1.(a) $E(S_1^2) = E\left(\frac{\sum_{j=1}^{n_1}(Y_{1j} - \bar{Y}_{1+})^2}{n_1 - 1}\right)$

$$= \frac{1}{n_1-1} E\left(\sum_{j=1}^{n_1} (Y_{1j} - \bar{Y}_{1+})^2\right)$$

$$= \frac{1}{n_1-1} E\left(\sum_{j=1}^{n_1} Y_{1j}^2 - 2\bar{Y}_{1+}Y_{1j} + \bar{Y}_{1+}^2\right)$$

$$= \frac{1}{n_1-1} E\left(\sum_{j=1}^{n_1} Y_{1j}^2 - 2\bar{Y}_{1+}\sum_{j=1}^{n_1} Y_{1j} + n_1\bar{Y}_{1+}^2\right)$$

$$= \frac{1}{n_1-1} E\left(\sum_{j=1}^{n_1} Y_{1j}^2 - 2n_1\bar{Y}_{1+}^2 + n_1\bar{Y}_{1+}^2\right)$$

$$= \frac{1}{n_1-1} E\left(\sum_{j=1}^{n_1} Y_{1j}^2 - n_1\bar{Y}_{1+}^2\right)$$

$$E(Y_{1j}^2) = Var(Y_{1j}) + [E(Y_{1j})]^2$$

$$= \sigma^2 + \mu_1^2 \quad \text{since } E(R_{1j}) = 0,\ Var(R_{1j}) = \sigma^2$$

$$E(\bar{Y}_{1+}^2) = Var(\bar{Y}_{1+}) + [E(\bar{Y}_{1+})]^2$$

$$= \frac{1}{n_1^2}\sum_{i=1}^{n_1} Var(\mu_1 + R_{1j}) + \left[\frac{1}{n_1}\sum_{i=1}^{n_1} E(\mu_1 + R_{1j})\right]^2 \quad \text{since } Y_i \perp\!\!\!\perp Y_j \ \forall i,j$$

$$= \frac{1}{n_1^2} n_1\sigma^2 + \mu_1^2 \qquad \text{since } Var(R_{1j}) = \sigma^2,\ E(R_{1j}) = 0$$

$$= \frac{1}{n_1}\sigma^2 + \mu_1^2$$

Hence $E(S_1^2) = \frac{1}{n_1-1} E\left(\sum_{j=1}^{n_1} Y_{1j}^2 - n_1\bar{Y}_{1+}^2\right)$

$$= \frac{1}{n_1-1}\left(\sum_{j=1}^{n_1} \mu_1^2 + \sigma^2 - n_1\left(\frac{1}{n_1}\sigma^2 + \mu_1^2\right)\right)$$

$$= \frac{1}{n_1-1}\left(n_1\mu_1^2 + n_1\sigma^2 - \sigma^2 - n\mu_1^2\right)$$

$$= \frac{1}{n_1-1}(n_1-1)\sigma^2$$

$$= \sigma^2$$

1.(b) Let $S_1^2 = \frac{\sum_{j=1}^{n_1}(Y_{1j} - \bar{Y}_{1+})^2}{n_1 - 1}$, $S_2^2 = \frac{\sum_{j=1}^{n_2}(Y_{2j} - \bar{Y}_{2+})^2}{n_2 - 1}$, $n = n_1 + n_2$

By part (a) $E(S_1^2) = \sigma^2$, we follow the same setp and get $E(S_2^2) = \sigma^2$

$$E\left(\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n-2}\right) = \frac{n_1-1}{n-2}E(S_1^2) + \frac{n_2-1}{n-2}E(S_2^2)$$

$$= \frac{n_1+n_2-2}{n-2}\sigma^2$$

$$= \sigma^2$$

2.(a) $W = \sum_{j=1}^{n_1}(y_{1j} - \mu_1)^2 + \sum_{j=1}^{n_2}(y_{2j} - \mu_2)^2$

$\frac{\partial W}{\partial \mu_1} = -2\sum_{j=1}^{n_1}(y_{1j} - \mu_1)$

$\frac{\partial W}{\partial \mu_2} = -2\sum_{j=1}^{n_2}(y_{2j} - \mu_2)$

$-2\sum_{j=1}^{n_1}(y_{1j} - \hat{\mu}_1) = 0$

$\sum_{j=1}^{n_1} y_{1j} = n_1 \hat{\mu}_1$

$\hat{\mu}_1 = \overline{y}_{1+}$

$\tilde{\mu}_1 = \overline{Y}_{1+}$

$-2\sum_{j=1}^{n_2}(y_{2j} - \hat{\mu}_2) = 0$

$\sum_{j=1}^{n_2} y_{2j} = n_2 \hat{\mu}_2$

$\hat{\mu}_2 = \overline{y}_{2+}$

$\tilde{\mu}_2 = \overline{Y}_{2+}$

2.(b) $\hat{\sigma}^2 = \frac{\sum_{j=1}^{n_1}(y_{1j} - \overline{y}_{1+})^2 + \sum_{j=1}^{n_2}(y_{2j} + \overline{y}_{2+})^2}{n_1+n_2-2}$

2.(c) $E(\tilde{\mu}_1) = E(\frac{\sum_{j=1}^{n_1} Y_{1j}}{n_1})$

$= \frac{1}{n_1}\sum_{j=1}^{n_1} E(\mu_1 + R_{1j})$

$= \frac{1}{n_1}\sum_{j=1}^{n_1} \mu_1$ since $E(R_{1j}) = 0$

$= \mu_1$

2.(d) We have $\hat{\mu}_1 = 75$, $\hat{\mu}_2 = 78$, $S_1 = 5$ $S_2^2 = 36$

$S_p^2 = \frac{(32-1)S_1^2 + (27-1)S_2^2}{32+27-2} = 30.018$

A 95% C.I. is $\hat{\mu}_1 - \hat{\mu}_2 \pm c S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$ $c \sim t_{32+27-2} = t_{57}$

$= 75 - 78 \pm 2.002\sqrt{30.018}\sqrt{\frac{1}{32}+\frac{1}{27}}$

$= [-5.866, -0.133]$

4.(a) $y_{ij} = \mu + \tau_i + \beta_j + R_{ij}, \quad R_{ij} \sim N(0, \sigma^2)$

$i = 1, 2, 3 \qquad \sum_{i=1}^{3} \tau_i = 0$

$j = 1, \cdots, 12 \qquad \sum_{i=1}^{12} \beta_i = 0$

4.(b) i $E(\tilde{\theta}) = E(\tilde{\tau}_{pc} - \frac{\tilde{\tau}_{coke} + \tilde{\tau}_{pepsi}}{2})$

$= E(\tilde{\tau}_{pc}) - \frac{1}{2} E(\tilde{\tau}_{coke}) - \frac{1}{2} E(\tilde{\tau}_{pepsi})$

$= \tau_{pc} - \frac{\tau_{coke} + \tau_{pepsi}}{2}$

$= \theta$

$Var(\tilde{\theta}) = Var(\tilde{\tau}_{pc} - \frac{\tilde{\tau}_{coke} + \tilde{\tau}_{pepsi}}{2})$

$= Var(\bar{Y}_{1+}) + \frac{1}{4} Var(\bar{Y}_{2+} + \bar{Y}_{3+})$

$= \frac{1}{12^2} Var(\sum_{j=1}^{12} Y_{1j}) + \frac{1}{4}\frac{1}{12^2} Var(\sum_{j=1}^{12} Y_{2j}) + \frac{1}{4}\frac{1}{12^2} Var(\sum_{j=1}^{12} Y_{3j}) \qquad \bar{Y}_{i+} = \frac{1}{12}\sum_{j=1}^{12} Y_{ij}$

$= \frac{1}{12}\sigma^2 + \frac{1}{48}\sigma^2 + \frac{1}{48}\sigma^2 \qquad$ since $Var(\bar{Y}_{ij}) = \sigma^2$ and $Y_{ij} \perp\!\!\!\perp Y_{kl} \ \forall i, j, k, l$

$= \frac{1}{8}\sigma^2$

Since τ_i is normally distributed, by theorem, θ is normally distributed.

$\tilde{\theta} \sim N(\theta, \frac{1}{8}\sigma^2)$

4.(b) ii

$\hat{\tau}_{pc} = 5.5 - \bar{y}, \quad \hat{\tau}_{coke} = 7.75 - \bar{y}, \quad \hat{\tau}_{pepsi} = 6.67 - \bar{y}$

$\hat{\theta} = 5.5 - \bar{y} - \frac{7.75 - \bar{y} + 6.67 - \bar{y}}{2} = -1.71$

$\hat{\sigma}^2 = 0.9217$

Since $\tilde{\theta} \sim N(\theta, \frac{1}{8}\sigma^2)$

A 95% C.I. is

$\hat{\theta} \pm C\frac{\hat{\sigma}}{4}, \quad C \sim t_{22}$

$= -1.71 \pm 2.074\sqrt{\frac{0.9217}{8}}$

$= [-2.208, -1.006]$

5. (a) $\hat{\sigma}^2 = \dfrac{\sum_{ijk}(\hat{y}_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\gamma}_j - R_{ijk})}{3\times3\times3-3-3-1+2}$

$= \dfrac{\sum_{ijk}(\hat{y}_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\gamma}_j - R_{ijk})}{22}$

5. (b) i $E(\tilde{\mu}) = E\left(\dfrac{\sum_{ijk}(\mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + R_{ijk})}{n}\right)$

$= \dfrac{n\mu + \sum_i \tau_i + \sum_j \gamma_j + \sum_{ij}(\tau\gamma)_{ij}}{n}$

$= \mu + \dfrac{\sum_{ij}(\tau\gamma)_{ij}}{n}$

Since $\tilde{\mu}$ is unbiased

$\sum_{ij}(\tau\gamma)_{ij} = 0$

$E(\tilde{\tau}_i) = E(\bar{Y}_{i++} - \bar{Y}_{+++})$

$= E\left(\dfrac{\sum_{jk}\mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + R_{ijk}}{9}\right) - \mu$

$= \mu + \tau_i + \dfrac{\sum_{jk}(\tau\gamma)_{ij}}{9} - \mu$

$= \tau_i + \dfrac{\sum_{jk}(\tau\gamma)_{ij}}{9}$

Since $E(\tilde{\tau}_i) = \tau_i$, $\sum_j (\tau\gamma)_{ij} = 0$

$E(\tilde{\gamma}_j) = E(\bar{Y}_{+j+} - \bar{Y}_{+++})$

$= \mu + \gamma_j + \dfrac{\sum_{ik}(\tau\gamma)_{ij}}{9} - \mu$

$= \gamma_j + \dfrac{\sum_{ik}(\tau\gamma)_{ij}}{9}$

Since $E(\tilde{\gamma}_j) = \gamma_j$, $\sum_i (\tau\gamma)_{ij} = 0$

Hence we need $\begin{cases} \sum_i (\tau\gamma)_{ij} = 0 \\ \sum_j (\tau\gamma)_{ij} = 0 \end{cases}$

(b)(ii) $\hat{\sigma}^2 = \dfrac{\sum_{ijk}(y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\gamma}_j - (\widehat{\tau\gamma})_{ij} - R_{ijk})}{3\times3\times3-3-3-9-1+2+6-1}$

$= \dfrac{\sum_{ijk}(y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\gamma}_j - (\widehat{\tau\gamma})_{ij} - R_{ijk})}{18}$

6.(a)

$\hat{\mu}_2 = 70.5$, $\hat{\sigma}_2^2 = 501.667$

A 95% C.I. is

$\hat{\mu}_2 \pm C\sqrt{1-\frac{n_2}{N_2}}\frac{\hat{\sigma}_2}{\sqrt{n_2}}$, $C \sim N(0,1)$

$= 70.5 \pm 1.96\sqrt{1-\frac{4}{28}}\sqrt{\frac{501.667}{4}}$

$= [50.178, 90.822]$

(b) $\hat{\pi} = \frac{3}{4}$,

$\hat{\theta} = \frac{42+175+95}{3} = 72$

$\hat{\sigma}_{ratio}^2 = \frac{(42-72)^2+66^2+(79-72)^2+(95-72)^2}{3} = 1944.667$

A 95% C.I. is

$\hat{\mu}_2 \pm \frac{C}{\hat{\pi}}\sqrt{1-\frac{n_2}{N_2}}\frac{\hat{\sigma}_{ratio}}{\sqrt{n_2}} = [18.6525, 125.3475]$

(C) $\hat{\mu}_1 = 74.67$ $\hat{\mu}_2 = 70.5$ $\hat{\mu}_3 = 66.5$

$\hat{\mu} = W_1\mu_1 + W_2\mu_2 + W_3\mu_3 = 70.167$

$\hat{\sigma}_1^2 = 72.33$ $\hat{\sigma}_2 = 501.667$ $\hat{\sigma}_3 = 264.5$

$\sum_{i=1}^{3} W_i^2 \frac{\sigma_i^2}{n_i}(1-\frac{n_i}{N_i}) = 31.66$

A 95% C.I. is

$\hat{\mu} \pm C\sqrt{\sum_{i=1}^{3} W_i^2 \frac{\sigma_i^2}{n_i}(1-\frac{n_i}{N_i})}$ $C \sim N(0,1)$

$= [59.139, 81.195]$