Problem Set 10

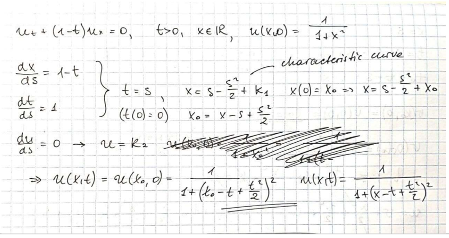
Problem 1

Solve the initial value problem for the advection equation

$$u_t + (1-t)u_x = 0, \quad t > 0, \quad x \in \mathbb{R}$$
$$u(x,0) = \frac{1}{1+x^2}.$$

Plot the characteristic curves as well as the solution $u(x,t)$ at several different times.

Solution


$u_t + (1-t)u_x = 0$, $t>0$, $x \in \mathbb{R}$, $u(x,0) = \frac{1}{1+x^2}$

characteristic curve

$\left.\begin{aligned} \frac{dx}{ds} &= 1-t \\ \frac{dt}{ds} &= 1 \end{aligned}\right\}$ $t = s$, $(t(0)=0)$ $x = s - \frac{s^2}{2} + k_1$ $x(0) = x_0 \Rightarrow x = s - \frac{s^2}{2} + x_0$

$x_0 = x - s + \frac{s^2}{2}$

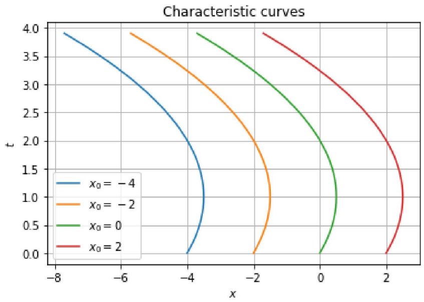
$\frac{du}{ds} = 0 \to u = k_2$

$\Rightarrow u(x,t) = u(x_0, 0) = \frac{1}{1+(x_0 - t + \frac{t^2}{2})^2}$ $u(x,t) = \frac{1}{1+(x-t+\frac{t^2}{2})^2}$

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.cm as cm
```

```
def fx(t, c):
    return t - t**2/2 + c

t = np.arange(0, 4, 0.1)
for c in range(-4, 4, 2):
    x = fx(t, c)
    plt.plot(x, t)

plt.title('Characteristic curves')
plt.xlabel('$x$')
plt.ylabel('$t$')
plt.grid()
plt.legend(tuple(["$x_0 = {}$".format(i) for i in range(-4, 4, 2)]))
```

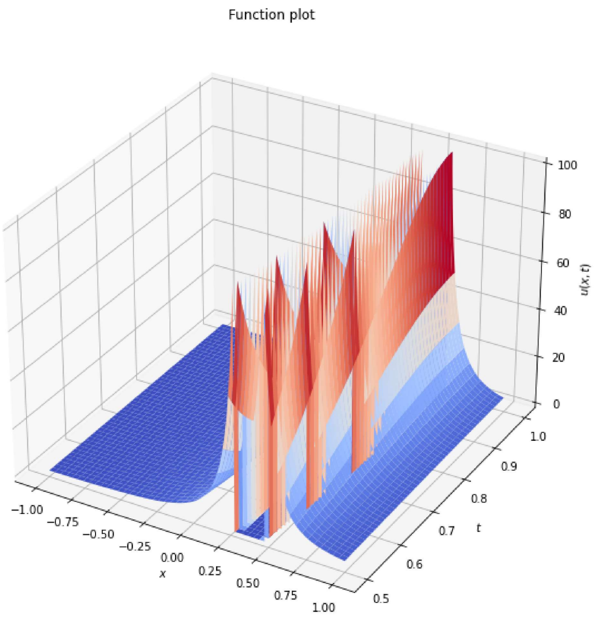
```
<matplotlib.legend.Legend at 0x1913a76fcc8>
```




```
def u(x, t):
    res = 1/(x - t + t**2/2)**2
    if (res >= 100):
        return 0
    else:
        return res


x = np.arange(-1, 1, 0.001)
t = np.arange(0.5, 1, 0.001)

X, T = np.meshgrid(x, t)
ni, nj = X.shape
U = np.zeros(X.shape)

for i in range(ni):
    for j in range(nj):
        U[i][j] = u(X[i][j], T[i][j])

ax = plt.subplot(projection='3d')
ax.plot_surface(X, T, U, cmap=cm.coolwarm)
plt.xlabel('$x$')
plt.ylabel('$t$')
plt.title('Function plot')
ax.set_zlabel('$u(x,t)$')
plt.gcf().set_size_inches(10, 10)
ax.set_zlim((0, 100))
```

```
(0.0, 100.0)
```

Contents



> ⚠ **Warning**
>
> Function is discontinuous at the points where the graph is weird

Problem 2

Solution

$u_t = t + x \qquad u = \frac{t^2}{2} + xt + f(x)$

$u_x = 0$

$u = g(t) \qquad \frac{t^2}{2} + xt + f(x) = g(t)$

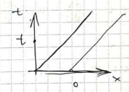
$v(x,0) = u(x,0) - \frac{x^2}{2} + c = -\frac{x^2}{2} + c$

$u_t + u_x = t + x \qquad u(x,0) = 0 \qquad u(0,t) = t \qquad v(0,t) = t - \frac{t^2}{2} + C$

$v = u - \frac{t^2}{2} - \frac{x^2}{2} + c$

$v'_t = u_t - t \qquad v'_x = u_x - x$

$v_t + v_x = 0$



$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = 1 \qquad \frac{dv}{ds} = 0 \rightarrow v = c$

1) $x > t$: $\quad u(s=0) = u(x_0, 0) = 0$

$v(s=0) = \cancel{u(x_0,0)}\ v(x_0, 0) = -\frac{x_0^2}{2} + c = k \qquad c = +\frac{x_0^2}{2} + k$

2) $x < t$: $\quad u(s=0) = u(0, t_0)$

$v(s=0) = v(0, t_0) = t_0 - \frac{t_0^2}{2} + c = k$

$\rightarrow t = x + t_0 \qquad t_0 = t - x$

$v = t - x - \frac{t^2 - 2tx + x^2}{2} + c = \underbrace{t - x + tx}_{u} - \frac{t^2}{2} - \frac{x^2}{2} + c$

$t - x + tx$

or $\quad x = t + x_0 \qquad x_0 = x - t$

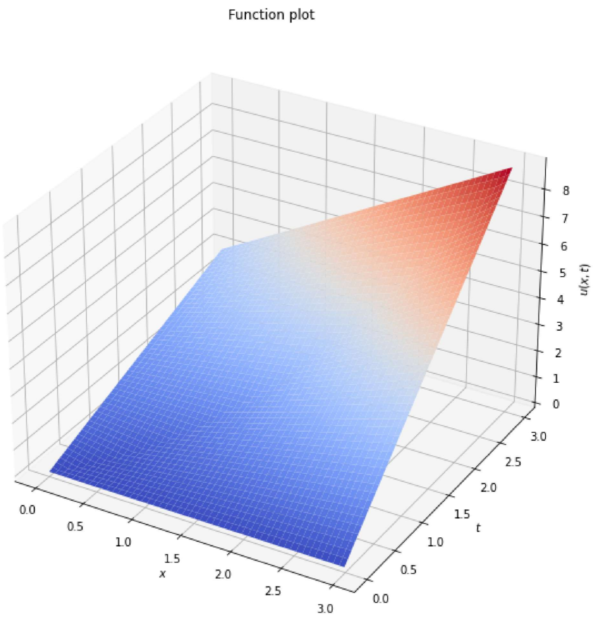
$v(s=0) = -\frac{x^2}{2} + \underbrace{xt} - \frac{t^2}{2} + c$

$u =$

1) $t > x \rightarrow u = t - x + tx \qquad x = t: \quad u = x - x + x^2 = x^2$

2) $t < x \rightarrow u = xt \qquad x = t: \quad u = x^2$

```
def u(x, t):
    if (x >= t):
        return x*t
    else:
        return x*t + t - x

x = np.arange(0, 3, 0.01)
t = np.arange(0, 3, 0.01)

X, T = np.meshgrid(x, t)
ni, nj = X.shape
U = np.zeros(X.shape)

for i in range(ni):
    for j in range(nj):
        U[i][j] = u(X[i][j], T[i][j])

ax = plt.subplot(projection='3d')
ax.plot_surface(X, T, U, cmap=cm.coolwarm)
plt.xlabel('$x$')
plt.ylabel('$t$')
plt.title('Function plot')
ax.set_zlabel('$u(x,t)$')
plt.gcf().set_size_inches(10, 10)
```



Problem 3

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Solution

Problem 3:

$u_t + \cancel{} = 0$ $x \in \mathbb{R}$ $u(x,0) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

uu_x

$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = u \quad \frac{du}{ds} = 0$

$t = s + k_1 \quad x = us + k_2 \quad u = k_3$

1) $x_0 \in [0,1]$: $\quad u(x_0, 0) = 1 \to k_3 = 1 \to u = 1$

$x_0 = k_2 \to x = us + x_0$

$t_0 = 0 \to k_1 = 0 \to t = s$

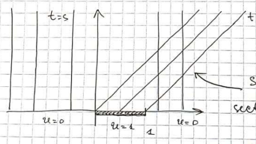
$x = us + k_2 = t + x_0 \qquad t = x - x_0$

2) $x_0 \in (-\infty, 0) \cup (1, \infty)$: $\quad u(x_0, 0) = 0 \to k_3 = 0$

$x_0 = k_2 \to x = us + x_0$

$t_0 = 0 \to k_1 = 0 \to t = s$

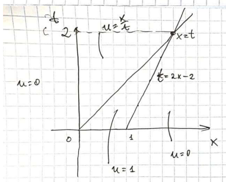
$x = us + k_2 = x_0 \qquad t = s$



Shock at the intersection of sol-ns:

$\dot{\sigma} = \frac{[F(u)]}{[u]} = \frac{\frac{(u_+)^2}{2} - \frac{(u_-)^2}{2}}{u_+ - u_-} = \frac{u_+ + u_-}{2} = \frac{1}{2}$

$\Rightarrow$ Shockwave: $\quad x - 1 = \frac{t}{2}$ or $t = 2x - 2$



$t = 2$:

$u_- = \frac{x}{t} \qquad u_+ = 0$

$\dot{\sigma} = \frac{x}{2t}$

Shock wave from $p(2,2)$

$x = \sqrt{2t}$ or $t = \frac{x^2}{2}$

Finally the sol-n:

$t \le 2$: $\quad u = \begin{cases} 0, & x < 0 \cup x > 1 + \frac{t}{2} \\ \frac{x}{t}, & 0 < x < t \\ 1, & t < x < 1 + \frac{t}{2} \end{cases}$

$t \ge 2$: $\quad u = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 < x < \sqrt{2t} \\ 0, & \sqrt{2t} < x \end{cases}$

Problem 4

Solution

Problem4

$$\left(u+\frac{u^2}{2}\right)_t = u_{xx}-u_x \qquad u(-\infty)=0 \quad u(\infty)=1 \quad u(0)=\frac{1}{2}$$

$u=u(z), \quad z=x-ct$

$u_x = u_z z_x = u_z$

$u_{xx} = (u_z z_x)_x = u_{zz} z_x^2 + u_z z_{xx} = u_{zz}$

$u_t = u_z z_t = -cu_z$

$uu_t = -cuu_z$

$\cancel{u_{xx} - u_x + cu_z + cuu_z = 0}$

$u_{zz} - u_z + cu_z + cuu_z = 0$

$$\int u_{zz}\,dz - \int u_z\,dz + \int cu_z\,dz + \underbrace{\int cuu_z\,dz}_{=\int cu\,du} = 0$$

$$u_z - u + cu + c\frac{u^2}{2} = K$$

$u(-\infty)=0$: assuming u is continuous, $u(-\infty)=0 \Rightarrow u_z(-\infty)=0$

then:

$0-0+0+0=K \Rightarrow k=0$

$u(+\infty)=0$: same logic $u_z(+\infty)=0$

$0-1+c+\frac{c}{2}=0 \rightarrow \frac{3c}{2}=1 \rightarrow c=\frac{2}{3}$

$$u_z = u - \frac{2}{3}u - \frac{1}{3}u^2 = \frac{1}{3}(u-u^2)$$

$$\frac{du}{u-u^2} = \frac{1}{3}dz \qquad \frac{1}{u-u^2} = \frac{1}{u(1-u)} = \frac{1}{u}+\frac{1}{1-u}$$

$$\frac{du}{u} + \frac{du}{1-u} = \frac{1}{3}dz \rightarrow \ln|u| - \int\frac{d(1-u)}{1-u} = \frac{1}{3}z + \tilde{C}$$

$$\ln|u| - \ln|1-u| = \frac{1}{3}z + \tilde{\tilde{C}}$$

$$\ln\left|\frac{u}{1-u}\right| = \frac{1}{3}z + \tilde{\tilde{C}} \qquad \frac{u}{1-u} = Ae^{\frac{z}{3}}$$

$$u = Ae^{z/3} - uAe^{z/3} \rightarrow u = \frac{Ae^{z/3}}{1+Ae^{z/3}} \qquad u(0)=\frac{1}{2} \rightarrow \frac{A}{1+A}=\frac{1}{2} \rightarrow A=1$$

$$u(x,t) = e^{\frac{x-ct}{3}}\Big/\left(1+e^{\frac{x-ct}{3}}\right) \leftarrow u=\frac{e^{z/3}}{1+e^{z/3}}$$

Problem 5

Solution

Problem 5

$$\begin{cases} u_t = \mathcal{D}u_{xx} + u + v & 0<x<\pi \\ v_t = 3u - v \end{cases}$$

a) $\mathcal{D}=0 \Rightarrow \begin{cases} u_t = u+v \\ v_t = 3u-v \end{cases}$

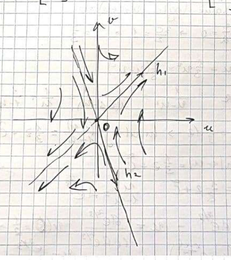
f.p.s: $\begin{cases} u=-v \\ 3u=v \end{cases} \rightarrow \underline{u=0, \ v=0}$

$$J = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \qquad \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0: \ (1+\lambda)(\lambda-1)-3=0$$

$\lambda^2-1-3=0 \quad \lambda^2=4 \quad \lambda=\pm 2 \Rightarrow$ unstable saddle

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{cases} x+y=\lambda x & -y = x-\lambda x = x(1-\lambda) \\ 3x-y=\lambda y & y = x(\lambda-1) \end{cases}$$

$$\lambda_1 = 2 \quad h_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2=-2 \quad h_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$



b) $\omega = \begin{bmatrix} u \\ v \end{bmatrix} = a(t)\cos nx \rightarrow a(t) = \begin{bmatrix} a_u(t) \\ a_v(t) \end{bmatrix}$

$u_t = a_u'(t)\cos nx \qquad u_{xx} = -n^2 a_u(t)\cos nx$

$v_t = a_v'(t)\cos nx$

$$\begin{cases} a_u'(t)\cancel{\cos nx} = -\mathcal{D}n^2 a_u(t)\cancel{\cos nx} + a_u(t)\cancel{\cos nx} + a_v(t)\cancel{\cos nx} \\ a_v'(t)\cancel{\cos nx} = 3a_u(t)\cancel{\cos nx} - a_v(t)\cancel{\cos nx} \end{cases}$$

$$\begin{cases} a_u'(t) = -\mathcal{D}n^2 a_u(t) + a_u(t) + a_v(t) = a_u(t)(1-\mathcal{D}n^2) + a_v(t) \\ a_v'(t) = 3a_u(t) - a_v(t) \end{cases}$$

$$J = \begin{bmatrix} 1-\mathcal{D}n^2 & 1 \\ 3 & -1 \end{bmatrix} \qquad \sigma: \begin{vmatrix} 1-\mathcal{D}n^2-\sigma & 1 \\ 3 & -1-\sigma \end{vmatrix} = 0$$

$$(1-\mathcal{D}n^2-\sigma)(1+\sigma)+3=0$$

$$\sigma^2 + \mathcal{D}n^2\sigma + \mathcal{D}n^2 + 4 = 0$$

$$\sigma = \frac{1}{2}\left(-\mathcal{D}n^2 \pm \sqrt{\mathcal{D}^2n^4 - 4\mathcal{D}n^2 + 16}\right)$$