**Mathematical Methods in Engineering and Applied Science**

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**Problem Set 2**. Due on Oct. 16 at 23:59. ..............................................

(1) Consider linear system

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 2 \\ x_2 + 2x_3 &= 3. \end{aligned}$$

  (a) Find the $LU$ factorization of the coefficient matrix $A$. Show that $U = DL^T$ with $D$ diagonal and thus $A = LDL^T$. Find the exact solution using the $LU$ factorization.
  (b) Solve the system using Jacobi and Gauss-Seidel iterations. How many iterations are needed to reduce the relative error of the solution to $10^{-8}$?
  (c) Plot in semilog scales the relative errors by both methods as a function of the number of iterations.
  (d) Explain the convergence rate. Which of the methods is better and why?

(2) Factor these two matrices $A$ into $S\Lambda S^{-1}$:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

Using that factorization, find for both: (a) $A^3$; (b) $A^{-1}$.

(3) Given a system $Ax = b$ with

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 2 & 3 & 4 \\ -2 & 1 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ a \\ -1 \end{bmatrix},$$

for which $a$ there is a solution? Find the general solution of the system for that $a$.

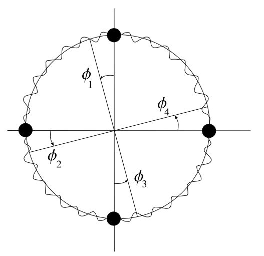
(4) Consider the system of linear differential equations

$$\frac{du}{dt} = Au, \quad \text{with } A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

  (a) Using the spectral factorization of $A$, determine the general solution of the system.
  (b) Find the behavior of the solution at large $t$ if the initial condition is $u(0) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$.

(5) For matrix $A = \begin{bmatrix} 2021 & 20 & 0 \\ 20 & 2021 & 21 \\ 0 & 21 & 2021 \end{bmatrix}$ and vector $b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, what is the most likely direction of vector $x = A^{2021} b$?

(6) Four unit masses are joined by springs of unit spring constant on a ring of unit radius as shown in the figure. Convince yourself that Newton's second law for the masses results in




$$\begin{aligned} \ddot{\phi}_1 &= -2\phi_1 + \phi_2 + \phi_4, \\ \ddot{\phi}_2 &= -2\phi_2 + \phi_3 + \phi_1, \\ \ddot{\phi}_3 &= -2\phi_3 + \phi_4 + \phi_2, \\ \ddot{\phi}_4 &= -2\phi_4 + \phi_1 + \phi_3, \end{aligned}$$

or $\ddot{u} = -K_4 u$, where $u = [\phi_1 \ \phi_2 \ \phi_3 \ \phi_4]^T$ and $K_4$ is a matrix of coefficients.

  (a) How can you see that $K_4$ is singular? What is the physical meaning of this fact?
  (b) Find the eigenvalues and eigenvectors of $K_4$ and using the spectral factorization, $K_4 = S\Lambda S^{-1}$, diagonalize and solve the system.
  (c) Describe the normal modes of the oscillations in terms of eigenvalues and eigenvectors of $K_4$. What are the largest and smallest frequencies and corresponding eigenvectors? Can you explain the physics of the largest frequency mode?
  (d) What is the solution that starts at $u(0) = [1 \ 0 \ -1 \ 0]^T$ and $\dot{u}(0) = 0$?