3.5 Absorbing DTMCs

Exercises

Exercise 3.5.1. Suppose that the TPM for a DTMC can be expressed as

$$P = \begin{bmatrix} Q & R \\ \mathbf{0} & I \end{bmatrix}.$$

Use mathematical induction to show that

$$P^{(n)} = \begin{bmatrix} Q^n & \sum_{i=0}^{n-1} Q^i R \\ \mathbf{0} & I \end{bmatrix}, \ n \in \mathbb{Z}^+.$$

Solution: First of all, for $n = 1$, we immediately obtain

$$P^{(1)} = \begin{bmatrix} Q^1 & \sum_{i=0}^{1-1} Q^i R \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} Q & Q^0 R \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} Q & R \\ \mathbf{0} & I \end{bmatrix} \text{ since } Q^0 = I.$$

Clearly, the result is true for $n = 1$. Assuming now that the result is true for $n = k$ (inductive hypothesis), we consider

$$\begin{aligned} P^{(k+1)} &= P^{(k)} P \\ &= \begin{bmatrix} Q^k & \sum_{i=0}^{k-1} Q^i R \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} Q & R \\ \mathbf{0} & I \end{bmatrix} \text{ by the inductive hypothesis} \\ &= \begin{bmatrix} Q^k Q + (\sum_{i=0}^{k-1} Q^i R)\mathbf{0} & Q^k R + (\sum_{i=0}^{k-1} Q^i R) I \\ \mathbf{0}Q + I\mathbf{0} & \mathbf{0}R + I^2 \end{bmatrix} \\ &= \begin{bmatrix} Q^{k+1} & \sum_{i=0}^{(k+1)-1} Q^i R \\ \mathbf{0} & I \end{bmatrix} \text{ since } I^2 = I. \end{aligned}$$

As the result is true for $n = k+1$, the result is therefore true in general by mathematical induction.

Exercise 3.5.2. Consider a DTMC $\{X_n, n \in \mathbb{N}\}$ with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.2 & 0 & 0.8 & 0 \\ 0.8 & 0 & 0.2 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Suppose that $X_0 = 2$ with probability 1.

(a) What is the probability that the DTMC ultimately ends up in state 3?

(b) Reorder the states of this DTMC and apply the result of the *Gambler's Ruin Problem* to answer part (a).

(c) How many transitions, on average, does it take to end up in one of the absorbing states?

(d) Determine the mean number of visits to state 0 prior to absorption.

(e) Determine the probability that the DTMC ever makes a visit to state 0.

Solution: (a) From the form of the given TPM, we note that

$$Q = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.8 & 0 & 0.2 \\ 0 & 0.8 & 0 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 0.8 & 0 \\ 0 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

We wish to calculate $U_{2,3}$. To do so, we first find $(I - Q)^{-1}$ by Gaussian elimination as follows:

$$\left[\, I - Q \,\middle|\, I \,\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -0.2 & 0 & 1 & 0 & 0 \\ -0.8 & 1 & -0.2 & 0 & 1 & 0 \\ 0 & -0.8 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+0.8R_1; R_3/0.8} \left[\begin{array}{ccc|ccc} 1 & -0.2 & 0 & 1 & 0 & 0 \\ 0 & 0.84 & -0.2 & 0.8 & 1 & 0 \\ 0 & -1 & 1.25 & 0 & 0 & 1.25 \end{array}\right]$$

$$\xrightarrow{0.84R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -0.2 & 0 & 1 & 0 & 0 \\ 0 & 0.84 & -0.2 & 0.8 & 1 & 0 \\ 0 & 0 & 0.85 & 0.8 & 1 & 1.05 \end{array}\right]$$

$$\xrightarrow{R_2/0.84;R_3/0.85} \left[\begin{array}{ccc|ccc} 1 & -0.2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -5/21 & 20/21 & 25/21 & 0 \\ 0 & 0 & 1 & 16/17 & 20/17 & 21/17 \end{array}\right]$$

$$\xrightarrow{R_2+5R_3/21} \left[\begin{array}{ccc|ccc} 1 & -0.2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 20/17 & 25/17 & 5/17 \\ 0 & 0 & 1 & 16/17 & 20/17 & 21/17 \end{array}\right]$$

$$\xrightarrow{R_1+R_2/5} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 21/17 & 5/17 & 1/17 \\ 0 & 1 & 0 & 20/17 & 25/17 & 5/17 \\ 0 & 0 & 1 & 16/17 & 20/17 & 21/17 \end{array}\right].$$

Therefore, we can now calculate the matrix $U = (I - Q)^{-1}R$:

$$U = \begin{bmatrix} 21/17 & 5/17 & 1/17 \\ 20/17 & 25/17 & 5/17 \\ 16/17 & 20/17 & 21/17 \end{bmatrix} \begin{bmatrix} 0.8 & 0 \\ 0 & 0 \\ 0 & 0.2 \end{bmatrix} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad 3 & \qquad\quad 4 \end{array} \\ \begin{bmatrix} 84/85 & 1/85 \\ 16/17 & 1/17 \\ 64/85 & 21/85 \end{bmatrix} \end{array}.$$

Thus, we see that $U_{2,3} = 64/85 \approx 0.753$.

(b) First of all, relabel the states of this DTMC as follows:

$$\begin{aligned} 0^* &\equiv \text{ state 3 in the original DTMC,} \\ 1^* &\equiv \text{ state 0 in the original DTMC,} \\ 2^* &\equiv \text{ state 1 in the original DTMC,} \\ 3^* &\equiv \text{ state 2 in the original DTMC,} \\ 4^* &\equiv \text{ state 4 in the original DTMC.} \end{aligned}$$

As a result, the "new" TPM corresponding to states $\{0^*, 1^*, 2^*, 3^*, 4^*\}$ looks like

$$P^* = \begin{array}{c} \\ 0^* \\ 1^* \\ 2^* \\ 3^* \\ 4^* \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0^* & 1^* & 2^* & 3^* & 4^* \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0.2 & 0 & 0 \\ 0 & 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 0.8 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array},$$

which is identical in form to the *Gambler's Ruin Problem* with $N = 4$, $p = 0.2$, and $q = 0.8$. Using the formula derived for the *Gambler's Ruin Problem*, we wish to calculate

$$1 - P(3^*) = 1 - \frac{1 - (0.8/0.2)^3}{1 - (0.8/0.2)^4} = 1 - \frac{21}{85} = \frac{64}{85},$$

which agrees with the result we found in part (a).

(c) We wish to compute v_2. Defining $\underline{v}' = (v_0, v_1, v_2)'$, we know that

$$\underline{v}' = (I - Q)^{-1}\underline{e}' = \begin{bmatrix} 21/17 & 5/17 & 1/17 \\ 20/17 & 25/17 & 5/17 \\ 16/17 & 20/17 & 21/17 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 27/17 \\ 50/17 \\ 57/17 \end{bmatrix}.$$

Therefore, we see that $v_2 = 57/17 \approx 3.353$.

(d) We wish to compute $S_{2,0}$. To do so, we know that

$$S = (I - Q)^{-1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 21/17 & 5/17 & 1/17 \\ 20/17 & 25/17 & 5/17 \\ 16/17 & 20/17 & 21/17 \end{bmatrix} \end{array}.$$

Therefore, we see that $S_{2,0} = 16/17 \approx 0.941$.

(e) We wish to compute $f_{2,0}$. Using the matrix S in part (d), we obtain

$$f_{2,0} = \frac{S_{2,0} - \delta_{2,0}}{S_{0,0}} = \frac{16/17 - 0}{21/17} = \frac{16}{21} \approx 0.762.$$

Exercise 3.5.3. Consider a DTMC on the state space $\{1, 2, 3, 4\}$ with TPM of the form

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.4 \\ 0.25 & 0.25 & 0.5 & 0 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{bmatrix} \end{array}.$$

Suppose that the DTMC begins in state 1 with probability 1.

(a) Calculate the probability that state 3 is encountered before state 4.

(b) Calculate the mean number of transitions until either state 3 or state 4 is entered.

Solution: (a) To handle this problem, we will treat states 3 and 4 as *absorbing states* and consider the alternative TPM:

$$P^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Note that the first two rows of P^* are identical to the first two rows of P, indicating that transitions emanating from either states 1 or 2 have not been altered probabilistically. Using this alternative TPM, we can easily identify

$$Q = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}.$$

To determine the desired probability, we wish to calculate $U_{1,3}$, which can be determined by solving the following linear system:

$$U_{1,3} = 0.2 + 0.4U_{1,3} + 0.3U_{2,3} \implies 0.6U_{1,3} - 0.3U_{2,3} = 0.2, \tag{3.2}$$
$$U_{2,3} = 0.2 + 0.2U_{1,3} + 0.2U_{2,3} \implies 0.2U_{1,3} - 0.8U_{2,3} = -0.2. \tag{3.3}$$

Then, (3.2) $- 3 \times$ (3.3) $\implies 2.1U_{2,3} = 0.8 \implies U_{2,3} = 8/21 \approx 0.381$. Thus,

$$\begin{aligned} 0.2U_{1,3} - 0.8\left(\frac{8}{21}\right) &= -0.2 \\ 0.2U_{1,3} &= \frac{11}{105} \\ U_{1,3} &= \frac{11}{21} \approx 0.524. \end{aligned}$$

(b) Again, we can use the alternative TPM P^* introduced in part (a) to determine w_1, the mean number of transitions until either state 3 or state 4 is entered (starting from transient state 1). The values w_1 and w_2 satisfy:

$$w_1 = 1 + 0.4w_1 + 0.3w_2 \implies 0.6w_1 - 0.3w_2 = 1, \tag{3.4}$$
$$w_2 = 1 + 0.2w_1 + 0.2w_2 \implies 0.2w_1 - 0.8w_2 = -1. \tag{3.5}$$

Then, (3.4) $- 3 \times$ (3.5) $\implies 2.1w_2 = 4 \implies w_2 = 4/2.1 = 40/21 \approx 1.905$. Thus,

$$\begin{aligned} 0.2w_1 - 0.8\left(\frac{40}{21}\right) &= -1 \\ 0.2w_1 &= \frac{32}{21} - 1 \\ w_1 &= \frac{55}{21} \approx 2.619. \end{aligned}$$

Exercise 3.5.4. Consider a DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.3 & 0.4 & 0.2 & 0 & 0.1 \\ 0.4 & 0.1 & 0.1 & 0.1 & 0.3 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Suppose that the DTMC is equally likely to begin in state 0 or state 1. What is the probability that the DTMC ultimately ends up in state i, $i = 2, 3, 4$?

Solution: Let us first group states 2 and 3 together as a single state, which we will denote by 3^*. As a result of this grouping, our revised TPM has the form

$$\begin{array}{c} \\ 0 \\ 1 \\ 3^* \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 3^* & 4 \end{array} \\ \left[\begin{array}{cc|cc} 0.3 & 0.4 & 0.2 & 0.1 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

From the above TPM, we identify

$$Q = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.3 & 0.4 \\ 0.4 & 0.1 \end{bmatrix} \end{array}, \quad R = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 3^* & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \end{array}, \text{ and } U = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 3^* & 4 \end{array} \\ \begin{bmatrix} U_{0,3^*} & U_{0,4} \\ U_{1,3^*} & U_{1,4} \end{bmatrix} \end{array}.$$

Since $U = (I - Q)^{-1}R$, we obtain

$$U = \frac{1}{0.63 - 0.16} \begin{bmatrix} 0.9 & 0.4 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 3^* & 4 \end{array} \\ \begin{bmatrix} 0.553191 & 0.446809 \\ 0.468085 & 0.531915 \end{bmatrix} \end{array}.$$

Let A_i denote the event that the DTMC ultimately ends up in state i, $i = 2, 3, 4$. We immediately see that

$$\begin{aligned} P(A_4) &= P(X_0 = 0)P(A_4|X_0 = 0) + P(X_0 = 1)P(A_4|X_0 = 1) \\ &= \left(\frac{1}{2}\right) U_{0,4} + \left(\frac{1}{2}\right) U_{1,4} \\ &= \left(\frac{1}{2}\right)(0.446809) + \left(\frac{1}{2}\right)(0.531915) \\ &= 0.489362. \end{aligned}$$

However, if absorbed into state 3^*, the DTMC will remain in recurrent class $\{2, 3\}$ with associated TPM

$$\begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \end{array}.$$

Clearly, the conditions of the BLT are met for this "smaller" DTMC, implying that the limiting probabilities π_2 and π_3 satisfy

$$\begin{aligned}
\pi_2 &= 0.8\pi_2 + 0.5\pi_3, \\
\pi_3 &= 0.2\pi_2 + 0.5\pi_3 \Longrightarrow \pi_3 = \frac{2}{5}\pi_2, \\
1 &= \pi_2 + \pi_3 \Longrightarrow \pi_2 = \frac{5}{7} \text{ and } \pi_3 = \frac{2}{7}.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
P(A_2) &= P(X_0 = 0)P(A_2|X_0 = 0) + P(X_0 = 1)P(A_2|X_0 = 1) \\
&= \left(\frac{1}{2}\right) U_{0,3^*} \cdot \pi_2 + \left(\frac{1}{2}\right) U_{1,3^*} \cdot \pi_2 \\
&= \left(\frac{1}{2}\right) (0.553191) \left(\frac{5}{7}\right) + \left(\frac{1}{2}\right) (0.468085) \left(\frac{5}{7}\right) \\
&= 0.364742
\end{aligned}$$

and

$$\begin{aligned}
P(A_3) &= P(X_0 = 0)P(A_3|X_0 = 0) + P(X_0 = 1)P(A_3|X_0 = 1) \\
&= \left(\frac{1}{2}\right) U_{0,3^*} \cdot \pi_3 + \left(\frac{1}{2}\right) U_{1,3^*} \cdot \pi_3 \\
&= \left(\frac{1}{2}\right) (0.553191) \left(\frac{2}{7}\right) + \left(\frac{1}{2}\right) (0.468085) \left(\frac{2}{7}\right) \\
&= 0.145897.
\end{aligned}$$

Exercise 3.5.5. Consider a DTMC $\{X_n, n \in \mathbb{N}\}$ with TPM

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & p & 0 & 1-p \\ q & 1-q & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & p & 1-p & 0 \end{bmatrix} \end{array},$$

where $0 < p < 1$ and $0 \leq q \leq 1$. Suppose that the DTMC is *equally likely* to begin at time 0 in either state 2 or state 3.

(a) Assuming that $q \neq 0$, prove that a limiting distribution exists for this DTMC.

(b) Assuming that $q = p$, determine the limiting probabilities (in terms of p) of this DTMC.

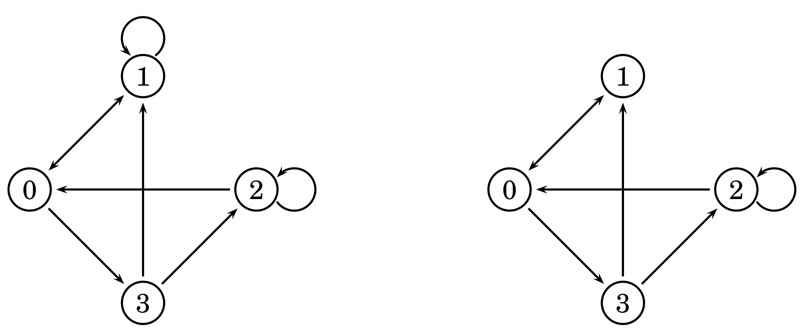
(c) Assuming that $q = 0$, determine an explicit expression (in terms of p) for the mean number of times state 0 is visited prior to reaching state 1.

(d) Let W represent the number of transitions needed to reach either state 0 or state 1, whichever happens first. Determine E$[W]$.

(e) Find the pmf of W from part (d) and then use it to determine E$[W]$.

Solution: (a)

State Transition Diagram



$q < 1$ $\qquad\qquad$ $q = 1$

In both cases above, all states communicate with each other. Moreover, since $P_{2,2} = p > 0$, it follows that $d(2) = 1$. Thus, the DTMC is aperiodic. Finally, since this is a finite-state DTMC, it is (positive) recurrent. By the BLT, a limiting probability distribution exists.

(b) Using the BLT with $q = p$, $\underline{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3)$ satisfies the following system of linear equations:

$$\begin{aligned}
\pi_0 &= p\pi_1 + (1-p)\pi_2, \\
\pi_1 &= p\pi_0 + (1-p)\pi_1 + p\pi_3 \Longrightarrow \pi_1 = \pi_0 + \pi_3 \\
\pi_2 &= p\pi_2 + (1-p)\pi_3 \Longrightarrow \pi_3 = \pi_2, \\
\pi_3 &= (1-p)\pi_0, \\
1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3.
\end{aligned}$$

Note that the last equation gives rise to

$$\begin{aligned}
\pi_0 + \pi_0 + \pi_3 + (1-p)\pi_0 + (1-p)\pi_0 &= 1 \\
2\pi_0 + (1-p)\pi_0 + 2(1-p)\pi_0 &= 1 \\
(5-3p)\pi_0 &= 1 \\
\pi_0 &= \frac{1}{5-3p}.
\end{aligned}$$

Therefore, it follows that $\underline{\pi} = \left(\frac{1}{5-3p}, \frac{2-p}{5-3p}, \frac{1-p}{5-3p}, \frac{1-p}{5-3p}\right)$.

(c) We will treat state 1 as absorbing and consider the modified TPM

$$P^* = \begin{array}{c} \\ 0 \\ 2 \\ 3 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 2 & 3 & 1 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 1-p & 0 & p \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array},$$

along with corresponding initial conditions given by $\underline{\alpha}_0^* = (0, 1/2, 1/2, 0)$. Now, we wish to calculate $S_{0,0}$, $S_{2,0}$, and $S_{3,0}$, which satisfy the linear system of equations:

$$\begin{aligned}
S_{0,0} &= 1 + (1-p)S_{3,0}, \\
S_{2,0} &= (1-p)S_{0,0} + pS_{2,0} \Longrightarrow S_{2,0} = S_{0,0}, \\
S_{3,0} &= (1-p)S_{2,0} \Longrightarrow S_{3,0} = (1-p)S_{0,0}.
\end{aligned}$$

Using the first equation, we obtain

$$S_{0,0} = 1 + (1-p)^2 S_{0,0} \Longrightarrow S_{0,0} = \frac{1}{1-(1-p)^2}.$$

and this subsequently leads to $S_{2,0} = \frac{1}{1-(1-p)^2}$ and $S_{3,0} = \frac{1-p}{1-(1-p)^2}$. Therefore,

$$\begin{aligned}
\text{Mean number of times state 0 is visited prior to reaching state 1} &= \frac{1}{2}S_{2,0} + \frac{1}{2}S_{3,0} \\
&= \frac{2-p}{2(1-(1-p)^2)} \\
&= \frac{2-p}{2(2p-p^2)} \\
&= \frac{1}{2p} \ .
\end{aligned}$$

(d) Since we are only concerned in the first time the DTMC enters either state 0 or state 1, let us combine both states into a single <u>absorbing</u> state and call it state 0^*. As such, the "new" TPM we wish to work with is of the form

$$P^{**} = \begin{array}{c|ccc} & 2 & 3 & 0^* \\ \hline 2 & p & 0 & 1-p \\ 3 & 1-p & 0 & p \\ 0^* & 0 & 0 & 1 \end{array}.$$

Using P^{**}, we wish to compute $\underline{w}' = (w_2, w_3)'$, since $\mathrm{E}[W] = \frac{1}{2}w_2 + \frac{1}{2}w_3$. To calculate $\underline{w}'$, we must solve the following system of linear equations:

$$\begin{aligned}
w_2 = 1 + pw_2 \implies w_2 &= \frac{1}{1-p}, \\
w_3 = 1 + (1-p)w_2 \implies w_3 &= 1 + \frac{1-p}{1-p} = 2.
\end{aligned}$$

Therefore, we simply obtain

$$\mathrm{E}[W] = \frac{1}{2} \cdot \frac{1}{1-p} + \frac{1}{2} \cdot 2 = \frac{1+2-2p}{2(1-p)} = \frac{3-2p}{2(1-p)}.$$

(e) We wish to determine $p_W(w) = P(W = w)$, $w = 1, 2, 3, \ldots$. To do so, we begin by conditioning on the state of the DTMC at time 0 to get

$$\begin{aligned}
P(W = w) &= P(X_0 = 2)P(W = w|X_0 = 2) + P(X_0 = 3)P(W = w|X_0 = 3) \\
&= \frac{1}{2}P(W = w|X_0 = 2) + \frac{1}{2}P(W = w|X_0 = 3).
\end{aligned}$$

We remark that $W|(X_0 = 2) \sim Y$ where $Y \sim \mathrm{GEO}(1-p)$ (since from state 2, you next go to state 0 or remain in state 2). As a result, we end up with

$$\begin{aligned}
P(W=w) &= \frac{1}{2}P(Y=w) + \frac{1}{2}\Big(P(X_1=1|X_0=3)P(W=w|X_1=1,X_0=3) \\
&\quad + P(X_1=2|X_0=3)P(W=w|X_1=2,X_0=3)\Big) \\
&= \frac{1}{2}p^{w-1}(1-p) + \frac{1}{2}\Big(pP(W=w|X_1=1,X_0=3) \\
&\quad + (1-p)P(W=w|X_1=2,X_0=3)\Big).
\end{aligned}$$

However, note that $W|(X_1=1,X_0=3)$ is equal to 1 with probability 1, which implies that $P(W=w|X_1=1,X_0=3)=\delta_{1,w}$. On the other hand, $W|(X_1=2,X_0=3)\sim 1+Y$ where $Y\sim \mathrm{GEO}(1-p)$ (since absorption to either state 0 or state 1 cannot possibly occur on the first transition). Therefore, we obtain

$$P(W=w) = \frac{1}{2}p^{w-1}(1-p) + \frac{1}{2}\Big(p\delta_{1,w} + (1-p)P(Y=w-1)\Big),\ w=1,2,3,\ldots\ .$$

This ultimately leads to

$$P(W=1) = \frac{1}{2}p^0(1-p) + \frac{1}{2}\Big(p\delta_{1,1} + (1-p)P(Y=0)\Big) = \frac{1-p}{2} + \frac{p}{2} = \frac{1}{2}$$

and

$$\begin{aligned}
P(W=w) &= \frac{1}{2}p^{w-1}(1-p) + \frac{1}{2}(1-p)p^{(w-1)-1}(1-p) \\
&= \frac{1}{2}p^{w-1}(1-p) + \frac{1}{2}(1-p)^2p^{w-2} \\
&= \frac{1}{2}p^{w-2}(1-p)(p+1-p) \\
&= \frac{1}{2}p^{w-2}(1-p),\ w=2,3,4,\ldots\ .
\end{aligned}$$

Therefore, using this pmf, we obtain

$$\begin{aligned}
\mathrm{E}[W] &= \sum_{w=1}^{\infty} wP(W = w) \\
&= 1 \cdot \frac{1}{2} + \sum_{w=2}^{\infty} w \cdot \frac{1}{2} p^{w-2}(1-p) \\
&= \frac{1}{2} + \frac{1}{2} \sum_{x=1}^{\infty} (x+1) p^{x-1}(1-p) && \text{with } x = w - 1 \\
&= \frac{1}{2} + \frac{1}{2}\mathrm{E}[Y+1] && \text{where } Y \sim \mathrm{GEO}(1-p) \\
&= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{1-p} + 1\right) \\
&= \frac{1 - p + 1 + 1 - p}{2(1-p)} \\
&= \frac{3 - 2p}{2(1-p)},
\end{aligned}$$

which agrees with our result in part (d).

Exercise 3.5.6. Consider a gambler who, at each play of a game, rolls a fair six-sided die. Assume that successive plays of the game are independent of each other. If the gambler rolls a 5 or a 6, one unit is won. If the gambler rolls a 2, 3, or 4, no units are won or lost. However, if a 1 is rolled, then the gambler loses **all** of their units and goes bankrupt. The gambler will continue playing the game until their fortune either reaches 0 units or N units ($N \in \mathbb{Z}^+$), where N represents the jackpot. For $n \in \mathbb{N}$, define X_n as the gambler's fortune after the n^{th} play of the game. Suppose that the gambler starts with one unit (i.e., $X_0 = 1$ with probability 1).

(a) Specify the form of the TPM for the DTMC $\{X_n, n \in \mathbb{N}\}$.

(b) What is the probability that the gambler wins the jackpot before going bankrupt?

(c) Assume now that the jackpot is $N = 3$ units. How many times should the gambler expect to have their fortune be at 2 units before they stop playing? How many games should the gambler expect to play in total before going bankrupt or winning the jackpot?

(d) Assuming again that $N = 3$, what is the probability that the gambler's fortune never increases higher than 1 unit?

Solution: (a) Based on the given information, we can form the TPM as follows:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-2 \\ N-1 \\ N \\ 0 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & \cdots & N-2 & N-1 & N & 0 \end{array} \\ \left[\begin{array}{ccccccccc} 1/2 & 1/3 & 0 & 0 & \cdots & 0 & 0 & 0 & 1/6 \\ 0 & 1/2 & 1/3 & 0 & \cdots & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 1/2 & 1/3 & \cdots & 0 & 0 & 0 & 1/6 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1/2 & 1/3 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

Although not necessary, note that, for convenience, we have reordered the states so that the two absorbing states are adjacent to each other in the TPM.

(b) Our goal is to calculate $U_{1,N}$, which represents the probability of interest. For a transient state i and an absorbing state k, we know that $U_{i,k} = R_{i,k} + \sum_{j=1}^{N-1} Q_{i,j} U_{j,k}$, resulting in the following linear system of equations:

$$\begin{aligned} U_{1,N} &= \frac{1}{2}U_{1,N} + \frac{1}{3}U_{2,N}, \\ U_{2,N} &= \frac{1}{2}U_{2,N} + \frac{1}{3}U_{3,N}, \\ U_{3,N} &= \frac{1}{2}U_{3,N} + \frac{1}{3}U_{4,N}, \\ &\vdots \\ U_{i,N} &= \frac{1}{2}U_{i,N} + \frac{1}{3}U_{i+1,N}, \;\; i = 1, 2, \ldots, N-2, \\ &\vdots \\ U_{N-1,N} &= \frac{1}{3} + \frac{1}{2}U_{N-1,N}. \end{aligned}$$

Clearly, from this last equation, we have that

$$\frac{1}{2}U_{N-1,N} = \frac{1}{3} \Longrightarrow U_{N-1,N} = \frac{2}{3}.$$

In addition, for $i = 1, 2, \ldots, N-2$, $\frac{1}{2}U_{i,N} = \frac{1}{3}U_{i+1,N}$ and so

$$U_{i,N} = \frac{2}{3}U_{i+1,N} = \left(\frac{2}{3}\right)^2 U_{i+2,N} = \cdots = \left(\frac{2}{3}\right)^{N-1-i} U_{i+(N-1-i),N} = \left(\frac{2}{3}\right)^{N-1-i} U_{N-1,N} = \left(\frac{2}{3}\right)^{N-i}.$$

Finally, letting $i = 1$, we find that $U_{1,N} = (2/3)^{N-1}$.

(c) We wish to calculate $S_{1,2}$ and w_1. When $N = 3$, our TPM in part (a) becomes

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 0 \end{array} \\ \begin{bmatrix} 1/2 & 1/3 & 0 & 1/6 \\ 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Recall that $S = (I - Q)^{-1}$, where

$$I - Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 1/3 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/3 \\ 0 & 1/2 \end{bmatrix}.$$

Therefore, it follows that

$$(I - Q)^{-1} = \frac{1}{(1/2)(1/2) - (-1/3)(0)} \begin{bmatrix} 1/2 & 1/3 \\ 0 & 1/2 \end{bmatrix} = \frac{1}{1/4} \begin{bmatrix} 1/2 & 1/3 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 2 & 4/3 \\ 0 & 2 \end{bmatrix},$$

and so $S_{1,2} = 4/3 \approx 1.333$. Next, recall that $\underline{w}' = (I - Q)^{-1}\underline{e}' = S\underline{e}'$, and so

$$\underline{w}' = \begin{bmatrix} 2 & 4/3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 2 \end{bmatrix}.$$

Thus, $w_1 = 10/3 \approx 3.333$.

(d) We wish to calculate $1 - f_{1,2}$. Using the results from part (c), note that

$$f_{1,2} = \frac{S_{1,2} - \delta_{1,2}}{S_{2,2}} = \frac{4/3 - 0}{2} = \frac{2}{3}.$$

Thus, the probability that the gambler's fortune never increases higher than 1 unit equals $1 - \frac{2}{3} = \frac{1}{3}$. Alternatively, this is the same as the probability that the DTMC is absorbed into state 0 before visiting state 2. Treating state 2 as an absorbing state, we consider the modified TPM

$$P^* = \begin{array}{c} \\ 1 \\ 2 \\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 0 \end{array} \\ \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}.$$

The probability of the gambler's fortune never increasing higher than 1 unit is the same as the probability of being absorbed into state 0 in this modified TPM, namely $U_{1,0}$. Therefore,

$$U_{1,0} = R_{1,0} + Q_{1,1}U_{1,0} = \frac{1}{6} + \frac{1}{2}U_{1,0} \Longrightarrow \frac{1}{2}U_{1,0} = \frac{1}{6} \Longrightarrow U_{1,0} = \frac{1}{3}.$$