a1q1

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q1 (a) Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^T$, $X = (x_1, x_2, \ldots, x_n)^T$ and Σ be variance-covariance matrix.

$$\begin{aligned}
E(\tilde{\beta}_{WLS}) &= E((X^TWX)^{-1}X^TWY) \\
&= (X^TWX)^{-1}X^TWE(Y) \\
&= (X^TWX)^{-1}X^TWE(X\beta + \epsilon) \\
&= (X^TWX)^{-1}X^TWXE(\beta) + (X^TWX)^{-1}XWE(\epsilon) \\
&= (X^TWX)^{-1}X^TWX\beta + 0 \qquad (E(\epsilon) = 0) \\
&= \beta
\end{aligned}$$

(b) Let $W = diag(\frac{1}{g(x_1)}, \frac{1}{g(x_n)}, \dots, \frac{1}{g(x_n)})$. So $W^T = W$.
Since $\Sigma = \sigma^2 diag(g(x_1), g(x_2), \dots, g(x_n))$, so $\Sigma \times W = \sigma^2$

$$\begin{aligned}
Var(\tilde{\beta}_{WLS}) &= Var((X^TWX)^{-1}XWY) \\
&= (X^TWX)^{-1}X^TW \times Var(Y) \times ((X^TWX)^{-1}X^TW)^T \\
&= (X^TWX)^{-1}X^TW \times Var(X\beta + \epsilon) \times ((X^TWX)^{-1}X^TW)^T \\
&= (X^TWX)^{-1}X^TW \times Var(\epsilon) \times W^TX(X^TW^TX)^{-1} \\
&= (X^TWX)^{-1}X^TW \times \Sigma \times W^TX(X^TW^TX)^{-1} \\
&= \sigma^2(X^TWX)^{-1}X^TWX(X^TW^TX)^{-1} \\
&= \sigma^2(X^TWX)^{-1}
\end{aligned}$$