Proof. We first prove $\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))=0$.
Let $P=\cup_{i=1}^{k}A_i$ where $A_i=\{u:u\in P,x_u=x_i\}$ and the unique value of $\mathbf{x}$ is $\mathbf{x}_1,\ldots,\mathbf{x}_k$.
$|A_i|=n_i$ for all $1\le i\le k$

$$
\begin{aligned}
&\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))\\
=&\frac{1}{N_{\mathcal{S}}N}\sum_{j=1}^{N_{\mathcal{S}}}[\sum_{i\in A_1}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))+\cdots+\sum_{i\in A_k}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))]
\end{aligned}
$$

For $i\in A_m$ where $1\le m\le k$, we have,

$$
\begin{aligned}
&\frac{1}{N_{\mathcal{S}}N}\sum_{j=1}^{N_{\mathcal{S}}}\sum_{i\in A_m}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))\\
=&\frac{1}{N_{\mathcal{S}}N}\sum_{j=1}^{N_{\mathcal{S}}}\sum_{i\in A_m}\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)\overline{\widehat{\mu}}(\mathbf{x}_i)-\frac{1}{N_{\mathcal{S}}N}\sum_{j=1}^{N_{\mathcal{S}}}\sum_{i\in A_m}\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)\mu(\mathbf{x}_i)\\
&-\frac{1}{N_{\mathcal{S}}N}\sum_{j=1}^{N_{\mathcal{S}}}\sum_{i\in A_m}\overline{\widehat{\mu}}(\mathbf{x}_i)^2+\frac{1}{N_{\mathcal{S}}N}\sum_{j=1}^{N_{\mathcal{S}}}\overline{\widehat{\mu}}(\mathbf{x}_i)\mu(\mathbf{x}_i)\\
=&\frac{1}{N}\sum_{i\in A_m}\frac{\overline{\widehat{\mu}}(\mathbf{x}_i)}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\frac{1}{N}\sum_{i\in A_m}\frac{\mu(\mathbf{x}_i)}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\frac{1}{N}\sum_{i\in A_m}\overline{\widehat{\mu}}(\mathbf{x}_i)^2+\frac{1}{N}\sum_{i\in A_m}\overline{\widehat{\mu}}(\mathbf{x}_i)\mu(\mathbf{x}_i)\\
=&\frac{1}{N}\sum_{i\in A_m}\overline{\widehat{\mu}}(\mathbf{x}_i)^2-\frac{1}{N}\sum_{i\in A_m}\overline{\widehat{\mu}}(\mathbf{x}_i)^2+\frac{1}{N}\sum_{i\in A_m}\overline{\widehat{\mu}}(\mathbf{x}_i)\mu(\mathbf{x}_i)-\frac{1}{N}\sum_{i\in A_m}\overline{\widehat{\mu}}(\mathbf{x}_i)\mu(\mathbf{x}_i)\\
=&0
\end{aligned}
$$

So, $\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))=0$
Hence, we have,

$$
\begin{aligned}
&\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\mu(\mathbf{x}_i))^2\\
=&\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i)+\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))^2\\
=&\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))^2+(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))^2+2(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))\\
=&\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\widehat{\mu}_{\mathcal{S}_j}(\mathbf{x}_i)-\overline{\widehat{\mu}}(\mathbf{x}_i))^2+\frac{1}{N_{\mathcal{S}}}\sum_{j=1}^{N_{\mathcal{S}}}\frac{1}{N}\sum_{i\in\mathcal{P}}(\overline{\widehat{\mu}}(\mathbf{x}_i)-\mu(\mathbf{x}_i))^2
\end{aligned}
$$

□