Problem Set 8

Contents

Problem 1

Plot the phase portrait and classify the fixed points for the following systems:

(a) $\dot{x} = -3x + 2y, \quad \dot{y} = x - 2y$;

(b) $\dot{x} = 5x + 10y, \quad \dot{y} = -x - y$;

(c) $\dot{x} = y, \quad \dot{y} = -x - 2y$.

Solution

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

We'll skip the process of finding the eigenvalues and eigenvectors of the 2×2 matrix at this point, the e-vals are discussed at the end of the problem.

```
import numpy as np
import sympy as sp
import scipy as scp
import matplotlib.pyplot as plt
%config InlineBackend.figure_format = 'jpg'
%matplotlib inline
```

```
a, b, c, d = sp.symbols('a, b, c, d')
t = sp.symbols('t')
x = sp.Function('x')(t)
y = sp.Function('y')(t)


A = sp.Matrix([
    [a, b],
    [c, d]
])

u = sp.Matrix([x, y])
```

(a), (b), (c)

$$A_a = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}, \quad A_b = \begin{bmatrix} 5 & 10 \\ -1 & -1 \end{bmatrix}, \quad A_c = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

The solution to these ODE systems are:

```
subs1 = {a: -3, b: 2, c: 1, d: -2}
subs2 = {a: 5, b: 10, c: -1, d: -1}
subs3 = {a: 0, b: 1, c: -1, d: -2}

sol1 = sp.solvers.ode.systems.dsolve_system(u.diff(t) - A.subs(subs1)*u, funcs=[x, y], t=t)
sol2 = sp.solvers.ode.systems.dsolve_system(u.diff(t) - A.subs(subs2)*u, funcs=[x, y], t=t)
sol3 = sp.solvers.ode.systems.dsolve_system(u.diff(t) - A.subs(subs3)*u, funcs=[x, y], t=t)

u1 = sp.Matrix([i.rhs for i in sol1[0]])
u2 = sp.Matrix([i.rhs for i in sol2[0]])
u3 = sp.Matrix([i.rhs for i in sol3[0]])
```

- System **(a)**:

```
u1
```

$$\begin{bmatrix} -2C_1e^{-4t} + C_2e^{-t} \\ C_1e^{-4t} + C_2e^{-t} \end{bmatrix}$$

- System **(b)**:

```
u2
```

$$\begin{bmatrix} (C_1 + 3C_2)e^{2t}\sin(t) - (3C_1 - C_2)e^{2t}\cos(t) \\ C_1e^{2t}\cos(t) - C_2e^{2t}\sin(t) \end{bmatrix}$$

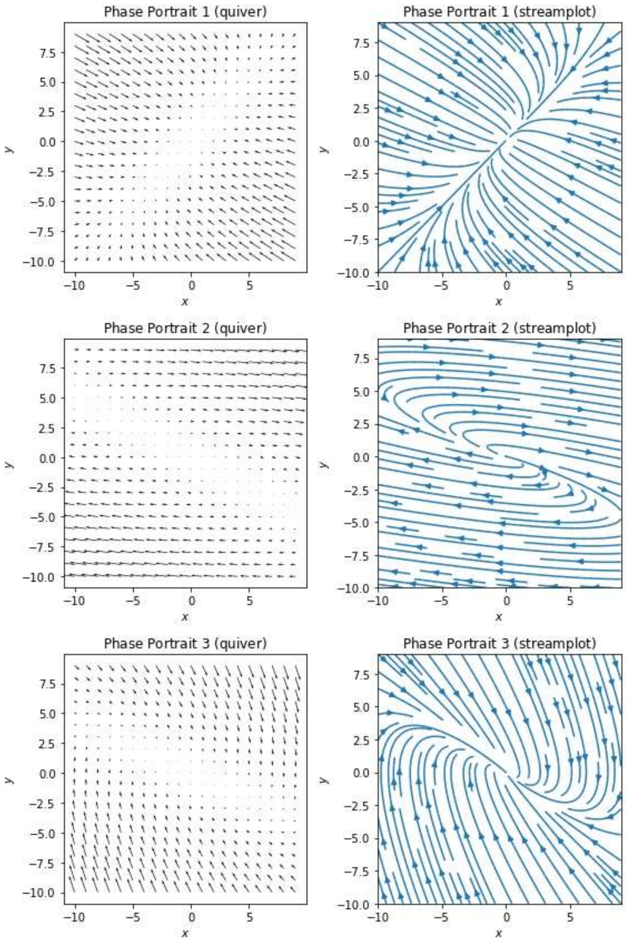
- System **(c)**:

```
u3
```

$$\begin{bmatrix}C_2te^{-t} + (C_1 + C_2)e^{-t}\\ -C_1e^{-t} - C_2te^{-t}\end{bmatrix}$$

Now let's plot the phase portraits:

```
Aa = np.array([
    [-3, 2],
    [1, -2]
])

Ab = np.array([
    [5, 10],
    [-1, -1]
])

Ac = np.array([
    [0, 1],
    [-1, -2]
])

x = np.arange(-10, 10, 1)
y = np.arange(-10, 10, 1)

X, Y = np.meshgrid(x, y)

ni, nj = X.shape

U1 = np.zeros(X.shape)
V1 = np.zeros(Y.shape)
U2 = np.zeros(X.shape)
V2 = np.zeros(Y.shape)
U3 = np.zeros(X.shape)
V3 = np.zeros(Y.shape)

for i in range(ni):
    for j in range(nj):
        U1[i][j], V1[i][j] = Aa.dot(np.array([X[i][j], Y[i][j]]))
        U2[i][j], V2[i][j] = Ab.dot(np.array([X[i][j], Y[i][j]]))
        U3[i][j], V3[i][j] = Ac.dot(np.array([X[i][j], Y[i][j]]))

Us = [U1, U2, U3]
Vs = [V1, V2, V3]

fig, (ax1, ax2, ax3) = plt.subplots(3, 2)
axes = (ax1, ax2, ax3)
for i, ax in enumerate(axes):
    ax[0].quiver(X, Y, Us[i], Vs[i])
    ax[1].streamplot(X, Y, Us[i], Vs[i])
    for a in ax:
        a.set_xlabel('$x$')
        a.set_ylabel('$y$')
    ax[0].set_title('Phase Portrait ${}$ (quiver)'.format(i+1))
    ax[1].set_title('Phase Portrait ${}$ (streamplot)'.format(i+1))

fig.set_size_inches(8, 12)
plt.tight_layout()
```



Now let's analyze the fixed points for each system of ODEs:

- **(a)** - This is a stable node, as the solution tends to 0 as $t \to 0$. x decreases as fast as y. This is also seen as $\lambda_{1,2} < 0 \in \mathbb{R}$.
- **(b)** - Spiral. The eigenvalues are complex conjugate (as we have both trigonometric and exponential functions in the solution, also checked via direct e-val computation). y and x are a combionation of oscillations with increasing amplitude over time.
- **(c)** - Degenerate node. The eigenvalues $\lambda_1 = \lambda_2 = 1$ are real and equal.

Problem 2

Suppose the relationship between Romeo and Juliet is such that

$$\dot{R} = aR + bJ, \quad \dot{J} = -bR - aJ$$

with positive a and b. Describe the type of the relationship and explain its fate depending on the initial conditions.

Solution

Let's write down the matrix equation:

$$\frac{d}{dt}\begin{bmatrix} R \\ J. \end{bmatrix} = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix}$$

While Romeo is sure about his feelings (aR) and responds to Juliet's attitude (bJ), Juliet is being a real woman - she hates him when he loves her and vice versa ($-bR$), also not knowing what to do with herself ($-aJ$). Once a wise man said, "Чем меньше женщину мы любим, тем больше меньше мы, чем тем".

The eigenvalues can be represented with trace τ

$$\tau = a - a = 0$$

and determinant Δ

$$\Delta = -a^2 + b^2$$

as follows:

$$\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) = \frac{\pm 1}{2}\sqrt{-4(b^2 - a^2)} = \pm\sqrt{a^2 - b^2}$$

with $a > 0$, $b > 0$.

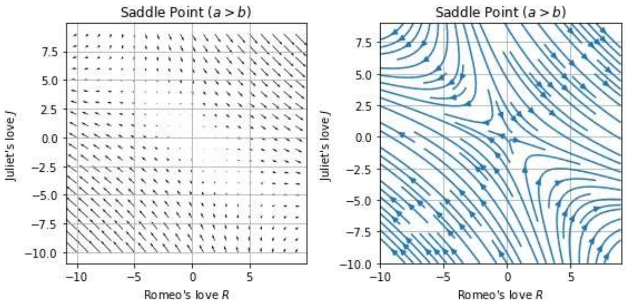
The (non-normalized) eigenvectors are:

$$s_{1,2} = \begin{bmatrix} -\frac{a}{b} \pm \frac{\sqrt{(a-b)(a+b)}}{b} \\ 1 \end{bmatrix}$$

Now, 3 different scenarios are possible:

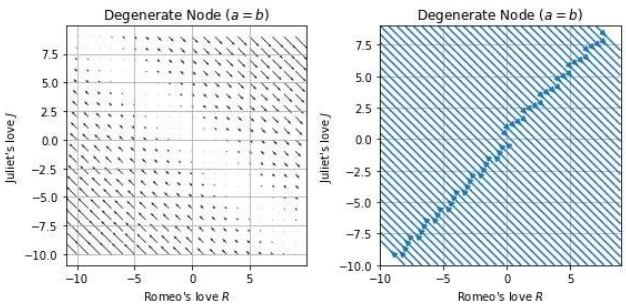
- $a > b$, then $\lambda_1 = -\lambda_2$ and $\lambda_{1,2} \in \mathbb{R}$. This will match the saddle case. Depending on the intial conditions J_0 and R_0, their relation will "follow" one of the main directions (eigenvectors). The eigenvectors will point to 2^{nd} and 4^{th} quadrants with corresponding eigenvalues being positive and negative, respectively. Thus, the love point will tend to 2^{nd} or 4^{th} quadrant, meaning that eventually either Juliet hates Romeo, but he keeps loving her more and more (poor guy =( ), or vice versa (poor girl =( ).

```
RJ = np.array([
    [5, 3],
    [-3, -5]
])

x = np.arange(-10, 10, 1)
y = np.arange(-10, 10, 1)

X, Y = np.meshgrid(x, y)

ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(Y.shape)

for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = RJ.dot(np.array([X[i][j], Y[i][j]]))

fig, (ax1, ax2) = plt.subplots(1, 2)
ax1.quiver(X, Y, U, V)
ax2.streamplot(X, Y, U, V)
ax1.grid()
ax1.set_title('Saddle Point ($a > b$)')
ax1.set_xlabel('Romeo\'s love $R$')
ax1.set_ylabel('Juliet\'s love $J$')
ax2.grid()
ax2.set_title('Saddle Point ($a > b$)')
ax2.set_xlabel('Romeo\'s love $R$')
ax2.set_ylabel('Juliet\'s love $J$')
fig.set_size_inches(8, 4)
plt.tight_layout()
```



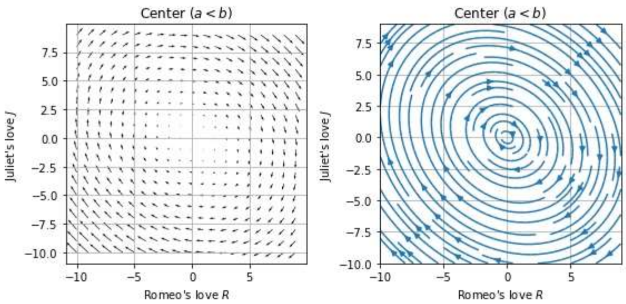
- $a = b$, thus, $\lambda_1 = \lambda_2 = 0$. This is a singular matrix ($rg(A) = 0$). This case is a degenerate node. There is one eigenvector: $(1, 1)^T$, which is the nullspace of the matrix. Romeo values his feeling as much as Juliet's: $aR + aJ$. Juliet hesitates both in her state and in Romeo's suspicious attention to her equally: $-aR - aJ$. Thus, when $R = -J$, their relationship will forever remain the same: one loves and another hates. Other than that case, whether $R > J$, Romeo will keep fallin in love while Juliet keep hating him more. Opposite in the other case.

```
RJ = np.array([
    [1, 1],
    [-1, -1]
])

for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = RJ.dot(np.array([X[i][j], Y[i][j]]))


fig, (ax1, ax2) = plt.subplots(1, 2)
ax1.quiver(X, Y, U, V)
ax2.streamplot(X, Y, U, V)
ax1.grid()
ax1.set_title('Degenerate Node ($a = b$)')
ax1.set_xlabel('Romeo\'s love $R$')
ax1.set_ylabel('Juliet\'s love $J$')
ax2.grid()
ax2.set_title('Degenerate Node ($a = b$)')
ax2.set_xlabel('Romeo\'s love $R$')
ax2.set_ylabel('Juliet\'s love $J$')
fig.set_size_inches(8, 4)
plt.tight_layout()
```



- $a < b$— $\lambda_1 = \lambda_2^*$, both are purely imaginary. This represents a center. With imaginary eigenvalues the matrix A represents pure rotation, thus, every change vector is perpendicular to the current love vector. The relationship will go roundabout an elliptic trajectory with certain radiuses, dependent on initial conditions.

```
RJ = np.array([
    [1, 5],
    [-5, -1]
])

for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = RJ.dot(np.array([X[i][j], Y[i][j]]))


fig, (ax1, ax2) = plt.subplots(1, 2)
ax1.quiver(X, Y, U, V)
ax2.streamplot(X, Y, U, V)
ax1.grid()
ax1.set_title('Center ($a < b$)')
ax1.set_xlabel('Romeo\'s love $R$')
ax1.set_ylabel('Juliet\'s love $J$')
ax2.grid()
ax2.set_title('Center ($a < b$)')
ax2.set_xlabel('Romeo\'s love $R$')
ax2.set_ylabel('Juliet\'s love $J$')
fig.set_size_inches(8, 4)
plt.tight_layout()
```



> **Note**
>
> Poor guys =(

Problem 3

For the system

$$\dot{x} = xy - 1, \quad \dot{y} = x - y^3$$

find the fixed points, classify them, sketch the neighbouring trajectories and try to fill in the rest of the phase plane.

Solution

Starting with the fixed points:

$$\begin{aligned} xy &= 1 \\ x &= y^3 \end{aligned}$$

We get 4 different fixed points:

$$\begin{aligned} y &= [1, -1, i, -i] \\ x &= [1, -1, -i, i] \\ p_1 = (1, 1), \quad p_2 = (-1, -1), \quad p_3 &= (i, -i), \quad p_4 = (-i, i). \end{aligned}$$

Assuming that x and y are real, we are left with the following two:

$$\begin{aligned} p_1 &= (1, 1) \\ p_2 &= (-1, -1). \end{aligned}$$

To classify them, let's linearize the equations:

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, t), \\ f(x, y, t) = f(x_0, y_0, t) + f_x'(x_0, y_0, t)(\Delta x) + f_y'(x_0, y_0, t)(\Delta y) &+ \ldots \\ \frac{dy}{dt} &= g(x, y, t), \\ g(x, y, t) = f(x_0, y_0, t) + g_x'(x_0, y_0, t)(\Delta x) + g_y'(x_0, y_0, t)(\Delta y) &+ \ldots \end{aligned}$$

This gives us

$$\begin{aligned} \dot{x} &= x_0 \Delta x + y_0 \Delta y \\ \dot{y} &= \Delta x + -3y_0^2 \Delta y \end{aligned}$$

or

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 & y_0 \\ 1 & -3y_0^2 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

> **Note**
>
> Here and afterwards x, y in the right hand side of the linearized equation are actually x' and y' that can be obtained as $x' = x_0 + \Delta x$, $y' = y_0 + \Delta y$.

- Starting with the first point, $(1, 1)$:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

```
A = sp.Matrix([
    [1, 1],
    [1, -3]
])

A.eigenvects()
```

```
[(-1 + sqrt(5),
  1,
  [Matrix([
   [2 + sqrt(5)],
   [          1]])]),
 (-sqrt(5) - 1,
  1,
  [Matrix([
   [2 - sqrt(5)],
   [          1]])])]
```

The eigenvalues are

$$\lambda_{1,2} = -1 \pm \sqrt{5}$$

and the eigenvectors:

$$s_{1,2} = \begin{bmatrix} 2 \pm \sqrt{5} \\ 1 \end{bmatrix}$$

This gives us the saddle fixed point.

- For the second point $(-1, -1)$:

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$$

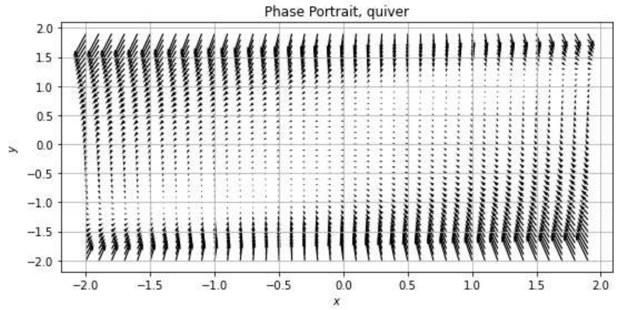
```
A = sp.Matrix([
    [-1, -1],
    [1, -3]
])

A.eigenvects()
```

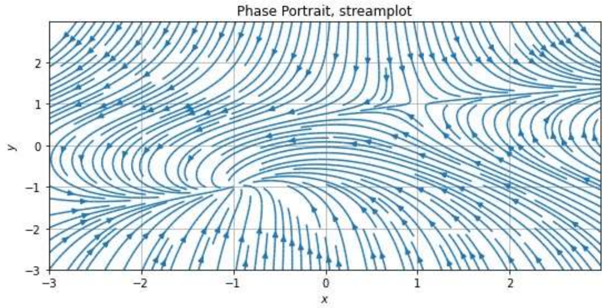
```
[(-2,
  2,
  [Matrix([
   [1],
   [1]])])]
```

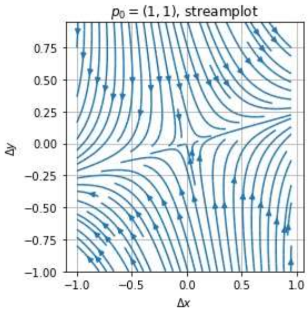
We have the eigenvalue $\lambda = 2$ with multiplicity 2 and the eigenvector $s = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which yields the degenerate node.

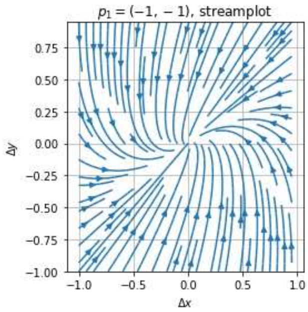
Now let's sketch the phase portrait. By the powers of **python** we can avoid sketching only the neighbourhoods and sketch the whole portrait:

```python
def dx(x, y, t):
    return x*y - 1

def dy(x, y, t):
    return x - y**3

A1 = np.array([
    [1, 1],
    [1, -3]
])
A2 = np.array([
    [-1, -1],
    [1, -3]
])

#quiver plot
x = np.arange(-2, 2, 0.1)
y = np.arange(-2, 2, 0.1)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)
for i in range(ni):
    for j in range(nj):
        U[i][j] = dx(X[i][j], Y[i][j], 0)
        V[i][j] = dy(X[i][j], Y[i][j], 0)
plt.subplot(3, 1, 1)
plt.quiver(X, Y, U, V)
plt.title('Phase Portrait, quiver')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()

#streamplot
x = np.arange(-3, 3, 0.01)
y = np.arange(-3, 3, 0.01)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)
for i in range(ni):
    for j in range(nj):
        U[i][j] = dx(X[i][j], Y[i][j], 0)
        V[i][j] = dy(X[i][j], Y[i][j], 0)
plt.subplot(3, 1, 2)
plt.streamplot(X, Y, U, V, density=2)
plt.title('Phase Portrait, streamplot')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()

#point (1, 1)
x = np.arange(-1, 1, 0.05)
y = np.arange(-1, 1, 0.05)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)
for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = A1.dot(np.array([X[i][j], Y[i][j]]))
plt.subplot(3, 2, 5)
plt.streamplot(X, Y, U, V)
plt.title('$p_0 = (1, 1)$, streamplot')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.grid()

#point (-1, -1)
x = np.arange(-1, 1, 0.05)
y = np.arange(-1, 1, 0.05)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)
for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = A2.dot(np.array([X[i][j], Y[i][j]]))
plt.subplot(3, 2, 6)
plt.streamplot(X, Y, U, V)
plt.title('$p_1 = (-1, -1)$, streamplot')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.grid()

fig = plt.gcf()
fig.set_size_inches(8, 12)
plt.tight_layout()
```









Problem 4

For the following model of rabbits and sheep, find the fixed points, investigate their stability and draw the phase portrait. Indicate the basins of attraction of any stable fixed point:

$$\dot{x} = x(3 - 2x - 2y), \quad \dot{y} = y(2 - x - y).$$

Solution

As always, we start with finding the fixed points:

```
x, y = sp.symbols('x, y')

eq1 = x*(3 - 2*x - 2*y)
eq2 = y*(2 - x - y)

eqs = [eq1, eq2]

sol = sp.solve(eqs, x, y)
sol
```

```
[(0, 0), (0, 2), (3/2, 0)]
```

Fixed points are:

$$p_1 = (0, 0)$$
$$p_2 = (0, 2)$$
$$p_3 = (\frac{3}{2}, 0)$$

Now let's linearize the equations around the fixed points:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 - 4x_0 & -2x_0 \\ -y_0 & 2 - 2y_0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

And find the types of the points:

```
x0, y0 = sp.symbols('x0, y0')
A = sp.Matrix([
    [3 - 4*x0, -2*x0],
    [-y0, 2 - 2*y0]
])

subs1 = {x0: 0, y0: 0}
subs2 = {x0: 0, y0: 2}
subs3 = {x0: sp.Rational(3, 2), y0: 0}
```

```
A.subs(subs1).eigenvects()
```

```
[(2,
  1,
  [Matrix([
   [0],
   [1]])]),
 (3,
  1,
  [Matrix([
   [1],
   [0]])])]
```

```
A.subs(subs2).eigenvects()
```

```
[(-2,
  1,
  [Matrix([
   [0],
   [1]])]),
 (3,
  1,
  [Matrix([
   [-5/2],
   [   1]])])]
```

```
A.subs(subs3).eigenvects()
```

```
[(-3,
  1,
  [Matrix([
   [1],
   [0]])]),
 (2,
  1,
  [Matrix([
   [-3/5],
   [   1]])])]
```

- p_1 = (0, 0): unstable node

$$\lambda_1 = 2, \quad \lambda_2 = 3,$$

$$s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- p_2 = (0, 2): saddle

$$\lambda_1 = -2, \quad \lambda_2 = 3,$$

$$s_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}$$

- p_3 = ($\frac{3}{2}$, 0): saddle

$$\lambda_1 = -3, \quad \lambda_2 = 2$$

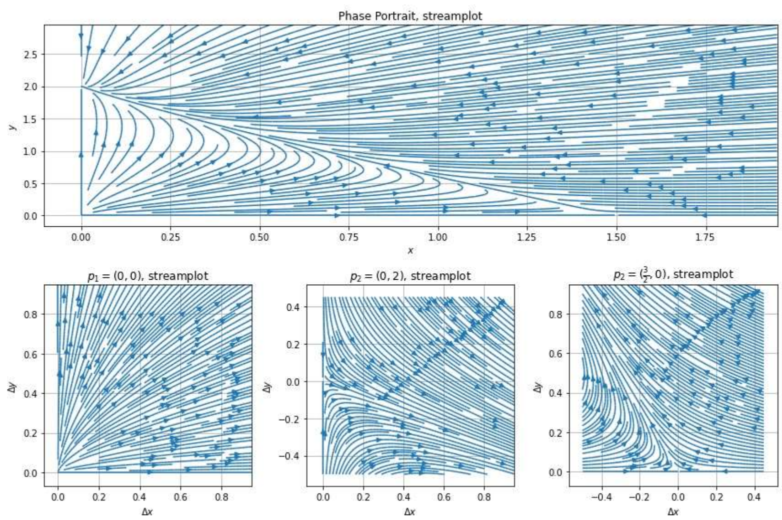
$$s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}$$

Now let's draw the phase portrait:

> **Note**
>
> From the formulation of the problem we are only interested in positive values of x and y, thus, phase portrait will only be drawn only for $x > 0$, $y > 0$.

> **Note**
>
> Point $(0, 0)$ on the phase portrait around the fixed point i coincides with the (x_0^i, y_0^i) point on the main phase portrait.

```python
def dx(x, y):
    return x*(3 - 2*x - 2*y)

def dy(x, y):
    return y*(2 - x - y)

A1 = np.array(A.subs(subs1), dtype=np.float64)
A2 = np.array(A.subs(subs2), dtype=np.float64)
A3 = np.array(A.subs(subs3), dtype=np.float64)

x = np.arange(0, 2, 0.05)
y = np.arange(0, 3, 0.05)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)
for i in range(ni):
    for j in range(nj):
        U[i][j] = dx(X[i][j], Y[i][j])
        V[i][j] = dy(X[i][j], Y[i][j])
plt.subplot(2, 1, 1)
plt.streamplot(X, Y, U, V, density=2)
plt.title('Phase Portrait, streamplot')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()

x = np.arange(0, 1, 0.05)
y = np.arange(0, 1, 0.05)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)

for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = A1.dot(np.array([X[i][j], Y[i][j]]))
plt.subplot(2, 3, 4)
plt.streamplot(X, Y, U, V, density=2)
plt.title('$p_1 = (0, 0)$, streamplot')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.grid()

x = np.arange(0, 1, 0.05)
y = np.arange(-0.5, 0.5, 0.05)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)

for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = A2.dot(np.array([X[i][j], Y[i][j]]))
plt.subplot(2, 3, 5)
plt.streamplot(X, Y, U, V, density=2)
plt.title('$p_2 = (0, 2)$, streamplot')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.grid()

x = np.arange(-0.5, 0.5, 0.05)
y = np.arange(0, 1, 0.05)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)

for i in range(ni):
    for j in range(nj):
        U[i][j], V[i][j] = A3.dot(np.array([X[i][j], Y[i][j]]))
plt.subplot(2, 3, 6)
plt.streamplot(X, Y, U, V, density=2)
#plt.plot(x, -5/3*(x-3/2))
plt.title('$p_2 = (\\frac{3}{2}, 0)$, streamplot')
plt.xlabel('$\Delta x$')
plt.ylabel('$\Delta y$')
plt.grid()
fig = plt.gcf()
fig.set_size_inches(12, 8)
plt.tight_layout()
```



As for the basins of attraction: neither of these points is a stable fixed point. However, we can predict the behaviour of the solutions at the large t.

For p_1, there is no basin of the attraction, as the node is unstable - every solution diverges from the point.

For p_2, we have two eigenvectors and eigenvalues with different signs, the first being negative and the second positive. This means that solutions will tend towards the second eigenvector $s_2 = \begin{bmatrix} -1 \\ \frac{2}{5} \end{bmatrix}$.

> **Warning**
>
> I don't quite understand what is going on on the plot for $p_2 = (0, 2)$, as it should point in the opposite direction.

For p_3 the same logic is applied with its eigenvectors. For $y = 0$, the solutions tend to the point $(\frac{3}{2}, 0)$. With $y_0 > 0$ they tend towards the eigenvector $s_2 = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}$.

Problem 5

Consider the system

$$\dot{x} = -y - x^3, \quad \dot{y} = x.$$

Show that the origin is a spiral, although the linearization predicts a center.

Solution

Fixed points:

$$\begin{aligned} -y - x^3 = 0, \quad x = 0 \\ \Rightarrow x = 0, \quad y = 0. \end{aligned}$$

Linearization:

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3x_0^2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

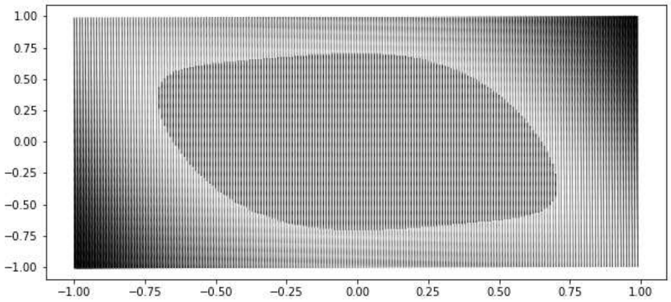
Let's find eigenvalues of the matrix $A|_{(0,0)}$:

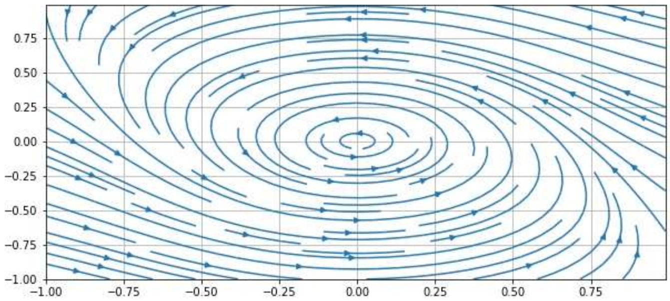
```
A = np.array([
    [0, -1],
    [1, 0]
])

sp.Matrix(A).eigenvects()
```

```
[(-I,
  1,
  [Matrix([
   [-I],
   [ 1]])]),
 (I,
  1,
  [Matrix([
   [I],
   [1]])])]
```

Pure imaginary λ's indeed tell us this should be a center (pure rotation of the vectors). However, let's plot the phase portrait aroun $(0, 0)$
:

```
def dx(x, y):
    return -y - x**3

def dy(x, y):
    return x

x = np.arange(-1, 1, 0.01)
y = np.arange(-1, 1, 0.01)
X, Y = np.meshgrid(x, y)
ni, nj = X.shape

U = np.zeros(X.shape)
V = np.zeros(X.shape)

for i in range(ni):
    for j in range(nj):
        U[i][j] = dx(X[i][j], Y[i][j])
        V[i][j] = dy(X[i][j], Y[i][j])

plt.subplot(211)
plt.quiver(X, Y, U, V)
plt.subplot(212)
plt.streamplot(X, Y, U, V)
plt.grid()
plt.gcf().set_size_inches(10, 10)
```





To explain this behaviour, let's move to polar coordinates:

$$r^2 = x^2 + y^2,$$
$$x = r\cos\varphi, \quad y = r\sin\varphi.$$

Then

$$\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi = -r\sin\varphi - r^3\cos^3\varphi,$$
$$\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi = r\cos\varphi$$

and

$$r\dot{\varphi} = r + r^3\cos^3\varphi\sin\varphi$$

Assuming we start not from the fixed point (i.e. $r \neq 0$), we get

$$\dot{\varphi} = 1 + r^2\cos^3\varphi\sin\varphi.$$

Substitute this to the first equation:

$$\dot{r}\sin\varphi\cos\varphi - r\dot{\varphi}\sin^2\varphi = -r\sin^2\varphi - r^3\cos^3\varphi\sin\varphi,$$
$$\dot{r} = -r^3\cos^4\varphi.$$

From here we see, recalling our assumption $r \neq 0$, that at $\forall\varphi \neq \pm\frac{\pi}{2}$ the $\dot{r}$ term is not zero, thus, the radius will change, meaning the fixed point is not center anymore (center means rotational movement). At $\varphi = \pm\frac{\pi}{2}$ $\dot{r} = 0$, which is seen on the phase portrait as the motion perpendicular to OY.

Problem 6

The Kermack-McKendrick model of an epidemic describes the population of helathy people $x(t)$ and sick people $y(t)$ in terms of the equations

$$\dot{x} = -kxy, \quad \dot{y} = kxy - ly,$$

where $k, l > 0$. Here, l is the death rate of the sick people in equation for $\dot{y}$ implies that people get sick at a rate proportional to their encounters (which itself is proportional to the product of the number of sick people y and healthy people x). The parameter k measures the probability of transmission of the disease during encounters.

(a) Find and classify the fixed points.

(b) Sketch the nullclines and the vector field.

(c) Find a conserved quantity for the system (hintL form an ODE for dy/dx and integrate it).

(d) Plot the phase portrait. What happens as $t \to \infty$?

(e) Let (x_0, y_0) be the initial condition. Under what conditions on (x_0, y_0) will the epidemic occur? (Epidemic occurs if $y(t)$ increases initially).

Solution

(a)

The fixed points are:

$$\begin{cases} -kxy = 0, \\ y(kx - l) = 0 \end{cases}$$

In this case there is only one possibility: $y = 0, \Rightarrow x \in \mathbb{R}$ - this is a line of fixed points in X direction.

Linearization gives us the following equation:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -kx \\ 0 & kx - l \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

The matrix A yields two eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = kx - l$$

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and corresponding eigenvectors:

$$s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} \frac{l}{kx} - 1 \\ 1 \end{bmatrix}$$

(b)

Let's elaborate on possible λ's:

1. $x < \frac{l}{k}$ - then λ_2 is negative and eigenvector s_2 points in the 1^{st} quadrant, the solutions tend to $y -> 0$.
2. $x = \frac{l}{k}$ - both eigenvalues are zero, nearby the fixed points line there is no change in the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ at the line $x = \frac{l}{k}$.
3. $x > \frac{l}{k}$ - λ_2 is positive, eigenvector s_2 points towards the 2^{nd} quadrant. The solutions tend towards the eigenvectors.

See the attached image for clarification:

The nullclines here are $y = 0$ ($\frac{dx}{dt} = 0$) and $x = \frac{l}{k}$ ($\frac{dy}{dt} = 0$).

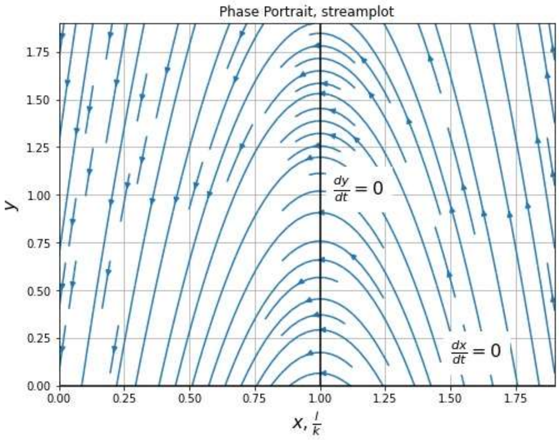
(e)

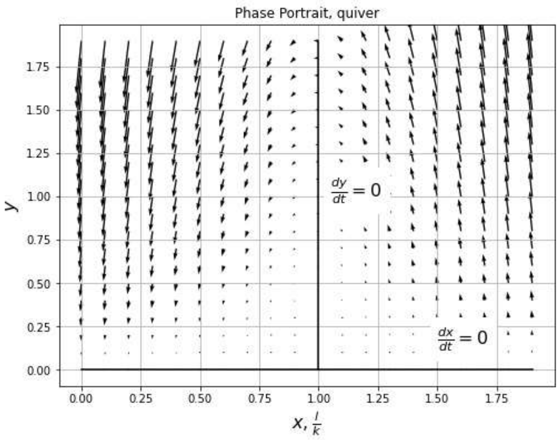
From this we can derive that epidemic occurs only if initial point (x_0, y_0) is to the right of the line $x = \frac{l}{k}$, i.e. $(x_0 > \frac{l}{k}, y_0 \neq 0)$

(d)

Now let's plot the phase portrait to confirm our expectations:

```
k = 0.1
l = 0.1

def dx(x, y):
    return -k*k*y

def dy(x, y):
    return k*x*y - l*y

x = np.arange(0, 2, 0.1)
y = np.arange(0, 2, 0.1)

X, Y = np.meshgrid(x, y)
ni, nj = X.shape
U = np.zeros(X.shape)
V = np.zeros(X.shape)
for i in range(ni):
    for j in range(nj):
        U[i][j] = dx(X[i][j], Y[i][j])
        V[i][j] = dy(X[i][j], Y[i][j])

plt.streamplot(X, Y, U, V)
plt.plot([l/k, l/k], [0, 1.9], color='black')
plt.plot([0, 1.9], [0, 0], color='black')
plt.xlabel('$x, \\frac{L}{k}$', fontsize=16)
plt.ylabel('$y$', fontsize=16)
plt.title('Phase Portrait, streamplot')
plt.text(x=1.05, y=1, s='$\\frac{dy}{dt}=0$', backgroundcolor='white', fontsize=16)
plt.text(x=1.5, y=0.15, s='$\\frac{dx}{dt}=0$', backgroundcolor='white', fontsize=16)
plt.gcf().set_size_inches(8, 6)
plt.grid()
plt.figure()
plt.quiver(X, Y, U, V)
plt.plot([l/k, l/k], [0, 1.9], color='black')
plt.plot([0, 1.9], [0, 0], color='black')
plt.xlabel('$x, \\frac{L}{k}$', fontsize=16)
plt.ylabel('$y$', fontsize=16)
plt.title('Phase Portrait, quiver')
plt.text(x=1.05, y=1, s='$\\frac{dy}{dt}=0$', backgroundcolor='white', fontsize=16)
plt.text(x=1.5, y=0.15, s='$\\frac{dx}{dt}=0$', backgroundcolor='white', fontsize=16)
plt.gcf().set_size_inches(8, 6)
plt.grid()
```





(c)

$$\begin{aligned} \frac{dx}{dt} &= -kxy, \\ \frac{dy}{dt} &= kxy - ly, \end{aligned}$$

Dividing the second equation by first we obtain:

$$\frac{dy}{dx} = \frac{ly - kxy}{kxy} = \frac{l}{kx} - 1$$

Integrating:

$$\begin{aligned} dy &= \frac{l}{k}\frac{dx}{x} - dx, \\ C = \frac{l}{k}\log x - x - y, \quad x, y > 0, \quad C &= Const \end{aligned}$$

By Arseniy Buchnev