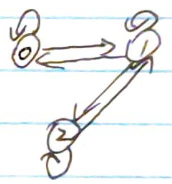
1. I have carefully read the integrity statement
Muchi Wang

2. (a) $\begin{cases} \frac{1}{2}\pi(0) + \frac{1}{4}\pi(1) = \pi(0) \\ \frac{1}{2}\pi(0) + \frac{2}{3}\pi(1) + \frac{1}{6}\pi(2) = \pi(1) \\ \frac{1}{12}\pi(1) + \frac{5}{6}\pi(2) = \pi(2) \\ \pi(0) + \pi(1) + \pi(2) = 1 \end{cases}$

$\Rightarrow \pi = [\frac{1}{4} \ \frac{1}{2} \ \frac{1}{4}]$

Since π is the only solution it is unique.

(b) Since $i \to j \ \forall i,j \in S$, the chain is irreducible



Since $P_{ii} > 0 \ \forall i \in S$, the chain is aperiodic

By part (a), stationary distribution exists

By theorem,

$\lim_{n\to\infty} P(X_n = j \mid X_0 = i) = \pi(j) \quad \forall j \in S$

$= \begin{cases} \frac{1}{4} & j=0 \\ \frac{1}{2} & j=1 \\ \frac{1}{4} & j=2 \end{cases}$

(c) Since I (irreducible), S (stationary distribution exists)

$E_y(T_y) = \frac{1}{\pi(y)}$

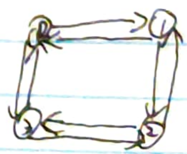
$E_0(T_0) = \frac{1}{\pi(0)} = 4$

3. (a) $P(X_0=0, X_3=1) = P(X_3=1 \mid X_0=0)P(X_0=0) = P_{01}^3 \mu(0) = \frac{77}{120} \times \frac{1}{2} = \frac{77}{240}$

(b) $P(X_{11}=0 \mid X_9=2, X_0=3) = P(X_{11}=0 \mid X_9=2) = P_{20}^2 = \frac{11}{20}$

Markov property

(c) Assume this MC is at state X_n,



X_{n+1} is even if X_n is odd

X_{n+1} is odd if X_n is even

$\sum_{n=0}^{9} X_n$ has odd number of odd terms if $X_0 = 3$

Hence $\sum_{n=0}^{9} X_n$ is odd

$P(\sum_{n=0}^{9} X_n \text{ is even} \mid X_0 = 3) = 0$

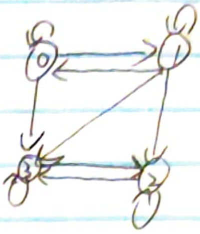
4. (a) $\{0,1\}$, $\{2,3\}$

~~{0,1} is transient~~

Since $\{2,3\}$ is closed and $P_{03}>0$

$P_{00}<1$ ($0\to 3$ and never exits $\{2,3\}$)

So $\{0,1\}$ is transient



Since $\{2,3\}$ is closed an finite,

$\{2,3\}$ is recurrent by corollary, $\{2,3\}$ is positive recurrent

(b) $A=\{2\}$, $B=\{3\}$, $C=\{0,1\}$

$$\begin{cases} h(0)=\frac{1}{2}h(0)+\frac{1}{3}h(1) \\ h(1)=\frac{1}{4}h(0)+\frac{1}{2}h(1)+\frac{1}{6} \end{cases}$$ where $h(2)=1$ and $h(3)=0$

$h=[\frac{1}{3}\quad \frac{1}{2}]$

The prob. is $\frac{1}{3}$

(c) $A=\{2,3\}$ $C=\{0,1\}$

$$\begin{cases} g(0)=1+\frac{1}{2}g(0)+\frac{1}{3}g(1) \\ g(1)=1+\frac{1}{4}g(0)+\frac{1}{2}g(1) \end{cases}$$

$$g=\begin{array}{cc} 0 & 1 \\ \end{array}$$
$$g=[5\quad \tfrac{9}{2}]$$

The expected time is $\frac{9}{2}$

(d) ~~Let~~ $A=\{3\}$ $C=\{0,1,2\}$

$$\begin{cases} g(0)=1+\frac{1}{2}g(0)+\frac{1}{3}g(1) \\ g(1)=1+\frac{1}{4}g(0)+\frac{1}{2}g(1)+\frac{1}{6}g(2) \\ g(2)=1+\frac{2}{3}g(2) \end{cases}$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ \end{array}$$
$$g=[6\quad 6\quad 3]$$

Let T_3^i be the i-th time that it visits state 3.

~~$E_1(T_3^3) = \ldots$~~

~~$= 6 + \frac{2}{3}E_2(T_3^2) + \frac{1}{3}E(T_3^2)$ Markov property~~

~~$+ \frac{2}{3}(3+1+\frac{3}{2}E_2(T_3^1))$~~

~~$= 7 + \frac{2}{3}(4+\frac{3}{2}\times 3)$~~

~~$= \frac{?}{6}$~~

$E_1(T_3^3) = 6 + \frac{1}{3}E_2(T_3^2) + \frac{2}{3}E_3(T_3^2) + 1$

$= 7 + \frac{1}{3}(3+1+\frac{1}{3}E_2(T_3)) + \frac{2}{3}(1+\frac{1}{3}E_2(T_3))$

$= 10$

5.(a) Assume $\rho_{ij} \neq 0$

since $\rho_{ij} > 0$ (ρ_{ij} is a probability, $\rho_{ij} \in [0,1]$)

and i is recurrent

$\rho_{jj} = 1$ by theorem

Since $\rho_{ij} > 0$ and $\rho_{jj} = 1 > 0$

i and j should be in the same class

But i is null recurrent and j is positive recurrent

$\Rightarrow i$ and j are not in the same class

By contradiction $\rho_{ij} = 0$

(b) $\rho_{jj} = P(T_j < \infty \mid X_0 = j)$

$= \sum_{i \in S \setminus \{j\}} P(T_j < \infty \mid X_0 = j, X_1 = i)\, P(X_1 = i \mid X_0 = j) + P(T_j < \infty \mid X_0 = j, X_1 = j)\, P(X_1 = j \mid X_0 = j)$

$= \sum_{i \in S \setminus \{j\}} P(T_j < \infty \mid X_1 = i)\, P_{ji} + P(T_j < \infty \mid X_0 = j, X_1 = j)\, P_{jj}$

If $P_{jj} = 0$

$\rho_{jj} = \sum_{i \in S \setminus \{j\}} \rho_{ij} P_{ji}$ Markov property

$= \sum_{i \in S \setminus \{j\}} P_{ji}$ ($\rho_{ij} = 1$)

$= 1$ by defn and $P_{jj} = 0$

If $P_{jj} > 0$

$\rho_{jj} = \sum_{i \in S \setminus \{j\}} \rho_{ij} P_{ji} + P_{jj}$ ($P(T_j < \infty \mid X_0 = j, X_1 = j) = 1$ if $P_{jj} > 0$)

$= \sum_{i \in S \setminus \{j\}} P_{ji} + P_{jj}$

$= \sum_{i \in S} P_{ji}$

$= 1$ by defn

Hence j is recurrent

6.(a) $P(X_1=0) \overset{\vee}{=} p^3$ (since independence)

(b) $E(Y) = 1\cdot r + 2\cdot q = r+2q$

~~Since extinc~~

extinction happens for sure $\Rightarrow E(Y) \leq 1$

$E(Y) = r+2q = 1-p-q+2q = 1-p+q$

$1-p+q \leq 1$

$p \geq q$

$\varphi(s) = p+rs+qs^2$

$p+rs+qs^2 = s$

$qs^2+(r-1)s+p=0$

$qs^2-(q+p)s+p=0$

$$s = \frac{q+p \pm \sqrt{(q+p)^2-4qp}}{2q}$$

$$= \frac{q+p \pm \sqrt{(q-p)^2}}{2q}$$

$$= \frac{q+p \pm (q-p)}{2q}$$ since $q-b>0$

$S = 1$ or $\frac{p}{q}$

since $s \in (0,1)$

$s = \frac{p}{q}$

(c) $S = \{0,1,2,4,6,\cdots\}$

$\{0\}$ is the obsorbing state

$\{0\}$ is recurrent

since $P_{11}=0$ (The # of offsprings is even)

$\{1\}$ is transient

Let $i \in S\setminus\{0,1\}$,

since $P_{i0}>0$ and 0 is the obsorbing state

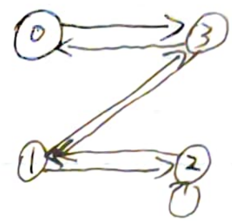
$P_{ii}<1$

so $\{2,4,6,\cdots\}$ is transient

7. (a) Let X_n be the # of umbrellas at current location

$S=\{0,1,2,3\}$

$$P=\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1-p & p \\ 2 & 0 & 1-p & p & 0 \\ 3 & 1-p & p & 0 & 0 \end{array}$$



$$\begin{cases} \pi(0)=(1-p)\pi(3) \\ \pi(1)=(1-p)\pi(2)+p\pi(3) \\ \pi(2)=(1-p)\pi(1)+p\pi(2) \\ \pi(3)=\pi(0)+p\pi(1) \\ \pi(0)+\pi(1)+\pi(2)+\pi(3)=1 \end{cases}$$

$\pi=\left[\frac{1-p}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}\right]$

Since $i \to j$ $\forall i,j \in S$,

this MC is irreducible

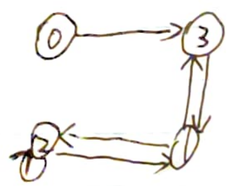
$I, S \Rightarrow R$

By theorem $\frac{N_n(0)}{n} = \pi(0) = \frac{1-p}{4-p}$

long-run percentage of time that the professor gets wet is

$p\frac{1-p}{4-p} = \frac{p-p^2}{4-p}$

(c) $$P=\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1-p & p \\ 2 & 0 & 1-p & p & 0 \\ 3 & 0 & 1 & 0 & 0 \end{array}$$



$$\begin{cases} \pi(0)=0 \\ \pi(1)=(1-p)\pi(2)+\pi(3) \\ \pi(2)=(1-p)\pi(1)+p\pi(2) \\ \pi(3)=\pi(0)+p\pi(1) \\ \pi(0)+\pi(1)+\pi(2)+\pi(3)=1 \end{cases}$$

$\pi(3)+p\pi(1)+p\pi(2)$

$=\frac{p}{2+p}+\frac{p}{2+p}+\frac{p}{2+p}$

$=\frac{3p}{2+p}$

$\pi=[0 \ \frac{1}{2+p} \ \frac{1}{2+p} \ \frac{p}{2+p}]$

Since the state 0 is only visited initially and it does not affect the long-run percentage, we can just remove it

$\pi=[\frac{1}{2+p} \ \frac{1}{2+p} \ \frac{p}{2+p}]$

since $i \to j$ $\forall i,j \in \{1,2,3\}$

this MC is irreducible

$I, S \Rightarrow R$

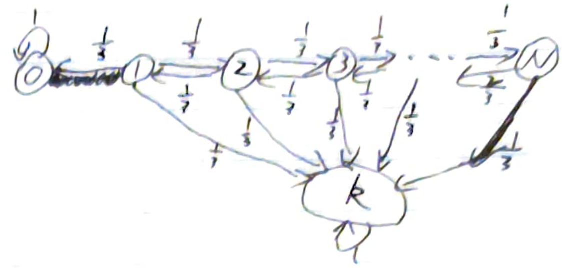
By theorem $\frac{N_n(i)}{n}=\pi(i)$

The long-run percentage of time that the professor ~~gets we~~ brings an umbrella is

8.(a) $\{0\}$ is positive recurrent since $P_{00}=1$ and $E_0(T_0)=1$

$\{k\}$ is positive recurrent since $P_{kk}=1$ and $E_k(T_k)=1$

$\{1,2,3,\cdots,N\}$ is transient since $\forall i\in\{1,2,\cdots,N\}$ $P_{ik}>0$ and k is absorbing so $P_{ii}<1$



(b) $P(X_4=1|X_0=1)=P_{11}^4=\sum_{i\in S} P_{1i}^2 P_{i1}^2$

$P_{1i}^2=[\frac{1}{3}\ \ \frac{1}{9}\ \ 0\ \ \frac{1}{9}\ \ 0\ \cdots\ 0\ \ \frac{4}{9}]$ if $N>3$

$P_{i1}^2=[0\ \ \frac{1}{9}\ \ 0\ \ \frac{1}{9}\ \ 0\ \cdots\ 0]^T$ if $N>3$

$P_{11}^4=\frac{1}{81}+\frac{1}{81}$ (if $N>3$)

$=\frac{2}{81}$

$P_{1i}^2=[\frac{1}{3}\ \ \frac{1}{9}\ \ 0\ \ \frac{1}{9}\ \ \frac{4}{9}]$ if $N=3$

$P_{i1}^2=[0\ \ \frac{1}{9}\ \ 0\ \ \frac{2}{9}\ \ 0]$ if $N=3$

$P_{11}^4=\frac{1}{81}+\frac{2}{81}$

$=\frac{1}{27}$

(c) $P=\begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & k \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ k \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

$A=\{0\}$, $B=\{k\}$, $C=\{1,2,3\}$

$\begin{cases} h(1)=\frac{1}{3}+\frac{1}{3}h(2) \\ h(2)=\frac{1}{3}h(1)+\frac{1}{3}h(3) \\ h(3)=\frac{2}{3}h(2) \end{cases}$ $(h(0)=1, h(k)=0)$

$h=\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ [\frac{7}{18}\ \ \frac{1}{6}\ \ \frac{1}{9}] \end{array}$

The probability is $\frac{1}{6}$

9. (a) $P(X_3 \neq 0, X_2 = 0 \mid X_0 = 1)$

$= P(X_3 = 1, X_2 = 0 \mid X_0 = 1) + P(X_3 = 2, X_2 = 0 \mid X_0 = 1)$

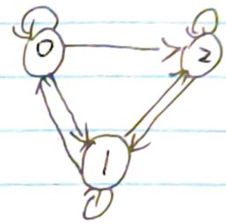
$= \dfrac{P(X_3 = 1, X_2 = 0, X_0 = 1) + P(X_3 = 2, X_2 = 0, X_0 = 1)}{P(X_0 = 1)}$

$= \dfrac{P(X_3 = 1 \mid X_2 = 0)\, P(X_2 = 0 \mid X_0 = 1)\, P(X_0 = 1) + P(X_3 = 2 \mid X_2 = 0)\, P(X_2 = 0 \mid X_0 = 1)\, P(X_0 = 1)}{P(X_0 = 1)}$ Markov Property

$= P_{01} P_{10}^2 + P_{02} P_{10}^2$

$= \frac{1}{4} \times \frac{3}{16} + \frac{1}{4} \times \frac{3}{16}$

$= \frac{3}{32}$



(b) $\begin{cases} \pi(0) = \frac{1}{2}\pi(0) + \frac{1}{4}\pi(1) \\ \pi(1) = \frac{1}{4}\pi(0) + \frac{1}{4}\pi(1) + \frac{5}{8}\pi(2) \\ \pi(2) = \frac{1}{4}\pi(0) + \frac{1}{2}\pi(1) + \frac{3}{8}\pi(2) \end{cases}$

$\pi = \begin{bmatrix} \pi(0) \\ \pi(1) \\ \pi(2) \end{bmatrix}^T = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} x \right)^T$ where $x \in \mathbb{R}$ (This gives all the stationary measure of P)

since this chain is irreducible ($i \to j \ \forall i, j \in S$)

and recurrent (finite and closed)

By theorem, $\mu_0(1) = E\left(\sum_{n=0}^{T_0 - 1} \mathbb{1}_{\{X_n = 1\}}\right)$,

$x = 1$ $\quad \mu_0(0) = 1$ $\qquad \pi(0) = 1 = \mu_0(0)$

since is still a stationary measure and

$E\left(\sum_{n=0}^{T_0 - 1} \mathbb{1}_{\{X_n = 1\}}\right) = 2$

(c) $P(X_n \neq 1, X_{n-1} \neq 1, \cdots, X_1 \neq 1 \mid X_0 = 0)$

$= \left(\frac{1}{2}\right)^n + \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \frac{1}{4} \left(\frac{3}{8}\right)^{n-k-1}$ (the chain stays at 0 all the time or stays at 0 k times, move to 2 and stays at 2 for the rest of the times)

$= 2^{-n} + \frac{1}{4}\left(\frac{3}{8}\right)^{n-1} \sum_{k=0}^{n-1} \left(\frac{4}{3}\right)^k$

$= 2^{-n} + \frac{1}{4}\left(\frac{3}{8}\right)^{n-1} \dfrac{1 - \left(\frac{4}{3}\right)^n}{-\frac{1}{3}}$

$= 2^{-n} - \frac{3}{4}\left(\frac{3}{8}\right)^{n-1} + 2^{-n+1}$

$= \left(\frac{1}{2}\right)^n \cdot 3 - \left(\frac{3}{4}\right)^n \cdot \frac{2}{2^n}$

$= \left(\frac{1}{2}\right)^n \left(3 - 2\left(\frac{3}{4}\right)^n\right)$

Hence $a = \frac{1}{2}$ $\quad b = \frac{3}{4}$

10.(a) The chain is ~~ined~~ ireducible
Since ~~the~~ S is finite
By theorem, at least one of the states is ~~re~~ positive recurrent
~~Hence~~ Since it is ireducible
S is positive recurrent
By observation, all the cycles have even lengths
So the # of steps it takes to go back to the same state is even
Hence the period of all states is 2
It is not aperiodic

(b) $A=\{7\}$ $\quad C=\{0,1,2,3,4,5,6\}$

$$\begin{cases} g(0)=1+\frac{1}{3}g(1)+\frac{1}{3}g(3)+\frac{1}{3}g(5) \\ g(1)=1+\frac{1}{3}g(0)+\frac{1}{3}g(2)+\frac{1}{3}g(4) \\ g(2)=1+\frac{1}{3}g(1)+\frac{1}{3}g(3) \\ g(3)=1+\frac{1}{3}g(0)+\frac{1}{3}g(2)+\frac{1}{3}g(6) \\ g(4)=1+\frac{1}{3}g(1)+\frac{1}{3}g(5) \\ g(5)=1+\frac{1}{3}g(0)+\frac{1}{3}g(4)+\frac{1}{3}g(6) \\ g(6)=1+\frac{1}{3}g(3)+\frac{1}{3}g(5) \\ g(7)=1 \end{cases}$$

$g(0)=10$ starting from 0

The expected number of steps until the chain reaches state 7 for the first time is 10

11.(a) Define a distribution π where $\pi(i) = P_{ai}$ $(a \in S)$

$\sum_{i \in S} \pi(i) P_{ij}$, $j \in S$

$= \sum_{i \in S} P_{ai} P_{ij}$

$= P^2_{aj}$

$= P_{aj}$ $(P = P^2)$

$= \pi(j)$

$\sum_{i \in S} \pi(i) = \sum_{i \in S} P_{ai} = 1$ by defn of DTMC

Hence π is a stationary distribution

Since it is irreducible

By theorem, $\{X_n\}_{n=0,1,2,\cdots}$ is positive recurrent

(b) Since $P = P^2$

$\Rightarrow P^n = P \quad \forall n \in \mathbb{R}$

By part (a), all states are positive recurrent

$\exists n, s.t. P^n_{ii} > 0 \quad \forall i \in S$

Hence $P_{ii} > 0 \quad \forall i \in S$

$\Rightarrow$ it is aperiodic

Since A (aperiodic), I (irreducible), S (stationary distribution exists from part (a))

$\lim_{n \to \infty} P^n_{xy} = \pi(y)$

$\pi(x) P_{xy} = \pi(x) \lim_{n \to \infty} P^n_{xy} = \pi(y)\pi(x) = \pi(y) \lim_{n \to \infty} P^n_{yx} = \pi(y) P_{yx}$

By theorem, it is time reversable

(c) By theorem, the chain can be partitioned into $T \cup R_1 \cup R_2 \cup \cdots$ where T are all the transient states and R_i, $i = 1, 2, \cdots$ are closed recurrent classes

① Assume $i, j \in S$ are not positive recurrent

$\Rightarrow \pi(i) = 0 = \pi(j)$ by theorem

$\Rightarrow \pi(i) P_{ij} = \pi(j) P_{ji}$

Which statisfies detail balance condition

② Assume i is positive recurrent and j is not

$\Rightarrow \pi(i) P_{ij} = \pi(i) 0 = 0$ since the class that i is in is closed $\Rightarrow i \not\to j$

$\Rightarrow \pi(j) P_{ji} = 0 P_{ji} = 0$ by theorem

③ Assume i and j are positive recurrent

If i and j are not in the same class, $\pi(i) P_{ij} = 0 = \pi(j) P_{ji}$ (closed)

else, since the class is irreducible, by part (b) it satisfies the detail balance condition

Hence, the chain is time reversable

12 (a) $\pi(i)=\sum_{j\in S}\pi(j)P_{ji}$

$$=\sum_{j=0}^{\infty}\pi(j)P_{ji}$$

Since $P_{ji}=p_{i-j}, 0<i\le j+1$, and $i>0$

$\Rightarrow P_{ji}=\begin{cases}p_{i-j} & -1<i-1\le j\\ 0 & \text{otherwise}\end{cases}$

$\Rightarrow \pi(i)=\sum_{j=i-1}^{\infty}\pi(j)p_{i-j}$

(b) $\Phi(\alpha)=q_0+q_1\alpha+q_2\alpha^2+\cdots$ is the generating function of $\{q_n\}$

$= p_1+p_0\alpha+p_{-1}\alpha^2+\cdots$

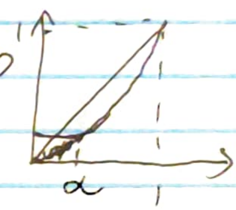
$\Rightarrow \Phi'(\alpha)=q_1+2q_2\alpha+3q_3\alpha^2+\cdots>0$ since all terms are non-negative and at least one term is positive, otherwise, $p_1, p_0, p_{-1},\cdots$ wouldn't be a distribution

$\Phi''(\alpha)=2q_2+6q_3\alpha+12q_4\alpha^2+\cdots>0$ for the same reason as above

$\Rightarrow \Phi(\alpha)$ is increasing and convex $\swarrow$ $\Phi(1)=\sum_{k=0}^{\infty}q_k=1$

$\Phi'(1)=p_0+2p_{-1}+3p_{-2}+\cdots=\sum_{k=0}^{\infty}kq_k>1$ since $k\in[0,\infty)$, $q_k>0$

Hence there exists an $\alpha\in(0,1)$, s.t. $\alpha=q_0+q_1\alpha+q_2\alpha^2+\cdots$



(c) Let $\pi\sim Geo(1-\alpha)$

By part (a) $\pi(i)=\sum_{j=1}^{\infty}\pi(j)p_{i-j}$, for $i>0$

$\pi(i)=\pi(i-1)p_1+\pi(i)p_0+\pi(i+1)p_{-1}+\cdots$

$(1-\alpha)\alpha^{i}=(1-\alpha)\alpha^{i-1}q_0+(1-\alpha)(1-\alpha)^{i}q_1+\cdots$

$\alpha=q_0+\alpha q_1+\alpha^2q_2+\cdots$

Since α is the solution, $\pi(i)=(\pi P)(i)$ for $i>0$

For $i=0$, $\pi(0)=\pi(0)\sum_{j\le 0}p_j+\pi(1)\sum_{j\le 0}p_{j-1}+\cdots$

$(1-\alpha)\sum_{j\le 0}p_j+(1-\alpha)\alpha\sum_{j\le 0}p_{j-1}+(1-\alpha)\alpha^2\sum_{j\le 0}p_{j-2}$

$=\sum_{j\le 0}\sum_{i=0}^{\infty}p_{j-i}\alpha^i\cdot(1-\alpha)$

$=\begin{bmatrix}p_0+p_{-1}\alpha+p_{-2}\alpha^2+p_{-3}\alpha^3+\cdots\\ +p_{-1}+p_{-2}\alpha+p_{-3}\alpha^2+\cdots\\ +p_{-2}+p_{-3}\alpha+\cdots\end{bmatrix}\cdot(1-\alpha)$

$=(1-\alpha)\sum_{i=1}^{\infty}q_i\sum_{v=0}^{i-1}\alpha^i$

$=(1-\alpha)\sum_{i=1}^{\infty}q_i\frac{1-\alpha^i}{1-\alpha}$

$=\sum_{i=1}^{\infty}q_i-\sum_{i=1}^{\infty}q_i\alpha^i$

$=1-q_0-(\alpha-q_0)$ by part (b)

$=1-\alpha$

$=\pi(0)$

Hence $\pi=\pi P$

Since π is already a distribution

$\sum_{i\in S}\pi(i)=1$

Hence $\pi\sim Geo(1-\alpha)$ is a stationary distribution

13. (a) $P(Y_{n+1}=j \mid Y_n=i)$

$$= P(X_{\varepsilon_1+\cdots+\varepsilon_n+\varepsilon_{n+1}}=j \mid X_{\varepsilon_1+\cdots+\varepsilon_n}=i)$$

$$= \sum_{k=1}^{\infty} P(X_{\varepsilon_1+\cdots+\varepsilon_n+\varepsilon_{n+1}}=j \mid X_{\varepsilon_1+\cdots+\varepsilon_n}=i, \varepsilon_{n+1}=k)\, P(\varepsilon_{n+1}=k)$$

$$= \sum_{k=1}^{\infty} P_{ij}^{k} p_k$$

(b) Since $\{X_n\}_{n=0,1,\cdots}$ is irreducible

$\exists\, n$, s.t. $P_{ij}^{n} > 0$, $\forall i,j \in S$

Let P' be the transition matrix of $\{Y_n\}_{n=0,1,\cdots}$

$P'_{ij} = \sum_{k=1}^{\infty} P_{ij}^{k} p_k \geq P_{ij}^{n} p_n > 0 \quad (p_n > 0)$

$i \to j$, $\forall i,j \in S \Longrightarrow \{Y_n\}_{n=0,1,\cdots}$ is irreducible

By the proof from part (c) and the chain is irreducible

$\Longrightarrow \{Y_n\}_{n=0,1,\cdots}$ is positive recurrent

(c) The stationary distribution of $\{Y_n\}_{n=0,1,\cdots}$ is also π

$$\sum_{i\in S} \pi(i) P'_{ij}$$

$$= \sum_{i\in S} \pi(i) \sum_{k=1}^{\infty} P_{ij}^{k} p_k$$

$$= \sum_{k=1}^{\infty} p_k \sum_{i\in S} \pi(i) P_{ij}^{k}$$

$$= \sum_{k=1}^{\infty} p_k \pi(j) \qquad (\pi = \pi P^n\ \forall n \in \mathbb{N})$$

$$= \pi(j) \qquad \left(\sum_{k=1}^{\infty} p_k = 1\right)$$

Clearly $\sum_{i\in S} \pi(i) = 1$

Hence, the stationary distribution of $\{Y_n\}_{n=0,1,\cdots}$ is π