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1. (a) Exercise 4.2.1

(1) $L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, let $|\psi\rangle = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$$L_z|\psi\rangle = \lambda|\psi\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \lambda\begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)\lambda(1+\lambda) = 0 \qquad \therefore \lambda_z = -1, 0, 1$$

(2) $L_z = 1 \Rightarrow m = 0, n = 0$ so $|\psi\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\langle L_x\rangle = \langle\psi|L_x|\psi\rangle = [1\ 0\ 0]\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}[1\ 0\ 0]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\langle L_x^2\rangle = \langle\psi|L_x^2|\psi\rangle = [1\ 0\ 0]\frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}$$

$$\Delta L_x = \sqrt{\langle L_x^2\rangle - \langle L_x\rangle^2} = \frac{1}{\sqrt{2}}$$

(3) $L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $|\psi\rangle = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$L_x|\psi\rangle = \lambda|\psi\rangle \Rightarrow \det(L_x - \lambda_z) = 0$

$\lambda_x - \lambda_x^3 = 0$, $\lambda_x = -1, 0, 1$

$$\begin{bmatrix} -\lambda_x & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda_x & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda_x \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$\begin{bmatrix} -\lambda_x l + \frac{1}{\sqrt{2}} m \\ \frac{1}{\sqrt{2}} l - \lambda_x m + \frac{1}{\sqrt{2}} n \\ \frac{1}{\sqrt{2}} m - \lambda_x m \end{bmatrix} = 0$$

when $\lambda_x = 1$, $|\psi\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

$\lambda_x = 0$, $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_x = -1$, $|\psi\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

(4) when $L_z = -1$ $\quad |\psi\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

probability of $L_x = 1$

$$\left| \langle \lambda_x = 1 | \psi \rangle \right|^2 = \left| \frac{1}{2} [1 \;\; \sqrt{2} \;\; 1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2$$

$$= \frac{1}{4}$$

probability of $L_x = 0$

$$\left| \langle \lambda_x = 0 | \psi \rangle \right|^2 = \left| \frac{1}{2} [1 \;\; 0 \;\; -1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{2}$$

probability of $L_x = -1$

$$\left| \langle \lambda_x = -1 | \psi \rangle \right|^2 = \left| \frac{1}{2} [1 \;\; -\sqrt{2} \;\; 1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{4}$$

(5) $|\psi\rangle = C_{-1}|\lambda_z=-1\rangle + C_0|\lambda_z=0\rangle + C_1|\lambda_z=1\rangle$

$$C_{-1} = \langle\lambda_z=-1|\psi\rangle = [0\ 0\ 1]\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$C_0 = \langle\lambda_z=0|\psi\rangle = [0\ 1\ 0]\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}$$

$$C_1 = \langle\lambda_z=1|\psi\rangle = [1\ 0\ 0]\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|\lambda_z=-1\rangle + \frac{1}{2}|\lambda_z=0\rangle + \frac{1}{2}|\lambda_z=1\rangle$$

Since L_z^2 is measured to be 1, λ_z can be 1 or -1

The probability of this result is $|C_{-1}|^2+|C_1|^2 = \frac{3}{4}$

After the measurement, the state is

$$|\psi'\rangle = C_{-1}'|\lambda_z=-1\rangle + C_1'|\lambda_z=1\rangle$$

$$C_1'/C_{-1}' = C_1/C_{-1} = \frac{1}{\sqrt{2}}$$

normalization $\Rightarrow C_1' = \sqrt{\frac{1}{3}}$ $\quad C_{-1}' = \sqrt{\frac{2}{3}}$

$$\Rightarrow |\psi'\rangle = \sqrt{\frac{2}{3}}|\lambda_z=-1\rangle + \sqrt{\frac{1}{3}}|\lambda_z=1\rangle$$

$L_z = -1$, probability $= \frac{2}{3}$

$L_z = 1$, 〃 $= \frac{1}{3}$

(6) In the L_z eigen basis

$$|L_z=1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad |L_z=0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix},\quad |L_z=-1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

let $|\psi\rangle = \begin{bmatrix}a\\b\\c\end{bmatrix}$

Then $P(L_z=1) = \frac{1}{4} = |\langle L_z=1|\psi\rangle|^2 = |a|^2$

$$P(L_z=0) = \frac{1}{2} = |\langle L_z=0|\psi\rangle|^2 = |b|^2$$

$$P(L_z=-1) = \frac{1}{4} = |\langle L_z=-1|\psi\rangle|^2 = |c|^2$$

$$\therefore\quad a = \frac{1}{2}e^{i\delta_1},\quad b = \frac{1}{\sqrt{2}}e^{i\delta_2},\quad c = \frac{1}{2}e^{i\delta_3}$$

for some arbitrary phase δ_i, which gives the desired answer.

The δ_i phase factors are not irrelevant.

for example

$$P(L_x=0) = |\langle \lambda_x=0|\psi\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}}[1\ 0\ -1]\,\frac{1}{2}\begin{bmatrix}e^{i\delta_1}\\ \sqrt{2}e^{i\delta_2}\\ e^{i\delta_3}\end{bmatrix} \right|^2$$

$$= \frac{1}{8}(e^{i\delta_1} - e^{i\delta_3})(e^{-i\delta_1} - e^{-i\delta_3})$$

$$= \frac{1}{4}(1 - \cos(\delta_3 - \delta_1))$$

So something measurable depends on the difference of the phases.

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1. (b) Exercise 4.2.2

$\psi^*(x) = \psi(x)$

$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x)\, dx$

$= -i\hbar \int_{-\infty}^{\infty} \psi(x) \frac{\partial}{\partial x} \psi(x)\, dx$

$= -\frac{i\hbar}{2} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \psi^2(x)\, dx$

$= -\frac{i\hbar}{2} \psi^2(x) \Big|_{-\infty}^{\infty} = 0$

for general, $\langle p \rangle = \int_{-\infty}^{\infty} [c\psi(x)]^* \left(-i\hbar \frac{\partial}{\partial x}\right) [c\psi(x)]\, dx$

$= -i\hbar |c|^2 \int_{-\infty}^{\infty} \psi(x) \frac{\partial}{\partial x} \psi(x)\, dx$

$= 0$

1. (c) Exercise 4.2.3.

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{\partial}{\partial x} \psi(x)\, dx$$

$$\langle p' \rangle = -i\hbar \int_{-\infty}^{\infty} \left[\psi^*(x) e^{-ip_0 x/\hbar}\right] \frac{\partial}{\partial x} \left[\psi(x) e^{ip_0 x/\hbar}\right] dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \left[\psi^*(x) e^{-ip_0 x/\hbar}\right]\left[\frac{\partial}{\partial x}\psi(x)\right] \cdot e^{ip_0 x/\hbar}\, dx$$

$$-i\hbar \int_{-\infty}^{\infty} \left[\psi^*(x) e^{-ip_0 x/\hbar}\right] \psi(x) \left[\frac{\partial}{\partial x}\left(e^{ip_0 x/\hbar}\right)\right] dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{\partial}{\partial x} \psi(x)\, dx - i\hbar \cdot ip_0/\hbar \int_{-\infty}^{\infty} \psi^*(x)\psi(x)\, dx$$

$$= \langle p \rangle + p_0$$

1. (d) Exercise 5.4.2

in the x basis

$$\frac{P^2}{2m}|E\rangle = E|E\rangle \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_E(x) = E\psi_E(x)$$

the solution can be written as

$$\psi_E(x) = Ae^{ikx} + Be^{-ikx} \quad \left(k = \frac{\sqrt{2mE}}{\hbar}\right)$$

$$\left(\text{and } \langle x|P\rangle = \psi_P(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{iPx/\hbar}\right)$$

$$\psi_E(x) = A\cdot\sqrt{2\pi\hbar}\cdot\psi_{P=\sqrt{2mE}}(x) + B\sqrt{2\pi\hbar}\,\psi_{P=-\sqrt{2mE}}(x)$$

$$\Rightarrow \psi_E(x) = \beta\langle x|p=\sqrt{2mE}\rangle + \gamma\langle x|p=-\sqrt{2mE}\rangle$$

$$\left(\beta = A\sqrt{2\pi\hbar}, \quad \gamma = B\sqrt{2\pi\hbar}\right)$$

2.

(a) $\psi(x) = A e^{-\alpha|x|}$

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = 1$$

$$|A|^2 \left[\int_{-\infty}^{0} e^{2\alpha x}\,dx + \int_{0}^{\infty} e^{-2\alpha x}\,dx \right] = 1$$

$$|A|^2 \left[\frac{1}{2\alpha} e^{2\alpha x} \Big|_{-\infty}^{0} - \frac{1}{2\alpha} e^{-2\alpha x} \Big|_{0}^{\infty} \right] = 1$$

$$|A|^2 \frac{1}{\alpha} = 1 \qquad A = \sqrt{\alpha}$$

(b)

$$\psi(p) = \langle p|\psi\rangle = \int \langle p x\rangle\langle x|\psi\rangle dx$$

$$(\langle p|x\rangle = \psi_p^*(x))$$

$$= \int \psi_p^*(x)\,\psi(x)\,dx$$

$$= \int \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \cdot \sqrt{\alpha} \cdot e^{-\alpha|x|}\,dx$$

$$= \sqrt{\frac{\alpha}{2\pi\hbar}} \int e^{-ipx/\hbar} e^{-\alpha|x|}\,dx$$

$$= \sqrt{\frac{\alpha}{2\pi\hbar}} \left\{ \int_{-\infty}^{0} e^{\left(\alpha - \frac{p}{\hbar} i\right) x} dx + \int_{0}^{\infty} e^{-\left(\alpha + \frac{p}{\hbar} i\right) x} dx \right\}$$

$$= \sqrt{\frac{\alpha}{2\pi\hbar}} \left\{ \frac{1}{\alpha - \frac{p}{\hbar} i} + \frac{1}{\alpha + \frac{p}{\hbar} i} \right\}$$

$$= \sqrt{\frac{\alpha}{2\pi\hbar}} \frac{2\alpha}{\alpha^2 + \frac{p^2}{\hbar^2}}$$

(c) $\langle x \rangle = \alpha \int_{-\infty}^{\infty} x e^{-2\alpha |x|} dx$

$$= \alpha \left\{ \int_{-\infty}^{0} x e^{2\alpha x} dx + \int_{0}^{\infty} x e^{-2\alpha x} dx \right\}$$

$$= \alpha \left\{ -\frac{1}{4\alpha^2} + \frac{1}{4\alpha^2} \right\} = 0$$

$$\langle x^2 \rangle = \alpha \int_{-\infty}^{\infty} x^2 e^{-2\alpha |x|} dx$$

$$= \alpha \left\{ \int_{-\infty}^{0} x^2 e^{2\alpha x} dx + \int_{0}^{\infty} x^2 e^{-2\alpha x} dx \right\}$$

$$= \alpha \left\{ \frac{1}{4\alpha^3} + \frac{1}{4\alpha^3} \right\} = \frac{1}{2\alpha^2}$$

$$\langle p \rangle = -i\hbar\,\alpha \int_{-\infty}^{\infty} e^{-\alpha|x|} \cdot \frac{\partial}{\partial x} e^{-\alpha|x|}\, dx$$

$$= -i\hbar\alpha \left\{ \int_{-\infty}^{0} \alpha \cdot e^{2\alpha x} dx + \int_{0}^{\infty} (-\alpha) e^{-2\alpha x} dx \right\}$$

$$= -i\hbar\,\alpha^2 \left\{ \frac{1}{2\alpha} - \frac{1}{2\alpha} \right\} = 0$$

$$\langle p^2 \rangle = \langle \psi | p^2 | \psi \rangle = \int \langle \psi | p | x \rangle \langle x | p | \psi \rangle dx$$

$$= \int \langle x | p | \psi \rangle^* \langle x | p | \psi \rangle dx$$

$$= \alpha \int i\hbar \frac{\partial}{\partial x} e^{-\alpha|x|} \cdot (-i\hbar) \frac{\partial}{\partial x} e^{-\alpha|x|} dx$$

$$= \hbar^2 \alpha \left\{ \int_{-\infty}^{0} \alpha^2 e^{2\alpha x} dx + \int_{0}^{\infty} \alpha^2 e^{-2\alpha x} dx \right\}$$

$$= \hbar^2 \alpha^2$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2}\,\alpha}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \hbar\alpha.$$

$$\Delta x\, \Delta p = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}$$

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3.

the case of particle with mass m

$$v = \frac{p}{m} \Rightarrow \Delta v = \frac{\Delta p}{m}$$

as time goes on wavepacket spreads.

but, the case of massless photon with $E = pc$

$$v = c \Rightarrow \Delta v = 0$$

$$\left(v = \frac{d\omega}{dk} = \frac{d(E/\hbar)}{d(p/\hbar)} = c \right)$$

as time goes on wavepacket does not spread.