Drivers in the Case of Two Assets

Future Returns Are Random In Nature

Optimizing Portfolio requires expectations:

- about average portfolio return (mean)
- about how far off it may be (variance)

Why?

Portfolio Return Is A Random Variable

Past Performance to Predictions

	Mean Portfolio Return
Computed on a sample of T Historical Returns	$\hat{\mu} = \frac{R_1 + R_2 + \ldots + R_T}{T}$
When the return is a random variable	$\mu = E[R]$

	Portfolio Return Variance
Computed on a sample of T Historical Returns	$\hat{\sigma}^2 = \frac{(R_1 - \hat{\mu})^2 + (R_2 - \hat{\mu})^2 + \ldots + (R_T - \hat{\mu})^2}{T - 1}$
When the return is a random variable	$\sigma^2 = E[(R - \mu)^2]$

Drivers of Mean & Variance

- Assume two assets:

Asset 1	Asset 2
Weight: w_1	Weight: w_2
Return: R_1	Return: R_2

- **Portfolio Return** $P = w_1 * R_1 + w_2 * R_2$
- Thus: $\boxed{E[P] = w_1 * E[R_1] + w_2 * E[R_2]}$

Portfolio Return Variance

Again, for a portfolio with 2 assets

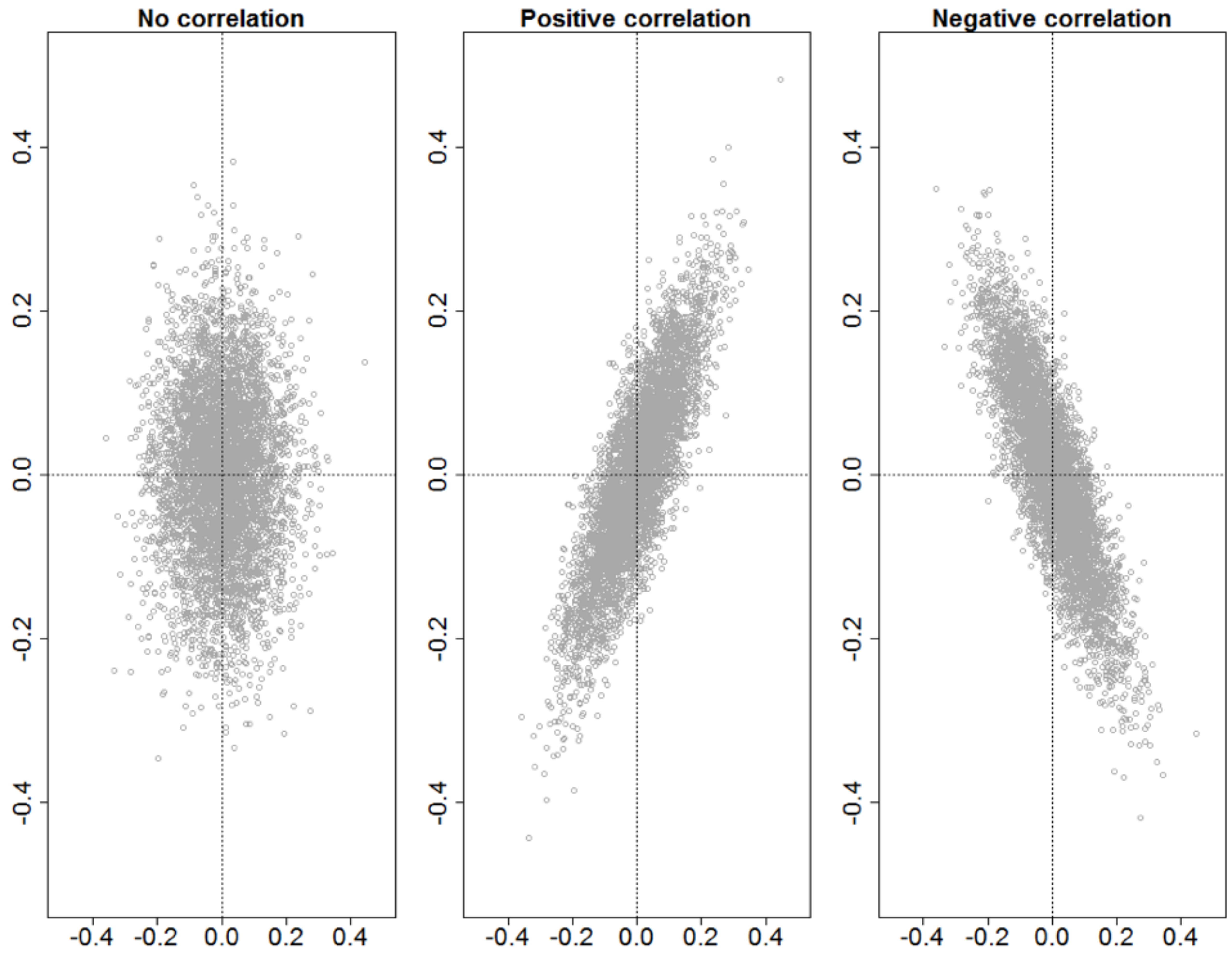
Variance of Portfolio Return

$$
\begin{aligned}
var(P) = \boxed{E[(P - E[P])^2]} &= w_1^2 * var(R_1) \\
&+ w_2 * var(R_2) \\
&+ 2 * w_1 * w_2 * \boxed{cov(R_1, R_2)}
\end{aligned}
$$

Covariance between return 1 and 2

$$
\begin{aligned}
Cov(R_1, R_2) &= \boxed{E[(R_1 - E[R_1])(R_2 - E(R_2))]} \\
&= StdDev(R_1) * StdDev(R_2) * corr(R_1, R_2)
\end{aligned}
$$

Correlations



Take Away Formulas

- E[Portfolio Return] = $E(P) = w_1 * E[R_1] + w_2 * E[R_2]$
- var(Portfolio Return) = $var(P) = w_1^2 * var(R_1)$ $+ w_2^2 * var(R_1)$ $+ 2 * w_1 * w_2 * cov(R_1, R_2)$

Let’s practice!

Using Matrix Notation

Variables at Stake for N Assets

- w: the N x 1 column-matrix of portfolio weights

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

- R: the N x 1 column-matrix of asset returns

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix}$$

- μ: the N x 1 column-matrix of expected returns

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

Variables at Stake for N Assets

- Σ: The N x N covariance matrix of the N asset returns:

σ_i^2 σ_{ij}

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{bmatrix}$$

Covariance: Outside Diagonal
Variance: On Diagonal

Generalizing from 2 to N Assets

Portfolio Return

$$\boxed{w_1 * R_1 + w_2 * R_2} \Rightarrow \boxed{w_1 * R_1 + \ldots + w_N * R_N}$$

Portfolio Expected Return

$$\boxed{w_1 * \mu_1 + w_2 * \mu_2} \Rightarrow \boxed{w_1 * \mu_1 + \ldots + w_N * \mu_N}$$

Portfolio Variance

$$\boxed{\begin{array}{c} w_1^2 * var(R_1) + w_2^2 * var(R_2) \\ +2 * w_1 * w_2 * cov(R_1, R_2) \end{array}} \Rightarrow \boxed{\begin{array}{l} w_1^2 + var(R_1) + \ldots + w_N^2 * var(R_N) \\ +2 * w_1 * w_2 * cov(R_1, R_2) + \ldots \\ +2 * w_{N-1} * w_N * cov(R_{N-1}, R_N) \end{array}}$$

Matrices Simplify the Notation

- Avoid large number of terms by using matrix notation
- We have 4 matrices:
  - weights (w), returns (R), expected returns (μ), and covariance matrix (Σ)

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \qquad w' = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix}$$

Simplifying the Notation

Portfolio Return

$$\boxed{w_1 * R_1 + \ldots + w_N * R_N} \Rightarrow \boxed{w'R}$$

Portfolio Expected Return

$$\boxed{w_1 * \mu_1 + \ldots + w_N * \mu_N} \Rightarrow \boxed{w'\mu}$$

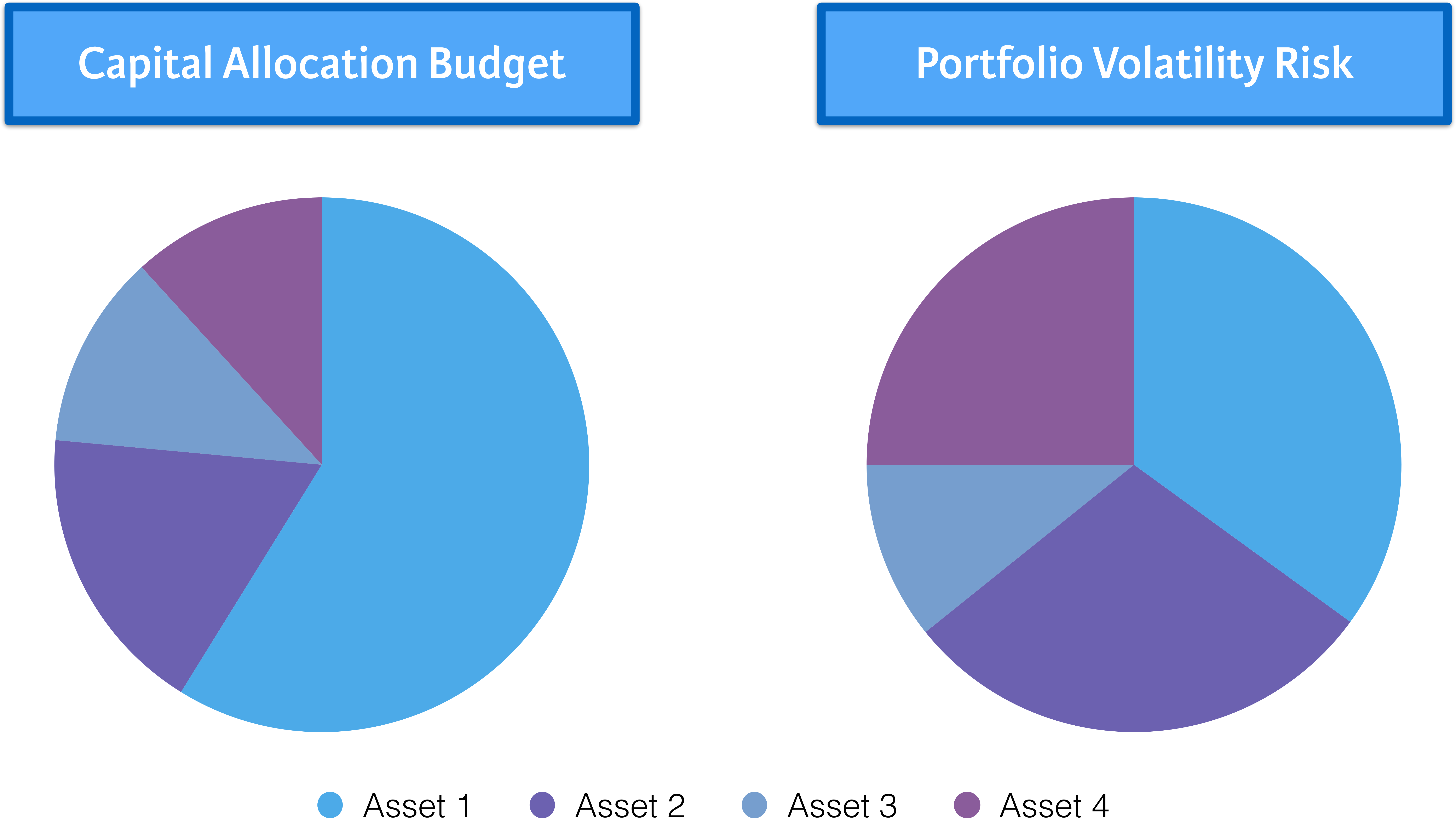
Portfolio Variance

$$\boxed{\begin{array}{l} w_1^2 + var(R_1) + \ldots + w_N^2 * var(R_N) \\ +2 * w_1 * w_2 * cov(R_1, R_2) + \ldots \\ +2 * w_{N-1} * w_N * cov(R_{N-1}, R_N) \end{array}} \Rightarrow \boxed{w'\Sigma w}$$

Let's practice!

Portfolio Risk Budget

Who Did It?


Capital Allocation Budget
Portfolio Volatility Risk
Asset 1
Asset 2
Asset 3
Asset 4

Portfolio Volatility In Risk Contribution

- Portfolio Volatility = $\sum_{i=1}^{N} RC_i$
    - Where: $$RC_i = \frac{w_i(\sum w)_i}{\sqrt{w' \sum w}}$$
- risk contribution of asset *i* depends on
    1. the complete matrix of weights w
    2. the full covariance matrix $\sum$

Percent Risk Contribution

$$\boxed{\%RC_i = \frac{RC_i}{\text{portfolio volatility}}}$$

where $\sum_{i=1}^{N} \%RC_i = 1$

Relatively more risky assets: $\%RC_i > w_i$

Relatively less risky assets: $\%RC_i < w_i$

Let's practice!