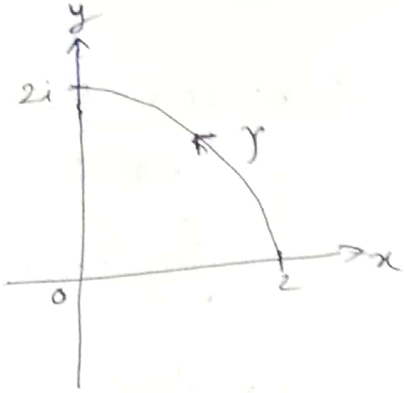
Mathematics-III

Assignment-4

Q.1. Use the ML-inequality to prove

(a) $\left|\int_{\gamma} \frac{1}{1+z^2}\,dz\right| \le \frac{\pi}{3}$, γ is the arc of $|z|=2$ from 2 to $2i$.

(b) $\left|\int_{|z|=R} \frac{\operatorname{Log} z}{z^2}\,dz\right| \le 2\pi\left(\frac{\pi+\ln R}{R}\right)$, $R>0$.



Solution (a) $|1+z^2| \ge \big|1-|z^2|\big| = \big|1-|z|^2\big|$

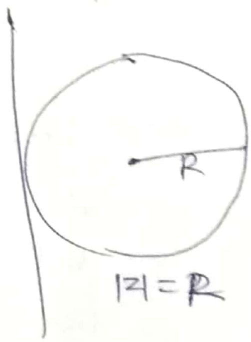
$= |1-2^2| = 3.$

Hence $\left|\frac{1}{1+z^2}\right| = \frac{1}{|1+z^2|} \le \frac{1}{3} = M.$

Length of $\gamma = L = \frac{2\pi \cdot 2}{4} = \pi.$

Therefore, $\left|\int_{\gamma}\frac{1}{1+z^2}\,dz\right| \le \int_{\gamma}\left|\frac{1}{1+z^2}\right| dz \le ML = \frac{\pi}{3}$

(b) $\left|\frac{\operatorname{Log} z}{z^2}\right| = \frac{|\operatorname{Log} z|}{|z|^2} = \frac{|\log|z| + i\operatorname{Arg} z|}{R^2}$ as $|z|=R$



$= \frac{|\log R + i\operatorname{Arg} z|}{R^2}$

$\le \frac{|\log R| + |i\operatorname{Arg} z|}{R^2}$

$\le \frac{|\log R| + |\operatorname{Arg} z|}{R^2}$

$\le \frac{\log R + \pi}{R^2} = M$ as $|\operatorname{Arg} z| \le \pi$.

$L = \cancel{2\pi(2) = 4\pi}.\ 2\pi R$

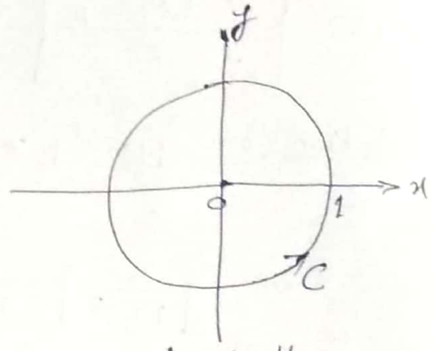
Hence $\left|\int_{|z|=R}\frac{\operatorname{Log} z}{z^2}\,dz\right| \le ML = \left(\frac{\log R+\pi}{R^2}\right)2\pi R$

$= 2\pi\left(\frac{\pi+\log R}{R}\right).$

— x —

Q.2. Apply the Cauchy-Goursat theorem to evaluate $\oint_C f(z)\,dz$, where the contour C is the unit circle $|z|=1$ in either direction and

(a) $f(z) = \dfrac{z^2}{(z-3)(z-5)}$ (b) $f(z) = \tan z$ (c) $f(z) = \log(z+2)$



Soln: (a) $f(z)$ is analytic on and inside C. Hence

$$\oint_C \frac{z^2}{(z-3)(z-5)}\,dz = 0.$$

(b) $\tan z$ is analytic everywhere except at the singularity $z = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \cdots$, but all these points are outside $|z|=1$. Hence $\tan z$ is analytic on C and inside C.

Hence $\oint_C \tan z\,dz = 0$.

(c) $\text{Log}(z+2)$ has singularity at $\text{Re}(z+2) \leq 0$ i.e. at $\text{Re}(z)+2 \leq 0$ i.e. at all z such that $\text{Re}(z) \leq -2$, but all such points will lie outside $C: |z|=1$.

Hence $\text{Log}(z+2)$ is analytic on and inside C.

Therefore, $\oint_C \text{Log}(z+2)\,dz = 0$.

Q.3. Let γ denote the positively oriented boundary of the square whose sides lie on the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the following integrals

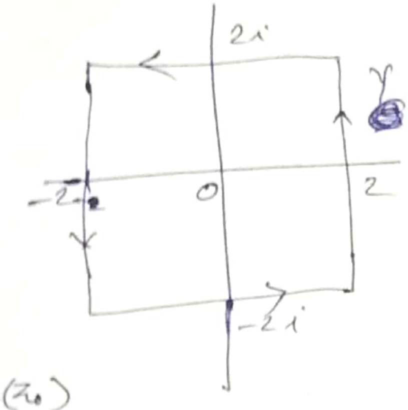
(a) $\displaystyle\int_\gamma \frac{e^{-z}}{z - i\frac{\pi}{2}}\,dz$ (b) $\displaystyle\int_\gamma \frac{\cos z}{z(z^2+8)}\,dz$ (c) $\displaystyle\int_\gamma \frac{z}{2z+1}\,dz.$

Soln:

(a) Let $f(z) = e^{-z}$

$z_0 = \frac{i\pi}{2}$.

Then $f(z)$ has singularity only at $z_0 = \frac{i\pi}{2}$ inside γ.



Hence

$$\int_\gamma \frac{e^{-z}}{z - \frac{i\pi}{2}}\,dz = \int \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

$$= 2\pi i\, e^{-z_0}$$

$$= 2\pi i\, e^{-\frac{i\pi}{2}} = 2\pi i(-i) = 2\pi.$$

(b) $$\int_\gamma \frac{\cos z}{z(z^2+8)}\,dz = \int_\gamma \frac{(\cos z)/(z^2+8)}{z-0}\,dz = 2\pi i\left[\frac{\cos z}{z^2+8}\right]_{z=0}$$

$$= 2\pi i\left(\frac{1}{8}\right) = \frac{i\pi}{4}.$$

(c) $$\int_\gamma \frac{z}{2z+1}\,dz = \frac{1}{2}\int_\gamma \frac{z}{z-(-\frac{1}{2})}\,dz = 2\pi i \frac{1}{2}\,[z]_{z=-\frac{1}{2}} = \pi i\left(\frac{-1}{2}\right) = \frac{-\pi i}{2}.$$

Q.4. Evaluate the integral $\displaystyle\int_C (z-z_0)^n\,dz$, $n = 0, \pm 1, \pm 2, \ldots$ where C denote the positively oriented circle $|z - z_0| = R$.

Soln: Use the parametrization $C: z(t) = z_0 + Re^{it}, 0 \le t \le 2\pi$ and show that

$$\int_C (z-z_0)^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1. \end{cases}$$

Q.5. Evaluate $\frac{1}{2\pi i}\int_C \frac{ze^z}{(z+1)^3}dz$, where C is a positively oriented simple closed curve enclosing $z=-1$.

Soln: Let $f(z) = ze^z$ and $z_0 = -1$. Then the above integral is

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3} = \frac{1}{2\pi i}\cdot\frac{2\pi i}{2} f^{(2)}(z_0) = \frac{1}{2}f^{(2)}(-1).$$

$$f(z) = ze^z \Rightarrow f'(z) = ze^z + e^z = e^z(z+1)$$

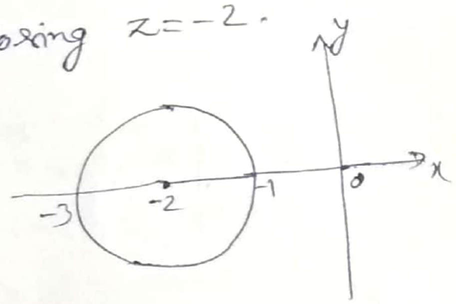
$$\Rightarrow f^{(2)}(z) = e^z + e^z(z+1) = e^z(z+2)$$

$$\Rightarrow f^{(2)}(-1) = e^{-1}(-1+2) = e^{-1}.$$

Hence $\frac{1}{2\pi i}\int_C \frac{ze^z}{(z+1)^3} = \frac{1}{2}e^{-1} = \frac{1}{2e}.$

Q.6. Evaluate $\frac{1}{2\pi i}\oint_C \frac{3z-1}{(z^3+2z^2)}dz$, where C is a positively oriented unit circle enclosing $z=-2$.

Soln: Take $f(z) = \frac{3z-1}{z}$ and $z_0 = -2$, and solve similar to Q. 5.



Q.7. Find the Taylor series expansion of

(a) $f(z) = \frac{6z+8}{(2z+3)(4z+5)}$ at $z=1$

(b) $f(z) = \frac{2i}{4+iz}$ at $z=-3i$

Soln: (a) Note that $\frac{6z+8}{(2z+3)(4z+5)} = \frac{1}{2z+3} + \frac{1}{4z+5}$

$$= \frac{1}{2(z-1)+3+2} + \frac{1}{4(z-1)+5+4} = \frac{1}{5}\cdot\frac{1}{1+\frac{2(z-1)}{5}} + \frac{1}{9}\cdot\frac{1}{1+\frac{4(z-1)}{9}}$$

$$= \frac{1}{5}\cdot\frac{1}{1-\left\{\frac{-2(z-1)}{5}\right\}} + \frac{1}{9}\cdot\frac{1}{1-\left\{\frac{-4(z-1)}{9}\right\}}$$

$$= \frac{1}{5}\left[1 - \frac{2(z-1)}{5} + \frac{2^2(z-1)^2}{5^2} - \frac{2^3(z-1)^3}{5^3} + \cdots\right]$$

$$+ \frac{1}{9}\left[1 - \frac{4(z-1)}{9} + \frac{4^2(z-1)^2}{9^2} - \frac{4^3(z-1)^3}{9^3} + \cdots\right]$$

Now simplify it.

(b) It is similar.

Q.8. Find the Laurant series expansions for the following functions aroung $z=0$.

(a) $f(z) = (z-3)^{-1}$ for $|z|>3$ (b) $f(z) = (z(z-1))^{-1}$ for $0<|z|<1$

(c) $f(z) = z^3 e^{1/z}$ for $|z|>0$.

Soln: (a) $f(z) = \dfrac{1}{z-3} = \dfrac{1}{z}\cdot\dfrac{1}{1-3/z}$.

Since $\left|\frac{3}{z}\right| = \frac{3}{|z|} < 1$, we have

$$f(z) = \frac{1}{z-3} = \frac{1}{z}\cdot\frac{1}{1-\frac{3}{z}} = \frac{1}{z}\left[1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots\right].$$

$$= \frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \frac{3^3}{z^4} + \cdots$$

(b) $$f(z) = \frac{1}{z(z-1)} = \frac{1}{z} - \frac{1}{z-1} = \frac{1}{z} + \frac{1}{1-z}$$

$$= \frac{1}{z} + 1 + z + z^2 + \cdots \quad \text{as } |z|<1$$

(c) $$f(z) = z^3 e^{\frac{1}{z}} = z^3\left[1+\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\cdots\right]$$

$$= z^3 + z^2 + \frac{1}{2!}z + \frac{1}{3!} + \frac{1}{4!}\cdot\frac{1}{z} + \cdots$$

Q.9. Find all possible Taylor and Laurant series expansions for the function $\dfrac{2z+3}{(z+1)(z+2)}$ around $z=1$ and specify the domain of validity.

Soln: $$\frac{2z+3}{(z+1)(z+2)} = \frac{1}{z+1} + \frac{1}{z+2}$$

$$\frac{1}{z+1} = \frac{1}{1-(-z)} = 1+(-z)+(-z)^2+\cdots \quad \text{if } |-z|<1$$
$$= 1-z+z^2-z^3+\cdots \quad \text{if } |z|<1.$$

$$\frac{1}{z+1} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\cdot\frac{1}{1-(\frac{-1}{z})} \quad \text{if } 0<|z| \text{ and } |\tfrac{-1}{z}|<1$$
$$= \frac{1}{z}\cdot\left[1+\left(\frac{-1}{z}\right)+\left(\frac{-1}{z}\right)^2+\left(\frac{-1}{z}\right)^3+\cdots\right]$$
if $0<|z|$ and $|z|>1$
$$= \frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right]$$
$$= \frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots \quad \text{if } |z|>1.$$

$$\frac{1}{z+2} = \frac{1}{2}\cdot\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\cdot\frac{1}{1-(\frac{-z}{2})}$$
$$= \frac{1}{2}\cdot\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right] \quad \text{if } \left|\frac{-z}{2}\right|<1$$
i.e. $|z|<2$

$$\frac{1}{z+2} = \frac{1}{z}\cdot\frac{1}{1+\frac{2}{z}} = \frac{1}{z}\cdot\frac{1}{1-(\frac{-2}{z})}$$
$$= \frac{1}{z}\left[1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right] \quad \text{if } \left|\frac{-2}{z}\right|<1$$
i.e. $|z|>2$.

Thus,
$$\frac{2z+3}{(z+1)(z+2)} = \frac{1}{z+1}+\frac{1}{z+2}$$
$$= (1-z+z^2-z^3+\cdots)+\frac{1}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right)$$
valid for $|z|<1$.

Now simplify.

$$\frac{2z+3}{(z+1)(z+2)} = \frac{1}{z+1}+\frac{1}{z+2}$$
$$= \left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right)+\frac{1}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right)$$
valid for $1<|z|<2$.

$$\frac{2z+3}{(z+1)(z+2)} = \frac{1}{z+1}+\frac{1}{z+2}$$
$$= \left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right)+\frac{1}{z}\left[1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right]$$
valid for $|z|>2$.

Q.10. For each function given below determine its isolated singular point and wheather that point is a pole, a removable singular point, or an essential singular point.

(a) $z^2 \exp(\frac{1}{z})$ (b) $\frac{\sin z}{\pi z}$ (c) $\frac{1-\cos z}{2z^2}$ (d) $\sec\frac{1}{z}$ (e) $\sin\frac{\pi}{z}$

Soln: (a) $z=0$.

$$z^2 \exp\left(\frac{1}{z}\right) = z^2\left(1+\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\cdots\right)$$

$$= z^2+z+\frac{1}{2!}+\frac{1}{3!}\cdot\frac{1}{z}+\cdots$$

$z=0$ is essential singularity.

(b) $z=0$ is only singularity and hence isolated.

$$\frac{\sin z}{\pi z} = \frac{1}{\pi z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)$$

$$= \frac{1}{\pi}\left[1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right]$$

Thus $z=0$ is a removable singularity.

(c) $$\frac{1-\cos z}{2z^2} = \frac{1}{2z^2}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)\right], \quad 0<|z|<\infty$$

$$= \frac{1}{2}\left[\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots\right]$$

Hence $z=0$ is an isolated singularity which is removable. When the value $f(0)=\frac{1}{2}$ is assigned, f becomes entire.

(d) $\sec\frac{1}{z}$

$z=0$ and z given by $\frac{1}{z}=(2n+1)\frac{\pi}{2}$, i.e. $z=\frac{2}{(2n+1)\pi}$, $n=0,\pm1,\pm2,\ldots$ are ~~single~~ singular

Each singular point $z=\frac{2}{(2n+1)\pi}$ is isolated singular points.

Q.11. Find the residue at $z=0$ of the following functions and indicate the type of singularity they have at $z=0$:

(a) $\frac{1}{z+z^2}$ (b) $z \cos \frac{1}{z}$ (c) $\frac{z-\sin z}{z}$ (d) $\frac{\cot z}{z^4}$

Soln: (a) Write

$$\frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{1+z} = \frac{1}{z}(1-z+z^2-z^3+\cdots), \quad 0<|z|<1$$

$$= \frac{1}{z} - 1 + z - z^2 + \cdots, \quad 0<|z|<1$$

Thus $z=0$ is a pole of order 1 i.e. simple pole.

$\underset{z=0}{\text{Res}}\left(\frac{1}{z+z^2}\right) = \text{coefficient of } \frac{1}{z} = 1.$

(b) $$z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^4} - \frac{1}{6!}\cdot\frac{1}{z^6} + \cdots\right)$$

$$= z - \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^3} - \frac{1}{6!}\cdot\frac{1}{z^5} + \cdots \quad (0<|z|<\infty)$$

Hence $z=0$ is an essential singularity and

$\underset{z=0}{\text{Res}}\left(z\cos\left(\frac{1}{z}\right)\right) = -\frac{1}{2!} = -\frac{1}{2}.$

(c) $$\frac{z-\sin z}{z} = \frac{1}{z}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right] = \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots,$$

$(0<|z|<\infty)$

Hence $z=0$ is a removable singularity and

$\underset{z=0}{\text{Res}}\left(\frac{z-\sin z}{z}\right) = 0.$

(d) For $0<|z|<\pi$, we have

$$\frac{\cot z}{z^4} = \frac{\cos z}{z^4 \sin z} = \frac{1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots}{z^4\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)}.$$

Now let

$$w = \frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}-\cdots.$$

Then for $z\neq 0$ such that $|w|<1$, we have

$$\frac{\cot z}{z^4} = \frac{1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots}{z^5(1-w)}$$

$$= \frac{1}{z^5}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)(1+w+w^2+w^3+\cdots).$$

Therefore, for $0<|z|<\pi$, we have

$$\frac{\cot z}{z^4} = \frac{1}{z^5}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)\left(1+\frac{z^2}{3!}+\left(\frac{1}{(3!)^2}-\frac{1}{5!}\right)z^4+\cdots\right),$$

that is,

$$\frac{\cot z}{z^4} = \frac{1}{z^5}-\left(\frac{1}{2!}-\frac{1}{3!}\right)\frac{1}{z^3}+\left[\frac{1}{(3!)^2}-\frac{1}{5!}+\frac{1}{4!}-\frac{1}{2!\,3!}\right]\frac{1}{z}+\cdots$$

for $0<|z|<w$, and

$$\operatorname*{Res}_{z=0}\left(\frac{\cot z}{z^4}\right) = \frac{1}{(3!)^2}-\frac{1}{5!}+\frac{1}{4!}-\frac{1}{2!\,3!} = -\frac{1}{45}.$$

Q.12. Use Cauchy's residue theorem to evaluate the integral of each of the following functions around the circle $|z|=\pi$ (Counterclockwise).

(a) $\frac{e^{-z}}{z^4}$ (b) $\frac{\cos z}{(z-1)^2}$ (c) $z^2 \sin\frac{1}{z}$.

Soln: (a) The function $\frac{e^{-z}}{z^2}$ has an isolated singularity at $z=0$ which is inside the circle $|z|=\pi$, and since

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2}\left\{1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\cdots\right\} = \frac{1}{z^2}-\frac{1}{z}+\frac{1}{2!}-\frac{z}{3!}+\cdots$$

for $0<|z|<\infty$, then

$$\operatorname*{Res}_{z=0}\left(\frac{e^{-z}}{z^2}\right) = -1.$$

Now, since $\frac{e^{-z}}{z^2}$ is analytic inside and on $|z|=\pi$, except at $z=0$, then

$$\oint_{|z|=\pi} \frac{e^{-z}}{z^2}\,dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{e^{-z}}{z^2}\right) = -2\pi i.$$

(b) $\cos z = \cos(z-1+1) = \cos(z-1)\cos 1 - \sin(z-1)\sin 1$

Therefore $\frac{\cos z}{(z-1)^2} = \frac{\cos 1 \cdot \cos(z-1)}{(z-1)^2} - \frac{\sin 1 \cdot \sin(z-1)}{(z-1)^2}$

$$= \frac{\cos 1}{(z-1)^2}\left[1 - \frac{(z-1)^2}{2!} + \frac{(z-1)^4}{4!} - \cdots\right] - \frac{\sin 1}{(z-1)^2}\left[(z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} \cdots\right]$$

$$= \left[\frac{\cos 1}{(z-1)^2} - \frac{(\cos 1)\cancel{(z-1)^2}}{2!} + \frac{\cos 1}{4!}(z-1)^2 - \cdots\right]$$
$$- \frac{\sin 1}{z-1} + \frac{\sin 1}{3!}(z-1) - \frac{\sin 1}{5!}(z-1)^3 + \cdots$$

$$= \frac{\cos 1}{(z-1)^2} - \frac{\sin 1}{z-1} - \frac{\cos 1}{2!} + \frac{\sin 1}{3!}(z-1) + \frac{\cos 1}{4!}(z-1)^2 + \cdots,$$
$$0<|z-1|<\infty$$

Clearly $\frac{\cos z}{(z-1)^2}$ has a ~~simple pole at z=1~~ pole of order 2 at $z=1$, ~~hence~~ which lies inside $|z|=\pi$,

and $\operatorname*{Res}_{z=1}\left(\frac{\cos z}{(z-1)^2}\right) = -\sin 1.$

Therefore. $\oint_{|z|=\pi} \frac{\cos z}{(z-1)^2}\,dz = 2\pi i \operatorname*{Res}_{z=1}\left(\frac{\cos z}{(z-1)^2}\right) = 2\pi i(-\sin 1)$
$= -2\pi i(\sin 1).$

(c) $z^2 \sin\frac{1}{z} = z^2\left(\frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \cdots\right)$, $0<|z|<\infty$

$= z - \frac{1}{3!\,z} + \frac{1}{5!\,z^3} - \cdots$, $0<|z|<\infty$

Clearly $z=0$ is a simple pole which lies inside $|z|=\pi$ and $\operatorname*{Res}_{z=0}\left(z^2\sin\frac{1}{z}\right) = -\frac{1}{3!} = -\frac{1}{6}.$

Therefore $\oint_{|z|=\pi} z^2 \sin\frac{1}{z}\,dz = 2\pi i \operatorname*{Res}_{z=0}\left(z^2\sin\frac{1}{z}\right) = 2\pi i\left(\frac{-1}{6}\right)$
$= \frac{-\pi i}{3}.$

Q.13 Evaluate the integral

(a) $\int_0^\infty \frac{dx}{1+x^4}$ (b) $\int_0^\infty \frac{x^2\,dx}{(x^2+1)^2}$

Soln: Theorem Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} f(z)\, e^{i\alpha z}\, dz \to 0$ as $R \to \infty$.

(a) The conditions of above theorem are satisfied. We know that $f(z) = \frac{1}{z^4+1}$ has simple poles in the upper half plane at $z_1 = e^{i\pi/4}$ and $z_2 = e^{i3\pi/4}$.

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}e^{-\frac{3\pi i}{4}} = -\frac{1}{4}e^{i\pi/4}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}e^{-i\pi/4}.$$

Hence $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx$ as $\frac{1}{1+x^4}$ is even,

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i \left[\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z)\right]$$

$$= \frac{2\pi i}{4}\left[-e^{i\pi/4} + e^{-i\pi/4}\right]$$

$$= -\frac{2\pi i}{4} \cdot 2i \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

(b) The conditions of above theorem are satisfied. Clearly the function $f(z) = \frac{z^2}{(z^2+1)^2}$ has pole of order 2 in the upper half plane at $z = i$.

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z)$$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{z^2}{(z^2+1)^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{z^2}{(z-i)^2(z+i)^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right]$$

$$= \lim_{z\to i} \left[\frac{(z+i)^2 \cdot 2z - z^2 \cdot 2(z+i)}{(z+i)^4}\right]$$

$$= \frac{4i^2 \cdot 2i - (-1)\,2(2i)}{16}$$

$$= \frac{-8i+4i}{16} = \frac{-4i}{16} = \frac{-i}{4}.$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2} = \text{P.V.} \int \frac{x^2}{(x^2+1)^2}\,dx$ as $\dfrac{x^2}{(x^2+1)^2}$ is even function.

$$= 2\pi i \operatorname*{Res}_{z=i} f(z)$$

$$= 2\pi i\left(\frac{-i}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}.$$

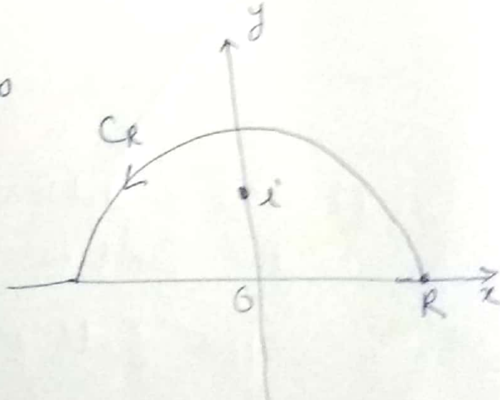
Therefore,

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{\pi}{4}.$$

Q.14 Show that

(a) $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}e^{-a} \quad (a \geq 0)$. (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+a^2)}\,dx = \pi e^{-a}$.

Soln: The function $f(z) = \frac{1}{z^2+1}$ has the singularities at $\pm i$; and so we may integrate around the closed contour shown here, where $R > 1$.



We start with

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}\,dx + \int_{C_R} f(z)\,e^{iaz} = 2\pi i B,$$

where

$$B = \operatorname*{Res}_{z=i}\left[f(z)\,e^{iaz}\right] = \left.\frac{e^{iaz}}{z+i}\right]_{z=i} = \frac{e^{-a}}{2i}$$

Hence

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a} - \int_{C_R} f(z)\,e^{iaz}\,dz$$

or

$$\int_{-R}^{R} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} - \operatorname{Re}\left(\int_{C_R} f(z)\,e^{iaz}\,dz\right)$$

Since $|f(z)| \le M_R$, where $M_R = \frac{1}{R^2-1}$,

we know that

$$\left|\operatorname{Re}\left(\int_{C_R} f(z)\,e^{iaz}\,dz\right)\right| \le \left|\int_{C_R} f(z)\,e^{iaz}\,dz\right| \le \frac{\pi R}{R^2-1}$$

$\longrightarrow 0$ as $R \to \infty$

Therefore

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} = \pi e^{-a}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\cos ax}{x^2+1} = \frac{\pi}{2} e^{-a}.$$

(b) Similar to (a).
Here poles of $\frac{1}{z^2+a^2}$ are at $\pm ai$, so that we can take the closed contour C consisting of the upper half ~~plane~~ C_R of the large circle $|z|=R$ and real axis from $-R$ to R. This R should be chosen in such a way that the pole ~~will~~ in upper half plane will line inside C.