**Mathematical Methods in Engineering and Applied Science**
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**Problem Set 8**. Due on Dec. 4 at 23:59

(1) Plot the phase portrait and classify the fixed points for the following systems:
  (a) $\dot{x} = -3x + 2y, \quad \dot{y} = x - 2y;$
  (b) $\dot{x} = 5x + 10y, \quad \dot{y} = -x - y;$
  (c) $\dot{x} = y, \quad \dot{y} = -x - 2y.$

(2) Suppose the relationship between Romeo and Juliet is such that

$$\dot{R} = aR + bJ, \quad \dot{J} = -bR - aJ$$

with positive $a$ and $b$. Describe the type of the relationship and explain its fate depending on the initial conditions.

(3) For the system

$$\dot{x} = xy - 1, \quad \dot{y} = x - y^3,$$

find the fixed points, classify them, sketch the neighboring trajectories and try to fill in the rest of the phase plane.

(4) For the following model of rabbits and sheep, find the fixed points, investigate their stability and draw the phase portrait. Indicate the basins of attraction of any stable fixed point:

$$\dot{x} = x\,(3 - 2x - 2y)\,, \quad \dot{y} = y\,(2 - x - y)\,.$$

(5) Consider the system

$$\dot{x} = -y - x^3, \quad \dot{y} = x.$$

Show that the origin is a spiral, although the linearization predicts a center.

(6) The Kermack-McKendrick model of an epidemic describes the population of healthy people $x\,(t)$ and sick people $y\,(t)$ in terms of the equations

$$\begin{aligned} \dot{x} &= -kxy \\ \dot{y} &= kxy - ly, \end{aligned}$$

where $k, l > 0$. Here, $l$ is the death rate of the sick people, and $kxy$ in equation for $\dot{y}$ implies that people get sick at a rate proportional to their encounters (which itself is proportional to the product of the number of sick people $y$ and healthy people $x$). The parameter $k$ measures the probability of transmission of the disease during the encounters.
  (a) Find and classify the fixed points.
  (b) Sketch the nullclines and the vector field.
  (c) Find a conserved quantity for the system (hint: form an ODE for $dy/dx$ and integrate it).
  (d) Plot the phase portrait. What happens as $t \to \infty$?
  (e) Let $(x_0, y_0)$ be the initial condition. Under what conditions on $(x_0, y_0)$ will the epidemic occur? (Epidemic occurs if $y\,(t)$ increases initially).