Homework 2

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1.

Consider a finite-difference formula

$$u'(x_i) \approx \frac{1}{h}\left(\frac{1}{36}u_{i-2} - \frac{2}{9}u_{i-1} - \frac{11}{9}u_i + \frac{20}{9}u_{i+1} - \frac{37}{36}u_{i+2} + \frac{2}{9}u_{i+3}\right), \tag{1}$$

where u_{i+s} denotes the value of a function $u(x)$ evaluated at a point $x_i + sh$, with $h \ll 1$ and s being an integer number.

1. *(max. 1 point)* Is (1) a consistent approximation to the first derivative of $u(x)$?
2. *(max. 1 point)* Use the Taylor series expansion to find the order of this approximation.

Solution:

$$u_{i+3} = u_i + 3hu_i' + \frac{9h^2}{2}u_i'' + O(h^3) \to \frac{27}{6}h^3 u'''(\xi_3)$$

$$u_{i+2} = u_i + 2hu_i' + 2h^2u_i'' + O(h^3) \to \frac{8}{6}h^3 u'''(\xi_2)$$

$$u_{i+1} = u_i + hu_i' + \frac{h^2}{2}u_i'' + O(h^3) \to \frac{1}{6}h^3 u'''(\xi_1)$$

$$u_{i-1} = u_i - hu_i' + \frac{h^2}{2}u_i'' - O(h^3) \to \frac{1}{6}h^3 u'''(\zeta_1)$$

$$u_{i-2} = u_i - 2hu_i' + 2h^2u_i'' - O(h^3) \to \frac{8}{6}h^3 u'''(\zeta_2)$$

$$u'(x_i) \approx u_i' \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{2h} \quad \text{- Central FD}$$

$$u'(x_i) \approx \frac{1}{h}\left(\frac{1}{36}u_{i-2} - \frac{2}{9}u_{i-1} - \frac{11}{9}u_i + \frac{20}{9}u_{i+1} - \frac{37}{36}u_{i+2} + \frac{2}{9}u_{i+3}\right) =$$

$$= \frac{1}{h}\Big[\frac{1}{36}u_i - \frac{2}{36}hu_i' + \frac{2}{36}h^2u_i'' - \frac{8}{6\cdot 36}h^3u'''(\zeta_2) -$$

$$- \frac{2}{9}u_i + \frac{2}{9}hu_i' - \frac{1}{9}h^2u_i'' - \frac{2}{9\cdot 6}h^3u'''(\zeta_1) -$$

$$- \frac{11}{9}u_i +$$

$$+ \frac{20}{9}u_i + \frac{20}{9}hu_i' + \frac{10}{9}h^2u_i'' + \frac{20}{9\cdot 6}h^3u'''(\xi_1) -$$

$$- \frac{37}{36}u_i - \frac{2\cdot 37}{36}hu_i' - \frac{2\cdot 37}{36}h^2u_i'' - \frac{8\cdot 37}{36}h^3u'''(\xi_2) +$$

$$+ \frac{2}{9}u_i + \frac{6}{9}hu_i' + h^2u_i'' + \frac{2\cdot 27}{6\cdot 9}h^3u'''(\xi_3)\Big] =$$

$$= \frac{1}{h}\left[0\cdot u_i + 1\cdot u_i'\cdot h + 0\cdot h^2\cdot u_i'' + u_i'''\cdot h^3\cdot C\right] =$$

$$= u_i' + \underline{\underline{Ch^2u_i'''}}$$

This approximation is consistent

Order of approx. is $\underline{\underline{2.}}$

2.

Consider the following ordinary differential equation:

$$u' + \frac{\cos x}{2 + \sin x} u = \frac{\cos 2x - 2\sin x}{2 + \sin x} \tag{2}$$

Let $u(x)$ be defined over the interval of x from 0 to 2π, and let us supplement (2) with periodic boundary conditions.

3. *(max. 1 point)* Find the exact analytical solution of this problem.
4. *(max. 1 point)* Use the finite-difference approximation (1) to solve this problem numerically. Plot the exact and the approximate $u(x)$ calculated on a uniform grid with step $h = \pi/10$ over the interval from 0 to 2π.
5. *(max. 1 point)* Vary h from a suitably small to a suitable large value to plot the convergence error norms (pointwise maximum and r.m.s.) as functions of h. Comment on the rate of convergence: compare your numerical observation with your theoretical result.

Solution:

$$u' + \frac{\cos x}{2+\sin x}\,u = \frac{\cos 2x - 2\sin x}{2+\sin x}$$

$$\cancel{u' + \frac{\cos x}{2+\sin x}\,u = \frac{\cos^2 x - \sin^2 x - 2\sin x}{2+\sin x}}$$

1) $u' + \frac{\cos x}{2+\sin x}\,u = 0$

$$\frac{u'}{u} = -\frac{\cos x}{2+\sin x}$$

$$\frac{du}{u} = -\frac{\cos x\,dx}{2+\sin x} = -\frac{d\sin x}{2+\sin x} = -\frac{d(\sin x+2)}{\sin x+2}$$

$$\int\frac{du}{u} = -\int\frac{d(\sin x+2)}{\sin x+2} + \hat{C}$$

$$\ln|u| = -\ln|\sin x+2| + \tilde{C}$$

$$u = \frac{C}{\sin x+2}$$

2) $C = C(x) \to u' = \frac{C'}{\sin x+2} + C\cdot\left(\frac{-1}{(\sin x+2)^2}\cdot\cos x\right)$.

$$\frac{C'}{\sin x+2} - \cancel{\frac{C\cos x}{(\sin x+2)^2}} + \cancel{\frac{C\cos x}{(2+\sin x)^2}} = \frac{\cos 2x - 2\sin x}{2+\sin x}$$

$$C' = \cos 2x - 2\sin x = \frac{dC}{dx}$$

$$dC = \cos 2x\,dx - 2\sin x\,dx = \frac{1}{2}\cos 2x\,d2x - 2\sin x\,dx$$

$$\int dC = \frac{1}{2}\int\cos 2x\,d2x - 2\int\sin x\,dx + \hat{D}$$

$$C = \frac{1}{2}\sin 2x + 2\cos x + D$$

$$u = \frac{\frac{1}{2}\sin 2x + 2\cos x + D}{\sin x + 2}$$

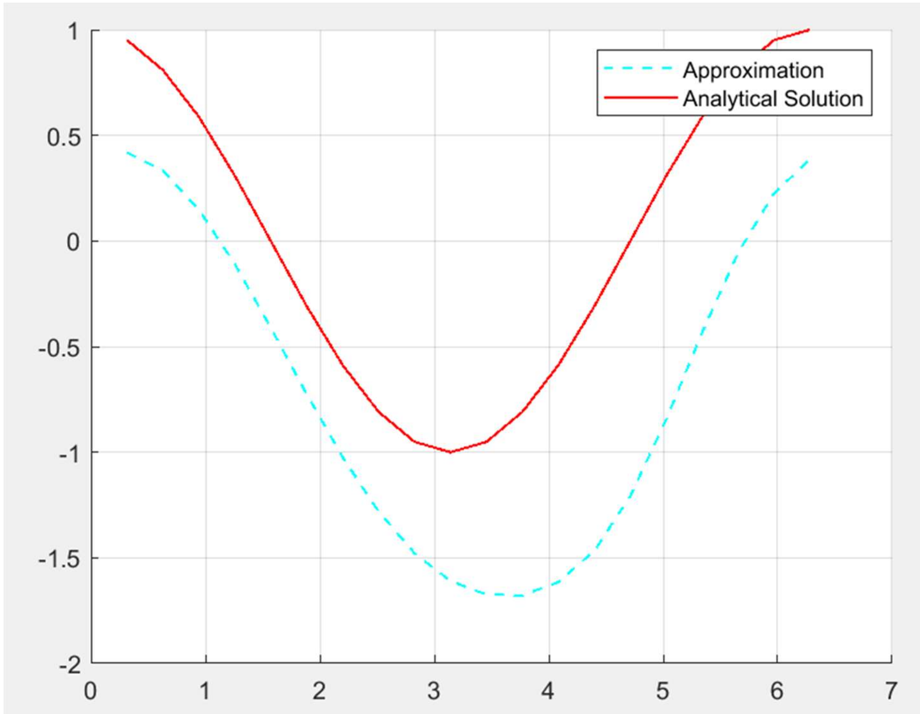
PBC: $u(0) = u(2\pi)$: $\cancel{\frac{2+D}{2} = \frac{2+D}{2}}$

$\cancel{u(0) = u(2\pi)}$

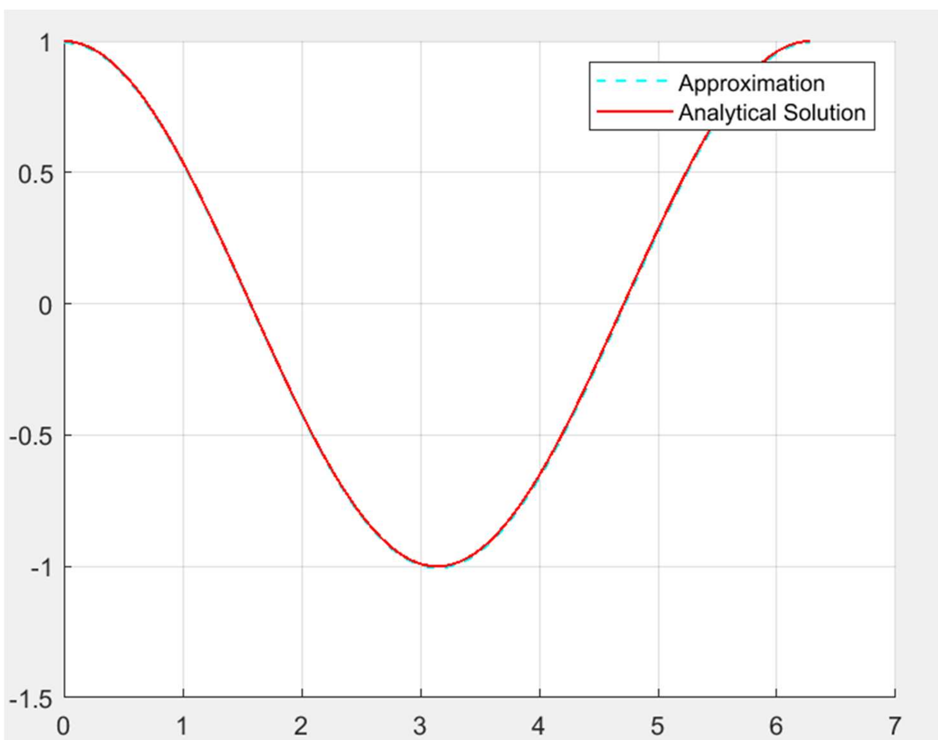
$u(0) = 1$; $\frac{2+D}{2} = 1 \Rightarrow D = 0$

$$u(x) = \frac{\frac{1}{2}\sin 2x + 2\cos x}{\sin x + 2} = \underline{\underline{\cos x}}$$

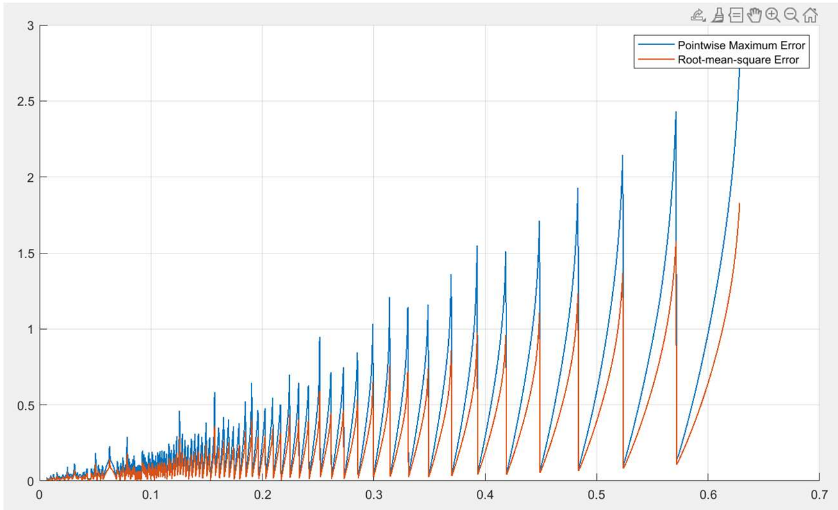
The graph of Approximation & Analytical solution with h = 2*pi/10 looks like this:



However, if we lower h to, say, 2*pi/1000, it starts to make more sense:



Let's plot the RMS and pointwise maximum errors of Numerical solutions vs h:



It shows an interesting behavior.

Sorry I'm failing to send it on time, so I will add comments in the comment section on Canvas.