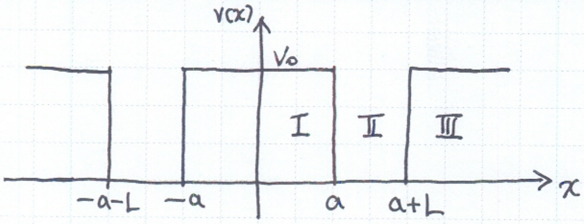
Homework #3 Solution.

1.



$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$$

$V(x)$: even function

$\Rightarrow$ We can create even or odd eigenstates.

(a) I : eigenstate is even

$\psi_I(x) = A\cosh(k_2 x)$ $\qquad k_2 = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

$\psi_{II}(x) = B\sin(k_1 x) + C\cos(k_1 x)$ $\qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$\psi_{III}(x) = De^{-k_2 x}$ $\qquad (e^{k_2 x}$ term is ignored $\because x\to\infty$ then $e^{k_2 x}\to\infty)$

boundary condition (ψ, ψ' : cont.)

$$A\cosh(k_2 a) = B\sin(k_1 a) + C\cos(k_1 a) \quad \text{—— ①}$$

$$Ak_2\sinh(k_2 a) = Bk_1\cos(k_1 a) - Ck_1\sin(k_1 a) \quad \text{—— ②}$$

] at $x=a$

$$B\sin(k_1(a+L)) + C\cos(k_1(a+L)) = De^{-k_2(a+L)} \quad \text{—— ③}$$

$$Bk_1\cos(k_1(a+L)) - Ck_1\sin(k_1(a+L)) = -Dk_2 e^{-k_2(a+L)} \quad \text{—— ④}$$

] at $x=a+L$

②/①

$$k_2\tanh(k_2 a) = \frac{Bk_1\cos(k_1 a) - Ck_1\sin(k_1 a)}{B\sin(k_1 a) + C\cos(k_1 a)}$$

$$C/B = \frac{k_1\cos(k_1 a) - k_2\sin(k_1 a)\tanh(k_2 a)}{k_1\sin(k_1 a) + k_2\cos(k_1 a)\tanh(k_2 a)} \quad \text{—— ⑤}$$

④/③

$$-k_2 = \frac{Bk_1\cos(k_1(a+L)) - Ck_1\sin(k_1(a+L))}{B\sin(k_1(a+L)) + C\cos(k_1(a+L))}$$

$$C/B = \frac{k_1\cos(k_1(a+L)) + k_2\sin(k_1(a+L))}{k_1\sin(k_1(a+L)) - k_2\cos(k_1(a+L))} \quad \text{—— ⑥}$$

⑤ = ⑥

$$\{k_1\cos(k_1 a) - k_2\sin(k_1 a)\tanh(k_2 a)\}\times\{k_1\sin(k_1(a+L)) - k_2\cos(k_1(a+L))\}$$
$$= \{k_1\sin(k_1 a) + k_2\cos(k_1 a)\tanh(k_2 a)\}\times\{k_1\cos(k_1(a+L)) + k_2\sin(k_1(a+L))\}$$

$$k_1^2\{\cos(k_1 a)\sin(k_1(a+L)) - \sin(k_1 a)\cos(k_1(a+L))\}$$
$$+ k_2^2\{\sin(k_1 a)\cos(k_1(a+L)) - \cos(k_1 a)\sin(k_1(a+L))\}\tanh(k_2 a)$$
$$- k_1 k_2\{\cos(k_1 a)\cos(k_1(a+L)) + \sin(k_1 a)\sin(k_1(a+L))\}$$
$$- k_1 k_2\{\sin(k_1 a)\sin(k_1(a+L)) + \cos(k_1 a)\cos(k_1(a+L))\}\tanh(k_2 a) = 0$$

$$k_1^2\sin(k_1 L) - k_2^2\sin(k_1 L)\tanh(k_2 a) - k_1 k_2\cos(k_1 L)(1+\tanh(k_2 a)) = 0$$

$$\therefore \tan(k_1 L) = \frac{k_1 k_2(1+(\tanh(k_2 a)))}{k_1^2 - k_2^2(\tanh(k_2 a))} \quad \text{——} \circledast$$

II. : eigenstate is odd.

$\psi_{I}(x) = A\sinh(k_2 x)$

$\psi_{II}(x) = B\sin(k_1 x) + C\cos(k_1 x)$

$\psi_{III}(x) = De^{-k_2 x}$

$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \frac{d}{dx}\sinh(x) = \cosh(x)$.

$\Rightarrow$ exchange $\cosh \to \sinh$ $\quad \sinh \to \cosh$. from $\circledast$.

$$\therefore \tan(k_1 L) = \frac{k_1 k_2(1+(\tanh(k_2 a))^{-1})}{k_1^2 - k_2^2(\tanh(k_2 a))^{-1}}$$

$$\therefore \tan(k_1 L) = \frac{k_1 k_2(1+(\tanh(k_2 a))^{\pm 1})}{k_1^2 - k_2^2(\tanh(k_2 a))^{\pm 1}} \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}},\ k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

(b) "Two wells get closer to each other" means $a \to 0$.

$$\lim_{a\to 0} \tanh(k_2 a) = \lim_{a\to 0} \frac{e^{k_2 a} - e^{-k_2 a}}{e^{k_2 a} + e^{-k_2 a}} = 0$$

$$\tan(k_1 L) \to \begin{cases} \dfrac{k_1 k_2 (1+0)}{k_1^2 - k_2^2 \cdot 0} = \dfrac{k_2}{k_1} & \text{for even} \\[2ex] \dfrac{k_1 k_2 (0+1)}{k_1^2 \cdot 0 - k_2^2} = -\dfrac{k_1}{k_2} & \text{for odd.} \end{cases}$$

$k_1 \tan(k_1 L) = k_2$ for even

$k_1 \cot(k_1 L) = -k_2$ for odd <u>same</u> with a single well case.

(c) $a \gg 1$

$$\tanh(k_2 a) = \frac{e^{k_2 a} - e^{-k_2 a}}{e^{k_2 a} + e^{-k_2 a}} = \frac{1 - e^{-2k_2 a}}{1 + e^{-2k_2 a}} \simeq 1 - 2e^{-2k_2 a}$$

$$\coth(k_2 a) \simeq 1 + 2e^{-2k_2 a}$$

$$\tan(k_1 L) \simeq \frac{k_1 k_2 (2 \mp 2e^{-2k_2 a})}{k_1^2 - k_2^2 \pm 2k_2^2 e^{-2k_2 a}} = \frac{2k_1 k_2}{k_1^2 - k_2^2} \frac{(1 \mp e^{-2k_2 a})}{\left(1 \pm \frac{2k_2^2}{k_1^2 - k_2^2} e^{-2k_2 a}\right)}$$

$$\simeq \frac{2k_1 k_2}{k_1^2 - k_2^2} (1 \mp e^{-2k_2 a}) \left(1 \mp \frac{2k_2^2}{k_1^2 - k_2^2} e^{-2k_2 a}\right)$$

$$= \frac{2k_1 k_2}{k_1^2 - k_2^2} \left(1 \mp \frac{k_1^2 + k_2^2}{k_1^2 - k_2^2} e^{-2k_2 a}\right)$$

When $a \to \infty$ $\tan(k_1 L) = \dfrac{2k_1 k_2}{k_1^2 - k_2^2}$ ← for even & odd cases.

⇒ No gap!

I failed to obtain the gap between two modes. π−π

2.

(5.3.2). $\psi = c\tilde{\psi}$ $\quad \tilde{\psi}$: real. $\quad c$: const.

$$j = \frac{\hbar}{2mi}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

$$= \frac{\hbar}{2mi}\left(c^*\tilde{\psi}\,\nabla(c\tilde{\psi}) - c\tilde{\psi}\,\nabla(c^*\tilde{\psi})\right)$$

$$= \frac{\hbar}{2mi}|c|^2\left(\tilde{\psi}\nabla\tilde{\psi} - \tilde{\psi}\nabla\tilde{\psi}\right) = 0$$

$j = 0$.

(5.3.4)

$$\psi = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$$

$$j = \frac{\hbar}{2mi}\cdot 2\,\mathrm{Im}\left(\psi^*\nabla\psi\right)$$

$$\psi^*\nabla\psi = \left(A^*e^{-ipx/\hbar} + B^*e^{ipx/\hbar}\right)\left(i\frac{p}{\hbar}\right)\left(Ae^{ipx/\hbar} - Be^{-ipx/\hbar}\right)$$

$$= i\frac{p}{\hbar}\Big(|A|^2 - |B|^2 + \underbrace{AB^*e^{2ipx/\hbar} - A^*Be^{-2ipx/\hbar}}_{\text{pure Imaginary}}\Big)$$

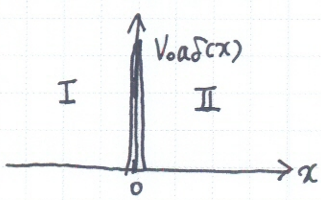
pure Imaginary ($\because$ $a - a^*$: pure imaginary)

$$\therefore j = \frac{\hbar}{2mi}\cdot 2\cdot\left(i\frac{p}{\hbar}\right)\left(|A|^2 - |B|^2\right)$$

$$= \frac{p}{m}\left(|A|^2 - |B|^2\right)$$: no cross term.!!!

(5.4.2)

(a) $V(x) = V_0 a\,\delta(x)$



$\psi_{I}(x) = e^{ikx} + re^{-ikx}$ $\qquad x < 0$

$\psi_{II}(x) = te^{ikx}$ $\qquad x > 0$.

e^{-ikx} : going left (unphysical)

boundary condition.

① continuous at $x=0$.

$$1+r=t \quad \text{——} ①$$

② ψ'

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = (E - V_0 a\,\delta(x))\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2}{dx^2}\psi(x)\,dx = \int_{-\epsilon}^{\epsilon}(E - V_0 a\,\delta(x))\,\psi(x)\,dx \qquad (\epsilon \to 0)$$

$$-\frac{\hbar^2}{2m}\left(\psi'(\epsilon)-\psi'(-\epsilon)\right) = \int_{-\epsilon}^{\epsilon}E\,\psi(x)\,dx - V_0 a\,\psi(0)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\psi'(0^+)-\psi'(0^-)\right) = -V_0 a\,\psi(0)$$

$$-\frac{\hbar^2}{2m}\left\{tik - ik(1-r)\right\} = -V_0 a t \quad \text{——} ②.$$

from ① & ②

$$-\frac{\hbar^2}{2m}\left\{ikt - ik(2-t)\right\} = -V_0 a t$$

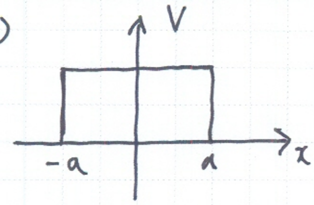
$$-\frac{\hbar^2}{2m}ik\cdot 2\cdot(t-1) = -V_0 a t$$

$$t = \frac{\hbar^2}{m}ik \Big/ \left(\frac{\hbar^2}{m}ik - V_0 a\right) = \frac{\hbar^2 k}{\hbar^2 k + i m V_0 a} \,//$$

$$T = \frac{\hbar^4 k^2}{\hbar^4 k^2 + m^2 V_0^2 a^2}$$

$$R = 1 - T = \frac{m^2 V_0^2 a^2}{\hbar^4 k^2 + m^2 V_0^2 a^2}$$

(b)



$0 < E < V_0$

$$\psi_I(x) = e^{ikx} + re^{-ikx} \qquad x<-a \qquad k=\sqrt{\frac{2mE}{\hbar^2}}$$

$$\psi_{II}(x) = Ae^{Kx} + Be^{-Kx} \qquad -a<x<a \qquad K=\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\psi_{III}(x) = te^{ikx} \qquad x>a$$

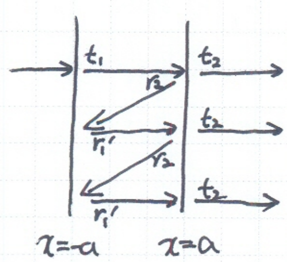
Boundary condition at $x=\pm a$

$$\begin{cases} e^{-ika}+re^{ika} = Ae^{-Ka}+Be^{Ka} \\ ik(e^{-ika}-re^{ika}) = K(Ae^{-Ka}-Be^{Ka}) \end{cases}$$

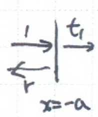
$$\begin{cases} Ae^{Ka}+Be^{-Ka} = te^{ika} \\ K(Ae^{Ka}-Be^{-Ka}) = ikte^{ika} \end{cases}$$

You can solve!

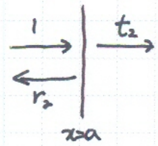
Another Method.



$$t = t_1t_2 + t_1r_2r_1't_2 + t_1r_2r_1'r_2r_1't_2 + \cdots$$
$$= t_1t_2(1+r_2r_1'+(r_2r_1')^2+\cdots)$$
$$= \frac{t_1t_2}{1-r_2r_1'}$$



$$e^{ik(-a)} + re^{-ik(-a)} = t_1e^{K(-a)}$$
$$ik(e^{ik(-a)}-re^{-ik(-a)}) = Kt_1e^{K(-a)}$$
$$(K+ik)t_1e^{-Ka} = 2ike^{-ika} \qquad \therefore t_1 = \frac{2ik}{K+ik}e^{-ika}e^{Ka}$$

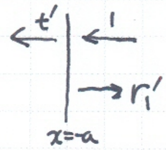


$$e^{Ka}+r_2e^{-Ka}=t_2e^{ika}$$

$$K(e^{Ka}-r_2e^{-Ka})=ikt_2e^{ika}$$

$$t_2=\frac{2K}{K+ik}e^{-ika}e^{Ka}$$

$$r_2=\frac{K-ik}{K+ik}e^{2Ka}$$



$$e^{-K(-a)}+r_1'e^{K(-a)}=t'e^{-ik(-a)}$$

$$K(-e^{-K(-a)}+r_1'e^{K(-a)})=-ikt'e^{-ik(-a)}$$

$$r_1'=\frac{K-ik}{K+ik}e^{2Ka}$$

$$\therefore t=\frac{1}{1-\left(\frac{K-ik}{K+ik}\right)^2e^{4Ka}}\times\left(\frac{4ikK}{(K+ik)^2}\right)e^{-2ika}e^{2Ka}$$

$$=\frac{4ikKe^{-2ika}}{(K+ik)^2e^{-2Ka}-(K-ik)^2e^{2Ka}}$$

$$=\frac{4ikKe^{-2ika}}{-(K^2-k^2)\cdot 2\cdot\sinh(2Ka)+4ikK\cosh(2Ka)}$$

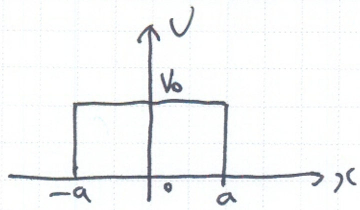
$$=\frac{2ikKe^{-2ika}}{(k^2-K^2)\sinh(2Ka)+2ikK\cosh(2Ka)}$$

$$T=|t|^2=\frac{4k^2K^2}{(k^2-K^2)^2\sinh^2(2Ka)+4k^2K^2\cosh^2(2Ka)}\qquad(\cosh^2x=1+\sinh^2x)$$

$$=\frac{4k^2K^2}{4k^2K^2+(k^2+K^2)^2\sinh^2(2Ka)}\qquad k=\sqrt{\frac{2mE}{\hbar^2}}\quad K=\sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$R=1-T=\frac{(k^2+K^2)^2\sinh^2(2Ka)}{4k^2K^2+(k^2+K^2)^2\sinh^2(2Ka)}$$

3.



$E > V_0$

from 5.4.2 (b) exchange $k \to ik'$ $\sinh(ik') = i\sin(k')$

$$T = \frac{-4k^2k'^2}{-4k^2k'^2 + (k^2-k'^2)^2(-\sin^2(2k'a))}$$

$$= \frac{4k^2k'^2}{4k^2k'^2 + (k^2-k'^2)^2\sin^2(2k'a)}$$

$$R = \frac{(k^2-k'^2)^2\sin^2(2k'a)}{4k^2k'^2 + (k^2-k'^2)^2\sin(2k'a)}$$

4.

(a) $$\begin{bmatrix} C \\ B \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} A \\ D \end{bmatrix}$$

→ concept of probability current. (Current conservation)

from (5.3.4)

$$j_{left} = \frac{\hbar k}{m}(|A|^2 - |B|^2)$$

$$j_{right} = \frac{\hbar k}{m}(|C|^2 - |D|^2)$$

$j_{left} = j_{right}$ $\Rightarrow$ $|A|^2 - |B|^2 = |C|^2 - |D|^2$

$\therefore |A|^2 + |D|^2 = |B|^2 + |C|^2$

$$\begin{bmatrix} C^* & B^* \end{bmatrix}\begin{bmatrix} C \\ B \end{bmatrix} = \begin{bmatrix} A^* & D^* \end{bmatrix}\begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} A \\ D \end{bmatrix} = \begin{bmatrix} A^* & D^* \end{bmatrix}\begin{bmatrix} A \\ D \end{bmatrix}$$

$\Rightarrow$ for arbitrary A, D.

$$\therefore \begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = I.$$

$S^\dagger S = I \Rightarrow S$: unitary matrix.

(b)

(i) $S_{11} = \dfrac{\hbar^2 k}{\hbar^2 k + imV_0 a}$ $\quad S_{12} = S_{11} - 1 = \dfrac{-imV_0 a}{\hbar^2 k + imV_0 a}$

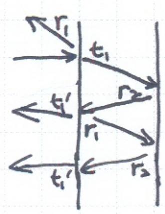
We can obtain S_{22} & S_{21} from exchanging $k \to -k$.

$S_{22} = \dfrac{\hbar^2 k}{-\hbar^2 k + imV_0 a}$ $\quad S_{21} = \dfrac{-imV_0 a}{-\hbar^2 k + imV_0 a}$

then we can see $S^\dagger S = I$ by simple calculation.

(ii) $$S_{11} = \frac{2ikK e^{-2ika}}{(k^2 - K^2)\sinh(2Ka) + 2ikK\cosh(2Ka)}$$

$$S_{12} = r_1 + t_1 r_2 t_1' + t_1 r_2 r_1' r_2 t_1' + \cdots$$



$$= r_1 + t_1 r_2 t_1' \left(1 + r_1' r_2 + (r_1' r_2)^2 + \cdots\right)$$

$$= r_1 + \frac{t_1 r_2 t_1'}{1 - r_1' r_2}$$

$$= \frac{(k^2 - K^2)\sinh(2Ka)\, e^{-2ika}}{(k^2 - K^2)\sinh(2Ka) + 2ikK\cosh(2Ka)}$$

We can obtain S_{22} & S_{21} from exchanging $k \to -k$, $K \to -K$.

$$S_{22} = \frac{2ikKe^{-2ika}}{-(k^2-K^2)\sinh(2Ka) + 2ikK\cosh(2Ka)}$$

$$S_{21} = \frac{-(k^2-K^2)\sinh(2Ka)e^{-2ika}}{-(k^2-K^2)\sinh(2Ka) + 2ikK\cosh(2Ka)}$$

$\Rightarrow$ $S^{+}S = I$ ...

(iii) from (ii) $K \to ik'$

$S^{+}S = I$.