3 Discrete-time Markov Chains

3.1 Definitions and Basic Concepts

Exercises

Exercise 3.1.1. Consider a box which initially contains a single blue ball and two yellow balls. A ball is drawn at random, removed, and replaced by a ball of the opposite colour, and this process repeats so that there are always exactly three balls in the box. Let X_n represent the number of blue balls in the box after n draws, with $X_0 = 1$. Is it reasonable to model $\{X_n, n \in \mathbb{N}\}$ as a DTMC? If so, specify the initial conditions $\underline{\alpha}_0$ and the TPM P.

Solution: Yes, it is reasonable to model $\{X_n, n \in \mathbb{N}\}$ as a DTMC. First of all, the possible values for X_n, $n \in \mathbb{N}$, are clearly 0, 1, 2, or 3. Moreover, we note that the composition of the contents in the box after the n^{th} draw is completely determined by simply knowing the contents of the box following the $(n-1)^{\text{th}}$ draw. In other words, the Markov property assumption is satisfied with this definition of $\{X_n, n \in \mathbb{N}\}$.

Since we always replace a drawn ball with one of the opposite colour (ensuring that there are always exactly three balls in the box), the initial conditions are simply given by

$$\underline{\alpha}_0 = (\alpha_{0,0}, \alpha_{0,1}, \alpha_{0,2}, \alpha_{0,3}) = (0, 1, 0, 0) \;\text{ since } X_0 = 1,$$

and the TPM would look like

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Exercise 3.1.2. Use mathematical induction and the definition of the Markov property to show that

$$P(X_{n+k} = x_{n+k}|X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = P(X_{n+k} = x_{n+k}|X_n = x_n),\ k \in \mathbb{Z}^+.$$

Solution: Applying mathematical induction, we start with the initial case of $k = 1$:

$$P(X_{n+1} = x_{n+1}|X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = P(X_{n+1} = x_{n+1}|X_n = x_n),$$

which simply holds by the Markov property. Thus, the result is true for $k = 1$. Now, we assume that the result is true for $k = m$ (inductive hypothesis). Consider

$$\begin{aligned}
&P(X_{n+m+1} = x_{n+m+1}|X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0)\\
&= \sum_{x_{n+m}} P(X_{n+m+1} = x_{n+m+1}, X_{n+m} = x_{n+m}|X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0)\\
&= \sum_{x_{n+m}} P(X_{n+m} = x_{n+m}|X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0)\\
&\times P(X_{n+m+1} = x_{n+m+1}|X_{n+m} = x_{n+m}, X_n = x_n, \dots, X_0 = x_0)\\
&= \sum_{x_{n+m}} P(X_{n+m} = x_{n+m}|X_n = x_n)\\
&\times P(X_{n+m+1} = x_{n+m+1}|X_{n+m} = x_{n+m}, X_n = x_n, \dots, X_0 = x_0) \qquad \text{by the inductive hypothesis}\\
&= \sum_{x_{n+m}} P(X_{n+m} = x_{n+m}|X_n = x_n)P(X_{n+m+1} = x_{n+m+1}|X_{n+m} = x_{n+m}) \qquad \text{by the Markov property}\\
&= \sum_{x_{n+m}} P^{(m)}_{x_n, x_{n+m}} P_{x_{n+m}, x_{n+m+1}}\\
&= P(X_{n+m+1} = x_{n+m+1}|X_n = x_n).
\end{aligned}$$

Thus, the result is true for $k = m + 1$. By mathematical induction, the result is true in general.

Exercise 3.1.3. In Example 3.1, determine the period of each state of the DTMC.

Solution: In Example 3.1, we showed that the DTMC is irreducible. This implies that each state has the same period. However, note that the diagonal elements of the TPM P are all positive. As such, each state clearly has a period of 1.

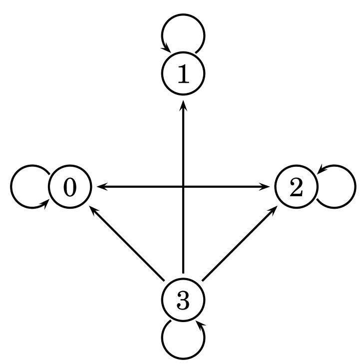
Exercise 3.1.4. Consider a DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} \end{array}.$$

Determine the communication classes of this DTMC and the period of each state.

Solution:

State Transition Diagram



From the above diagram, the communication classes are $\{0, 2\}$, $\{1\}$, and $\{3\}$. Next, we note that the main diagonal components of P are positive, implying immediately that $d(0) = d(1) = d(2) = d(3) = 1$.

Exercise 3.1.5. At the start of an experiment, three red and three blue balls are randomly distributed into two boxes in such a way that each box contains three balls. Following this, we draw one ball from each box and place the ball drawn from box 1 into box 2, and conversely with the ball drawn from box 2. We continue this process and let X_n represent how many red balls are in box 1 after the n^{th} draw, $n \in \mathbb{Z}^+$. Explain why $\{X_n, n \in \mathbb{N}\}$ is a DTMC and determine its initial conditions and TPM.

Solution: Since the determination of the number of red balls in box 1 after the n^{th} draw (either 0, 1, 2, or 3) is only dependent on the same information after the $(n-1)^{\text{th}}$ draw, the process $\{X_n, n \in \mathbb{N}\}$ satisfies the Markov property assumption, and hence is a DTMC.

Prior to drawing balls from each box, suppose that box 1 contains i red balls and $3-i$ blue balls, whereas box 2 contains $3-i$ red balls and i blue balls. Thus, there is a total of $3^2 = 9$ different (equally likely) combinations of possible ball transfers after a draw is made from each box. Grouping these events based on the colours of the drawn balls (R for red and B for blue), we find:

Box 1 Draw	Box 2 Draw	X_n	$\to$	X_{n+1}	
R	B	i	$\to$	$i-1$	with probability $\frac{i}{3} \cdot \frac{i}{3} = \frac{i^2}{9}$,
R	R	i	$\to$	i	with probability $\frac{i}{3} \cdot \frac{3-i}{3} = \frac{i(3-i)}{9}$,
B	B	i	$\to$	i	with probability $\frac{3-i}{3} \cdot \frac{i}{3} = \frac{i(3-i)}{9}$,
B	R	i	$\to$	$i+1$	with probability $\frac{3-i}{3} \cdot \frac{3-i}{3} = \frac{(3-i)^2}{9}$.

For $i = 0, 1, 2, 3$, we therefore obtain

$$P_{i,\min\{i-1,0\}} = \frac{i^2}{9}, \; P_{i,i} = \frac{2i(3-i)}{9}, \text{ and } P_{i,\max\{i+1,3\}} = \frac{(3-i)^2}{9},$$

and this leads to the TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Finally, three red and three blue balls are equally (and randomly) split into the two boxes at the start of the experiment. As such, the initial conditions of the DTMC, governed by the row vector $\underline{\alpha}_0 = (\alpha_{0,0}, \alpha_{0,1}, \alpha_{0,2}, \alpha_{0,3})$, are given by

$$\alpha_{0,i} = P(X_0 = i) = \frac{\binom{3}{i}\binom{3}{3-i}}{\binom{6}{3}}, \; i = 0, 1, 2, 3.$$

This immediately yields $\underline{\alpha}_0 = (1/20, 9/20, 9/20, 1/20)$.

Exercise 3.1.6. Prove that if the number of states in a DTMC is N, and if state j can be reached from state i, then it can be reached in N transitions or less.

Solution: Suppose that the number of states in a DTMC is N and $i \to j$, $i \neq j$. Since $i \to j$, $\exists\, n \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$. Let us choose this value of n to be the <u>smallest</u> such positive integer satisfying $P_{i,j}^{(n)} > 0$. We wish to prove that $n \leq N$. To prove this, we assume that $n > N$ and try to get a contradiction. Without loss of generality, assume that $X_0 = i$ and state j is entered in the least amount of transitions, namely n transitions. With $X_0 = i$ and $X_n = j$, states i and j cannot appear in the intermediate $n-1$ time periods corresponding to $X_1, X_2, \ldots, X_{n-1}$, as that would contradict our assumption about n. Moreover, any repetition of states for $X_1, X_2, \ldots, X_{n-1}$ would also imply that $i \to j$ in less than n steps. Excluding states i and j, there are $N-2$ states to place in $n-1$ locations. But $N < n$ implies that $N-2 < n-1$, and so at least one state will be repeated among $X_1, X_2, \ldots, X_{n-1}$, contrary to what we have assumed. Thus, it must be that $i \to j$ in N transitions or less. Finally, the case when $i = j$ is easily adapted from the above arguments.

Exercise 3.1.7. Consider a basketball player who makes a shot with the following probabilities: 1/2 if he has missed the last two times, 2/3 if he has hit one of his last two shots, and 4/5 if he has hit both of his last two shots.

(a) Formulate a DTMC to model the basketball player's shooting (making sure to incorporate the outcomes of his last two shots) and write down the corresponding transition probability matrix.

(b) Suppose that the basketball player misses his first two shots in a game. If he ended up taking 11 total shots in the game, what is the probability that he made the final shot he took?

(c) How would your answer in part (b) change if he had made his first two shots of the game? Are you surprised by the result?

Solution: (a) First of all, let X_n denote whether the basketball player makes his n^{th} shot. In particular,

$$X_n = \begin{cases} 0 & \text{, if } n^{\text{th}} \text{ shot is missed,} \\ 1 & \text{, if } n^{\text{th}} \text{ shot is made.} \end{cases}$$

For $n \geq 2$, define $Y_n = (X_n, X_{n-1})$. In other words, Y_n is able to keep track of the outcomes of the last two shots. Based on the information given, $\{Y_n, n \geq 2\}$ is a DTMC with state space $\{(0,0), (0,1), (1,0), (1,1)\}$ and associated TPM

$$P = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \left[\begin{array}{cccc} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{5} & 0 & \frac{4}{5} \end{array}\right] \end{array}.$$

(b) We wish to calculate $P(X_{11}=1)=P(Y_{11}=(1,0))+P(Y_{11}=(1,1))$. Note that $\underline{\alpha}_2=(1,0,0,0)$ and we need to find $P^{(9)}$. We end up calculating:

$$P^{(2)}=PP \quad = \begin{bmatrix} 0.25 & 0.166667 & 0.25 & 0.333333 \\ 0.166667 & 0.222222 & 0.166667 & 0.444444 \\ 0.111111 & 0.133333 & 0.222222 & 0.533333 \\ 0.0666667 & 0.16 & 0.133333 & 0.64 \end{bmatrix},$$

$$P^{(4)}=P^{(2)}P^{(2)} = \begin{bmatrix} 0.140278 & 0.16537 & 0.190278 & 0.504704 \\ 0.126852 & 0.170494 & 0.175 & 0.527654 \\ 0.110247 & 0.163111 & 0.170494 & 0.556148 \\ 0.100815 & 0.166844 & 0158296 & 0.574044 \end{bmatrix},$$

$$P^{(8)}=P^{(4)}P^{(4)} = \begin{bmatrix} 0.112451 & 0.166531 & 0.167866 & 0.553152 \\ 0.111911 & 0.166626 & 0.167336 & 0.554127 \\ 0.111021 & 0.166641 & 0.166626 & 0.555712 \\ 0.11063 & 0.166714 & 0.166238 & 0.556418 \end{bmatrix},$$

and

$$P^{(9)}=P^{(8)}P = \begin{bmatrix} 0.111736 & 0.166586 & 0.167246 & 0.554432 \\ 0.111497 & 0.166604 & 0.167039 & 0.554859 \\ 0.111057 & 0.166685 & 0.166604 & 0.555654 \\ 0.110886 & 0.166696 & 0.166458 & 0.55596 \end{bmatrix}.$$

Then,

$$P(Y_{11}=(1,0)) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.167246 \\ 0.167039 \\ 0.166604 \\ 0.166458 \end{bmatrix} = 0.167246$$

and

$$P(Y_{11}=(1,1)) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.554432 \\ 0.554859 \\ 0.555654 \\ 0.55596 \end{bmatrix} = 0.554432.$$

Thus, $P(X_{11}=1)=0.167246+0.554432=0.721678$.

(c) In this particular case, $\underline{\alpha}_2=(0,0,0,1)$. Therefore,

$$P(Y_{11}=(1,0)) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.167246 \\ 0.167039 \\ 0.166604 \\ 0.166458 \end{bmatrix} = 0.166458$$

and

$$P(Y_{11} = (1,1)) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.554432 \\ 0.554859 \\ 0.555654 \\ 0.55596 \end{bmatrix} = 0.55596.$$

Thus, $P(X_{11} = 1) = 0.166458 + 0.55596 = 0.722418$. The two answers are practically identical, implying that the outcomes of the first 2 shots are, in some sense, meaningless when the basketball player ends up taking 9 more shots.

Exercise 3.1.8. A taxi driver alternates between three locations in the city. When the taxi reaches location 1, it is equally likely to go on the next call to either location 2 or location 3. When it reaches location 2, it will go on the next call to location 1 with probability 1/4, remain at location 2 with probability 1/4, or go to location 3 with probability 1/2. From location 3, it will go on the next call to location 1 with probability 1/2, to location 2 with probability 1/4, or remain at location 3 with probability 1/4. Let $\{X_n, n \in \mathbb{N}\}$ be the DTMC denoting the location of the taxi following the n^{th} call. Assume that, initially, the taxi is equally likely to begin in any of the three locations.

(a) Determine the TPM associated with this DTMC.

(b) Calculate $P(X_7 = 3 | X_5 = 2)$.

(c) Calculate $P(X_3 = 1, X_1 = 2)$.

(d) If after the first call the taxi is at location 3, what is the probability that the next visit to location 3 occurs right after the fourth call?

Solution: (a) From the given information, the TPM is given by

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \end{array}.$$

(b) We have that

$$\begin{aligned} P(X_7 = 3 | X_5 = 2) &= \sum_{i=1}^{3} P(X_7 = 3, X_6 = i | X_5 = 2) \\ &= \sum_{i=1}^{3} P_{2,i} P_{i,3} \quad \text{via the Markov property} \\ &= \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) \\ &= \frac{3}{8} = 0.375. \end{aligned}$$

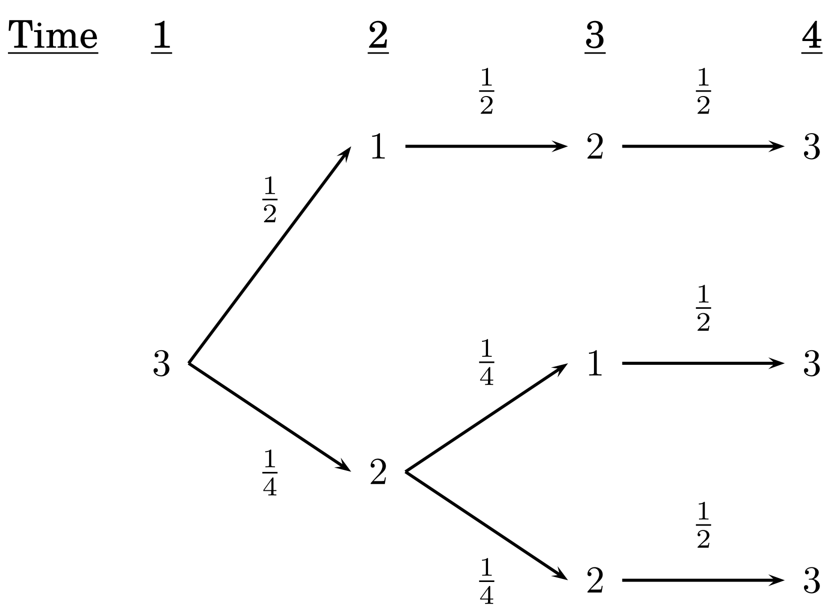
(c) First of all, we have that

$$P(X_1 = 2) = \underline{\alpha}_0 \begin{bmatrix} P_{1,2} \\ P_{2,2} \\ P_{3,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}\,.$$

Therefore,

$$\begin{aligned}
P(X_3 = 1, X_1 = 2) &= \sum_{k=1}^{3} P(X_3 = 1, X_2 = k, X_1 = 2) \\
&= P(X_1 = 2) \sum_{k=1}^{3} P_{2,k} P_{k,1} \quad \text{via the Markov property} \\
&= \frac{1}{3} \left(\frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \right) \quad \text{using the result from part (b)} \\
&= \frac{5}{48} \approx 0.104.
\end{aligned}$$

(d) Consider the following transition diagram:



Using the above diagram, we wish to calculate

$$\begin{aligned}
P(X_4 = 3, X_3 \neq 3, X_2 \neq 3 | X_1 = 3) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) \\
&= \frac{3}{16} = 0.1875.
\end{aligned}$$

Exercise 3.1.9. A box contains 5 balls, numbered $1, 2, 3, 4, 5$. Suppose that balls are drawn from the box one at a time **with replacement**. As balls are drawn, a stochastic process is formulated to keep track of the *largest* number observed. Specifically, let X_n be the largest number drawn after n draws, $n \in \mathbb{Z}^+$.

(a) Explain why the process $\{X_n, n \in \mathbb{Z}^+\}$ is a DTMC.

(b) Specify the TPM for this DTMC.

(c) Use this DTMC to calculate the probability that after 6 draws, the largest number observed is smaller than 4.

(d) Suppose instead that balls are drawn from the box one at a time **without replacement**. Does $\{X_n, n \in \mathbb{Z}^+\}$ satisfy the stationarity assumption? Justify your response.

Solution: (a) First of all, the possible values that X_n can take on lie in the set $\{1, 2, 3, 4, 5\}$. Secondly, to determine the largest number observed after n draws, we simply need to keep track of the largest number drawn after $n - 1$ draws. Thus, the Markov property assumption is satisfied by $\{X_n, n \in \mathbb{Z}^+\}$, and hence is a DTMC.

(b) To determine the TPM, we make the following observation: Suppose that the largest number observed after n draws is i, $i = 1, 2, 3, 4, 5$. Therefore, on the next draw, if you should obtain a ball of value i or less, then the largest number observed will not change. On the other hand, it is possible that you could draw a ball with value $i + 1, i + 2, \ldots, 5$, each with equal probability. As a result, the TPM is given by

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

In terms of the initial conditions, each of the 5 balls has an equal probability of being selected on the very first draw. Therefore, we immediately obtain $\underline{\alpha}_1 = (1/5, 1/5, 1/5, 1/5, 1/5)$.

(c) We wish to calculate $P(X_6 < 4) = 1 - P(X_6 = 4) - P(X_6 = 5)$, and so we need to find the distribution of X_6, which can be obtained via $\underline{\alpha}_6 = \underline{\alpha}_1 P^{(5)}$. Therefore,

$$P^{(2)} = PP = \begin{bmatrix} \frac{1}{25} & \frac{3}{25} & \frac{1}{5} & \frac{7}{25} & \frac{9}{25} \\ 0 & \frac{4}{25} & \frac{1}{5} & \frac{7}{25} & \frac{9}{25} \\ 0 & 0 & \frac{9}{25} & \frac{7}{25} & \frac{9}{25} \\ 0 & 0 & 0 & \frac{16}{25} & \frac{9}{25} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P^{(4)} = P^{(2)}P^{(2)} = \begin{bmatrix} \frac{1}{625} & \frac{3}{125} & \frac{13}{125} & \frac{7}{25} & \frac{369}{625} \\ 0 & \frac{16}{625} & \frac{13}{125} & \frac{7}{25} & \frac{369}{625} \\ 0 & 0 & \frac{81}{625} & \frac{7}{25} & \frac{369}{625} \\ 0 & 0 & 0 & \frac{256}{625} & \frac{369}{625} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$P^{(5)} = P^{(4)}P = \begin{bmatrix} \frac{1}{3125} & \frac{31}{3125} & \frac{211}{3125} & \frac{781}{3125} & \frac{2101}{3125} \\ 0 & \frac{32}{3125} & \frac{211}{3125} & \frac{781}{3125} & \frac{2101}{3125} \\ 0 & 0 & \frac{243}{3125} & \frac{781}{3125} & \frac{2101}{3125} \\ 0 & 0 & 0 & \frac{1024}{3125} & \frac{2101}{3125} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$\begin{aligned} \underline{\alpha}_6 &= \underline{\alpha}_1 P^{(5)} \\ &= (1/5, 1/5, 1/5, 1/5, 1/5) \begin{bmatrix} \frac{1}{3125} & \frac{31}{3125} & \frac{211}{3125} & \frac{781}{3125} & \frac{2101}{3125} \\ 0 & \frac{32}{3125} & \frac{211}{3125} & \frac{781}{3125} & \frac{2101}{3125} \\ 0 & 0 & \frac{243}{3125} & \frac{781}{3125} & \frac{2101}{3125} \\ 0 & 0 & 0 & \frac{1024}{3125} & \frac{2101}{3125} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \left(\frac{1}{15625}, \frac{63}{15625}, \frac{133}{3125}, \frac{3367}{15625}, \frac{11529}{15625}\right), \end{aligned}$$

and so

$$P(X_6 < 4) = 1 - P(X_6 = 4) - P(X_6 = 5) = 1 - \frac{3367}{15625} - \frac{11529}{15625} = \frac{729}{15625} \approx 0.047.$$

(d) If balls are now drawn without replacement, note the following one-step transition probabilities:

$$P(X_2 = 3 | X_1 = 3) = P(\text{ball 1 or 2 is chosen among balls 1, 2, 4, and 5}) = \frac{2}{4} = \frac{1}{2}$$

and

$$P(X_3 = 3|X_2 = 3) = P(\text{ball 1 (or 2) is chosen among 3 balls including 4 and 5}) = \frac{1}{3}\,.$$

These probabilities are not equal to each other, implying that the stationary assumption does not hold for this discrete-time stochastic process.

Exercise 3.1.10. Suppose that in the lobby of a particular building, some number (say, N) of people board an elevator that goes to some number (say, M) of floors. It is possible that there may be more people than floors, or more floors than people. Each person is equally likely to choose any floor, independently of one another. When a floor button is pushed, it will light up. The main interest is in determining the expected number of lit buttons when the elevator begins its ascent.

(a) How might one formulate this problem as a DTMC?

(b) Specify the TPM and initial conditions for your DTMC.

(c) Suppose that $M = 5$ and $N = 9$. Use your DTMC to calculate the expected number of lit buttons when the elevator begins its ascent.

Solution: (a) For $n \in \mathbb{Z}^+$, let X_n represent how many distinct floor buttons have been pushed following the choice of the n^{th} person boarding the elevator. Clearly, each X_n takes on values in the state space $\{1, 2, \ldots, M\}$. Moreover, we recognize that the value of X_n either equals the value of X_{n-1} (if the n^{th} person selects an already chosen floor) or equals the value of $\max\{X_{n-1} + 1, M\}$ (if the n^{th} person selects an unchosen floor). Since the Markov property holds true, $\{X_n, n \in \mathbb{Z}^+\}$ is a DTMC.

(b) The TPM of the DTMC in part (a) is of the form

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & \cdots & M-2 & M-1 & M \end{array} \\ \begin{bmatrix} \frac{1}{M} & \frac{M-1}{M} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2}{M} & \frac{M-2}{M} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{M} & \frac{M-3}{M} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{M-1}{M} & \frac{1}{M} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

At the start, there will surely be one lit button after the first person selects his/her floor. As a result, the initial conditions are simply given by

$$\underline{\alpha}_1 = (\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,M}) = (1, 0, \ldots, 0).$$

(c) With $M = 5$, our TPM in part (b) becomes

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

With $N = 9$, we first wish to compute $\underline{\alpha}_9 = \underline{\alpha}_1 P^{(8)}$. Therefore, we compute

$$P^{(2)} = PP \qquad = \begin{bmatrix} \frac{1}{25} & \frac{12}{25} & \frac{12}{25} & 0 & 0 \\ 0 & \frac{4}{25} & \frac{3}{5} & \frac{6}{25} & 0 \\ 0 & 0 & \frac{9}{25} & \frac{14}{25} & \frac{2}{25} \\ 0 & 0 & 0 & \frac{16}{25} & \frac{9}{25} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P^{(4)} = P^{(2)}P^{(2)} = \begin{bmatrix} \frac{1}{625} & \frac{12}{125} & \frac{12}{25} & \frac{48}{125} & \frac{24}{625} \\ 0 & \frac{16}{625} & \frac{39}{125} & \frac{66}{125} & \frac{84}{625} \\ 0 & 0 & \frac{81}{625} & \frac{14}{25} & \frac{194}{625} \\ 0 & 0 & 0 & \frac{256}{625} & \frac{369}{625} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$P^{(8)} = P^{(4)}P^{(4)} = \begin{bmatrix} \frac{1}{390625} & \frac{204}{78125} & \frac{1452}{15625} & \frac{37296}{78125} & \frac{166824}{390625} \\ 0 & \frac{256}{390625} & \frac{3783}{78125} & \frac{31602}{78125} & \frac{213444}{390625} \\ 0 & 0 & \frac{6561}{390625} & \frac{4718}{15625} & \frac{266114}{390625} \\ 0 & 0 & 0 & \frac{65536}{390625} & \frac{325089}{390625} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, we immediately obtain

$$\underline{\alpha}_9 = \underline{\alpha}_1 P^{(8)} = \left(\frac{1}{390625}, \frac{204}{78125}, \frac{1452}{15625}, \frac{37296}{78125}, \frac{166824}{390625} \right).$$

Finally, the expected number of lit buttons is simply the mean of the distribution of

X_9 (which we now have above), and this can be calculated as follows:

$$\begin{aligned}
\mathrm{E}[X_9] &= \sum_{i=1}^{5} i\alpha_{9,i} \\
&= 1\cdot\frac{1}{390625} + 2\cdot\frac{204}{78125} + 3\cdot\frac{1452}{15625} + 4\cdot\frac{37296}{78125} + 5\cdot\frac{166824}{390625} \\
&= \frac{1690981}{390625} \approx 4.329.
\end{aligned}$$

Exercise 3.1.11. A DTMC $\{X_n, n \in \mathbb{N}\}$ on the state space $\{1, 2, 3\}$ has TPM given by

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/2 \\ 2/3 & 1/6 & 1/6 \end{bmatrix} \end{array}.$$

Assume that, initially, the DTMC is equally likely to start in any of the three states.

(a) Calculate the two-step TPM of this DTMC.

(b) Calculate $P(X_3 = 1, X_1 = 2)$.

(c) Calculate $P(X_6 = 2, X_4 = 3 | X_2 = 2)$.

(d) Determine the pmf of X_3.

(e) Calculate $P(X_4 = 3, X_3 \neq 3, X_2 \neq 3 | X_1 = 3)$.

Solution: (a) We wish to calculate

$$P^{(2)} = P^2 = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{11}{36} & \frac{11}{36} \\ \frac{5}{36} & \frac{19}{36} & \frac{1}{3} \end{bmatrix} \end{array}.$$

(b) We wish to calculate

$$\begin{aligned}
P(X_3 = 1, X_1 = 2) &= P(X_1 = 2)P(X_3 = 1 | X_1 = 2) \\
&= \alpha_{1,2} P_{2,1}^{(2)} \\
&= \underline{\alpha}_0 \begin{bmatrix} P_{1,2} \\ P_{2,2} \\ P_{3,2} \end{bmatrix} \left(\frac{7}{18}\right) \\
&= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} \left(\frac{7}{18}\right) \\
&= \left(\frac{2}{9} + \frac{1}{9} + \frac{1}{18}\right)\left(\frac{7}{18}\right) \\
&= \frac{49}{324} \approx 0.151.
\end{aligned}$$

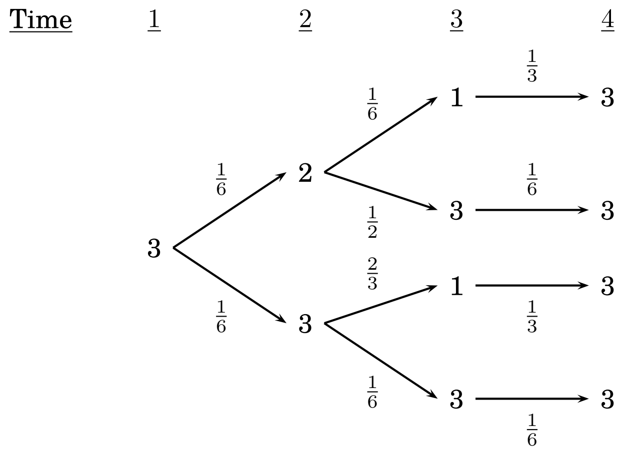
(c) We wish to calculate

$$\begin{aligned}
P(X_6 = 2, X_4 = 3 | X_2 = 2) &= P(X_4 = 3 | X_2 = 2) P(X_6 = 2 | X_4 = 3, X_2 = 2) \\
&= P_{2,3}^{(2)} P_{3,2}^{(2)} \quad \text{by the Markov property} \\
&= \left(\frac{11}{36}\right)\left(\frac{19}{36}\right) \\
&= \frac{209}{1296} \approx 0.161.
\end{aligned}$$

(d) We wish to find

$$\begin{aligned}
\underline{\alpha}_3 &= \begin{bmatrix} \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{bmatrix} \\
&= \underline{\alpha}_0 P^{(3)} \\
&= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{11}{36} & \frac{11}{36} \\ \frac{5}{36} & \frac{19}{36} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{11}{36} & \frac{41}{108} & \frac{17}{54} \\ \frac{55}{216} & \frac{89}{216} & \frac{1}{3} \\ \frac{67}{216} & \frac{35}{108} & \frac{79}{216} \end{bmatrix} \\
&= \begin{bmatrix} 47/162 & 241/648 & 73/216 \end{bmatrix} \approx \begin{bmatrix} 0.290 & 0.372 & 0.338 \end{bmatrix}.
\end{aligned}$$

(e) Consider the following transition diagram:



Using the above diagram, we wish to calculate

$$
\begin{aligned}
&P(X_4 = 3, X_3 \neq 2, X_2 \neq 1 | X_1 = 3) \\
&= \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \\
&= \frac{7}{108} \approx 0.065.
\end{aligned}
$$

3.2 Transience and Recurrence

Exercises

Exercise 3.2.1. In Exercise 3.1.4, determine whether each state of the DTMC is transient or recurrent.

Solution: In Exercise 3.1.4, we found the communication classes to be $\{0,2\}$, $\{1\}$, and $\{3\}$. Note that:

$$f_{0,0} = \frac{1}{2} + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)^{n-2}\left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{1-3/4} = 1 \implies \{0,2\} \text{ is recurrent,}$$

$$\sum_{n=1}^{\infty} P_{1,1}^{(n)} = \sum_{n=1}^{\infty} 1 = \infty \implies \{1\} \text{ is recurrent,}$$

$$\sum_{n=1}^{\infty} P_{3,3}^{(n)} = \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n = \frac{1/8}{1-1/8} = \frac{1}{7} < \infty \implies \{3\} \text{ is transient.}$$

Exercise 3.2.2. In Example 3.8, show that

$$f_{-1,0} = P(\text{DTMC ever makes a future visit to state } 0 | X_0 = -1) = \frac{1 - |q-p|}{2q}.$$

Solution: Conditioning on the state of the DTMC at time 1, we have that

$$f_{-1,0} = q f_{-2,0} + p(1) = p + q f_{-1,0}^2, \tag{3.1}$$

where we recognize that $f_{-2,0} = f_{-2,-1} f_{-1,0} = f_{-1,0}^2$ since in order for the DTMC to ever go from state -2 to state 0, it first has to go to state -1 (which occurs with probability $f_{-2,-1} = f_{-1,0}$) and if it does eventually go to state -1, then it must still go to state 0 (and the conditional probability of that happening is also $f_{-1,0}$). Therefore, rewriting (3.1), we end up with

$$q f_{-1,0}^2 - f_{-1,0} + p = 0,$$

which is a quadratic equation in $f_{-1,0}$. Applying the well-known quadratic formula, the two roots of the above equation simplify to become:

$$\begin{aligned} f_{-1,0} &= \frac{1 \pm \sqrt{1-4pq}}{2q} \\ &= \frac{1 \pm \sqrt{(p+q)^2 - 4pq}}{2q} \\ &= \frac{1 \pm \sqrt{q^2 + 2pq + p^2 - 4pq}}{2q} \\ &= \frac{1 \pm \sqrt{(q-p)^2}}{2q} \\ &= \frac{1 \pm |q-p|}{2q}. \end{aligned}$$

There can only be one unique solution for $f_{-1,0}$, which means that one of

$$r_1 = \frac{1+|q-p|}{2q} \text{ or } r_2 = \frac{1-|q-p|}{2q}$$

must be inadmissible. To determine which one it is, suppose that $p > q$. Then, $|q-p| = -(q-p)$ and the two respective roots become

$$r_1 = \frac{1-(q-p)}{2q} = \frac{1-q+p}{2q} = \frac{2p}{2q} = \frac{p}{q} > 1$$

and

$$r_2 = \frac{1+(q-p)}{2q} = \frac{1-p+q}{2q} = \frac{2q}{2q} = 1.$$

Since $0 \le f_{-1,0} \le 1$, the root r_1 must be inadmissible, thereby implying that

$$f_{-1,0} = \frac{1-|q-p|}{2q}.$$

Exercise 3.2.3. In Example 3.8, for $p < q$, show that

$$f_{0,0} = P(\text{DTMC ever makes a future visit to state } 0 | X_0 = 0) = 2p < 1.$$

Solution: Recall that we derived the general formula for $f_{0,0}$, namely

$$f_{0,0} = 1 - \frac{1}{2}\left(|q-p| + |p-q|\right).$$

If $p < q$ (i.e., $p < 1-p \Rightarrow 2p < 1$), then the above equation simplifies to become

$$f_{0,0} = 1 - \frac{1}{2}\big((q-p) - (p-q)\big) = 1 - \frac{1}{2}(2q-2p) = 1-(q-p) = 1-q+p = 2p < 1.$$

Exercise 3.2.4. Specify the communication classes of the DTMCs with the following TPMs, and determine whether they are transient or recurrent:

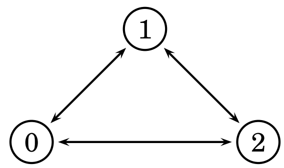
$$P_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}, \qquad P_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

$$P_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}, \qquad P_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$P_5 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}, \qquad P_6 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 \end{bmatrix} \end{array}.$$

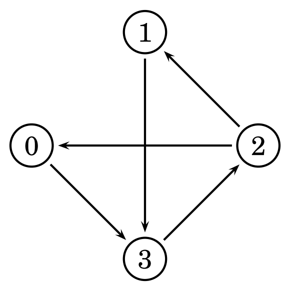
Solution:

State Transition Diagram for P_1



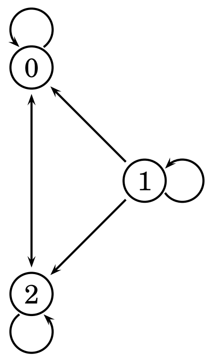
From the above diagram, the only communication class is $\{0, 1, 2\}$. Since P_1 is the TPM of a finite-state irreducible DTMC, $\{0, 1, 2\}$ is recurrent.

State Transition Diagram for P_2



From the above diagram, the only communication class is $\{0, 1, 2, 3\}$. Since P_2 is the TPM of a finite-state irreducible DTMC, $\{0, 1, 2, 3\}$ is recurrent.

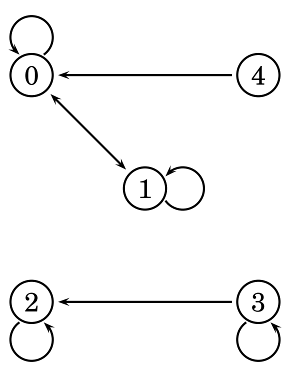
State Transition Diagram for P_3



From the above diagram, the communication classes are $\{0,2\}$, $\{1\}$, and $\{3,4\}$. Note that:

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{0,0}^{(n)} &= \sum_{n=1}^{\infty} \frac{1}{2} && = \infty \Longrightarrow \{0,2\} \text{ is recurrent,} \\
\sum_{n=1}^{\infty} P_{1,1}^{(n)} &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n && < \infty \Longrightarrow \{1\} \text{ is transient,} \\
\sum_{n=1}^{\infty} P_{3,3}^{(n)} &= \sum_{n=1}^{\infty} \frac{1}{2} && = \infty \Longrightarrow \{3,4\} \text{ is recurrent.}
\end{aligned}$$

State Transition Diagram for P_4



From the above diagram, the communication classes are $\{0,1\}$, $\{2\}$, $\{3\}$, and $\{4\}$.

Note that:

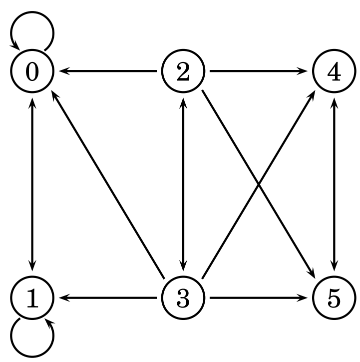
$$f_{0,0} = \frac{1}{4} + \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)^{n-2}\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1/2}{1-1/2} = 1 \implies \{0,1\} \text{ is recurrent,}$$

$$\sum_{n=1}^{\infty} P_{2,2}^{(n)} = \sum_{n=1}^{\infty} 1 = \infty \implies \{2\} \text{ is recurrent,}$$

$$\sum_{n=1}^{\infty} P_{3,3}^{(n)} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} = \frac{2/3}{1-2/3} = 2 < \infty \implies \{3\} \text{ is transient,}$$

$$\sum_{n=1}^{\infty} P_{4,4}^{(n)} = \sum_{n=1}^{\infty} 0 = 0 < \infty \implies \{4\} \text{ is transient.}$$

State Transition Diagram for P_5



From the above diagram, the communication classes are $\{0,1\}$, $\{2,3\}$, and $\{4,5\}$. First of all, we see that

$$f_{0,0} = \frac{1}{2} + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)^{n-2}\left(\frac{1}{3}\right) = \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{1-2/3} = 1 \implies \{0,1\} \text{ is recurrent,}$$

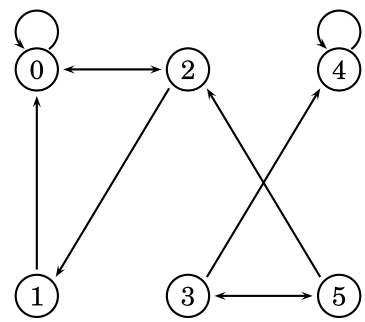
Looking at state 2, note that $P_{2,2}^{(2n-1)} = 0$ and $P_{2,2}^{(2n)} = (1/3)^n(1/6)^n = (1/18)^n$ for $n \in \mathbb{Z}^+$. Thus,

$$\sum_{n=1}^{\infty} P_{2,2}^{(n)} = \sum_{n=1}^{\infty} \left(\frac{1}{18}\right)^{n} = \frac{1/18}{1-1/18} = \frac{1}{17} < \infty \implies \{2,3\} \text{ is transient,}$$

Finally, note that $P_{4,4}^{(2n-1)} = 0$ and $P_{4,4}^{(2n)} = 1$ for $n \in \mathbb{Z}^+$. Thus,

$$\sum_{n=1}^{\infty} P_{4,4}^{(n)} = \sum_{n=1}^{\infty} 1 = \infty \implies \{4,5\} \text{ is recurrent.}$$

State Transition Diagram for P_6



From the above diagram, the communication classes are $\{0, 1, 2\}$, $\{3, 5\}$, and $\{4\}$. Looking at state 4, we note that since $P_{4,4} = 1$, then $P_{4,4}^{(n)} = 1 \; \forall \; n \in \mathbb{Z}^+$. Thus,

$$\sum_{n=1}^{\infty} P_{4,4}^{(n)} = \sum_{n=1}^{\infty} 1 = \infty \Longrightarrow \{4\} \text{ is recurrent.}$$

Next, we prove that state 5 is transient by assuming that it is recurrent and finding a contradiction. Assuming that state 5 is recurrent, note that it does not communicate with state 2. Therefore, by Theorem 3.5, we must have $P_{5,2} = 0$. However, we actually have $P_{5,2} = 1/5 > 0$, which is a contradiction. Thus, state 5 must be transient, thereby implying that $\{3, 5\}$ is a transient class.

Finally, we consider the values of $f_{2,2}^{(n)}$ for $n \in \mathbb{Z}^+$:

$$
\begin{aligned}
f_{2,2}^{(1)} &= P(X_1 = 2 | X_0 = 2) = 0, \\
f_{2,2}^{(2)} &= P(X_2 = 2, X_1 \neq 2 | X_0 = 2) = P(X_2 = 2, X_1 = 0 | X_0 = 2) = P_{2,0} P_{0,2} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}, \\
f_{2,2}^{(3)} &= P(X_3 = 2, X_2 \neq 2, X_1 \neq 2 | X_0 = 2) \\
&= P(X_3 = 2, X_2 \neq 2, X_1 = 0 | X_0 = 2) + P(X_3 = 2, X_2 \neq 2, X_1 = 1 | X_0 = 2) \\
&= P(X_3 = 2, X_2 = 0, X_1 = 0 | X_0 = 2) + P(X_3 = 2, X_2 = 0, X_1 = 1 | X_0 = 2) \\
&= P_{2,0} P_{0,0} P_{0,2} + P_{2,1} P_{1,0} P_{0,2} \\
&= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot 1 \cdot \frac{1}{2} \\
&= \frac{1}{12} + \frac{1}{3} = \frac{5}{12}.
\end{aligned}
$$

In fact, any path of length 3 or more starting at state 2 and returning to state 2 at time n (for the first time) takes the form

$$2 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 2 \quad \text{or} \quad 2 \to 1 \to 0 \to 0 \to \cdots \to 0 \to 2.$$

Therefore, for $n = 3, 4, 5, \ldots$, we have

$$\begin{aligned} f_{2,2}^{(n)} &= P_{2,0}(P_{0,0})^{n-2}P_{0,2} + P_{2,1}P_{1,0}(P_{0,0})^{n-3}P_{0,2} \\ &= \frac{1}{3}\left(\frac{1}{2}\right)^{n-2}\frac{1}{2} + \frac{2}{3}\left(\frac{1}{2}\right)^{n-3}\frac{1}{2} \\ &= \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} + \frac{2}{3}\left(\frac{1}{2}\right)^{n-2}. \end{aligned}$$

Thus, we see that

$$\begin{aligned} f_{2,2} &= \sum_{n=1}^{\infty} f_{2,2}^{(n)} \\ &= \frac{1}{6} + \sum_{n=3}^{\infty}\left\{\frac{1}{3}\left(\frac{1}{2}\right)^{n-1} + \frac{2}{3}\left(\frac{1}{2}\right)^{n-2}\right\} \\ &= \frac{1}{6} + \frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} \\ &= \frac{1}{6} + \frac{1}{12}\left(\frac{1}{1-\frac{1}{2}}\right) + \frac{1}{3}\left(\frac{1}{1-\frac{1}{2}}\right) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{2}{3} \\ &= 1, \end{aligned}$$

and so we conclude that state 2 is recurrent. Hence, $\{0, 1, 2\}$ is also a recurrent class.