1.(a) Assume $Y_n = i$ where $i > 0$

$Y_{n+1} = i+1$ if $X_{n+1} = i+1$ or $-i-1$

$Y_{n+1} = i+1$ if $X_{n+1} = i-1$ or $-i+1$

$P(Y_{n+1} = 1 | Y_n = 0) = 1$

For $i > 0$

$P(Y_{n+1} = i+1 | Y_n = i) = P(|X_n| = i+1 | |X_n| = i)$

$= \begin{cases} P & \text{if } X_n = i \\ 1-P & \text{if } X_n = -i \end{cases}$ ← only depends on X_n and Y_n

$P(Y_{n+1} = i-1 | Y_n = i) = P(|X_n| = i-1 | |X_n| = i)$

$= \begin{cases} 1-P & \text{if } X_n = i \\ P & \text{if } X_n = i \end{cases}$

If $\{Y_n\}_{n=0,1,2,\cdots}$ is a DTMC, $1-P = P \Rightarrow P = \frac{1}{2}$

~~since $P_{i,i+1}$ ... and ...~~

(b) $P((X_{n+1}, X_{n+2}) = (j_1, j_2) \mid (X_n, X_{n+1}) = (i_1, i_2), (X_{n-1}, X_n) = (i_{n-1}', i_n'), \cdots, (X_0, X_1) = (i_0', i_1'))$

$= \begin{cases} 0 & i_2 \neq j_1 \\ P_{i_2 j_2} (= P_{j_1 j_2}) & \text{o.w.} \end{cases}$

since the Prob. only depends on (X_n, X_{n+1}), $\{(X_n, X_{n+1})\}_{n=0,1,\cdots}$ is also a DTMC

Let S be the state space of $\{X_n\}_{n=0,1,\cdots}$

S' be the state space of $\{(X_n, X_{n+1})\}_{n=0,1,\cdots}$

$\pi((i,j)) = \sum_{(m,n) \in S'} \pi((m,n)) P'_{(m,n)(i,j)}$

$= \sum_{k \in S} \pi((k,i)) P_{ij}$

$= P_{ij} \sum_{k \in S} \pi((k,i))$

Let

$\pi((i,j)) = \pi(i) P_{ij}$

$P_{ij} \sum_{k \in S} \pi((k,i)) = P_{ij} \sum_{k \in S} \pi(k) P_{ki} = P_{ij} \pi(i)$ by defn

$\sum_{(i,j) \in S'} \pi((i,j)) = \sum_{(i,j) \in S'} \pi(i) P_{ij} = \sum_{j \in S} \sum_{i \in S} \pi(i) P_{ij} = \sum_{j \in S} \pi(j) = 1$

Hence $\pi((i,j)) = \pi(i) P_{ij}$ is a stationary distribution of $\{(X_n, X_{n+1})\}_{n=0,1,\cdots}$

2. (a) consider a DTMC with transition matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Both states have period 2

so $P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

clearly, P is reducible

(b) since $\gcd\{n \geq 1 : P^n_{xx} > 0\} = d$ $\forall x \in S$ (irreducible)

$\Rightarrow \gcd\{n \geq 1 : P^{d\frac{n}{d}}_{xx} > 0\} = d$ (All $\frac{n}{d}$ are co-prime, since we factor out the greatest common divisor)

$\gcd\{\frac{n}{d} \geq 1 : (P^d)^{\frac{n}{d}}_{xx} > 0\} = 1$ by the property of gcd

By defn for all the states in $\{Y_n\}_{n=0,1,\cdots}$, the period is 1.

3. $S = \{1, 2, 3\}$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.8 & 0.1 & 0.1 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.4 & 0.3 & 0.3 \end{array}$$

Since $P_{ij} > 0$ $\forall i, j \in S$, the chain is irreducible

Since the chain is finite and irreducible, all states are recurrent

~~By theorem~~

$$\begin{cases} \pi(1) = 0.8\pi(1) + 0.6\pi(2) + 0.4\pi(3) \\ \pi(2) = 0.1\pi(1) + 0.2\pi(2) + 0.3\pi(3) \\ \pi(3) = 0.1\pi(1) + 0.2\pi(2) + 0.3\pi(3) \\ \pi(1) + \pi(2) + \pi(3) = 1 \end{cases} \Rightarrow \begin{cases} \pi(1) = \frac{5}{7} \\ \pi(2) = \frac{1}{7} \\ \pi(3) = \frac{1}{7} \end{cases}$$

Hence stationary distribution exists

By theorem

$$\frac{N_n(y)}{n} \to \pi(y)$$

so

$$\begin{cases} \frac{N_n(1)}{n} \to \frac{5}{7} \\ \frac{N_n(1)}{n} \to \frac{1}{7} \\ \frac{N_n(1)}{n} \to \frac{1}{7} \end{cases}$$

4. (a) Since this chain has a tridiagonal transition matrix, it is detail balanced. By theorem, this chain is time-reversible

(b) since it is time-reversible

$\{Y_m\}_{m=0,1,2}$ has the same distribution as $\{X_n\}_{n=0,1,2}$ where $Y_m = X_{2-m}$

$$P(X_1=2 \mid X_2=3) = P(Y_1=2 \mid Y_0=3) = P_{32} = 0.6$$

(c) $P(X_2=3, X_4=1 \mid X_3=2)$

$$= P(Y_2=3, Y_0=1 \mid Y_1=2)$$
$$= \frac{P(Y_2=3, Y_1=2, Y_0=1)}{P(Y_1=2)}$$
$$= \frac{P_{12}P_{23}\pi(1)}{\pi(2)}$$
$$= \frac{0.2\cdot 0.3\,\pi(1)}{\pi(2)}$$
$$= \frac{0.06\,\pi(1)}{\pi(2)}$$

By part B:

$$\frac{\pi(1)}{\pi(2)} = \frac{0.1\pi(0)+0.7\pi(2)}{\pi(2)} = \frac{0.1\pi(0)}{\pi(2)}+0.7 = \frac{0.8\pi(1)}{\pi(2)}+0.7 = \frac{2.8\pi(2)}{\pi(2)}+0.7 = 3.5$$

Hence $P(X_2=3, X_4=1 \mid X_3=2) = 3.5\times 0.06 = 0.21$

5. (a) This chain is irreducible

since $i \to j \ \forall i,j \in S$

~~This chain can be decided into 2 classes~~ $d(S)=1$ since $P_{44}>0$

~~$\{0,1,2,4\}$ $d(\{0,1,2,4\})=1$ since $P_{24}>0$~~

(b)
$$\begin{cases} \pi(0) = \pi(3)+\frac{1}{2}\pi(4) \\ \pi(1) = \frac{1}{3}\pi(0) \\ \pi(2) = \frac{2}{3}\pi(0) \\ \pi(3) = \frac{1}{2}\pi(1)+\frac{1}{4}\pi(2) \\ \pi(4) = \frac{1}{2}\pi(1)+\frac{3}{4}\pi(2)+\frac{1}{2}\pi(4) \\ \sum_{i=0}^{4}\pi(i) = 1 \end{cases}$$

$$\Rightarrow \pi = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \left[\begin{matrix} \frac{3}{11} & \frac{1}{11} & \frac{2}{11} & \frac{1}{11} & \frac{4}{11} \end{matrix}\right] \end{array}$$

(c) since I, S

By theorem $E_0(T_0) = \frac{1}{\pi_0} = \frac{11}{3}$

(d) $C=\{3\}$ $A=\{0,1,2,4\}$

$$\begin{cases} g(0) = 1+\frac{1}{3}g(1)+g(2)\cdot\frac{2}{3} \\ g(1) = 1+\frac{1}{2}g(4) \\ g(2) = 1+\frac{3}{4}g(4) \end{cases}$$

$$\begin{cases} g(3)=0 \\ g(4)=1+\frac{1}{2}g(0)+\frac{1}{2}g(4) \end{cases}$$

$\Rightarrow$ $g = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ [\begin{matrix} 10 & 6 & \frac{17}{2} & 0 & 12 \end{matrix}] \end{array}$

$E_0(V_A) = 10$

(e) $A=\{4\}$, $B=\{2\}$, $C=\{0,1,3\}$

$$\begin{cases} h(0) = \frac{1}{3}h(1) \\ h(1) = \frac{1}{2}h(3)+1\cdot\frac{1}{2} \\ h(2)=0 \\ h(3) = h(0) \\ h(4)=1 \end{cases}$$

$h = \begin{array}{c} \begin{matrix} 0 & 1 & 3 \end{matrix} \\ [\begin{matrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{matrix}] \end{array}$

$P(V_A < V_B \mid X_0 = 0) = \frac{1}{5}$

~~6. Assume that the expected number of visits to state j before the first revisit to state i is k k≠1~~

~~Case 1. k=0~~

~~contradicts the fact that~~

6. $E_i(N(j) \; T_j^{(N(j))} < T_i) = \sum_{n=1}^{\infty} n \cdot P(T_j^{(n)} < T_i \mid X_0 = i)$

$= P(T_j < T_i \mid X_0=i)\, P(T_j > T_i \mid X_0 = j) + 2\cdot P(T_j<T_i \mid X_0=i)\, P(T_j<T_i \mid X_0=j)\, P(T_j > T_i \mid X_0=j) + \cdots$ by Makrov Proterty

$= P(T_j<T_i \mid X_0=i)^2 + 2\cdot P(T_j<T_i \mid X_0=i)^2 P(T_j<T_i \mid X_0=j) + \cdots$

since i, j are symmetric

$1-P(T_j<T_i \mid X_0=i) = P(T_i<T_j \mid X_0=j)$

$P(T_i<T_j \mid X_0=i) = P(T_j<T_i \mid X_0=j)$ since $P(T_i=T_j)=0$

$\Rightarrow \sum_{n=1}^{\infty} P(T_j<T_i \mid X_0=i)^2 \, n \cdot P(T_i<T_j \mid X_0=i)^{n-1}$

$= P(T_j<T_i \mid X_0=i)^2 \sum_{n=1}^{\infty} n\cdot P(T_i<T_j \mid X_0=i)^{n-1}$ Let $P(T_i<T_j \mid X_0=i) = P$

$= (1-P)^2 \left(\begin{matrix} 1 \\ +P+P \\ +P^2+P^2+\cdots \end{matrix} \right)$ each column is a geometric series with $0 \le P \le 1$

$= (1-P)^2 \sum_{n=0}^{\infty} \frac{P^n}{1-P} = (1-P)\sum_{n=0}^{\infty} P^n = (1-P)\frac{1}{1-P} = 1$