Mathematical Methods in Engineering and Applied Science
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Problem Set 10. Due on Dec. 19 at 23:59.

(1) Solve the initial value problem for the advection equation

$$u_t + (1-t)\,u_x = 0, \quad t > 0, x \in \mathbb{R}$$
$$u\,(x,0) = \frac{1}{1+x^2}.$$

Plot the characteristic curves as well as the solution $u\,(x,t)$ at several different times.

(2) Use the method of characteristics to solve the initial-boundary value problem:

$$u_t + u_x = t + x, \quad t > 0, x > 0$$
$$u\,(x,0) = 0$$
$$u\,(0,t) = t.$$

Plot the characteristic curves as well as the solution $u\,(x,t)$ at several different times.

(3) Solve the initial value problem for the Hopf equation:

$$u_t + uu_x = 0, \;\; x \in \mathbb{R}, \;\; t > 0, \quad u(x,0) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(4) Determine the traveling wave solutions $u = U\,(z)$, $z = x - ct$, of the problem

$$\left(u + \frac{u^2}{2}\right)_t = u_{xx} - u_x,$$

with $U\,(-\infty) = 0$, $U\,(\infty) = 1$, $U\,(0) = 1/2$.

(5) Consider the reaction-diffusion system

$$u_t = Du_{xx} + u + v, \quad 0 < x < \pi$$
$$v_t = 3u - v,$$

with no-flux boundary conditions.

(a) Analyze first the spatially homogeneous case with $D = 0$.

(b) Determine the growth rate σ of the normal modes, $w = [u,\ v]^T = a\,(t)\cos nx$, $n = 0, 1, 2, ...$

(c) For a given D, which modes are unstable? Discuss the behavior at large D and at small D.

(d) What is the largest value of D such that spatially non-uniform perturbations grow with time?

(e) Plot the neutral curve, i.e. $D\,(n)$ dependence for zero growth rate, $Re(\sigma) = 0$, and indicate the regions in the $D - n$ plane where the solution is stable and where it is unstable.

(6) (*Extra credit*). Equation $u_t + c\,(x,t)\,u_x = 0$ describes variable-speed advection in a certain non-uniform medium. Explain how to solve it by the method of characteristics for general $c\,(x,t)$ and initial data $u\,(x,0) = f\,(x)$. Next, specialize to $c = 1 + \epsilon \sin x$ with small parameter $\epsilon \to 0$ and find an explicit form of the solution including terms up to $O\,(\epsilon)$. Let $f = e^{-x^2}$ and plot the solution at $\epsilon = 0.1$ at several different t or as a surface in the (x,t)-plane. What happens if $c = 1 + \sin x$?