Problem Set 9

Contents

Problem 1

For the equation $\ddot{x} + \mu(x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$, where $\mu > 0$:

(a) Find and classify all the fixed points.

(b) Show that the system has a circular limit cycle and find its amplitude and period.

(c) Determine the stability of the limit cycle. Give an argument which shows that the limit cycle is unique, i.e., there are no other periodic trajectories.

Solution

Let's rewrite the equation as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2 - y^2)y - x \end{cases}$$

The fixed points are:

$$\begin{cases} y = 0, \\ \mu(1 - x^2 - y^2)y - x = 0 \end{cases} = \begin{cases} y = 0 \\ x = 0 \end{cases}$$

So, out fixed point is:

$$p_0 = (0, 0)$$

Next, we perform linearization:

$$J = \begin{bmatrix} 0 & 1 \\ -2\mu xy - 1 & -\mu(x^2 + 3y^2 - 1) \end{bmatrix}$$

Substituting the fixed point into the Jacobian:

$$J|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

(a)

```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
```

```
x, y, mu = sp.symbols('x, y, mu')

J = sp.Matrix([
    [0, 1],
    [-1, mu]
])

J.eigenvects()
```

```
[(mu/2 - sqrt((mu - 2)*(mu + 2))/2,
  1,
  [Matrix([
   [mu/2 + sqrt((mu - 2)*(mu + 2))/2],
   [                               1]])]),
 (mu/2 + sqrt((mu - 2)*(mu + 2))/2,
  1,
  [Matrix([
   [mu/2 - sqrt((mu - 2)*(mu + 2))/2],
   [                               1]])])]
```

Eigenvalues and eigenvectors of the matrix $J|_{(0,0)}$ are:

$$\lambda_{1,2} = \frac{(\mu \pm \sqrt{\mu^2 - 4})}{2},$$

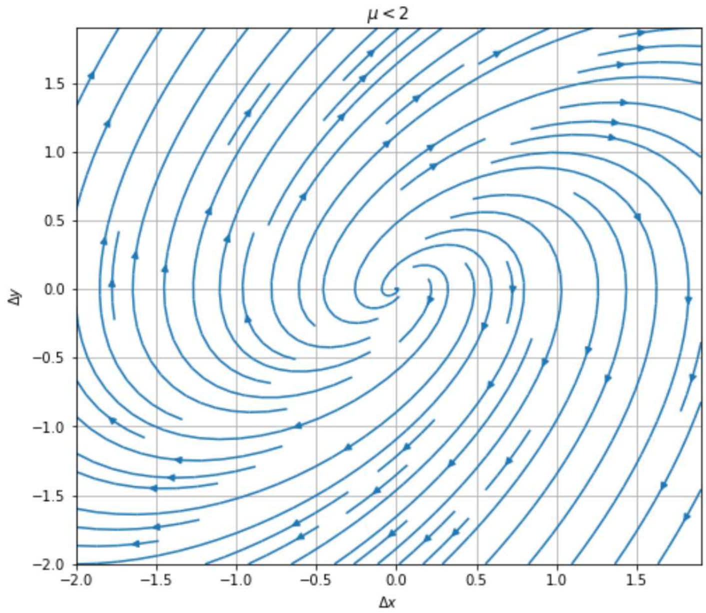
$$s_{1,2} = \begin{bmatrix} 1 \\ \frac{(\mu \pm \sqrt{\mu^2 - 4})}{2} \end{bmatrix}$$

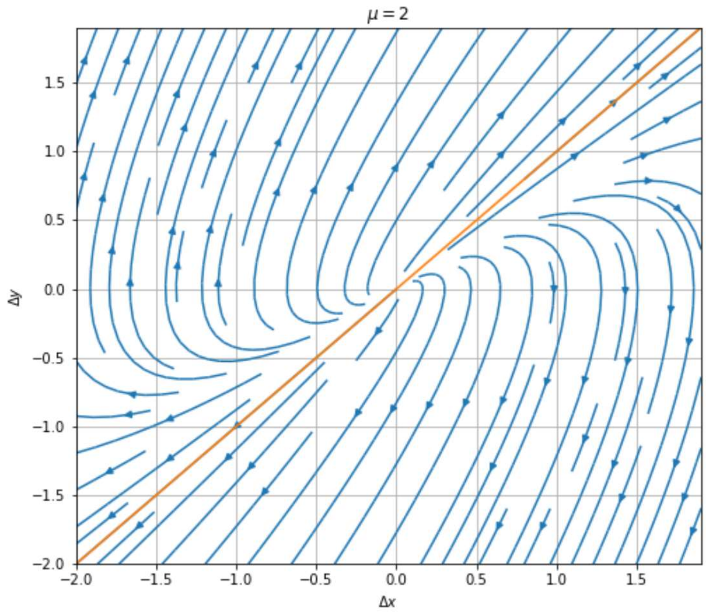
Now let's elaborate on different μ:

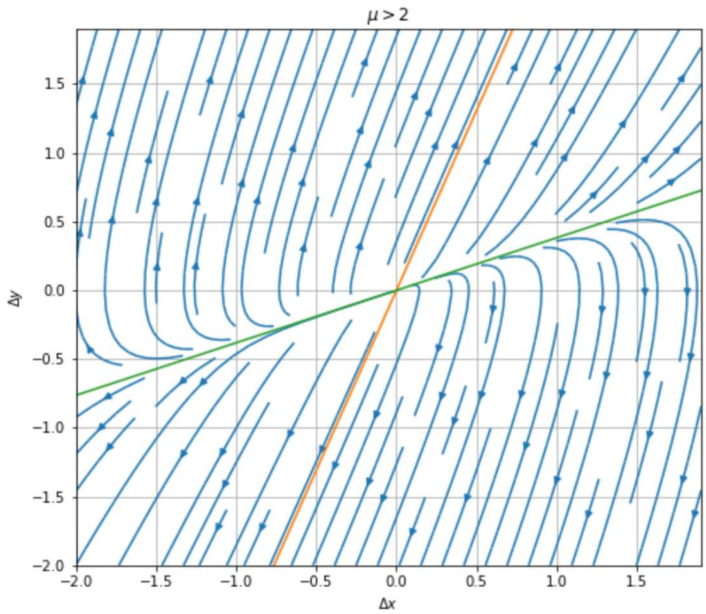
1. $\mu < 2$, then $\lambda_{1,2}$ are complex conjugate - the fixed point will be the unstable spiral, since $Re(\lambda_i) > 0$.
2. $\mu = 2$, then $\lambda_1 = \lambda_2 \in \mathbb{R}$, also $s_1 = s_2$, this is a degenerate node.
3. $\mu > 2$, then $\lambda_1 > \lambda_2 > 0$, this yields an unstable node.

Let's plot the phase portrait around the fixed point for different λ's:

```
J1 = np.array([
    [0, 1],
    [-1, 1]
])
J2 = np.array([
    [0, 1],
    [-1, 2]
])
J3 = np.array([
    [0, 1],
    [-1, 3]
])

s2 = np.array([1, 1])
s31 = np.array([2, 3 + np.sqrt(9 - 4)])/2
s32 = np.array([2, 3 - np.sqrt(9 - 4)])/2

x = np.arange(-2, 2, 0.1)
y = np.arange(-2, 2, 0.1)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U1 = np.zeros(X.shape)
V1 = np.zeros(X.shape)
U2 = np.zeros(X.shape)
V2 = np.zeros(X.shape)
U3 = np.zeros(X.shape)
V3 = np.zeros(X.shape)

for i in range(ni):
    for j in range(nj):
        U1[i][j], V1[i][j] = J1.dot(np.array([X[i][j], Y[i][j]]))
        U2[i][j], V2[i][j] = J2.dot(np.array([X[i][j], Y[i][j]]))
        U3[i][j], V3[i][j] = J3.dot(np.array([X[i][j], Y[i][j]]))

plt.subplot(311)
plt.streamplot(X, Y, U1, V1)
plt.title('$\mu < 2$')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.ylim((-2, 1.9))
plt.grid()
plt.subplot(312)
plt.streamplot(X, Y, U2, V2)
plt.title('$\mu = 2$')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.plot([-2, 2], s2[1]*[-2, 2])
plt.xlim((-2, 1.9))
plt.ylim((-2, 1.9))
plt.grid()
plt.subplot(313)
plt.streamplot(X, Y, U3, V3)
plt.title('$\mu > 2$')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.plot([-2, 2], s31[1]*np.array([-2, 2]))
plt.plot([-2, 2], s32[1]*np.array([-2, 2]))
plt.xlim((-2, 1.9))
plt.ylim((-2, 1.9))
plt.grid()
plt.gcf().set_size_inches(8, 24)
```







(b)

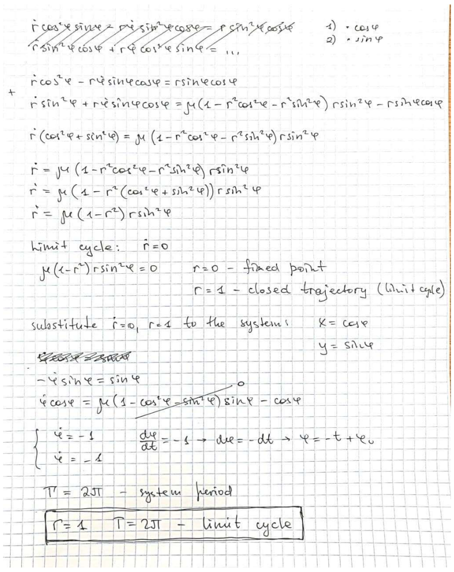
To obtain the limit cycle we move to polar coordinates:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \quad \begin{cases} \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{cases}$$

and the system is rewritten as follows:

$$\begin{cases} \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi = \sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi = \mu(1 - r^2\cos^2\varphi - r^2\sin^2\varphi)r\sin\varphi - r\cos\varphi \end{cases}$$

See the attached image:



$\cancel{\dot{r}\cos^2\varphi\sin\varphi - r\dot{\varphi}\sin^2\varphi\cos\varphi = r\sin^2\varphi\cos\varphi}$ 1) $\cdot\cos\varphi$

$\cancel{\dot{r}\sin^2\varphi\cos\varphi + r\dot{\varphi}\cos^2\varphi\sin\varphi = \ldots}$ 2) $\cdot\sin\varphi$

$\dot{r}\cos^2\varphi - r\dot{\varphi}\sin\varphi\cos\varphi = r\sin\varphi\cos\varphi$

$+$

$\dot{r}\sin^2\varphi + r\dot{\varphi}\sin\varphi\cos\varphi = \mu(1 - r^2\cos^2\varphi - r^2\sin^2\varphi)r\sin^2\varphi - r\sin\varphi\cos\varphi$

$\dot{r}(\cos^2\varphi + \sin^2\varphi) = \mu(1 - r^2\cos^2\varphi - r^2\sin^2\varphi)r\sin^2\varphi$

$\dot{r} = \mu(1 - r^2\cos^2\varphi - r^2\sin^2\varphi)r\sin^2\varphi$

$\dot{r} = \mu(1 - r^2(\cos^2\varphi + \sin^2\varphi))r\sin^2\varphi$

$\dot{r} = \mu(1 - r^2)r\sin^2\varphi$

Limit cycle: $\dot{r} = 0$

$\mu(1 - r^2)r\sin^2\varphi = 0$ $r = 0$ – fixed point

$r = 1$ – closed trajectory (limit cycle)

substitute $\dot{r} = 0$, $r = 1$ to the system: $x = \cos\varphi$, $y = \sin\varphi$

$-\dot{\varphi}\sin\varphi = \sin\varphi$

$\dot{\varphi}\cos\varphi = \mu(\underbrace{1 - \cos^2\varphi - \sin^2\varphi}_{0})\sin\varphi - \cos\varphi$

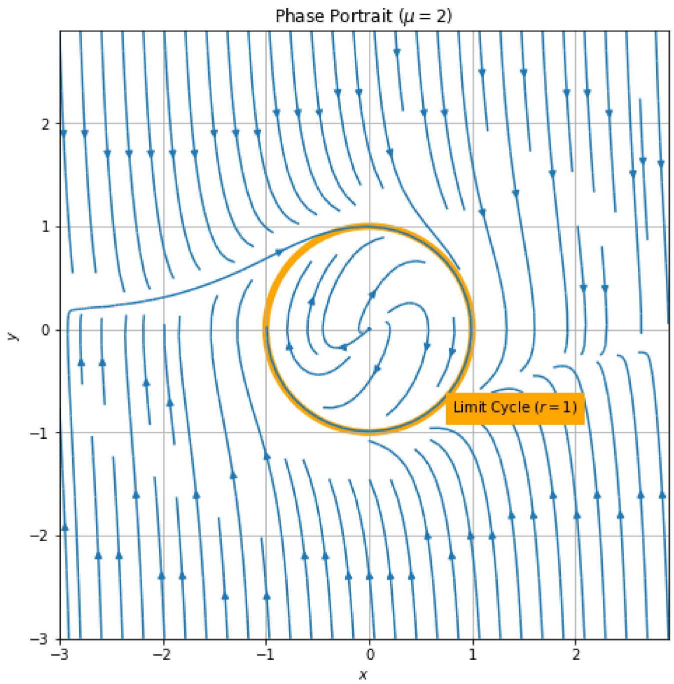
$\begin{cases} \dot{\varphi} = -1 \\ \dot{\varphi} = -1 \end{cases}$ $\frac{d\varphi}{dt} = -1 \to d\varphi = -dt \to \varphi = -t + \varphi_0$

$T = 2\pi$ – system period

$\boxed{r = 1 \quad T = 2\pi \text{ – limit cycle}}$

Now let's build the full phase portrait:

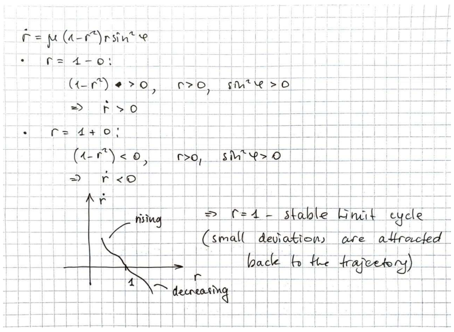
```
def dx(x, y):
    return y

def dy(x, y):
    return -mu*(x**2 + y**2 - 1)*y - x

x = np.arange(-3, 3, 0.1)
y = np.arange(-3, 3, 0.1)

X, Y = np.meshgrid(x, y)

ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

mu = 2

for i in range(ni):
  for j in range(nj):
    U[i][j] = dx(X[i][j], Y[i][j])
    V[i][j] = dy(X[i][j], Y[i][j])

plt.streamplot(X, Y, U, V)
ax = plt.gca()
ax.axis('square')
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.title('Phase Portrait ($\mu = 2$)')
ax.add_patch(plt.Circle((0, 0), 1, fill=None, edgecolor='orange', alpha=1, visible=True, lw=5))
plt.gcf().set_size_inches(8, 8)
tmp = plt.text(0.8, -0.8, s="Limit Cycle ($r = 1$)", backgroundcolor='orange')
```



(c)

The limit cycle is unique as there is only one condition that satisfied $\dot{r} = 0$, that is $r = \pm 1 = 1,\ r > 0$.

The cycle is stable, see the attached image:



As we see, all deviations from the stable cycle tend to be eliminated as the $\dot{r}$ changes its sign contrary to the sign of the deviation. That is, $\frac{dr}{dt}$ is strictly positive when $r < 1$ and stricly negative when $r > 1$. Thus, this is analogous to a global minima of some function, for example, $(r-1)^2$. That also proves the uniqueness of the cycle.

Problem 2

Investigate the phase plane of the system $\dot{x} = y, \quad \dot{y} = x(\mu - x^2)$, for $\mu < 0$, $\mu = 0$, and $\mu > 0$. Describe the bifurcation as μ increases through zero.

Solution

Fixed points:

$$\begin{cases} y = 0 \\ x(\mu - x^2) = 0 \end{cases}$$

From here we derive:

$$p_1 = (0, 0)$$
$$p_{2,3} = (0, \pm\sqrt{\mu}), \quad \mu \geq 0$$

Linearization:

$$\begin{bmatrix} 0 & 1 \\ \mu - 3x^2 & 0 \end{bmatrix}$$

Let's investigate on μ with different signs:

1. $\mu > 0$, then $p_{2,3} = (0, \pm\sqrt{\mu})$ and

$$J|_{p_1} = \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}, \quad J|_{p_{2,3}} = \begin{bmatrix} 0 & 1 \\ -2\mu & 0 \end{bmatrix}$$

In this case there are 3 different fixed points. Let's determine their characteristics:

```
x, mu = sp.symbols('x, mu')
```

```
subs1 = {mu - 3*x**2: mu}

J1 = J.subs(subs1)

J1.eigenvects()
```

```
[(mu/2 - sqrt((mu - 2)*(mu + 2))/2,
  1,
  [Matrix([
   [mu/2 + sqrt((mu - 2)*(mu + 2))/2],
   [                               1]])]),
 (mu/2 + sqrt((mu - 2)*(mu + 2))/2,
  1,
  [Matrix([
   [mu/2 - sqrt((mu - 2)*(mu + 2))/2],
   [                               1]])])]
```

\

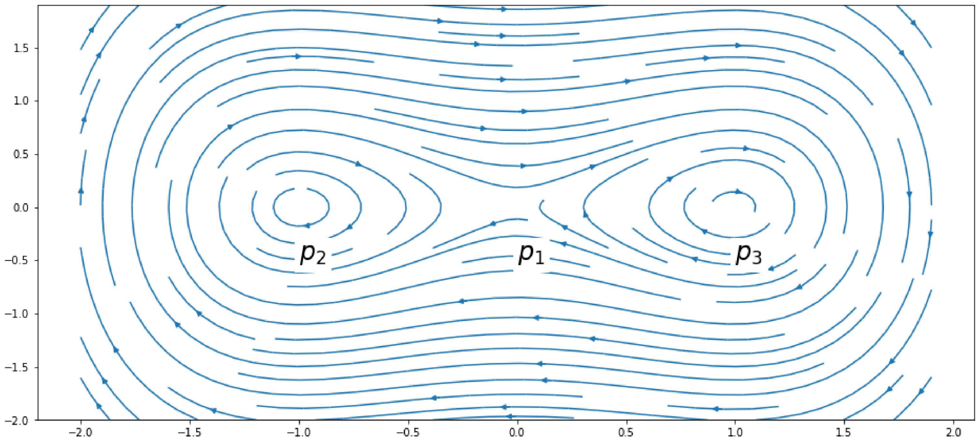
```
subs2 = {mu - 3*x**2: -2*mu}

J2 = J.subs(subs2)

J2.eigenvects()
```

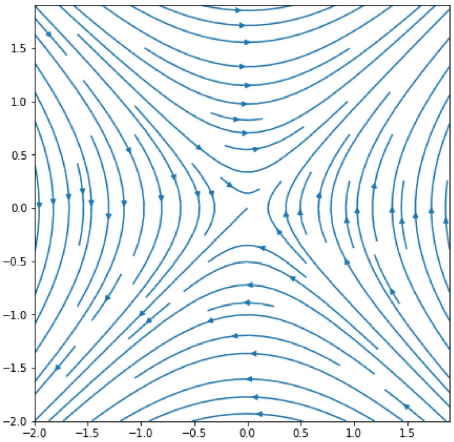
```
[(mu/2 - sqrt((mu - 2)*(mu + 2))/2,
  1,
  [Matrix([
   [mu/2 + sqrt((mu - 2)*(mu + 2))/2],
   [                               1]])]),
 (mu/2 + sqrt((mu - 2)*(mu + 2))/2,
  1,
  [Matrix([
   [mu/2 - sqrt((mu - 2)*(mu + 2))/2],
   [                               1]])])]
```

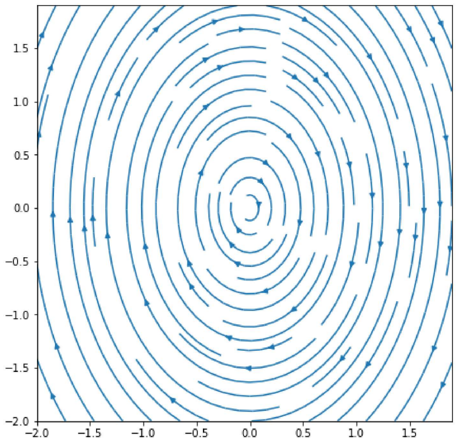
For the point p_1 we get $\lambda_1 = -\lambda_2$ and this is a saddle. For the points $p_{2,3}$ we obtain $\lambda_1 = \lambda_2^*$, complex conjugate eigenvalues, these are centers.

Let's plot the phase portraits for $\mu = 1$:

```python
mu = 1

def dx(x, y):
    return y

def dy(x, y):
    return x * (mu - x**2)

J1 = np.array([
    [0, 1],
    [1, 0]
])

J2 = np.array([
    [0, 1],
    [-2, 0]
])

x = np.arange(-2, 2, 0.1)
y = np.arange(-2, 2, 0.1)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

U1 = np.zeros(X.shape)
V1 = np.zeros(X.shape)

U2 = np.zeros(X.shape)
V2 = np.zeros(X.shape)

for i in range(ni):
  for j in range(nj):
    arr = np.array([X[i][j], Y[i][j]])
    U[i][j] = dx(X[i][j], Y[i][j])
    V[i][j] = dy(X[i][j], Y[i][j])
    U1[i][j], V1[i][j] = J1.dot(arr)
    U2[i][j], V2[i][j] = J2.dot(arr)


plt.subplot(211)
plt.streamplot(X, Y, U, V)
plt.text(-1, -0.5, s='$p_2$', fontsize=24, backgroundcolor='white')
plt.text(1, -0.5, s='$p_3$', fontsize=24, backgroundcolor='white')
plt.text(0, -0.5, s='$p_1$', fontsize=24, backgroundcolor='white')
plt.subplot(223)
plt.streamplot(X, Y, U1, V1)
plt.gca().axis('square')
plt.subplot(224)
plt.streamplot(X, Y, U2, V2)
plt.gca().axis('square')
plt.gcf().set_size_inches(16,16)
```







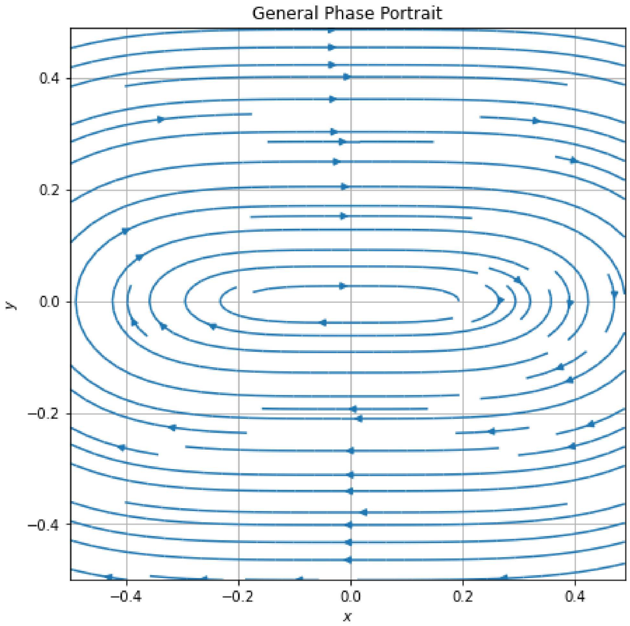
As an act of empiric thinking we may predict what will happen when we decrease μ to zero: $\pm\sqrt{\mu}$ defines the coordinates of the fixed points $p_{2,3}$ on the X axis. Gradually decreasing μ will lead to the fixed points collapsing into single point at $(0, 0)$. Let's observe that:

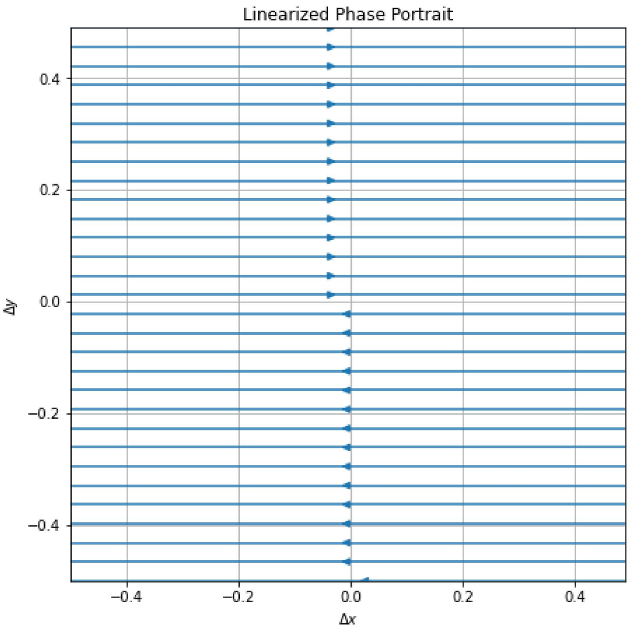
1. $\mu = 0$, then $p = (0, 0)$. Let's determine the type of the fixed point:

$$J|_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 0$ - this is a degenerate node by linearization. Let's plot both linearized around the p and general phase portraits:

```python
mu = 0

J21 = np.array([
    [0, 1],
    [0, 0]
])

x = np.arange(-0.5, 0.5, 0.01)
y = np.arange(-0.5, 0.5, 0.01)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

U1 = np.zeros(X.shape)
V1 = np.zeros(X.shape)

for i in range(ni):
  for j in range(nj):
    arr = np.array([X[i][j], Y[i][j]])
    U[i][j] = dx(X[i][j], Y[i][j])
    V[i][j] = dy(X[i][j], Y[i][j])
    U1[i][j], V1[i][j] = J21.dot(arr)

plt.subplot(211)
plt.streamplot(X, Y, U, V)
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.title('General Phase Portrait')
plt.grid()
plt.gca().axis('square')
plt.xlim((-0.5, 0.49))
plt.ylim((-0.5, 0.49))
plt.subplot(212)
plt.streamplot(X, Y, U1, V1)
plt.xlabel('$\\Delta x$')
plt.ylabel('$\\Delta y$')
plt.title('Linearized Phase Portrait')
plt.grid()
plt.gca().axis('square')
plt.xlim((-0.5, 0.49))
plt.ylim((-0.5, 0.49))
plt.gcf().set_size_inches(8, 16)
```



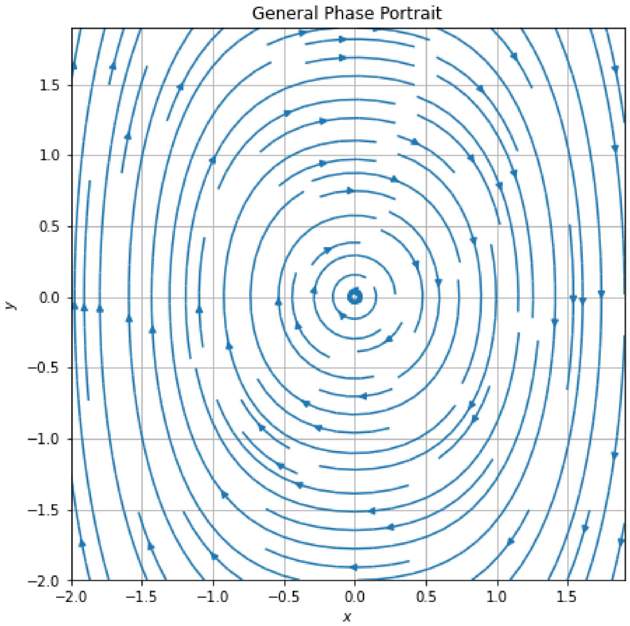


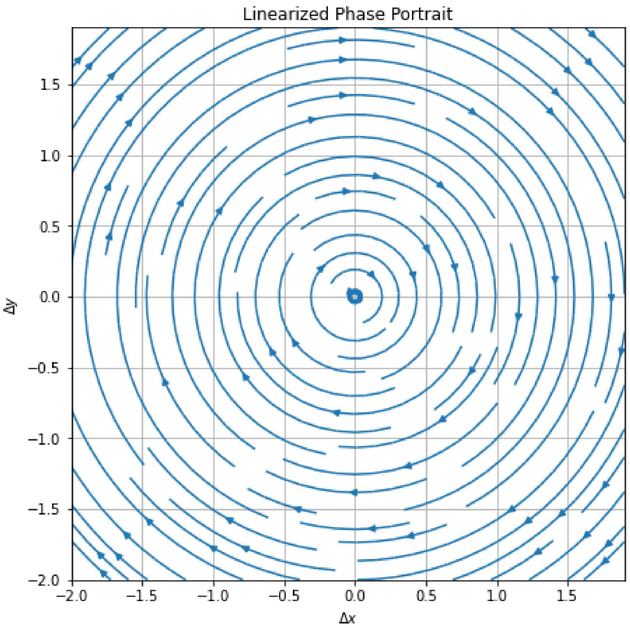
We see that indeed the fixed point with $\mu = 0$ becomes a degenerate node.

1. $\mu < 0$, then $p = (0, 0)$ as x and $y \in \mathbb{R}$, but the solutions change:

$$J|_p = \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}$$

```
mu = -1

J22 = np.array([
    [0, 1],
    [mu, 0]
])

x = np.arange(-2, 2, 0.1)
y = np.arange(-2, 2, 0.1)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

U1 = np.zeros(X.shape)
V1 = np.zeros(X.shape)

for i in range(ni):
  for j in range(nj):
    arr = np.array([X[i][j], Y[i][j]])
    U[i][j] = dx(X[i][j], Y[i][j])
    V[i][j] = dy(X[i][j], Y[i][j])
    U1[i][j], V1[i][j] = J22.dot(arr)

plt.subplot(211)
plt.streamplot(X, Y, U, V)
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.title('General Phase Portrait')
plt.grid()
plt.gca().axis('square')
plt.xlim((-2, 1.9))
plt.ylim((-2, 1.9))
plt.subplot(212)
plt.streamplot(X, Y, U1, V1)
plt.xlabel('$\\Delta x$')
plt.ylabel('$\\Delta y$')
plt.title('Linearized Phase Portrait')
plt.grid()
plt.gca().axis('square')
plt.xlim((-2, 1.9))
plt.ylim((-2, 1.9))
plt.gcf().set_size_inches(8, 16)
```






```
mu = sp.symbols('mu')

JJJ = sp.Matrix([
    [0, 1],
    [mu, 0]
])

JJJ.eigenvects()
```

```
[(-sqrt(mu),
  1,
  [Matrix([
   [-1/sqrt(mu)],
   [          1]])]),
 (sqrt(mu),
  1,
  [Matrix([
   [1/sqrt(mu)],
   [         1]])])]
```

λ's are complex conjugate, this is a center.

And thems the facts:

As μ approaches zero, the fixed points (two different centers and central saddle) collapse, transitioning through a degenerate node at $\mu = 0$ and transforming to center fixed point at $\mu < 0$. The bifurcation occurs at $\mu = 0$.

Problem 3

Consider the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$, where $f = -1$ for $|x| < 1$ and $f = 1$ for $|x| \geq 1$.

(a) Show the system is equivalent to $\dot{x} = \mu(y - F(x))$, $\dot{y} = -x/\mu$ where

$$F = \begin{cases} x + 2, & x \leq -1 \\ -x, & |x| < 1 \\ x - 2, & x \geq 1. \end{cases}$$

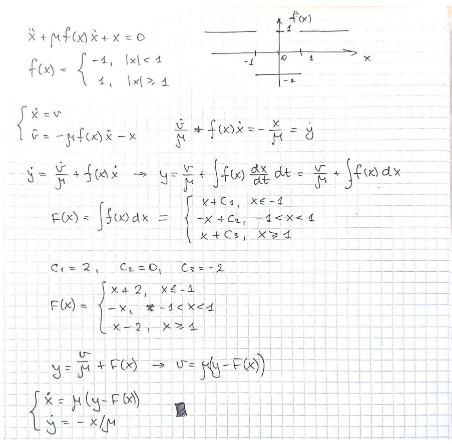
(b) Graph the nullclines.

(c) Show that the sytem exhibits relaxation oscillations for $\mu \geq 1$, and plot the limit cycle in the (x, y) plane.

(d) Estimate the period of the limit cycle for $\mu \gg 1$.

Solution

(a)



$$\ddot{x} + \mu f(x)\dot{x} + x = 0$$

$$f(x) = \begin{cases} -1, & |x| < 1 \\ 1, & |x| \geqslant 1 \end{cases}$$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\mu f(x)\dot{x} - x \end{cases} \qquad \frac{\dot{v}}{\mu} + f(x)\dot{x} = -\frac{x}{\mu} = \dot{y}$$

$$\dot{y} = \frac{\dot{v}}{\mu} + f(x)\dot{x} \rightarrow y = \frac{v}{\mu} + \int f(x)\frac{dx}{dt}dt = \frac{v}{\mu} + \int f(x)\,dx$$

$$F(x) = \int f(x)\,dx = \begin{cases} x + C_1, & x \leq -1 \\ -x + C_2, & -1 < x < 1 \\ x + C_3, & x \geqslant 1 \end{cases}$$

$$C_1 = 2, \quad C_2 = 0, \quad C_3 = -2$$

$$F(x) = \begin{cases} x + 2, & x \leq -1 \\ -x, & -1 < x < 1 \\ x - 2, & x \geqslant 1 \end{cases}$$

$$y = \frac{v}{\mu} + F(x) \rightarrow v = \mu(y - F(x))$$

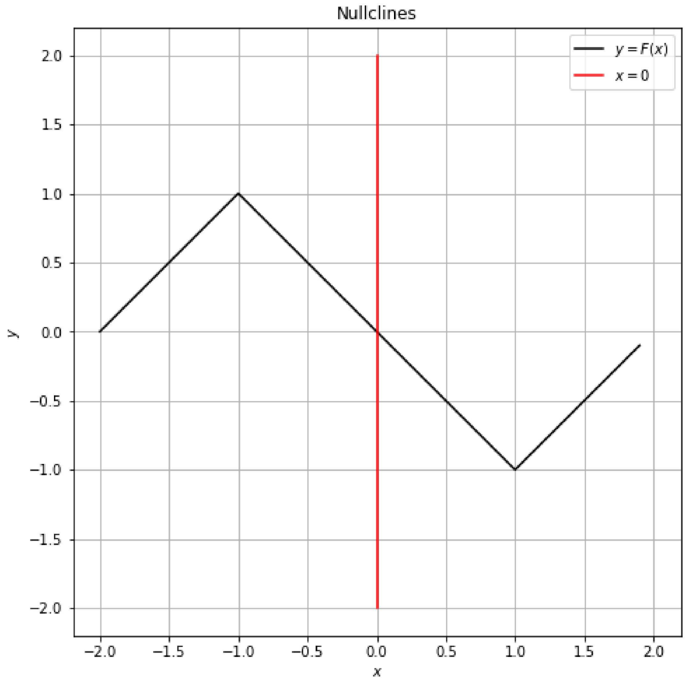
$$\begin{cases} \dot{x} = \mu(y - F(x)) \\ \dot{y} = -x/\mu \end{cases}$$

(b)

The nullclines are:

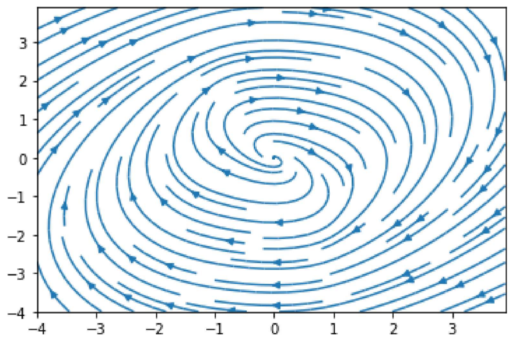
$$\begin{cases} \dot{x} = 0 = \mu(y - F(x)) \\ \dot{y} = 0 = -\frac{x}{\mu} \end{cases} = \begin{cases} y = F(x) \\ x = 0 \end{cases}$$

```python
def F(x):
    if (x <= -1):
        return x + 2
    elif (-1 < x < 1):
        return -x
    elif (1 <= x):
        return x - 2
    else:
        print('Error')
        return 0

x = np.arange(-2, 2, 0.1)
y = np.array([F(x_) for x_ in x])

nc1 = plt.plot(x, y, color='black', label='$y = F(x)$')
nc2 = plt.plot([0, 0], [-2, 2], color='red', label='$x = 0$')
plt.grid()
plt.gcf().set_size_inches(8, 8)
plt.gca().axis('square')
plt.legend()
plt.title('Nullclines')
plt.xlabel('$x$')
tmp = plt.ylabel('$y$')
```




```
mu = 1
def dx(x, y):
  return mu * (y - F(x))
def dy(x, y):
  return -x / mu

x = np.arange(-4, 4, 0.1)
y = np.arange(-4, 4, 0.1)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

for i in range(ni):
  for j in range(nj):
    U[i][j] = dx(X[i][j], Y[i][j])
    V[i][j] = dy(X[i][j], Y[i][j])

plt.streamplot(X, Y, U, V)
```

```
<matplotlib.streamplot.StreamplotSet at 0x2ef987004c8>
```



(c)

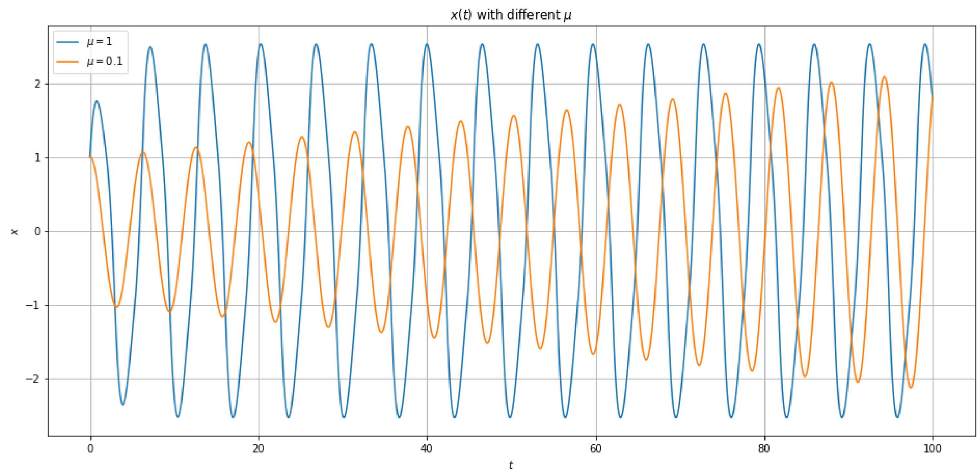
To see that the system exhibits relaxation let's look at the original equation:

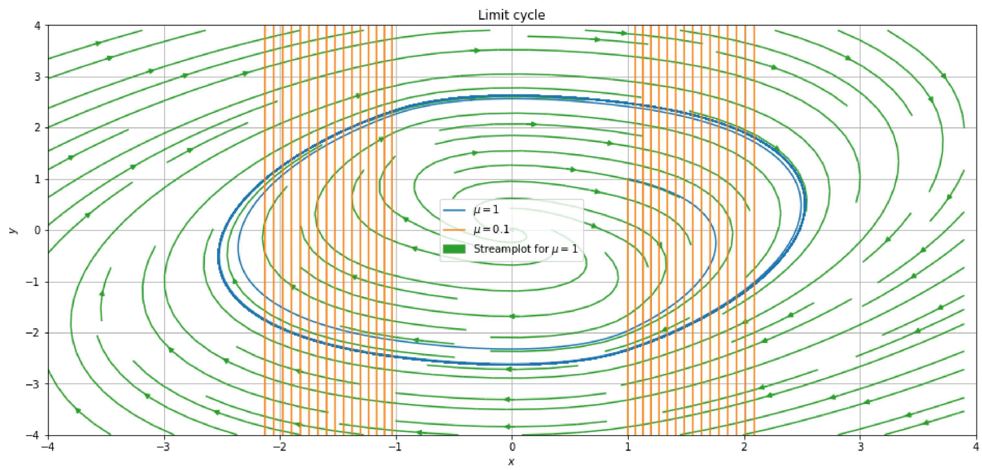
$$\ddot{x} + \mu f(x)\dot{x} + x = 0$$

and take into account the form of the function $f(x)$. The term $\mu f(x)$ represents damping, and it is negative (e.g. increasing the signal) for $|x| < 1$, as $f(x) = -1,\ |x| < 1$ and positive otherwise. Thus, the system will approach some sustained oscillation.

To find the limit cycle, we have to numerically integrate the equation. We'll start with $t_0 = 0,\ (x_0, y_0) = (1, 1)$:

```
interval = np.arange(0, 100, 0.01)
z = np.zeros(interval.shape)
v = np.zeros(interval.shape)
z0 = 1
v0 = 1
z[0] = z0
v[0] = v0

mu = 1

for i in range(len(z)-1):
    z[i+1] = z[i] + (interval[i+1] - interval[i])*dx(z[i], v[i])
    v[i+1] = v[i] + (interval[i+1] - interval[i])*dy(z[i], v[i])

plt.subplot(211)
plt.plot(interval, z, label='$\mu = 1$')
plt.grid()
plt.title('$x(t)$ with different $\mu$')
plt.xlabel('$t$')
plt.ylabel('$x$')

plt.subplot(212)
plt.plot(z, v, label='$\mu = 1$')
plt.grid()
plt.title('Limit cycle')
plt.xlabel('$x$')
plt.ylabel('$y$')

mu = 0.01

for i in range(len(z)-1):
    z[i+1] = z[i] + (interval[i+1] - interval[i])*dx(z[i], v[i])
    v[i+1] = v[i] + (interval[i+1] - interval[i])*dy(z[i], v[i])

plt.subplot(211)
plt.plot(interval, z, label='$\mu = 0.1$')
plt.legend()

plt.subplot(212)
plt.plot(z, v, label='$\mu = 0.1$')
plt.streamplot(X, Y, U, V)
#plt.gca().axis('square')
plt.xlim((-4, 4))
plt.ylim((-4, 4))
plt.legend(('$\mu = 1$', '$\mu = 0.1$', 'Streamplot for $\mu = 1$'))

plt.gcf().set_size_inches(16, 16)
```





Problem 4

Consider the system:

$$\dot{x} = -y + \mu x + xy^2, \quad \dot{y} = x + \mu y - x^2.$$

(a) Linearize about the origin and determine the type of the fixed point.

(b) Write down the system to find all the fixed points. Eliminate y to find the equation for $x_c(\mu)$. Make a plot of this function to find out how many fixed points there are for given μ.

(c) Investigate numerically the nature of the solutions on a phase plane as μ varies about $\mu = 0$.

(d) What is the type of the bifurcation that takes place as μ crosses 0?

(e) Rewrite the system in the polar coordinates $x = r\cos\upsilon, \ y = r\sin\upsilon$ and approximate the system assuming r small. Show that to leading order the system becomes

$$\dot{r} = \mu r + \frac{1}{8}r^3, \quad \dot{\Theta} = 1,$$

and hence one can expect a limit cycle of radius $r \approx \sqrt{-8\mu}$ when $\mu < 0$. Confirm this numerically.

Solution

(a)

Linearization:

$$J = \begin{bmatrix} \mu + y^2 & -1 + 2xy \\ 1 - 2x & \mu \end{bmatrix}$$

Fixed points:

$$\begin{cases} -y + \mu x + xy^2 = 0 \\ x + \mu y - x^2 = 0 \end{cases}$$

$p = (0, 0)$ satisfies the condition.

$$J|_p = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

```
mu = sp.symbols('mu')

J = sp.Matrix([
    [mu, -1],
    [1, mu]
])

J.eigenvects()
```

```
[(mu - I,
  1,
  [Matrix([
   [-I],
   [ 1]])]),
 (mu + I,
  1,
  [Matrix([
   [I],
   [1]])])]
```

The eigenvalues are complex conjugate with $Re(\lambda_i) \neq 0$, hence, this is a spiral.

(b)

For $\mu \neq 0$:

$$\begin{cases} \dot{x} = -y + \mu x + xy^2 \\ \dot{y} = x + \mu y - x^2 \end{cases}$$

$$\begin{cases} -y + \mu x + xy^2 = 0 \\ x + \mu y - x^2 = 0 \end{cases} \longrightarrow y = \frac{x^2 - x}{\mu}$$

$$\frac{x - x^2}{\mu} + \mu x + \frac{x}{\mu^2}(x^2 - x)^2 = 0$$

$$x\mu - x^2\mu + \mu^2 x + x^5 - 2x^4 + x^3 = 0$$

$$x^5 - 2x^4 + x^3 - \mu x^2 + \mu(\mu + 1)x = 0$$

> **Warning**
>
> μ^3 instead of μ^2 at the term x

$$x^5 - 2x^4 + x^3 - \mu x^2 + \mu(\mu^2 + 1)x = 0$$

For $\mu = 0$ we can directly find the fixed points:

$$\begin{cases} -y + xy^2 = 0 \\ x - x^2 = 0 \end{cases} = \begin{cases} y = xy^2 \\ x(x-1) = 0 \end{cases}$$

From here follows that

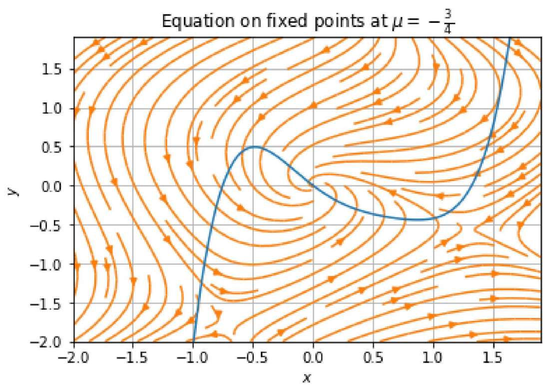
$$\begin{aligned} p_1 &= (0,0) \\ p_2 &= (1,0) \\ p_3 &= (1,1) \end{aligned}$$

Let's plot the function for $x_c(\mu)$ when $\mu \neq 0$ and search for solutions other than $x = 0$, as we already know that $(0,0)$ is a fixed point $\forall \mu$. For that we can split the equation into two parts and search for the intersection points:

$$x^4 = 2x^3 - x^2 + \mu x - \mu(\mu + 1)$$

> **Warning**
>
> This did no good

```
x = np.arange(-3, 3, 0.01)
m = -0.75
def part1(x):
    return x**4

def part2(x):
    return 2*x**3 - x**2 + m*x - m*(m+1)

def xx(x):
    return x**5 - 2*x**4 + x**3 - m*x**2 + m*(m**2 + 1)*x

y = xx(x)

plt.plot(x, y)

plt.ylim((-2, 1.9))
plt.xlim((-2, 1.9))
plt.grid()

def dx(x, y):
  return -y + m*x + x*y**2
def dy(x, y):
  return x + m*y - x**2

x = np.arange(-2, 2, 0.1)
y = np.arange(-2, 2, 0.1)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

for i in range(ni):
  for j in range(nj):
    U[i][j] = dx(X[i][j], Y[i][j])
    V[i][j] = dy(X[i][j], Y[i][j])

plt.streamplot(X, Y, U, V)
plt.title('Equation on fixed points at $\\mu = -\\frac{3}{4}$')
plt.xlabel('$x$')
tmp = plt.ylabel('$y$')
```



We can clearly see that fixed points appear at coordinate x where the function crosses 0. The y coordinate can be derived substituting the x coordinate into the system for fixed points.

(c), (d)

The eigenvalues are $\mu \pm i$, thus, when μ crosses 0, the fixed point undergoes a transition from an unstable spiral ($\mu > 0$) to a center ($\mu = 0$) and to a stable spiral ($\mu < 0$). The eigenvalues cross the imaginary axis on a complex plane. This is a Hopf bifurcation. From the point **(b)** we see thatwe have a stable limit cycle with $\mu > 0$, thus, it is supercritical.

(e)

$$\begin{cases} \dot r\cos\varphi - r\dot\varphi\sin\varphi = -r\sin\varphi + \mu r\cos\varphi + r^3\cos\varphi\sin^2\varphi & \cdot\sin\varphi \\ \dot r\sin\varphi + r\dot\varphi\cos\varphi = r\cos\varphi + \mu r\sin\varphi - r^2\cos^2\varphi & \cdot\cos\varphi \end{cases}$$

$$\dot r\sin\varphi\cos\varphi - r\dot\varphi\sin^2\varphi = -r\sin^2\varphi + \mu r\sin\varphi\cos\varphi + r^2\cos\varphi\sin^3\varphi$$

$$- \quad \dot r\sin\varphi\cos\varphi + r\dot\varphi\cos^2\varphi = r\cos^2\varphi + \mu r\sin\varphi\cos\varphi + r^2\cos^3\varphi$$

$2\cdot\cancel{\dot r\sin\varphi\cos\varphi}$

$$r\dot\varphi = r - r^2\cos^3\varphi - r^2\cos\varphi\sin^3\varphi$$

$$r\dot\varphi = r - r^2\cos\varphi(\cos^2\varphi + \sin^3\varphi) \quad |:r$$

$$\dot\varphi = 1 - r\cos\varphi(\cos^2\varphi + \sin^3\varphi) \qquad r \text{ small} \rightarrow \dot\varphi = 1 \quad \varphi = t + \varphi_0$$

$$\dot r\cos^2\varphi\sin\varphi - r\dot\varphi\sin^2\varphi\cos\varphi = -r\sin^2\varphi\cos\varphi + \mu r\cos^2\varphi\sin\varphi + r^2\cos^2\varphi\sin^3\varphi$$

$$\dot r\sin^2\varphi\cos\varphi + r\dot\varphi\sin\varphi\cos^2\varphi = r\sin\varphi\cos^2\varphi + \mu r\sin^2\varphi\cos\varphi - r^2\cos^3\varphi\sin\varphi$$

$$\dot r\cos^2\varphi - r\dot\varphi\sin\varphi\cancel{\cos\varphi} = -r\sin\varphi\cancel{\cos\varphi} + \mu r\cos^2\varphi + r^3\cos^2\varphi\sin^2\varphi$$

$$+ \quad \dot r\sin^2\varphi + r\dot\varphi\sin\varphi\cancel{\cos\varphi} = r\sin\varphi\cancel{\cos\varphi} + \mu r\sin^2\varphi - r^2\cos^2\varphi\sin\varphi$$

$$\dot r = \mu r + r^2\cos^2\varphi(r\sin^2\varphi - \sin\varphi) = \mu r + r^3\sin^2\varphi\cos^2\varphi - r^2\sin\varphi\cos^2\varphi$$

$$r^3\sin^2\varphi\cos^2\varphi = r^3\cdot\frac{\sin^2 2\varphi}{4} = r^3\cdot\frac{1}{8}(-\cos 4\varphi)$$

$$r^2\sin\varphi\cos^2\varphi = r^2\cdot\frac{1}{2}\sin 2\varphi\cos\varphi = r^2\cdot\frac{1}{4}(\sin 3\varphi + \sin\varphi)$$

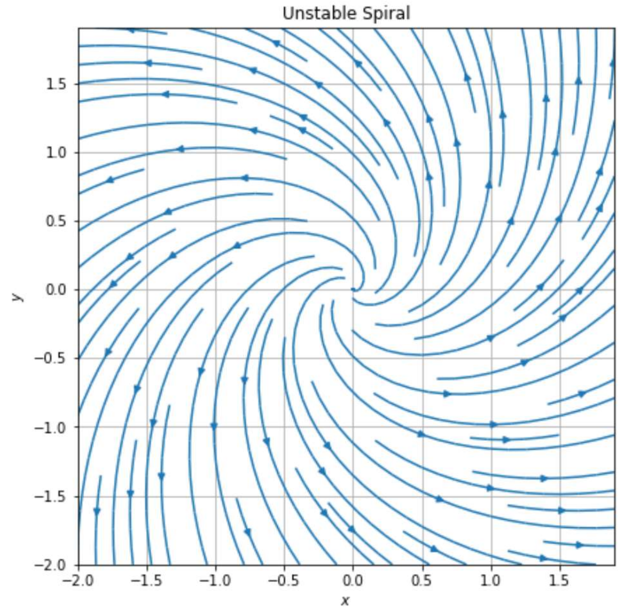
$$\dot r = \mu r - \frac{1}{8}r^3\cos 4\varphi - \underbrace{\frac{1}{4}r^2\sin 3\varphi - \frac{1}{4}r^2\sin\varphi}_{?}$$

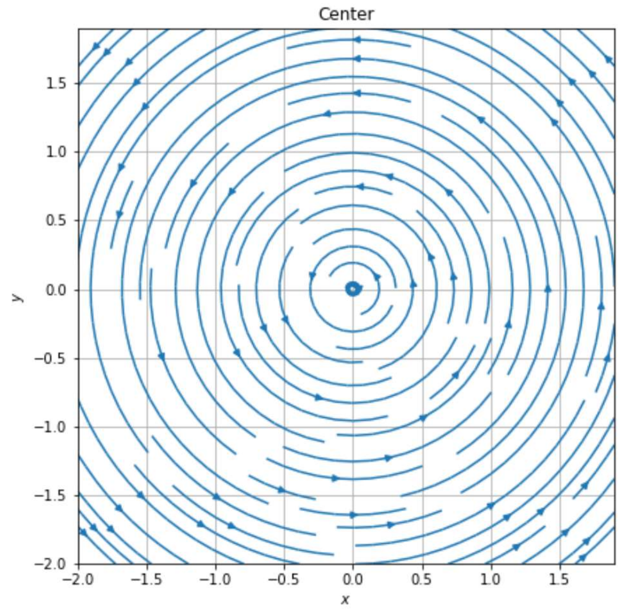
~~higher~~ ~~frequency~~

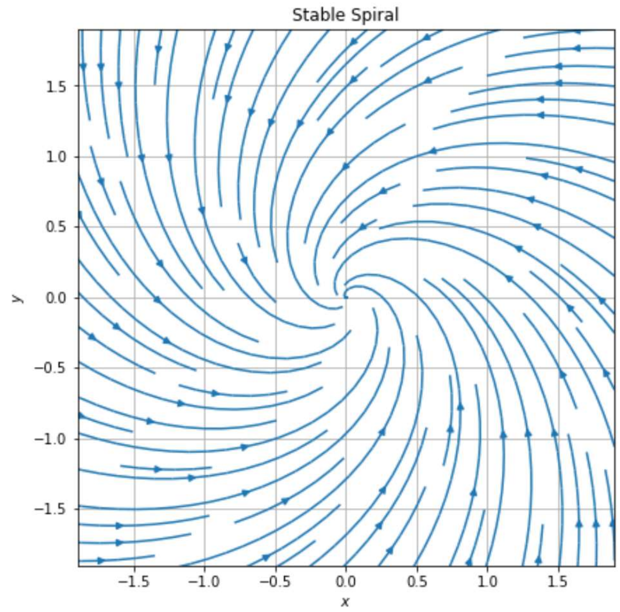
$$\begin{cases} \dot\varphi \approx 1 \\ \dot r \approx \mu r + \frac{1}{8}r^3 \end{cases} \qquad \dot r = 0 \Rightarrow \mu\cancel{r} = \frac{1}{8}r^{\cancel{3}\,2} \Rightarrow \underline{r = \pm\sqrt{-8\mu}}$$

```
J1 = np.array([
    [1, -1],
    [1, 1]
])
J2 = np.array([
    [0, -1],
    [1, 0]
])
J3 = np.array([
    [-1, -1],
    [1, -1]
])

x = np.arange(-2, 2, 0.1)
y = np.arange(-2, 2, 0.1)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U1 = np.zeros(X.shape)
V1 = np.zeros(X.shape)
U2 = np.zeros(X.shape)
V2 = np.zeros(X.shape)
U3 = np.zeros(X.shape)
V3 = np.zeros(X.shape)

for i in range(ni):
  for j in range(nj):
    arr = np.array([X[i][j], Y[i][j]])
    U1[i][j], V1[i][j] = J1.dot(arr)
    U2[i][j], V2[i][j] = J2.dot(arr)
    U3[i][j], V3[i][j] = J3.dot(arr)


plt.subplot(311)
plt.streamplot(X, Y, U1, V1)
plt.gca().axis('square')
plt.title('Unstable Spiral')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()
plt.xlim((-2, 1.9))
plt.ylim((-2, 1.9))
plt.subplot(312)
plt.streamplot(X, Y, U2, V2)
plt.gca().axis('square')
plt.title('Center')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()
plt.subplot(313)
plt.streamplot(X, Y, U3, V3)
plt.gca().axis('square')
plt.title('Stable Spiral')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()
plt.xlim((-1.9, 1.9))
plt.ylim((-1.9, 1.9))
plt.gcf().set_size_inches(8, 24)
```

Print to PDF


Unstable Spiral
y
x
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


Center
y
x
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


Stable Spiral
y
x
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

(d)

Problem 5

Consider the system:

$$\dot{x} = y, \quad \dot{y} = x^2 - y - \mu.$$

(a) Analyze the fixed points of the system at all possible μ.

(b) What type of bifurcation takes plase as μ crosses 0?

(c) Draw the bifurcation diagram in the space of x_c vs μ, where x_c is the critical point.

(d) Plot the phase plane at $\mu = 0.01$.

Solution

(a)

Fixed points:

$$\begin{cases} y = 0 \\ x^2 - y - \mu = 0 \end{cases} = \begin{cases} y = 0 \\ x^2 = \mu \end{cases}$$

This yields the following fixed points:

$$\begin{aligned} p_{1,2} &= (0, \pm\sqrt{\mu}), \quad \mu > 0 \\ p_3 &= (0,0), \quad \mu = 0 \\ p_4 &= \text{\empty}, \quad \mu < 0 \end{aligned}$$

Linearization:

$$J = \begin{bmatrix} 0 & 1 \\ 2x_f & -1 \end{bmatrix}$$

Eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}(\pm\sqrt{8x_f + 1} - 1)$$

For $\mu > 0$, $x_f = \pm\mu$, and there are two fixed points. Let's discuss them separately:

1. $x_f = \sqrt{\mu}$, then $\lambda_{1,2} = \frac{1}{2}(\pm\sqrt{8\sqrt{\mu} + 1} - 1)$. The undersquare expression is strictly > 0, thus, we have two lambdas of different sign - this is a saddle.
2. $x_f = -\sqrt{\mu}$, then $\lambda_{1,2} = \frac{1}{2}(\pm\sqrt{1 - 8\sqrt{\mu}} - 1)$. There are three cases:

$$\begin{cases} \sqrt{\mu} = \frac{1}{8}, & \lambda_1 = \lambda_2 = -1 \;\; (1) \\ \sqrt{\mu} < \frac{1}{8}, & \lambda_2 < \lambda_1 < 0 \;\; (2) \\ \sqrt{\mu} > \frac{1}{8}, & \lambda_1 = \lambda_2^* \;\; (3) \end{cases}$$

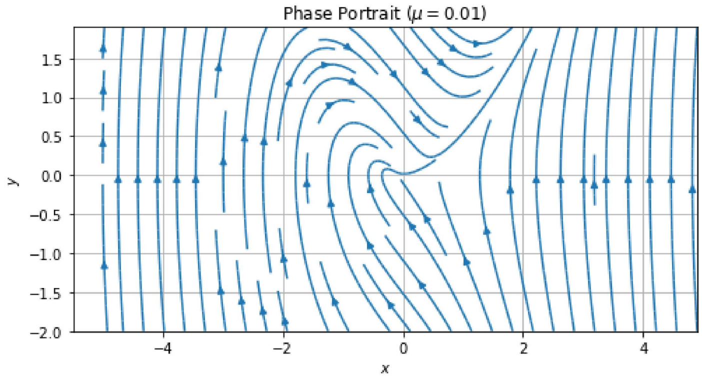
(1) - degenerate node, (2) - stable node, (3) - stable spiral.

For $\mu = 0$, there is only one fixed point $(0,0)$, and it yields $\lambda_1 = -1$, $\lambda_2 = 0$. This is a saddle.

For $\mu < 0$, there are no fixed points $\Rightarrow$ bifurcation occurs there.

(d)

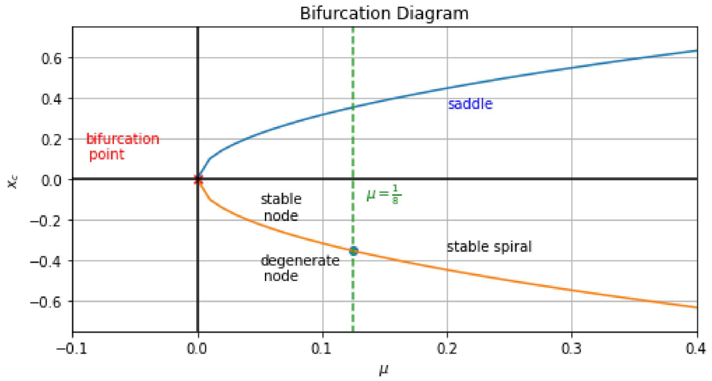
```
mu = -0.01

def dx(x, y):
  return y
def dy(x, y):
  return x**2 - y - mu

x = np.arange(-5, 5, 0.1)
y = np.arange(-2, 2, 0.1)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

for i in range(ni):
  for j in range(nj):
    U[i][j] = dx(X[i][j], Y[i][j])
    V[i][j] = dy(X[i][j], Y[i][j])

plt.streamplot(X, Y, U, V)
plt.title('Phase Portrait ($\\mu = 0.01$)')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()
plt.gcf().set_size_inches(8, 4)
```



(c)

With transition of μ through 0 the Jacobian experiences zero eigenvalue, thus, $\mu = 0$ is a steady state saddle-node bifurcation point.

```
mu = np.arange(0, 0.41, 0.01)
x1 = np.sqrt(mu)
x2 = -np.sqrt(mu)

plt.plot(mu, x1)
plt.plot(mu, x2)

plt.plot([-1, 4], [0, 0], color='black')
plt.plot([0, 0], [-3, 3], color='black')
plt.plot([1/8, 1/8], [-3, 3], linestyle='--')
plt.xlim((-0.1, 0.4))
plt.ylim((-0.75, 0.75))
plt.rcParams['text.color'] = 'black'
plt.title('Bifurcation Diagram')
plt.xlabel('$\\mu$')
plt.ylabel('$x_c$')
plt.rcParams['text.color'] = 'blue'
plt.text(0.2, 0.35, s='saddle')
plt.rcParams['text.color'] = 'black'
plt.text(0.2, -0.35, s='stable spiral')
plt.text(0.05, -0.5, s='degenerate\n node')
plt.text(0.05, -0.2, s='stable\n node')
plt.scatter(1/8, -np.sqrt(1/8))
plt.scatter(0, 0, color='red', marker='x')
plt.rcParams['text.color'] = 'red'
plt.text(-0.09, 0.1, s='bifurcation\n point')
plt.rcParams['text.color'] = 'green'
plt.text(1/8+0.01, -0.1, s='$\\mu = \\frac{1}{8}$')
plt.rcParams['text.color'] = 'black'
plt.grid()
plt.gcf().set_size_inches(8, 4)
```


Bifurcation Diagram
bifurcation
point
saddle
stable
node
$\mu = \frac{1}{8}$
degenerate
node
stable spiral
x_c
μ
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.1
0.0
0.1
0.2
0.3
0.4