1.(a) $S=\{0,1,2,\cdots,M\}$

communicating classes: $\{0\}, \{M\}, \{1,\cdots,M-1\}$

since $\{0\}$ and $\{M\}$ are finite and closed, they are recurrent

Since $p_{01}>0$ and $\{0\}$ is closed, 1 is transient

$\{1,\cdots,M-1\}$ is transient

$d(0)=1$ since $p_{00}>0$

$d(M)=1$ since $p_{MM}>0$

For the other states, it takes some steps to go to a different state and takes the same steps to go back to that state

$d(1)=\cdots=d(M-1)=2$

(b) $A=\{0\}$, $B=\{M\}$, $C=\{1,\cdots,M-1\}$

$h(0)=1$
$h(M)=0$ boundary condition

$h(1)=ph(2)+1-p$

$h(2)=(1-p)h(1)+ph(3)$

$h(3)=(1-p)h(2)+ph(4)$

$\vdots$

$h(M-1)=(1-p)h(M-2)$

(c)
$$\begin{cases} h(1)=ph(2)+1-p \\ h(2)=(1-p)h(1)+0 \\ h(0)=1 \\ h(3)=0 \end{cases}$$

$h(1)=p[(1-p)h(1)]+1-p$

$(1-p+p^2)h(1)=1-p$

$$h(1)=\frac{1-p}{1-p+p^2}$$

2. (a) $P(X_{n+1}=j \mid X_n=i)$

$= P(\sum_{m=1}^{i} Y_m^{(n)} = j \mid X_n = i)$

Let $Z^{(n)} = \sum_{m=1}^{X_n} Y_m^{(n)}$

$Z^n \sim Bin(iM, p)$

so $P(X_{n+1}=j \mid X_n=i)$

$= P(Z_n = j \mid X_n = i)$

$= \binom{iM}{j} p^j (1-p)^{iM-j}$

(b) $\varphi(s) = \sum_{k=0}^{\infty} \binom{M}{k} p^k (1-p)^{M-k} s^k$

$= \sum_{k=0}^{M} \binom{M}{k} (ps)^k (1-p)^{M-k}$ since $\binom{M}{k}=0 \ \forall k>M$

$= [(1-p)+sp]^M$

(c) $E(Y) = \varphi'(1) = Mp[(1-p)+p]^{M-1} = Mp$

$E(Y) \le 1 \longrightarrow \mu_\infty = 1$

$Mp \le 1$

$p \le \frac{1}{M}$

(d) $u_1 = P(N \le 1 \mid X_0=1) = P(X_1=0 \mid X_0=1) = \binom{3}{0} p^0 (1-p)^3 = (1-p)^3 = \frac{1}{8}$

$u_2 = \varphi(u_1) = \varphi((1-p)^3) = [(1-p)+p(1-p)^3]^3 = (1-p)^3(1+p(1-p)^2)^3 = (1-p)^3(p^3-2p^2+p+1)^3$

$= \frac{729}{4096}$

$P(N=2 \mid X_0=1) = P(N \le 2 \mid X_0=1) - P(N \le 1 \mid X_0=1) = u_2 - u_1$

$= \frac{217}{4096}$

(e) since $E(Y) \le 1 \longrightarrow \mu_\infty = 1$

$E(Y) > 1 \longrightarrow \mu_\infty < 1$

$\mu_\infty = 1$ if $p \le \frac{1}{M}$

$\mu_\infty < 1$ if $p > \frac{1}{M}$

(f) If $P(N<\infty \mid X_0=5)=1$, for each individual among those 5 has to extinct independently.

$P(N<\infty \mid X_0=5) = (P(N<\infty \mid X_0=1))^5$

since $E(Y) = Mp = \frac{3}{2} > 1$

$\varphi(s) = s$

$(\frac{1}{4} + \frac{3}{4}s)^2 = s$ $\Rightarrow$ $s = \frac{1}{9}$ so $P(N<\infty \mid X_0=5) = \frac{1}{9^5}$

$(9s-1)(s-1)=0$

3. Let T_i be the # of steps it takes from state i

so $T_i = i$

$E(T) = \sum_{i=1}^{\infty} E(T_i) P_i = \sum_{i=1}^{\infty} i P_i$

$E(T) \to \infty \iff \sum_{i=1}^{\infty} i P_i \to \infty$

since $i \to \infty$

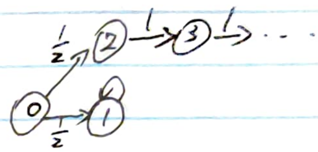
$\sum_{i=1}^{\infty} i P_i < \infty$ if $P_m = 0\ \forall m > M$ where M is an integer

since I and R (positive recurrement) $E(T) < \infty$

$\pi(y) = \frac{M_0(y)}{E_0(T_0)} = \frac{\sum_{i=y}^{\infty} P_i}{E(T)}$ by theorem

4. Let $S = \{0, 1, 2, \cdots\}$

$P_{01} = P_{02} = \frac{1}{2}$, $P_{11} = 1$, $P_{i,i+1} = 1\ \forall i \in \{2, 3, \cdots\}$



with initial distribution that always starts from state 0.

$\lim_{n\to\infty} P(X_n = 1) = \frac{1}{2} = P_{02}$, $\lim_{n\to\infty} P(X_n = i) = 0\ \forall i \in \{0, 2, 3, \cdots\}$ since all other states are going to the next one. Hence the sum is $\frac{1}{2}$.

MC has to have a infinite state space and reducible.

If MC is finite, then the chain must end up with some state when $n \to \infty$, the sum would be 1. If the chain is irreducible, $i \leftrightarrow j\ \forall i, j \in S$, so the sum must be 1 if states are pos. rec. and 0 otherwise.

5. Let (X, Y) = (# of pairs of shoes at front door, # of pairs of shoes at back door)

(a)

$$P = \begin{array}{c|ccc} & (0,2) & (1,1) & (2,0) \\ \hline (0,2) & \frac{3}{4} & \frac{1}{4} & 0 \\ (1,1) & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ (2,0) & 0 & \frac{1}{4} & \frac{3}{4} \end{array}$$

This chain is irreducible

proportion of barefoot is $\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}$

$$\begin{cases} \pi(0) = \frac{3}{4}\pi(0) + \frac{1}{4}\pi(1) \\ \pi(1) = \frac{1}{4}\pi(0) + \frac{1}{2}\pi(1) + \frac{1}{4}\pi(2) \\ \pi(2) = \frac{1}{4}\pi(1) + \frac{3}{4}\pi(2) \\ \pi(0) + \pi(1) + \pi(2) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi(0) = \frac{1}{3} \\ \pi(1) = \frac{1}{3} \\ \pi(2) = \frac{1}{3} \end{cases}$$

stationary distribution exists, $I, S \Rightarrow R$

By theorem $\frac{N_n(0,2)}{n} = \frac{N_n(2,0)}{n} \to \frac{1}{3}$

(b)
$$P = \begin{array}{c|cccc} & (3,0) & (2,1) & (1,2) & (0,3) \\ \hline (3,0) & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ (2,1) & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ (1,2) & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ (0,3) & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{array}$$

Irreducible

$$\begin{cases} \pi(0) = \frac{3}{4}\pi(0) + \frac{1}{4}\pi(1) \\ \pi(1) = \frac{1}{4}\pi(0) + \frac{1}{2}\pi(1) + \frac{1}{4}\pi(2) \\ \pi(2) = \frac{1}{4}\pi(1) + \frac{1}{2}\pi(2) + \frac{1}{4}\pi(3) \\ \pi(3) = \frac{1}{4}\pi(2) + \frac{3}{4}\pi(3) \\ \pi(0) + \pi(1) + \pi(2) + \pi(3) = 1 \end{cases}$$

$\pi(0) = \pi(1) = \pi(2) = \pi(3) = \frac{1}{4}$

since stationary distribution exists and $I, S \Rightarrow R$

By theorem $\frac{N_n((0,3))}{n} = \frac{N_n((3,0))}{n} \to \frac{1}{4}$

Proportion of barefoot is $\frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} = \frac{1}{4}$

b. Since I and stationary distribution exists

$\pi(y) = \frac{1}{E_y(T_y)}$ by theorem

$E_y(T_y) = \frac{1}{\pi(y)}$

$$\begin{aligned} E(T) &= E(E(T_{x_0} \mid X_0)) \\ &= E(E_{x_0}(T_{x_0})) \\ &= \sum_{x_0=1}^{M} \pi(x_0) \frac{1}{\pi(x_0)} \\ &= M \end{aligned}$$