Mathematical Methods in Engineering and Applied Science

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Problem Set 3. Due on Oct. 23 at 23:59. ....

(1) The e-value problem for longitudinal waves in an elastic rod is:

$$u'' + k^2 u = 0,$$
$$u(0) = 0, \quad u_x(l) = 0.$$

(a) Explain the physical meaning of the second boundary condition.
(b) Solve the problem analytically to find k and u.
(c) Plot the eigenfunctions that correspond to the lowest three eigenvalues.

(2) Consider elastic waves in a nonuniform rod between $x = 0$ and $x = l$, as in the lecture notes. The density of the rod varies with space as $\rho = \rho_0 \left(1 - \epsilon \frac{x(l-x)}{l^2}\right)$, where ρ_0 is constant and $\epsilon \geq 0$ is some number. The corresponding eigenvalue problem is:

$$\phi'' + k_0^2 \left(1 - \epsilon \frac{x(l-x)}{l^2}\right) \phi = 0, \quad \phi(0) = \phi(l) = 0.$$

Here $k_0^2 = \rho_0 \omega^2 / E = \omega^2 / c_0^2$. After rescaling x with l, the equation can be written as

$$\phi'' + \lambda^2 (1 - \epsilon x (1 - x)) \phi = 0, \quad \phi(0) = \phi(1) = 0.$$

where $\lambda^2 = k_0^2 l^2 = \omega^2 l^2 / c_0^2$. Next, the problem is to find the solution at various values of ϵ. Note that if $\epsilon = 0$, then we have the analytical solution: $\lambda_n = n\pi$ and $\phi_n = \sin n\pi x$, $n \in \mathbb{N}$.

(a) Find the eigenvalues λ^2 numerically using the second derivative approximation $\phi'' \approx (\phi_{i-1} - 2\phi_i + \phi_{i+1}) / h^2$, with the grid points $x = x_i = ih$, $i = 0, 1, 2, ..., n+1$ and the grid size $h = l/(n+1)$. Take n sufficiently large so that the computed eigenvalues and eigenvectors are accurate. How do you choose that number?
(b) Plot the eigenvectors that correspond to the lowest five eigenvalues at $\epsilon = 0.1$ and on a separate plot show the first ten numerically found eigenvalues λ. How do they compare with those at $\epsilon = 0$?
(c) How do eigenvalues and eigenfunctions change when ϵ increases? Plot $\lambda(\epsilon)$ for the first five eigenvalues.
(d) What happens when ϵ becomes close to 4?

(3) Let $A = \begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix}$.

(a) Find the SVD of A and illustrate on paper how $A = U\Sigma V^T$ transforms a vector x into Ax by a sequence of three transformations.
(b) Compare the result from (a) with how $A = S\Lambda S^{-1}$ transforms x into Ax.

(4) Given column vectors a and b, find the SVD of:

(a) $A = ab^T$.
(b) $B = ab^T + ba^T$ when $a^T b = 0$.

(5) For the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

(a) using SVD, find the orthonormal basis for the column space, $C(A)$, and complete it with the basis for the left nullspace, $N(A^T)$, to form a basis for $\mathbb{R}^4$. Write down the 4×4 orthogonal matrix U.
(b) Also using SVD, find the orthonormal basis for the row space, $R(A)$, and complete it with the basis for the nullspace, $N(A)$, to form a basis for $\mathbb{R}^3$. Write down the 3×3 orthogonal matrix V.
(c) Write the full SVD $A = U\Sigma V^T$.
(d) What is the best rank-1 approximation of A?