1.(a) Assume $Y_n = i$ where $i > 0$

$Y_{n+1} = i+1$ if $x_{n+1} = i+1$ or $-i-1$

$Y_{n+1} = i+1$ if $x_{n+1} = i-1$ or $-i+1$

$P(Y_{n+1} = 1 \mid Y_n = 0) = 1$

For $i > 0$

$P(Y_{n+1} = i+1 \mid Y_n = i) = P(|x_n| = i+1 \mid |x_n| = i)$

$$= \begin{cases} P & \text{if } x_n = i \\ 1-P & \text{if } x_n = -i \end{cases} \quad \leftarrow \text{only depends on } x_n \text{ and } Y_n$$

$P(Y_{n+1} = i-1 \mid Y_n = i) = P(|x_n| = i-1 \mid |x_n| = i)$

$$= \begin{cases} 1-P & \text{if } x_n = i \\ P & \text{if } x_n = i \end{cases} \checkmark$$

If $\{Y_n\}_{n=0,1,2,\cdots}$ is a DTMC, $1-P=P \Rightarrow P = \frac{1}{2}$

~~since $P_{i,i+1}$ $P_{i-1,i+1}$ and ... i,j,i or $i-1$~~

(b) $P((x_{n+1}, x_{n+2}) = (j_1, j_2) \mid (x_n, x_{n+1}) = (i_1, i_2), (x_{n-1}, x_n) = \cdots (x_0, x_1) = (i_0', i_1'))$
$(i_{n-1}, i_n)_{j}$

$$= \begin{cases} 0 & \text{if } j_2 \neq j_1 \\ P_{i_2 j_2} (=P_{i_1 j_2}) & \text{o.w.} \end{cases}$$

since the prob. only depends on (x_n, x_{n+1}), $\{(x_n, x_{n+1})\}_{n=0,1,\cdots}$ is also a DTMC

Let S be the state space of $\{x_n\}_{n=0,1,\cdots}$

S' be the state space of $\{(x_n, x_{n+1})\}_{n=0,1,\cdots}$

$\pi((i,j)) = \sum_{(m,n) \in S'} \pi((m,n)) P'_{(m,n)(i,j)}$

$= \sum_{k \in S} \pi((k,i)) P_{ij}$

$= P_{ij} \sum_{k \in S} \pi((k,i))$

Let ~~...~~

$\pi((i,j)) = \pi(i) P_{ij}$

$P_{ij} \sum_{k \in S} \pi((k,i)) = P_{ij} \sum_{k \in S} \pi(k) P_{ki} = P_{ij} \pi(i)$ by defn

$\sum_{(i,j) \in S'} \pi((i,j)) = \sum_{(i,j) \in S'} \pi(i) P_{ij} = \sum_{j \in S} \sum_{i \in S} \pi(i) P_{ij} = \sum_{v \in S} \pi(j) = 1$

Hence $\pi((i,j)) = \pi(i) P_{ij}$ is a stationary distribution of $\{(x_n, x_{n+1})\}_{n=0,1,\cdots}$

bradley

2. (a) consider a DTMC with transition matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Both states have period 2

so $P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

clearly, p is reducible

(b) since $\gcd\{n \geq 1 : p_{xx}^n\} = d \quad \forall x \in S$ (irreducible)

$\Rightarrow \gcd\{n \geq 1 : p_{xx}^{nd}\} = d$

$\gcd\{n \geq 1 : (p_{xx}^d)^{\frac{n}{d}}\} = 1$ by the property of gcd

By defn for all the states in $\{Y_n\}_{n=0,1,\dots}$, the period is 1.

3. $S = \{1, 2, 3\}$

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \end{array}$$

Since $p_{ij} > 0 \quad \forall i, j \in S$, the chain is irreducible

Since the chain is finite and irreducible, all states are recurrent

By theorem

$$\begin{cases} \pi(1) = 0.8\pi(1) + 0.6\pi(2) + 0.4\pi(3) \\ \pi(2) = 0.1\pi(1) + 0.2\pi(2) + 0.1\pi(3) \\ \pi(3) = 0.1\pi(1) + 0.2\pi(2) + 0.3\pi(3) \\ \pi(1) + \pi(2) + \pi(3) = 1 \end{cases} \implies \begin{cases} \pi(1) = \frac{5}{7} \\ \pi(2) = \frac{1}{7} \\ \pi(3) = \frac{1}{7} \end{cases}$$

Hence stationary distribution exists

By theorem

$$\frac{N_n(y)}{n} \to \pi(y)$$

so
$$\begin{cases} \frac{N_n(1)}{n} \to \frac{5}{7} \\ \frac{N_n(2)}{n} \to \frac{1}{7} \\ \frac{N_n(3)}{n} \to \frac{1}{7} \end{cases}$$

4.(a) Since this chain has a tridiagonal ^{transition} matrix, it is detail balanced.
By theorem, this chain is time-reversible

(b) since it is time-reversible

$\{Y_m\}_{m=0,1,2}$ has the same distribution as $\{X_n\}_{n=0,1,2}$ where $Y_m = X_{2-m}$

$P(X_1 = 2 | X_2 = 3) = P(Y_1 = 2 | Y_0 = 3)$

$\qquad = P_{32}$

$\qquad = 0.6$

(c) $P(X_2 = 3, X_4 = 1 | X_3 = 2)$

$\quad = P(Y_2 = 3, Y_0 = 1 | Y_1 = 2)$

$\quad = \dfrac{P(Y_2 = 3, Y_1 = 2, Y_0 = 1)}{P(Y_1 = 2)}$

$\quad = \dfrac{P_{12} P_{23} \, \pi(1)}{\pi(2)}$

$\quad = \dfrac{0.2 \cdot 0.3 \, \pi(1)}{\pi(2)}$

$\quad = \dfrac{0.06 \, \pi(1)}{\pi(2)}$

By part B:

$\dfrac{\pi(1)}{\pi(2)} = \dfrac{0.1 \pi(0) + 0.7 \pi(2)}{\pi(2)}$

$\qquad = \dfrac{0.1 \pi(0)}{\pi(2)} + 0.7$

$\qquad = \dfrac{0.8 \pi(2)}{\pi(2)} + 0.7$

$\qquad = \dfrac{2.8 \pi(2)}{\pi(2)} + 0.7$

$\qquad = 3.5$

Hence $P(X_2 = 3, X_4 = 1 | X_3 = 2) = 3.5 \times 0.06 = 0.21$

5. (a) This chain is irreducible
since $i \to j \; \forall \, i, j \in S$
~~This chain can be divided into 2 classes~~ $d(\{S\}) = 1$ since $P_{44} > 0$
~~$\{0,1,2,4\}$~~ ~~$d(\{0,1,2,4\}) = 1$ since~~

(b) $\begin{cases} \pi(0) = \pi(3) + \frac{1}{2}\pi(4) \\ \pi(1) = \frac{1}{3}\pi(0) \\ \pi(2) = \frac{2}{3}\pi(0) \\ \pi(3) = \frac{1}{2}\pi(1) + \frac{1}{4}\pi(2) \\ \pi(4) = \frac{1}{2}\pi(1) + \frac{3}{4}\pi(2) + \frac{1}{2}\pi(4) \\ \sum_{i=0}^{4} \pi(i) = 1 \end{cases}$

$\Rightarrow \pi = \begin{bmatrix} \frac{3}{11} & \frac{1}{11} & \frac{2}{11} & \frac{1}{11} & \frac{4}{11} \end{bmatrix}$

(c) Since I, S
By theorem $E_0(T_0) = \frac{1}{\pi_0} = \frac{11}{3}$

(d) $C = \{3\}$ $A = \{0,1,2,4\}$

$\begin{cases} g(0) = 1 + \frac{1}{3}g(1) + g(2) \cdot \frac{2}{3} \\ g(1) = 1 + \frac{1}{2}g(4) \\ g(2) = 1 + \frac{3}{4}g(4) \end{cases}$

$$\begin{cases} g(3) = 0 \\ g(4) = 1 + \frac{1}{2}g(0) + \frac{1}{2}g(4) \end{cases}$$

$$\Rightarrow g = \qquad g = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 10 & 6 & \frac{7}{2} & 0 & 12 \end{bmatrix}$$

$E_0(V_A) = 10$

(e) $A = \{4\}$, $B = \{2\}$, $C = \{0, 1, 3\}$

$$\begin{cases} h(0) = \frac{1}{3}h(1) \\ h(1) = \frac{1}{2}h(3) + 1 \cdot \frac{1}{2} \\ h(2) = 0 \\ h(3) = \cdots h(0) \\ h(4) = 1 \end{cases}$$

$$h = \begin{bmatrix} 0 & 1 & 3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

$P(V_A < V_B \mid X_0 = 0) = \frac{1}{5}$

~~b) Assume that the expected number of visits to state j before the first visit to state i is k with $k \neq 1$. Case 1: $k \neq 0$ contradicts the fact that~~

6. $E_i(N(j) \mid T_j^{(N(j))} < T_i) = \sum_{n=1}^{\infty} n \cdot P(T_j^{(n)} < T_i \mid X_0 = i)$

$= P(T_j < T_i \mid X_0 = i) P(T_j > T_i \mid X_0 = j) + 2 \cdot P(T_j < T_i \mid X_0 = i) P(T_j < T_i \mid X_0 = j)$

$P(T_j > T_i \mid X_0 = j) + \cdots$ by Markov Property

$= P(T_j < T_i \mid X_0 = i)^2 + 2 \cdot P(T_j < T_i \mid X_0 = i)^2 P(T_j < T_i \mid X_0 = j) + \cdots$

since i, j are symmetric

$1 - P(T_j < T_i \mid X_0 = i) = P(T_i < T_j \mid X_0 = j)$

$P(T_i < T_j \mid X_0 = i) = P(T_j < T_i \mid X_0 = j)$ since $P(T_i = T_j) = 0$

$\Rightarrow \sum_{n=1}^{\infty} P(T_j < T_i \mid X_0 = i)^2 \, n \cdot P(T_j < T_j \mid X_0 = j)^{n-1}$

$= P(T_j < T_i \mid X_0 = i)^2 \sum_{n=1}^{\infty} n \cdot P(T_j < T_j \mid X_0 = i)^{n-1}$ Let $P(T_i < T_j \mid X_0 = i) = P$

$(1-P)^2 (1 + P + P$ each column is a geometric series

$+ P^2 + P^2 + \cdots)$ with $0 \leq P \leq 1$

$= (1-P)^2 \sum_{n=0}^{\infty} \frac{P^n}{1-P} = (1-P) \sum_{n=0}^{\infty} P^n = (1-P)\frac{1}{1-P} = 1$