4.3 Further Properties of the Poisson Process

Exercises

Exercise 4.3.1. Customers arrive to a bank at a Poisson rate of λ per hour. Suppose two customers arrived during the first hour.

(a) What is the probability that both customers arrived during the first 20 minutes?

(b) What is the probability that at least one customer arrived during the first 20 minutes?

Solution: (a) Given $N(1) = 2$, we know that S_1 and S_2 are distributed as the order statistics $Y_{(1)}$ and $Y_{(2)}$ of 2 iid $\mathrm{U}(0,1)$ random variables. Recall that the joint pdf of $Y_{(1)}$ and $Y_{(2)}$ is given by

$$g(y_1, y_2) = \frac{2!}{1^2} = 2, \ 0 < y_1 < y_2 < 1.$$

Therefore, we can directly calculate

$$\begin{aligned}
P(\text{both arrived during first 20 minutes}) &= P\left(0 < S_1 < S_2 \leq \frac{1}{3} \middle| N(1) = 2\right) \\
&= P\left(0 < Y_{(1)} < Y_{(2)} \leq \frac{1}{3}\right) \\
&= \int_0^{1/3} \int_0^{y_2} 2 dy_1 dy_2 \\
&= \int_0^{1/3} 2y_2 dy_2 \\
&= \left(y_2^2\right) \Big|_{y_2=0}^{y_2=1/3} \\
&= \left(\frac{1}{3}\right)^2 \\
&= \frac{1}{9} \ .
\end{aligned}$$

(b) We wish to calculate

$$
\begin{aligned}
P(\text{at least 1 arrived during first 20 minutes}) &= 1 - P(\text{both arrived during last 40 minutes}) \\
&= 1 - P\left(\frac{1}{3} < S_1 < S_2 \leq 1 \middle| N(1) = 2\right) \\
&= 1 - P\left(\frac{1}{3} < Y_{(1)} < Y_{(2)} \leq 1\right) \\
&= 1 - \int_{1/3}^{1} \int_{1/3}^{y_2} 2 dy_1 dy_2 \\
&= 1 - \int_{1/3}^{1} 2\left(y_2 - \frac{1}{3}\right) dy_2 \\
&= 1 - \left(y_2^2 - \frac{2}{3} y_2\right)\Big|_{y_2=1/3}^{y_2=1} \\
&= 1 - \left[\left(1 - \frac{2}{3}\right) - \left(\frac{1}{9} - \frac{2}{9}\right)\right] \\
&= 1 - \frac{1}{3} - \frac{1}{9} \\
&= \frac{5}{9}\ .
\end{aligned}
$$

Exercise 4.3.2. Suppose that people arrive at a bus stop in accordance with a Poisson process $\{N(t), t \geq 0\}$ with rate λ. The bus is set to depart at time $s > 0$. Let X represent the total amount of waiting time of all those who board the bus at time s.

(a) Determine $\mathrm{E}[X|N(s)]$.

(b) Determine $\mathrm{Var}(X|N(s))$.

(c) Determine $\mathrm{Var}(X)$.

Solution: (a) Clearly, the total amount of waiting time is given by $X = \sum_{i=1}^{N(t)}(t - S_i)$. Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of iid $\mathrm{U}(0, t)$ random variables. Applying Theorem 4.6, we get

$$\begin{aligned}
\mathrm{E}[X|N(t) = n] &= \mathrm{E}\left[\sum_{i=1}^{N(t)}(t - S_i)\middle| N(t) = n\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{n}(t - S_i)\middle| N(t) = n\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{n}(t - Y_{(i)})\right] \text{ where } \{Y_{(i)}\}_{i=1}^{n} \text{ are the corresponding order statistics} \\
&= \mathrm{E}\left[\sum_{i=1}^{n}(t - Y_i)\right] \text{ since } \sum_{i=1}^{n} Y_{(i)} = \sum_{i=1}^{n} Y_i \\
&= n\mathrm{E}[t - Y_1] \text{ since } \{Y_i\}_{i=1}^{n} \text{ are iid } \mathrm{U}(0, t) \text{ random variables} \\
&= n\int_0^t (t - s) \cdot \frac{1}{t} ds \\
&= \frac{nt}{2} .
\end{aligned}$$

Therefore, $\mathrm{E}[X|N(t)] = \mathrm{E}[X|N(t) = n]|_{n=N(t)} = N(t)t/2$.

(b) Applying Theorem 4.6 once again, we obtain

$$
\begin{aligned}
\mathrm{Var}(X|N(t)=n) &= \mathrm{Var}\left(\sum_{i=1}^{n}(t-Y_{(i)})\right) \\
&= \mathrm{Var}\left(nt-\sum_{i=1}^{n}Y_{(i)}\right) \\
&= \mathrm{Var}\left(\sum_{i=1}^{n}Y_{(i)}\right) \quad \text{since } nt \text{ is a constant} \\
&= \mathrm{Var}\left(\sum_{i=1}^{n}Y_{i}\right) \quad \text{since } \sum_{i=1}^{n}Y_{(i)}=\sum_{i=1}^{n}Y_{i} \\
&= n\mathrm{Var}(Y_1) \text{ due to the independence of } \{Y_i\}_{i=1}^{n} \\
&= \frac{nt^2}{12}\ .
\end{aligned}
$$

Hence, $\mathrm{Var}(X|N(t)) = \mathrm{Var}(X|N(t)=n)|_{n=N(t)} = N(t)t^2/12$.

(c) Using parts (a) and (b) and the conditional variance formula, we get

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathrm{Var}(\mathrm{E}[X|N(t)]) + \mathrm{E}[\mathrm{Var}(X|N(t))] \\
&= \mathrm{Var}\left(\frac{N(t)t}{2}\right) + \mathrm{E}\left[\frac{N(t)t^2}{12}\right] \\
&= \frac{t^2}{4}\cdot \mathrm{Var}(N(t)) + \frac{t^2}{12}\cdot \mathrm{E}[N(t)] \\
&= \frac{t^2}{4}\cdot \lambda t + \frac{t^2}{12}\cdot \lambda t \\
&= \frac{\lambda t^3}{4} + \frac{\lambda t^3}{12} \\
&= \frac{\lambda t^3}{3}\ .
\end{aligned}
$$

Exercise 4.3.3. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let S_n denote the arrival time of the n^{th} event.

(a) Calculate $P(N(1/4) = 1|N(1) = 4)$.

(b) If V is a continuous rv on $(0, \infty)$ with mgf $\phi_V(t) = \mathrm{E}[e^{tV}]$ and V is independent of the Poisson process, show that $P(N(V) > 0) = 1 - \phi_V(-\lambda)$.

(c) For $n \in \mathbb{Z}^+$, determine $\mathrm{E}[S_n - S_1|N(1) = n]$.

Solution: (a) Since $N(1/4)|(N(1) = 4) \sim \mathrm{BIN}(4, 1/4)$, we simply obtain

$$P(N(1/4) = 1|N(1) = 4) = \binom{4}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3 = \frac{27}{64} \approx 0.422.$$

(b) Letting $f_V(v)$ represent the pdf of V, note that

$$\begin{aligned}
P(N(V) > 0) &= \int_0^\infty P(N(V) > 0|V = v)f_V(v)dv \\
&= \int_0^\infty P(N(v) > 0|V = v)f_V(v)dv \\
&= \int_0^\infty P(N(v) > 0)f_V(v)dv \quad \text{since } V \text{ is independent of the Poisson process} \\
&= \int_0^\infty [1 - P(N(v) = 0)]f_V(v)dv \\
&= \int_0^\infty (1 - e^{-\lambda v})f_V(v)dv \\
&= \int_0^\infty f_V(v)dv - \int_0^\infty e^{(-\lambda)v}f_V(v)dv \\
&= 1 - \phi_V(-\lambda).
\end{aligned}$$

(c) Given $N(1) = n$, $S_1 = \min\{Y_1, Y_2, \ldots, Y_n\}$ and $S_n = \max\{Y_1, Y_2, \ldots, Y_n\}$ where $Y_1, Y_2, \ldots, Y_n$ are iid $\mathrm{U}(0, 1)$ random variables with cdf $P(Y_i \leq y) = 1 - P(Y_i > y) = y$, $0 \leq y \leq 1$. First of all, for $0 \leq y \leq 1$, note that

$$\begin{aligned}
P(S_1 > y|N(1) = n) &= P(\min\{Y_1, Y_2, \ldots, Y_n\} > y) \\
&= P(Y_1 > y, Y_2 > y, \ldots, Y_n > y) \\
&= P(Y_1 > y)P(Y_2 > y)\cdots P(Y_n > y) \quad \text{due to independence} \\
&= (1 - y)^n.
\end{aligned}$$

Thus, the corresponding pdf is

$$g_1(y) = \frac{d}{dy}[1 - P(S_1 > y)] = n(1 - y)^{n-1}, \ 0 < y < 1.$$

Similarly, for $0 \leq y \leq 1$, we have that

$$
\begin{aligned}
P(S_n \leq y|N(1)=n) &= P(\max\{Y_1, Y_2, \ldots, Y_n\} \leq y) \\
&= P(Y_1 \leq y, Y_2 \leq y, \ldots, Y_n \leq y) \\
&= P(Y_1 \leq y)P(Y_2 \leq y) \cdots P(Y_n \leq y) \quad \text{due to independence} \\
&= y^n.
\end{aligned}
$$

Thus, the corresponding pdf is

$$
g_n(y) = \frac{d}{dy}\left[P(S_n \leq y)\right] = ny^{n-1}, \; 0 < y < 1.
$$

Hence,

$$
\begin{aligned}
\mathrm{E}[S_n - S_1|N(1)=n] &= \mathrm{E}[S_n|N(1)=n] - \mathrm{E}[S_1|N(1)=n] \\
&= \int_0^1 y \cdot ny^{n-1}dy - \int_0^1 y \cdot n(1-y)^{n-1}dy \\
&= n\left[\int_0^1 y^n dy - \int_0^1 y(1-y)^{n-1}dy\right] \\
&= n\left[\left(\frac{y^{n+1}}{n+1}\right)\Big|_{y=0}^{y=1} - \int_0^1 (1-z)z^{n-1}dz\right] \quad \text{where } z = 1-y \\
&= n\left(\frac{1}{n+1}\right) - n\left(\frac{z^n}{n} - \frac{z^{n+1}}{n+1}\right)\Big|_{z=0}^{z=1} \\
&= \frac{n}{n+1} - 1 + \frac{n}{n+1} \\
&= \frac{n-1}{n+1}\,.
\end{aligned}
$$

Exercise 4.3.4. Suppose that traffic at a certain point along Westmount Road can be described by a Poisson process at rate $\lambda = 2$ per minute and that 60% of the vehicles are cars, 30% are trucks, and 10% are buses. Furthermore, it is assumed that vehicles are independent of one another.

(a) What is the joint probability that exactly 3 trucks and exactly 2 buses pass by this point in a 5-minute span of time?

(b) Calculate the probability that 2 trucks will pass by this point before two vehicles that are not trucks pass by there.

(c) What is the probability that the length of time between the first and third arrivals of any vehicle passing this point exceeds 45 seconds?

(d) Given that 5 cars passed by this point over a 4-minute time period, what is the variance of the total number of vehicles that passed by this point during these 4 minutes?

Solution: (a) Let $N_T(t)$ denote the number of trucks passing by the point by time t. Likewise, let $N_B(t)$ represent the number of buses passing by the point by time t. It follows that:

$$N_T(5) \sim \text{POI}(2 \cdot 0.3 \cdot 5 = 3),$$
$$N_B(5) \sim \text{POI}(2 \cdot 0.1 \cdot 5 = 1).$$

We wish to calculate

$$\begin{aligned}
&P(N_T(5) = 3, N_B(5) = 2) \\
&= P(N_T(5) = 3)P(N_B(5) = 2) \text{ since } N_T(5) \text{ and } N_B(5) \text{ are independent} \\
&= \frac{e^{-3}3^3}{3!} \cdot \frac{e^{-1}1^2}{2!} \\
&= \frac{9}{4}e^{-4} \approx 0.0412.
\end{aligned}$$

(b) First of all, let $N_1(t)$ and $N_2(t)$ represent the number of trucks and "non-trucks" (i.e., cars or buses) passing by the point by time t, respectively. Let $S_n^{(1)}$ be the arrival time of the n^{th} truck. Likewise, let $S_m^{(2)}$ be the arrival time of the m^{th} non-truck. We wish to calculate

$$\begin{aligned}
P(\text{2 trucks pass by before 2 non-trucks}) &= P(S_2^{(1)} < S_2^{(2)}) \\
&= \sum_{j=0}^{2-1} \binom{2+j-1}{2-1} \left(\frac{2(0.3)}{2(0.3)+2(0.7)}\right)^2 \left(\frac{2(0.7)}{2(0.3)+2(0.7)}\right)^j \\
&= \sum_{j=0}^{1} \binom{j+1}{1} \left(\frac{0.3}{0.3+0.7}\right)^2 \left(\frac{0.7}{0.3+0.7}\right)^j \\
&= (0.3)^2 + 2(0.3)^2(0.7) \\
&= 0.216.
\end{aligned}$$

(c) Let X be the time between the 1$^{\text{st}}$ and 3$^{\text{rd}}$ vehicle arrivals. If T_i denotes the i^{th} interarrival time, then $X = T_2 + T_3$ where T_2 and T_3 are iid EXP(2) random variables. Thus, $X \sim \text{Erlang}(2, 2)$. Since 45 seconds is equivalent to 3/4 of a minute, it follows that

$$P(X > 3/4) = e^{-2(3/4)} \sum_{i=0}^{2-1} \frac{[2(3/4)]^i}{i!} = e^{-3/2}\left(1 + \frac{3}{2}\right) = \frac{5}{2}e^{-3/2} \approx 0.558.$$

(d) Let $N_C(t)$ denote the number of cars passing by the point by time t. Also, let $N(t)$ denote the number of vehicles passing by the point by time t. Since $N(t) = N_C(t) + N_T(t) + N_B(t)$, it follows that

$$\begin{aligned}
\text{Var}(N(4)|N_C(4) = 5) &= \text{Var}(N_C(4) + N_T(4) + N_B(4)|N_C(4) = 5) \\
&= \text{Var}(5 + N_T(4) + N_B(4)|N_C(4) = 5) \\
&= \text{Var}(N_T(4) + N_B(4)|N_C(4) = 5) \\
&= \text{Var}(N_T(4) + N_B(4)) \;\; \text{since } N_C(4) \text{ is independent of } N_T(4) \text{ and } N_B(4) \\
&= \text{Var}(Y) \;\; \text{where } Y = N_T(4) + N_B(4) \sim \text{POI}(2 \cdot 0.4 \cdot 4 = 3.2) \\
&= 3.2.
\end{aligned}$$

Exercise 4.3.5. Suppose that the number of visits to a certain website can be modelled by a Poisson process $\{N(t), t \geq 0\}$ with rate λ per hour. In addition, define

$$Y(t) = \begin{cases} 1 & , \text{ if } N(t) = 0, 2, 4, \ldots \\ 0 & , \text{ if } N(t) = 1, 3, 5, \ldots \end{cases}.$$

(a) Determine $P(N(4) - N(1) > 2 | N(1) = 2)$.

(b) Show that

$$P(Y(t) = 1) = \frac{1 + e^{-2\lambda t}}{2}.$$

(*Hint*: The determination of $P_t(z) = E[z^{N(t)}] = \sum_{n=0}^{\infty} z^n P(N(t) = n)$ may prove to be useful.)

(c) Determine $P(N(t) = 0 | Y(t) = 1)$.

(d) Determine $\mathrm{E}[Y(s) | N(t) = 1]$ where $0 < s < t$.

(e) Calculate the probability that there have been more visits from 1:00 pm to 2:00 pm than from 2:00 pm to 3:00 pm (of the same day), given that there have been 10 visits in total from 1:00 pm to 3:00 pm.

(f) Let S_n, $n \in \mathbb{Z}^+$, represent the arrival time of the n^{th} website visit. Calculate

$$P(S_1 \leq t/2, S_2 \leq 2t/3 | N(t) = 2).$$

Solution: (a) We wish to determine

$$\begin{aligned} P(N(4) - N(1) > 2 | N(1) = 2) &= P(N(4) - N(1) > 2) \text{ due to independent increments} \\ &= P(N(3) > 2) \text{ due to stationary increments} \\ &= 1 - \sum_{n=0}^{2} P(N(3) = n) \\ &= 1 - \sum_{n=0}^{2} \frac{e^{-3\lambda}(3\lambda)^n}{n!} \\ &= 1 - e^{-3\lambda} - 3\lambda e^{-3\lambda} - \frac{9\lambda^2}{2} e^{-3\lambda} \\ &= 1 - e^{-3\lambda}\left(1 + 3\lambda + \frac{9\lambda^2}{2}\right). \end{aligned}$$

(b) First of all, we find

$$\begin{aligned} P_t(z) &= E[z^{N(t)}] \\ &= \sum_{n=0}^{\infty} z^n P(N(t)=n) \\ &= \sum_{n=0}^{\infty} z^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\ &= e^{-\lambda t}\sum_{n=0}^{\infty} \frac{(z\lambda t)^n}{n!} \\ &= e^{-\lambda t}e^{z\lambda t} \\ &= e^{\lambda t(z-1)},\ z \in \mathbb{R}. \end{aligned} \tag{4.2}$$

Note the following results hold true:

$$\begin{aligned} P_t(1) &= \sum_{n=0}^{\infty} 1^n P(N(t)=n) \\ &= P(N(t)=0)+P(N(t)=1)+P(N(t)=2)+P(N(t)=3)+\cdots \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} P_t(-1) &= \sum_{n=0}^{\infty} (-1)^n P(N(t)=n) \\ &= P(N(t)=0)-P(N(t)=1)+P(N(t)=2)-P(N(t)=3)+\cdots. \end{aligned} \tag{4.4}$$

Adding (4.3) and (4.4) together, we obtain

$$\begin{aligned} P_t(1)+P_t(-1) &= 2\big(P(N(t)=0)+P(N(t)=2)+P(N(t)=4)+\cdots\big) \\ &= 2P(N(t)=0,2,4,\ldots) \\ &= 2P(Y(t)=1). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} P(Y(t)=1) &= \frac{P_t(1)+P_t(-1)}{2} \\ &= \frac{e^{\lambda t(1-1)}+e^{\lambda t(-1-1)}}{2} \quad \text{by (4.2)} \\ &= \frac{1+e^{-2\lambda t}}{2}. \end{aligned}$$

(c) Using the result from part (b), we obtain

$$\begin{aligned}
P(N(t)=0|Y(t)=1) &= \frac{P(N(t)=0, Y(t)=1)}{P(Y(t)=1)} \\
&= \frac{P(N(t)=0, N(t)=0,2,4,\ldots)}{P(Y(t)=1)} \\
&= \frac{P(N(t)=0)}{P(Y(t)=1)} \\
&= \frac{e^{-\lambda t}(\lambda t)^0/0!}{(1+e^{-2\lambda t})/2} \\
&= \frac{2e^{-\lambda t}}{1+e^{-2\lambda t}}\ .
\end{aligned}$$

(d) We may apply two possible solution methods, as follows:

<u>Method 1</u>: Note that

$$\begin{aligned}
&\mathrm{E}[Y(s)|N(t)=1] \\
&= 0 \cdot P(Y(s)=0|N(t)=1) + 1 \cdot P(Y(s)=1|N(t)=1) \\
&= P(Y(s)=1|N(t)=1) \\
&= \frac{P(Y(s)=1, N(t)=1)}{P(N(t)=1)} \\
&= \frac{P(N(s)=0, N(t)=1)}{P(N(t)=1)} \text{ since } \{Y(s)=1\} \cap \{N(t)=1\} = \{N(s)=0\} \cap \{N(t)=1\} \\
&= \frac{P(N(s)=0, N(t)-N(s)=1)}{P(N(t)=1)} \\
&= \frac{e^{-\lambda s} \cdot e^{-\lambda(t-s)}\lambda(t-s)/1!}{e^{-\lambda t}\lambda t/1!} \text{ by independent increments} \\
&= 1 - \frac{s}{t}.
\end{aligned}$$

<u>Method 2</u>: Recall that for $0 < s < t$, we know that $N(s)|(N(t)=n) \sim \mathrm{BIN}(n, s/t)$. Thus,

$$\begin{aligned}
\mathrm{E}[Y(s)|N(t)=1] &= P(Y(s)=1|N(t)=1) \\
&= P(N(s)=0,2,4,\ldots|N(t)=1) \\
&= P(N(s)=0|N(t)=1) \\
&= \binom{1}{0}\left(\frac{s}{t}\right)^0\left(1-\frac{s}{t}\right)^1 \\
&= 1 - \frac{s}{t}.
\end{aligned}$$

(e) We may apply two possible solution methods, as follows:

Method 1: Suppose that 1 pm represents time 0. We are given that $N(2) = 10$. We wish to calculate $P(N(1) > N(2) - N(1)|N(2) = 10)$, which we will denote by x for convenience. However, since 1 to 2 pm and 2 to 3 pm are non-overlapping time intervals, we know that $(N(1), N(2) - N(1))|(N(2) = 10) \sim \text{MN}(10, 1/2, 1/2)$. Equivalently, we have $N(1)|(N(2) = 10) \sim \text{BIN}(10, 1/2)$. Therefore, it follows that

$$\begin{aligned}
1 &= P(N(1) > N(2) - N(1)|N(2) = 10) \\
&\quad + P(N(1) = N(2) - N(1)|N(2) = 10) \\
&\quad + P(N(1) < N(2) - N(1)|N(2) = 10) \\
&= x + P(N(1) = N(2) - N(1)|N(2) = 10) + x \quad \text{by symmetry (since } p_1 = p_2 = 1/2) \\
&= x + P(N(1) = 5|N(2) = 10) + x \\
&= 2x + \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 \implies x = \frac{1 - \binom{10}{5}\left(\frac{1}{2}\right)^{10}}{2} = \frac{193}{512} \approx 0.377.
\end{aligned}$$

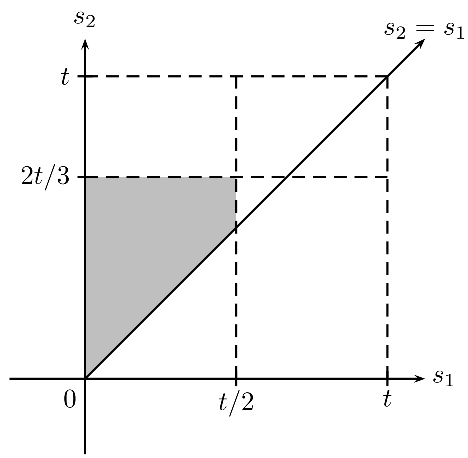
Method 2: With $N(1)|(N(2) = 10) \sim \text{BIN}(10, 1/2)$, we remark that there will be more visits from 1 to 2 pm than from 2 to 3 pm if $N(1) = 6, 7, 8, 9$, or 10. Therefore,

$$\begin{aligned}
x &= P(N(1) \geq 6|N(2) = 10) \\
&= \sum_{i=6}^{10} P(N(1) = i|N(2) = 10) \\
&= \sum_{i=6}^{10} \binom{10}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{10-i} \\
&= \left(\frac{1}{2}\right)^{10}\left\{\binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right\} \\
&= \frac{1}{1024}(210 + 120 + 45 + 10 + 1) \\
&= \frac{386}{1024} \\
&= \frac{193}{512} \approx 0.377.
\end{aligned}$$

(f) Given $N(t) = 2$, the conditional joint pdf of S_1 and S_2 is given by

$$g(s_1, s_2) = \frac{2!}{t^2} = \frac{2}{t^2},\ 0 < s_1 < s_2 < t,$$

which we wish to integrate over the shaded region below:



Thus,

$$\begin{aligned}
P(S_1 \leq t/2, S_2 \leq 2t/3 | N(t) = 2) &= \int_0^{t/2} \int_{s_1}^{2t/3} \frac{2}{t^2} ds_2 ds_1 \\
&= \frac{2}{t^2} \int_0^{t/2} \left(\frac{2t}{3} - s_1 \right) ds_1 \\
&= \frac{2}{t^2} \left(\frac{2t}{3} s_1 - \frac{s_1^2}{2} \right) \Bigg|_{s_1=0}^{s_1=t/2} \\
&= \frac{2}{t^2} \left(\frac{2t}{3} \cdot \frac{t}{2} - \frac{(t/2)^2}{2} \right) \\
&= \frac{2}{t^2} \left(\frac{t^2}{3} - \frac{t^2}{8} \right) \\
&= \frac{2}{t^2} \cdot \frac{5t^2}{24} \\
&= \frac{5}{12} \approx 0.417.
\end{aligned}$$

Exercise 4.3.6. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. Suppose that each time an event occurs, it is a type I event with probability p and a type II event with probability $1-p$. Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ denote the counting processes associated with type I and type II events, respectively. Assume that $0 < s < t$.

(a) Determine $P(N_1(s) = 1,\ N_2(t) = 2)$.

(b) Determine $P(N_1(s) = 1,\ N_1(t) = 3)$.

(c) Determine $\mathrm{Var}(N_1(s)|N_1(t) = 5)$.

(d) Determine $\mathrm{E}[N_1(t)|N_1(s) = 2]$.

(e) Determine $\mathrm{E}[N(t)|N_1(s) = 2]$.

(f) Let $S_n^{(2)}$, $n \in \mathbb{Z}^+$, represent the arrival time of the n^{th} type II event. Calculate

$$P(S_1^{(2)} > t/3, S_2^{(2)} > 3t/4|N_2(t) = 2).$$

Solution: (a) Since $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes, we obtain

$$\begin{aligned} P(N_1(s) = 1, N_2(t) = 2) &= P(N_1(s) = 1)P(N_2(t) = 2) \\ &= \frac{e^{-\lambda ps}(\lambda ps)^1}{1!} \cdot \frac{e^{-\lambda(1-p)t}[\lambda(1-p)t]^2}{2!} \\ &= \frac{1}{2}\lambda^3 p(1-p)^2 st^2 e^{-\lambda[ps+(1-p)t]}. \end{aligned}$$

(b) We wish to determine

$$\begin{aligned} P(N_1(s) = 1, N_1(t) = 3) &= P(N_1(s) = 1, N_1(t) - N_1(s) = 2) \\ &= P(N_1(s) = 1)P(N_1(t) - N_1(s) = 2) \text{ by independent increments} \\ &= \frac{e^{-\lambda ps}(\lambda ps)^1}{1!} \cdot \frac{e^{-\lambda p(t-s)}[\lambda p(t-s)]^2}{2!} \\ &= \frac{1}{2}\lambda^3 p^3 s(t-s)^2 e^{-\lambda pt}. \end{aligned}$$

(c) For $s < t$, recall that $N_1(s)|(N_1(t) = 5) \sim \mathrm{BIN}(5, s/t)$. Thus, we immediately get

$$\mathrm{Var}(N_1(s)|N_1(t) = 5) = 5\left(\frac{s}{t}\right)\left(1 - \frac{s}{t}\right).$$

(d) We wish to determine

$$\begin{aligned}
\mathrm{E}[N_1(t)|N_1(s)=2] &= \mathrm{E}[N_1(s)+N_1(t)-N_1(s)|N_1(s)=2] \\
&= \mathrm{E}[2+N_1(t)-N_1(s)|N_1(s)=2] \\
&= 2+\mathrm{E}[N_1(t)-N_1(s)|N_1(s)=2] \\
&= 2+\mathrm{E}[N_1(t)-N_1(s)] \ \text{ by independent increments} \\
&= 2+\lambda p(t-s) \ \text{ since } N_1(t)-N_1(s) \sim \mathrm{POI}(\lambda p(t-s)).
\end{aligned}$$

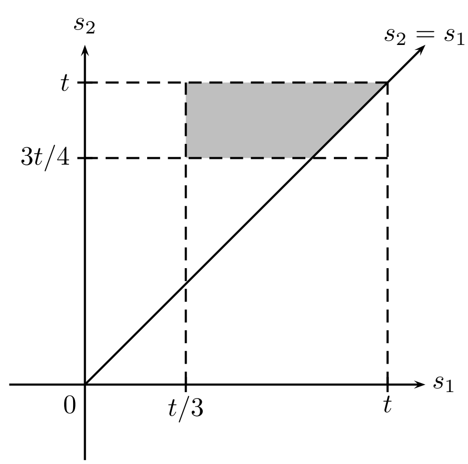
(e) We wish to determine

$$\begin{aligned}
\mathrm{E}[N(t)|N_1(s)=2] &= \mathrm{E}[N_1(t)+N_2(t)|N_1(s)=2] \\
&= \mathrm{E}[N_1(t)|N_1(s)=2]+\mathrm{E}[N_2(t)|N_1(s)=2] \\
&= 2+\lambda p(t-s)+\mathrm{E}[N_2(t)|N_1(s)=2] \ \text{ from part (d)} \\
&= 2+\lambda p(t-s)+\mathrm{E}[N_2(t)] \ \text{ since } N_1(s) \text{ and } N_2(t) \text{ are independent} \\
&= 2+\lambda p(t-s)+\lambda(1-p)t \ \text{ since } N_2(t) \sim \mathrm{POI}(\lambda(1-p)t) \\
&= 2+\lambda(t-ps).
\end{aligned}$$

(f) Given $N_2(t)=2$, the conditional joint pdf of $S_1^{(2)}$ and $S_2^{(2)}$ is given by

$$g(s_1,s_2)=\frac{2!}{t^2}=\frac{2}{t^2},\ 0<s_1<s_2<t,$$

which we wish to integrate over the shaded region below:



Thus,

$$
\begin{aligned}
P(S_1^{(2)} > t/3, S_2^{(2)} > 3t/4 | N_2(t) = 2) &= \int_{3t/4}^{t} \int_{t/3}^{s_2} \frac{2}{t^2} ds_1 ds_2 \\
&= \frac{2}{t^2} \int_{3t/4}^{t} \left(s_2 - \frac{t}{3} \right) ds_2 \\
&= \frac{2}{t^2} \left(\frac{s_2^2}{2} - \frac{t}{3} s_2 \right) \Bigg|_{3t/4}^{t} \\
&= \frac{2}{t^2} \left(\frac{t^2}{2} - \frac{t^2}{3} - \frac{9t^2}{32} + \frac{t}{3} \cdot \frac{3t}{4} \right) \\
&= \frac{2}{t^2} \left(\frac{t^2}{6} - \frac{t^2}{32} \right) \\
&= 2 \left(\frac{1}{6} - \frac{1}{32} \right) \\
&= \frac{13}{48} \approx 0.271.
\end{aligned}
$$