Mathematical Methods in Engineering and Applied Science.
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Problem Set 9. Due on Dec. 12 at 23:59.

(1) For the equation $\ddot{x} + \mu\left(x^2 + \dot{x}^2 - 1\right)\dot{x} + x = 0$, where $\mu > 0$:
  (a) Find and classify all the fixed points.
  (b) Show that the system has a circular limit cycle and find its amplitude and period.
  (c) (*Extra credit*). Determine the stability of the limit cycle. Give an argument which shows that the limit cycle is unique, i.e., there are no other periodic trajectories.

(2) Investigate the phase plane of the system $\dot{x} = y$, $\dot{y} = x\left(\mu - x^2\right)$, for $\mu < 0$, $\mu = 0$, and $\mu > 0$. Describe the bifurcation as μ increases through zero.

(3) Consider the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$, where $f = -1$ for $|x| < 1$ and $f = 1$ for $|x| \geq 1$.
  (a) Show the system is equivalent to $\dot{x} = \mu(y - F(x))$, $\dot{y} = -x/\mu$ where

$$F = \begin{cases} x+2, & x \leq -1 \\ -x, & |x| < 1 \\ x-2, & x \geq 1. \end{cases}$$

  (b) Graph the nullclines.
  (c) Show that the system exhibits relaxation oscillations for $\mu \geq 1$, and plot the limit cycle in the (x, y) plane.
  (d) (*Extra credit*). Estimate the period of the limit cycle for $\mu \gg 1$.

(4) Consider the system

$$\dot{x} = -y + \mu x + xy^2, \quad \dot{y} = x + \mu y - x^2.$$

  (a) Linearize about the origin and determine the type of the fixed point.
  (b) Write down the system to find all the fixed points. Eliminate y to find the equation for $x_c(\mu)$. Make a plot of this function to find out how many fixed points there are for given μ.
  (c) Investigate numerically the nature of the solutions on a phase plane as μ varies about $\mu = 0$.
  (d) What is the type of the bifurcation that takes place as μ crosses 0?
  (e) Rewrite the system in polar coordinates $x = r\cos\vartheta$, $y = r\sin\vartheta$ and approximate the system assuming r small. Show that to leading order the system becomes

$$\dot{r} = \mu r + \frac{1}{8}r^3, \quad \dot{\theta} = 1,$$

  and hence one can expect a limit cycle of radius $r \approx \sqrt{-8\mu}$ when $\mu < 0$. Confirm this numerically.

(5) (*Extra credit*). Consider the system

$$\dot{x} = y, \quad \dot{y} = x^2 - y - \mu.$$

  (a) Analyze the fixed points of the system at all possible μ.
  (b) What type of bifurcation takes place as μ crosses 0?
  (c) Draw the bifurcation diagram in the space of x_c vs μ, where x_c is the critical point.
  (d) Plot the phase plane at $\mu = 0.01$.