = (0, 4)$, calculate the first two steps of the gradient descent explicitly and indicate on a single plot both the positions and the gradient vectors at those positions. Also plot the level curves of f going through these points.

(d) Implement the descent algorithm in Matlab or Python and starting with the same initial condition as in (c) find the minimum within a tolerance of $tol = 10^{-6}$. How many iterations does it take to reach the minimum?