3.3 Limiting Behaviour of DTMCs

Exercises

Exercise 3.3.1. In Example 3.1, convince yourself that by taking several higher powers of P^n,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.47619 & 0.238095 & 0.285714 \\ 0.47619 & 0.238095 & 0.285714 \\ 0.47619 & 0.238095 & 0.285714 \end{bmatrix} \end{array}.$$

Verify that the conditions of the BLT hold true for this DTMC, and show that the limiting probabilities obtained using the BLT agree with the above limiting TPM.

Solution: Let us take several higher powers of P^n:

$$P^{(2)} = PP = \begin{bmatrix} 0.54 & 0.26 & 0.2 \\ 0.2 & 0.36 & 0.44 \\ 0.6 & 0.1 & 0.3 \end{bmatrix},$$

$$P^{(4)} = P^{(2)}P^{(2)} = \begin{bmatrix} 0.4636 & 0.254 & 0.2824 \\ 0.444 & 0.2256 & 0.3304 \\ 0.524 & 0.222 & 0.254 \end{bmatrix},$$

$$P^{(8)} = P^{(4)}P^{(4)} = \begin{bmatrix} 0.475679 & 0.23775 & 0.286572 \\ 0.479134 & 0.23702 & 0.283845 \\ 0.47459 & 0.239567 & 0.285842 \end{bmatrix},$$

$$P^{(16)} = P^{(8)}P^{(8)} = \begin{bmatrix} 0.476188 & 0.238097 & 0.285715 \\ 0.476189 & 0.238093 & 0.285719 \\ 0.476195 & 0.238094 & 0.28571 \end{bmatrix},$$

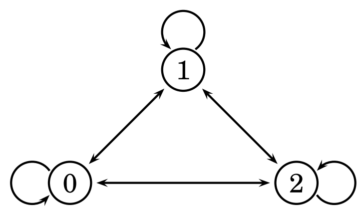
$$P^{(32)} = P^{(16)}P^{(16)} = \begin{bmatrix} 0.47619 & 0.238095 & 0.285714 \\ 0.47619 & 0.238095 & 0.285714 \\ 0.47619 & 0.238095 & 0.285714 \end{bmatrix},$$

and

$$P^{(64)} = P^{(32)}P^{(32)} = \begin{bmatrix} 0.47619 & 0.238095 & 0.285714 \\ 0.47619 & 0.238095 & 0.285714 \\ 0.47619 & 0.238095 & 0.285714 \end{bmatrix}.$$

Note that $P^{(32)}$ and $P^{(64)}$ are identical (to 6 decimal places of accuracy), and so this would appear to be the limiting TPM.

State Transition Diagram



From the above diagram, all states are accessible from each other, which implies that the DTMC is irreducible. Also, note that $P_{i,i} > 0$, $i = 0, 1, 2$, which immediately implies that all states are aperiodic. This is a finite-state, irreducible DTMC, and so the DTMC is also (positive) recurrent. Since the conditions of the BLT hold true, a limiting probability distribution $\underline{\pi} = (\pi_0, \pi_1, \pi_2)$ exists and satisfies the following system of linear equations:

$$\begin{aligned} \pi_0 &= 0.7\pi_0 + 0.5\pi_2 \Longrightarrow \pi_2 = \frac{3}{5}\pi_0, \\ \pi_1 &= 0.2\pi_0 + 0.6\pi_1 \Longrightarrow \pi_1 = \frac{1}{2}\pi_0, \\ \pi_2 &= 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2, \\ 1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3. \end{aligned}$$

Note that the last equation gives rise to

$$\left(1 + \frac{1}{2} + \frac{3}{5}\right)\pi_0 = 1 \Longrightarrow \pi_0 = \frac{10}{21}\ .$$

With $\pi_0 = \frac{10}{21} \approx 0.47619$, it immediately follows that $\pi_1 = \frac{5}{21} \approx 0.238095$ and $\pi_2 = \frac{6}{21} \approx 0.285714$. Note that these values are in agreement with the rows of the above limiting TPM.

Exercise 3.3.2. For the DTMC in Example 3.9, show that there exists an infinite number of stationary distributions.

Solution: With $\underline{p} = (p_0, p_1, p_2)$, we consider the system of linear equations given by $\underline{p} = \underline{p}P$:

$$\begin{aligned} p_0 &= p_2, \\ p_1 &= p_1, \\ p_2 &= p_0. \end{aligned}$$

But $p_0 + p_1 + p_2 = 1$, and so we simply get $2p_0 + p_1 = 1$. Let $p_1 = \alpha$ for any $\alpha \in (0, 1)$ so that $p_0 = p_2 = \frac{1-\alpha}{2}$. Thus, there are an infinite number of stationary distributions of the form

$$\underline{p} = \left(\tfrac{1-\alpha}{2}, \quad \alpha, \quad \tfrac{1-\alpha}{2} \right), \ \alpha \in (0, 1).$$

Exercise 3.3.3. Suppose that the TPM of a two-state DTMC is given by

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} \end{array},$$

where $0 < \alpha < 1$ and $0 < \beta < 1$.

(a) Assuming that $\alpha = \beta = p$, show by mathematical induction that

$$P^{(n)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{bmatrix} \end{array}.$$

(b) In part (a), what happens to $P^{(n)}$ as $n \to \infty$?

(c) Find the limiting probability distribution of this DTMC.

(d) Show that the first return time distribution to state 0 is given by $f_{0,0}^{(1)} = \alpha$ and $f_{0,0}^{(n)} = (1-\alpha)(1-\beta)\beta^{n-2}$, $n = 2, 3, 4, \ldots$.

(e) Use part (d) to calculate $m_{0,0}$, the mean recurrent time for state 0, and then verify that $\pi_0 = 1/m_{0,0}$ where π_0 represents the long-run mean fraction of time the DTMC is in state 0.

Solution: (a) With $\alpha = \beta = p$, the TPM becomes

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \end{array}.$$

Using mathematical induction, for $n = 1$, we immediately obtain

$$P^{(1)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^1 & \frac{1}{2} - \frac{1}{2}(2p-1)^1 \\ \frac{1}{2} - \frac{1}{2}(2p-1)^1 & \frac{1}{2} + \frac{1}{2}(2p-1)^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + p - \frac{1}{2} & \frac{1}{2} - p + \frac{1}{2} \\ \frac{1}{2} - p + \frac{1}{2} & \frac{1}{2} + p - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

Thus, the result holds true for $n = 1$. Assuming now that the result is true for $n = k$ (inductive hypothesis), we consider

$$P^{(k+1)} = P^{(k)}P = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^k & \frac{1}{2} - \frac{1}{2}(2p-1)^k \\ \frac{1}{2} - \frac{1}{2}(2p-1)^k & \frac{1}{2} + \frac{1}{2}(2p-1)^k \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad 0 & \quad\quad 1 \end{array} \\ \begin{bmatrix} P_{0,0}^{(k+1)} & P_{0,1}^{(k+1)} \\ P_{1,0}^{(k+1)} & P_{1,1}^{(k+1)} \end{bmatrix} \end{array}.$$

Applying basic matrix multiplication, it follows that

$$\begin{aligned} P_{0,0}^{(k+1)} &= \sum_{l=0}^{1} P_{0,l}^{(k)} P_{l,0} \\ &= \left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right)(p) + \left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right)(1-p) \\ &= \frac{1}{2}p + \frac{1}{2}(1-p) + \frac{1}{2}(2p-1)^k p + \frac{1}{2}(2p-1)^k(p-1) \\ &= \frac{1}{2} + \frac{1}{2}(2p-1)^k(2p-1) = \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1}, \end{aligned}$$

$$\begin{aligned} P_{0,1}^{(k+1)} &= \sum_{l=0}^{1} P_{0,l}^{(k)} P_{l,1} \\ &= \left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right)(1-p) + \left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right)(p) \\ &= \frac{1}{2}(1-p) + \frac{1}{2}p - \frac{1}{2}(2p-1)^k(p-1) - \frac{1}{2}(2p-1)^k p \\ &= \frac{1}{2} - \frac{1}{2}(2p-1)^k(2p-1) = \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1}, \end{aligned}$$

$$\begin{aligned} P_{1,0}^{(k+1)} &= \sum_{l=0}^{1} P_{1,l}^{(k)} P_{l,0} \\ &= \left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right)(p) + \left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right)(1-p) \\ &= \frac{1}{2}p + \frac{1}{2}(1-p) - \frac{1}{2}(2p-1)^k p - \frac{1}{2}(2p-1)^k(p-1) \\ &= \frac{1}{2} - \frac{1}{2}(2p-1)^k(2p-1) = \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1}, \end{aligned}$$

and

$$
\begin{aligned}
P_{1,1}^{(k+1)} &= \sum_{l=0}^{1} P_{1,l}^{(k)} P_{l,1} \\
&= \left(\frac{1}{2} - \frac{1}{2}(2p-1)^k\right)(1-p) + \left(\frac{1}{2} + \frac{1}{2}(2p-1)^k\right)(p) \\
&= \frac{1}{2}(1-p) + \frac{1}{2}p + \frac{1}{2}(2p-1)^k(p-1) + \frac{1}{2}(2p-1)^k p \\
&= \frac{1}{2} + \frac{1}{2}(2p-1)^k(2p-1) = \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1}.
\end{aligned}
$$

Therefore,

$$
P^{(k+1)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1} & \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1} & \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1} \end{bmatrix},
$$

and so the result is true for $n = k+1$. Hence, by mathematical induction, the result is true in general.

(b) Note that

$$
0 < p < 1 \Longrightarrow 0 < 2p < 2 \Longrightarrow -1 < 2p - 1 < 1.
$$

Therefore,

$$
\begin{aligned}
\lim_{n\to\infty} P^{(n)} &= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\lim\limits_{n\to\infty}(2p-1)^n & \frac{1}{2} - \frac{1}{2}\lim\limits_{n\to\infty}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}\lim\limits_{n\to\infty}(2p-1)^n & \frac{1}{2} + \frac{1}{2}\lim\limits_{n\to\infty}(2p-1)^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(0) & \frac{1}{2} - \frac{1}{2}(0) \\ \frac{1}{2} - \frac{1}{2}(0) & \frac{1}{2} + \frac{1}{2}(0) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.
\end{aligned}
$$

(c) With $0 < \alpha < 1$ and $0 < \beta < 1$, it is clear that the DTMC is irreducible and aperiodic. Since this is a finite-state DTMC, it is also positive recurrent. Therefore, since the conditions of the BLT are met, the limiting probability distribution $\underline{\pi} = (\pi_0, \pi_1)$ exists and satisfies the following system of linear equations:

$$
\begin{aligned}
\pi_0 &= \alpha\pi_0 + (1-\beta)\pi_1 \Longrightarrow \pi_0 = \frac{1-\beta}{1-\alpha}\pi_1, \\
\pi_1 &= (1-\alpha)\pi_0 + \beta\pi_1, \\
1 &= \pi_0 + \pi_1.
\end{aligned}
$$

Note that the last equation gives rise to

$$\frac{1-\beta}{1-\alpha}\pi_1 + \pi_1 = 1 \Longrightarrow \pi_1 = \frac{1-\alpha}{2-\alpha-\beta} \ .$$

As an immediate consequence, we obtain $\pi_0 = \frac{1-\beta}{2-\alpha-\beta}$.

(d) Clearly, $f_{0,0}^{(1)} = P_{0,0} = \alpha$. For $n = 2, 3, 4, \ldots$, note that

$$\begin{aligned}
f_{0,0}^{(n)} &= P(X_n = 0, X_{n-1} \neq 0, \ldots, X_2 \neq 0, X_1 \neq 0 | X_0 = 0) \\
&= P(X_n = 0, X_{n-1} = 1, \ldots, X_2 = 1, X_1 = 1 | X_0 = 0) \\
&= P_{0,1}\left(\prod_{i=1}^{n-2} P_{1,1}\right) P_{1,0} \\
&= P_{0,1}P_{1,0}P_{1,1}^{n-2} \\
&= (1-\alpha)(1-\beta)\beta^{n-2}.
\end{aligned}$$

(e) Using the result of part (d), we have that

$$\begin{aligned}
m_{0,0} &= \sum_{n=1}^{\infty} n f_{0,0}^{(n)} \\
&= \alpha + (1-\alpha)\sum_{n=2}^{\infty} n(1-\beta)\beta^{n-2} \\
&= \alpha + (1-\alpha)\ \underbrace{\sum_{y=1}^{\infty}(y+1)(1-\beta)\beta^{y-1}}_{=\mathrm{E}[Y+1] \text{ where } Y\sim \mathrm{GEO}(1-\beta)} \\
&= \alpha + (1-\alpha)\left(\frac{1}{1-\beta} + 1\right) \\
&= \alpha + \frac{(1-\alpha)(2-\beta)}{1-\beta} \\
&= \frac{\alpha(1-\beta) + 2 - \beta - 2\alpha + \alpha\beta}{1-\beta} \\
&= \frac{2-\alpha-\beta}{1-\beta} \\
&= \frac{1}{\pi_0} \text{ from part (c).}
\end{aligned}$$

Exercise 3.3.4. Let $\{X_i\}_{i=1}^{\infty}$ be an iid sequence of Bernoulli(p) random variables, where $p \in (0,1)$. For $n \in \mathbb{Z}^+$, let $Y_n = \sum_{i=1}^{n} X_i$.

(a) Justify why the stochastic process $\{Y_n, n \in \mathbb{Z}^+\}$ is a DTMC.

(b) Determine the TPM of this DTMC.

(c) Specify the communication classes of this DTMC. For each class of this DTMC, determine whether it is recurrent or transient. Justify your response.

(d) Does a limiting distribution for this DTMC exist? If so, what is the limiting distribution? Justify your response.

(e) For each $k \in \mathbb{Z}^+$, let $T_k = \min\{n \in \mathbb{Z}^+ : Y_n = k\}$. Determine the pmf of the random variable T_k.

Solution: (a) First of all, note that Y_n can take on values in the countable set $\{0, 1, 2, \ldots\}$. Moreover, note that $Y_{n+1} = \sum_{i=1}^{n+1} X_i = \sum_{i=1}^{n} X_i + X_{n+1} = Y_n + X_{n+1}$, implying that the value of Y_{n+1} can be determined from knowledge of Y_n and an independent random variable X_{n+1}. Thus, the Markov property is satisfied and $\{Y_n, n \in \mathbb{Z}^+\}$ is a DTMC.

(b) From any given state i, $i \in \mathbb{N}$, the DTMC either remains in state i (with failure probability $1-p$) or transitions to state $i+1$ (with success probability p). As a result, it follows that

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} 1-p & p & 0 & 0 & 0 & \cdots \\ 0 & 1-p & p & 0 & 0 & \ddots \\ 0 & 0 & 1-p & p & 0 & \ddots \\ 0 & 0 & 0 & 1-p & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \end{array}.$$

(c) For each $i \in \mathbb{N}$, we note that once the process leaves state i, it is impossible to return back to state i. Thus, no two distinct states communicate with each other. Hence, $\{0\}, \{1\}, \{2\}, \ldots$ are the communication classes of this DTMC. Moreover, for each $i \in \mathbb{N}$, note that

$$\sum_{n=1}^{\infty} P_{i,i}^{(n)} = \sum_{n=1}^{\infty} (1-p)^n = \frac{1-p}{1-(1-p)} = \frac{1-p}{p} < \infty \text{ since } p \in (0,1).$$

Thus, each state of this DTMC must be transient.

(d) Since all states are transient, a limiting probability distribution does exist for this DTMC. In particular, by Theorem 3.6 we know that

$$\lim_{n\to\infty} P_{i,j}^{(n)} = 0 \quad \forall\, i, j \in \mathbb{N}.$$

(e) Let R_i, $i \in \mathbb{N}$, represent the number of transitions it takes for the DTMC to go from state i to state $i+1$. As a result, we have that $T_k = \sum_{i=0}^{k-1} R_i$, $k \in \mathbb{Z}^+$. Based on the form of the TPM, we deduce that each $R_i \sim \text{GEO}(p)$. Moreover, since the sequence of random variables $\{R_i\}_{i=0}^{\infty}$ is independent, it follows that $T_k \sim \text{NB}(k, p)$, with pmf

$$p_{T_k}(x) = P(T_k = x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, \quad x = k, k+1, k+2, \ldots.$$

Exercise 3.3.5. Consider a DTMC $\{X_n, n \in \mathbb{N}\}$ with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1-p_0 & p_0 & 0 & 0 \\ 1-p_1 & 0 & p_1 & 0 \\ 1-p_2 & 0 & 0 & p_2 \\ 1-p_3 & 0 & 0 & p_3 \end{bmatrix} \end{array}.$$

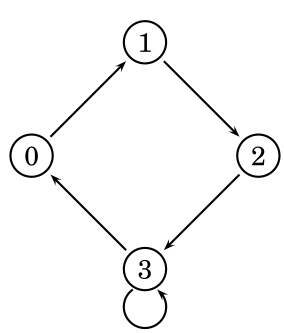
(a) Suppose that $p_0 = p_1 = p_2 = 1$ but $0 < p_3 < 1$. Does the DTMC have a limiting probability distribution? If so, find it.

(b) Suppose now that $(p_0, p_1, p_2, p_3) = (1/2, 1/3, 1/3, 1/2)$. Does the DTMC have a limiting probability distribution? If so, find it.

(c) Suppose once again that $(p_0, p_1, p_2, p_3) = (1/2, 1/3, 1/3, 1/2)$, and that the initial conditions are governed by the stationary distribution of the DTMC. Assuming that $n = 2, 3, 4, \ldots$, calculate $P(X_{n-1} = 1 \mid X_n = 0)$.

Solution: (a)

State Transition Diagram



From the above diagram, all states are accessible from each other, which implies that the DTMC is irreducible. Also, note that $P_{3,3} = p_3 > 0$, which immediately implies that state 3 (and all other states) is aperiodic. This is a finite-state, irreducible DTMC, and so the DTMC is also (positive) recurrent. By the BLT, a limiting probability distribution $\underline{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3)$ is known to exist and satisfies the following system of linear equations:

$$\begin{aligned}
\pi_0 &= (1-p_3)\pi_3 \Longrightarrow \pi_3 = \frac{\pi_0}{1-p_3}, \\
\pi_1 &= \pi_0, \\
\pi_2 &= \pi_1, \\
\pi_3 &= \pi_2 + p_3\pi_3, \\
1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3.
\end{aligned}$$

Note that the last equation gives rise to

$$\left(1 + 1 + 1 + \frac{1}{1-p_3}\right)\pi_0 = 1 \Longrightarrow \pi_0 = \frac{1-p_3}{4-3p_3}\ .$$

Thus, it immediately follows that $\underline{\pi} = \left(\frac{1-p_3}{4-3p_3}, \frac{1-p_3}{4-3p_3}, \frac{1-p_3}{4-3p_3}, \frac{1}{4-3p_3}\right)$.

(b) With $(p_0, p_1, p_2, p_3) = (1/2, 1/3, 1/3, 1/2)$, the TPM becomes

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \end{array}.$$

It is evident that this DTMC is also irreducible, aperiodic, and positive recurrent. Therefore, the BLT can again be used to find the limiting probability distribution. Specifically, we solve the system of linear equations defined by $\underline{\pi} = \underline{\pi}P$:

$$\begin{aligned}
\pi_0 &= \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1 + \frac{2}{3}\pi_2 + \frac{1}{2}\pi_3, \\
\pi_1 &= \frac{1}{2}\pi_0, \\
\pi_2 &= \frac{1}{3}\pi_1 \Longrightarrow \pi_2 = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\pi_0 = \frac{1}{6}\pi_0, \\
\pi_3 &= \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 \Longrightarrow \pi_3 = 2\left(\frac{1}{3}\right)\pi_2 = \left(\frac{2}{3}\right)\left(\frac{1}{6}\right)\pi_0 = \frac{1}{9}\pi_0.
\end{aligned}$$

Using the condition that $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$, we can solve for π_0 as follows:

$$\begin{aligned}
\pi_0 + \frac{1}{2}\pi_0 + \frac{1}{6}\pi_0 + \frac{1}{9}\pi_0 &= 1 \\
\left(\frac{18+9+3+2}{18}\right)\pi_0 &= 1 \Longrightarrow \pi_0 = \frac{18}{32} = \frac{9}{16}\ .
\end{aligned}$$

It immediately follows that $\pi_1 = (1/2)(9/16) = 9/32$, $\pi_2 = (1/6)(9/16) = 3/32$, and $\pi_3 = (1/9)(9/16) = 1/16$. Thus, $\underline{\pi} = (9/16, 9/32, 3/32, 1/16)$.

(c) Recall that the limiting probability distribution in part (b) is the unique stationary distribution, implying that $P(X_m = i) = \pi_i \ \forall \ m \in \mathbb{N}$ and $i = 0, 1, 2, 3$. For $n = 2, 3, 4, \ldots$, we obtain

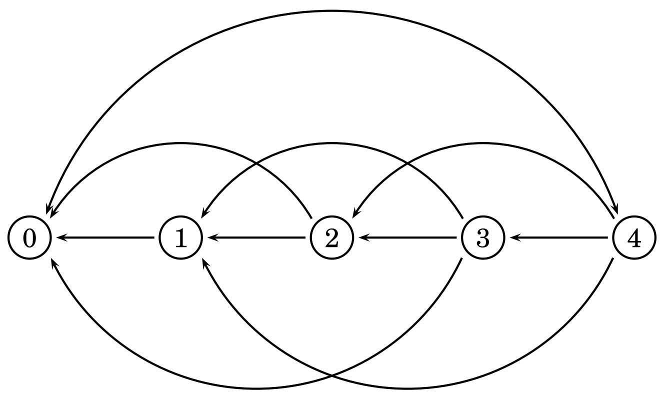
$$\begin{aligned}
P(X_{n-2} = 1 | X_n = 0) &= \frac{P(X_n = 0, X_{n-2} = 1)}{P(X_n = 0)} \\
&= \frac{P(X_{n-2} = 1)P(X_n = 0 | X_{n-2} = 1)}{\pi_0} \\
&= \frac{\pi_1}{\pi_0}\sum_{j=0}^{3} P_{1,j}P_{j,0} \\
&= \frac{9/32}{9/16}\Big[(2/3)(1/2) + (0)(2/3) + (1/3)(2/3) + (0)(1/2)\Big] \\
&= \frac{1}{2}\left(\frac{1}{3} + \frac{2}{9}\right) \\
&= \frac{5}{18} \approx 0.278.
\end{aligned}$$

Exercise 3.3.6. Consider a DTMC with state space $\{0, 1, 2, 3, 4\}$. Let $P_{0,4} = 1$, and suppose that when the chain is in state $i, i > 0$, the next state to be visited is equally likely to be any of the states $0, 1, \ldots, i-1$. Find the limiting probabilities of this DTMC.

Solution: Based on the given information, we can construct the TPM as follows:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix} \end{array}.$$

State Transition Diagram



From the above diagram, this finite-state DTMC is irreducible and therefore positive recurrent. Moreover,

$$d(0) = \gcd\{n \in \mathbb{Z}^+ : P_{0,0}^{(n)} > 0\} = \gcd\{2, 3, 4, \ldots\} = 1,$$

and so the DTMC is also aperiodic. By the BLT, the limiting probabilities π_0, π_1, π_2, π_3, and π_4 satisfy:

$$\begin{aligned}
\pi_4 &= \pi_0, \\
\pi_3 &= \frac{1}{4}\pi_4 \implies \pi_3 = \frac{1}{4}\pi_0, \\
\pi_2 &= \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \implies \pi_2 = \frac{1}{12}\pi_0 + \frac{1}{4}\pi_0 = \frac{1}{3}\pi_0, \\
\pi_1 &= \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \implies \pi_1 = \frac{1}{6}\pi_0 + \frac{1}{12}\pi_0 + \frac{1}{4}\pi_0 = \frac{1}{2}\pi_0, \\
\pi_0 &= \pi_1 + \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4, \\
1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4.
\end{aligned}$$

Note that the last equation gives rise to

$$\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+1\right)\pi_0 = 1 \Longrightarrow \pi_0 = \frac{12}{37}\ .$$

Thus, we immediately obtain $\pi_0 = \pi_4 = 12/37$, $\pi_1 = 6/37$, $\pi_2 = 4/37$, and $\pi_3 = 3/37$.

Exercise 3.3.7. Each morning an individual starts his day by getting a latte at the local coffee shop. As he prepares to go out, he is equally likely to leave either from the front or back door of his house. The individual owns m pairs of shoes which he removes at whichever door he happens to enter back through. In a similar fashion, he is equally likely to come back either through the front or back door. However, if there are no shoes at the door from which he decides to leave from, he simply strolls over barefoot to his neighbour's house and enjoys a latte with the neighbour on his outdoor patio.

(a) Describe how you might model this scenario as a DTMC and specify its associated TPM.

(b) What is the long-run proportion of the time the individual enjoys a latte with his neighbour?

Solution: (a) Let X_n denote the number of pairs of shoes at the front door at beginning of day n. Clearly, to determine the value of X_n, all we require is knowledge of X_{n-1}. As a result, the stochastic process $\{X_n, n \in \mathbb{Z}^+\}$ satisfies the Markov property, and so is a DTMC on the finite state space $\{0, 1, \ldots, m\}$ with TPM of the form

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ m-2 \\ m-1 \\ m \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \ldots & m-2 & m-1 & m \end{array} \\ \begin{bmatrix} 3/4 & 1/4 & 0 & \ldots & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & \ldots & 0 & 0 & 0 \\ 0 & 1/4 & 1/2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1/2 & 1/4 & 0 \\ 0 & 0 & 0 & \ldots & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & \ldots & 0 & 1/4 & 3/4 \end{bmatrix} \end{array} .$$

The form of P stems from the fact that if $X_n \in \{1, 2, \ldots, m-1\}$, then there will be at least one pair of shoes at whatever door he chooses to leave from. As such, he will have a fifty percent chance of returning the shoes to the same door he left from (i.e., $P_{i,i} = 1/2, i = 1, 2, \ldots, m-1$). Conversely, there is a fifty percent chance that he returns the shoes to the opposite door, thereby either incrementing or decrementing X_n by 1 with equal probabilities:

$$P_{i,i+1} = P(\text{leave from back door, return through front door}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\ ,$$
$$P_{i,i-1} = P(\text{leave from front door, return through back door}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\ .$$

If $X_n = 0$, then he will leave from the front door (with probability $1/2$) and be forced to walk barefoot to his neighbour's house, so the number of shoes at the front door will not change no matter which door he enters back through. On the other hand, if he leaves from the back door (with probability $1/2$), then he will either return through the back door (and this does not change the value of X_n, with total probability $(1/2)^2 = 1/4$) or X_n will increment by 1 if he returns through the front door (also with total probability $1/4$). Thus, $P_{0,0} = 1/2+1/4 = 3/4$ and $P_{0,1} = 1/4$. Symmetrically, by applying similar arguments, it is straightforward to justify that $P_{m,m} = 3/4$ and $P_{m,m-1} = 1/4$.

(b) Looking at the structure of P, it is clear that all states communicate with each other, implying that the DTMC is irreducible. The DTMC is also aperiodic (due to positive main diagonal probabilities in the TPM) and positive recurrent (since the DTMC has a finite state space of size $m + 1$). Therefore, by the BLT, a limiting distribution exists and is unique. Moreover, note that the DTMC is *doubly stochastic* $\implies \pi_j = \frac{1}{m+1}$, $j = 0, 1, \ldots, m$. Thus, we ultimately obtain

$$\begin{aligned}
&\text{Long-run proportion of the time he enjoys a latte with his neighbour} \\
&= P(\text{he chooses front door to leave from}) \cdot \pi_0 + P(\text{he chooses back door to leave from}) \cdot \pi_m \\
&= \frac{1}{2}\pi_0 + \frac{1}{2}\pi_m \\
&= \frac{1}{2}\left(\frac{1}{m+1} + \frac{1}{m+1}\right) \\
&= \frac{1}{m+1}\ .
\end{aligned}$$

Exercise 3.3.8. Consider a dam with a maximum capacity of $h \in \mathbb{Z}^+$ units of water. Let X_n represent the amount of water in the dam at the **end** of the n^{th} day. In addition, suppose that the following facts are known:

- The daily inputs to the dam are independent of each other, and on a given day,

$$P(\text{input is } j \text{ units}) = 2^{-(j+1)}, \ j = 0, 1, 2, \dots \ .$$

- Any *overflow* (when the total amount of water exceeds the maximum capacity of h units) is regarded as lost.

- Provided the dam is not empty, one unit of water is released at the end of the day.

- The value of X_n is the amount of water in the dam following the one unit release of water at the end of the day.

Thus, $\{X_n, n \in \mathbb{Z}^+\}$ is a discrete-time stochastic process with state space $\{0, 1, \dots, h-1\}$.

(a) Justify why the stochastic process $\{X_n, n \in \mathbb{Z}^+\}$ is a DTMC.

(b) Prove that the DTMC is irreducible and aperiodic.

(c) Assuming that $h = 5$, determine the associated TPM and solve for the limiting probabilities of the DTMC.

Solution: (a) To determine the amount of water in the dam at the end of any given day, we only need to keep track of its contents 24 hours earlier, which implies that the Markov property assumption is satisfied by $\{X_n, n \in \mathbb{Z}^+\}$. Thus, $\{X_n, n \in \mathbb{Z}^+\}$ is a DTMC.

(b) From any state i, $i = 0, 1, \dots, h-1$, it is always possible to visit state $j \geq i$ in the next transition. This is because the daily input could be (with positive probability) of the amount $j - i + 1 \geq 1$, thereby raising its capacity to $j + 1$ before releasing one unit to bring the amount back down to j. However, it is also possible to visit state $\max\{i-1, 0\}$ in the next transition, and this happens when the daily input is equal to 0 (with positive probability). Therefore, there is always a path connecting any states i and j of this DTMC. By definition, all states communicate with each other and the DTMC is irreducible. Moreover, note that

$$P_{0,0} = P(\text{daily input is 0 or 1 unit}) = 2^{-1} + 2^{-2} = \frac{3}{4} > 0.$$

Thus, state 0 is aperiodic, and since the DTMC is irreducible, all states are aperiodic.

(c) With $h = 5$, the TPM is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}.$$

Since this DTMC is also positive recurrent (as there are a finite number of states), the BLT ensures that the limiting probabilities satisfy:

$$\begin{aligned}
\pi_0 &= \frac{3}{4}\pi_0 + \frac{1}{2}\pi_1 \Longrightarrow \pi_1 = \frac{1}{2}\pi_0, \\
\pi_1 &= \frac{1}{8}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 \Longrightarrow \pi_2 = \frac{3}{2}\pi_1 - \frac{1}{4}\pi_0 = \frac{1}{2}\pi_0, \\
\pi_2 &= \frac{1}{16}\pi_0 + \frac{1}{8}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 \Longrightarrow \pi_3 = \frac{3}{2}\pi_2 - \frac{1}{4}\pi_1 - \frac{1}{8}\pi_0 = \frac{1}{2}\pi_0, \\
\pi_3 &= \frac{1}{32}\pi_0 + \frac{1}{16}\pi_1 + \frac{1}{8}\pi_2 + \frac{1}{4}\pi_3 + \frac{1}{2}\pi_4 \Longrightarrow \pi_4 = \frac{3}{2}\pi_3 - \frac{1}{16}\pi_0 - \frac{1}{8}\pi_1 - \frac{1}{4}\pi_2 = \frac{1}{2}\pi_0.
\end{aligned}$$

Using the condition that $\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$, we can solve for π_0 as follows:

$$\left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right)\pi_0 = 1 \Longrightarrow \pi_0 = \frac{1}{3}\ .$$

Therefore, we immediately obtain $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 1/6$.

3.4 Two Interesting Applications

Exercises

Exercise 3.4.1. Consider a *Galton-Watson Branching Process* with $X_0 = 1$ and $\mu > 1$. Use mathematical induction to show that $z^* \geq P(X_n = 0)$, $n \in \mathbb{N}$, where $z = z^*$ is any non-negative solution satisfying $z = \sum_{j=0}^{\infty} z^j \alpha_j$.

> **Solution:** We prove this using mathematical induction. Starting with $n = 0$, since $X_0 = 1$ (with probability 1), $P(X_0) = 0 \leq z^*$. Therefore, the result is true for $n = 0$. Assuming the result is true for $n = k$ (i.e., $P(X_k = 0) \leq z^*$), we consider
>
> $$\begin{aligned} P(X_{k+1} = 0) &= \sum_{j=0}^{\infty} P(X_{k+1} = 0 | X_1 = j) P(X_1 = j | X_0 = 1) \\ &= \sum_{j=0}^{\infty} P(X_{k+1} = 0 | X_1 = j) \alpha_j \\ &= \sum_{j=0}^{\infty} P(X_k = 0 | X_0 = j) \alpha_j \text{ due to the stationary assumption} \\ &= \sum_{j=0}^{\infty} [P(X_k = 0 | X_0 = 1)]^j \alpha_j \\ &= \sum_{j=0}^{\infty} [P(X_k = 0)]^j \alpha_j \text{ suppressing the condition ``} X_0 = 1\text{''} \\ &\leq \sum_{j=0}^{\infty} (z^*)^j \alpha_j \text{ by the inductive hypothesis} \\ &= z^* \text{ by the definition of } z^*. \end{aligned}$$
>
> Thus, $P(X_{k+1} = 0) \leq z^*$ and the result is true for $n = k + 1$. Hence, by mathematical induction, the result is true in general.

Exercise 3.4.2. For a *Galton-Watson Branching Process* having $X_0 = 1$, calculate the probability that the population will eventually die out when the number of offspring (produced by any one individual) has probability distribution $\{\alpha_m\}_{m=0}^{\infty}$ given by

(a) $\alpha_0 = 1/2,\ \alpha_1 = 1/5,\ \alpha_2 = 1/5,\ \alpha_3 = 1/10$

(b) $\alpha_0 = 2/5,\ \alpha_1 = 3/10,\ \alpha_2 = 1/5,\ \alpha_3 = 1/10$

(c) $\alpha_0 = 1/4,\ \alpha_1 = 1/3,\ \alpha_2 = 1/4,\ \alpha_3 = 1/6$

(d) $\alpha_0 = 1/6,\ \alpha_1 = 1/3,\ \alpha_2 = 5/12,\ \alpha_3 = 1/12$

(e) $\alpha_0 = 3/7,\ \alpha_1 = 1/35,\ \alpha_2 = 12/35,\ \alpha_3 = 2/35,\ \alpha_4 = 1/7$

Solution: (a) We first calculate

$$\mu = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{5}\right) + 2\left(\frac{1}{5}\right) + 3\left(\frac{1}{10}\right) = \frac{9}{10}\ .$$

Since $\mu < 1$, the population will die out with probability 1.

(b) We first calculate

$$\mu = 0\left(\frac{2}{5}\right) + 1\left(\frac{3}{10}\right) + 2\left(\frac{1}{5}\right) + 3\left(\frac{1}{10}\right) = 1.$$

Since $\mu = 1$, the population will die out with probability 1.

(c) We begin by calculating

$$\mu = 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{6}\right) = \frac{4}{3}\ .$$

Since $\mu > 1$, $\pi_0 \in (0, 1)$ is known to exist. To find π_0, we solve the equation

$$\begin{aligned} z &= \sum_{j=0}^{\infty} z^j \alpha_j \\ &= \frac{1}{4} + z\left(\frac{1}{3}\right) + z^2\left(\frac{1}{4}\right) + z^3\left(\frac{1}{6}\right), \end{aligned}$$

giving rise to the cubic equation

$$3 - 8z + 3z^2 + 2z^3 = 0,$$

or equivalently,

$$(2z - 1)(z + 3)(z - 1) = 0.$$

The three roots are $z = 1/2$, $z = 1$, and $z = -3$. Thus, $\pi_0 = 1/2$.

(d) We begin by calculating

$$\mu = 0\left(\frac{1}{6}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{5}{12}\right) + 3\left(\frac{1}{12}\right) = \frac{17}{12}\,.$$

Since $\mu > 1$, $\pi_0 \in (0,1)$ is known to exist. To find π_0, we solve the equation

$$\begin{aligned} z &= \sum_{j=0}^{\infty} z^j \alpha_j \\ &= \frac{1}{6} + z\left(\frac{1}{3}\right) + z^2\left(\frac{5}{12}\right) + z^3\left(\frac{1}{12}\right), \end{aligned}$$

giving rise to the cubic equation

$$\begin{aligned} 0 &= 2 - 8z + 5z^2 + z^3 \\ &= (z^2 + 6z - 2)(z-1), \end{aligned}$$

or equivalently,

$$(z + 3 - \sqrt{11})(z + 3 + \sqrt{11})(z-1) = 0.$$

The three roots are $z = -3 + \sqrt{11}$, $z = 1$, and $z = -3 - \sqrt{11}$. Thus, $\pi_0 = -3 + \sqrt{11} \approx 0.317$.

(e) We begin by calculating

$$\mu = 0\left(\frac{3}{7}\right) + 1\left(\frac{1}{35}\right) + 2\left(\frac{12}{35}\right) + 3\left(\frac{2}{35}\right) + 4\left(\frac{1}{7}\right) = \frac{51}{35}\,.$$

Since $\mu > 1$, $\pi_0 \in (0,1)$ is known to exist. To find π_0, we solve the equation

$$\begin{aligned} z &= \sum_{j=0}^{\infty} z^j \alpha_j \\ &= \frac{3}{7} + z\left(\frac{1}{35}\right) + z^2\left(\frac{12}{35}\right) + z^3\left(\frac{2}{35}\right) + z^4\left(\frac{1}{7}\right), \end{aligned}$$

giving rise to the quartic equation

$$\begin{aligned} 0 &= 15 - 34z + 12z^2 + 2z^3 + 5z^4 \\ &= (5z^3 + 7z^2 + 19z - 15)(z-1) \\ &= (5z-3)(z^2 + 2z + 5)(z-1), \end{aligned}$$

or equivalently,

$$(5z-3)(z + 1 - 2i)(z + 1 + 2i)(z-1) = 0.$$

The four roots are $z = 3/5$, $z = 1$, $z = -1 - 2i$, and $z = -1 + 2i$. Thus, $\pi_0 = 3/5$.

Exercise 3.4.3. Consider a *Galton-Watson Branching Process* $\{X_n, n = 0, 1, 2, \ldots\}$ with $X_0 = 1$. Suppose that the number of offspring (produced by any one individual) has probability distribution $\{\alpha_m\}_{m=0}^{\infty}$ which satisfies

$$\alpha_0 = 1 - q, \quad \alpha_1 = (1-q)q, \quad \alpha_2 = q^2, \quad \alpha_m = 0 \;\forall\; m \geq 3 \text{ where } 0 < q < 1.$$

(a) Find a condition on q which ensures the probability that the population will eventually die out is strictly less than 1.

(b) Assuming that the condition on q in part (a) holds true, determine the probability that the population will eventually die out.

Solution: (a) We must ensure that $\mu = \sum_{m=0}^{\infty} m\alpha_m > 1$. Note that

$$\mu = 0 \cdot (1-q) + 1 \cdot (1-q)q + 2 \cdot q^2 = q + q^2 = q(1+q).$$

Therefore, we require

$$\begin{aligned} q(1+q) &> 1 \\ q^2 + q - 1 &> 0 \\ (q - r_1)(q - r_2) &> 0, \end{aligned}$$

where (via the quadratic formula)

$$r_1 = \frac{-1 + \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{-1 + \sqrt{5}}{2}$$

and

$$r_2 = \frac{-1 - \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{-1 - \sqrt{5}}{2}\,.$$

Clearly, $q > r_2$. Thus, we must have that $q - r_1 > 0$, or simply, $q > \frac{-1+\sqrt{5}}{2} \approx 0.618$.

(b) To determine π_0, we must find the roots of the equation

$$\begin{aligned} \sum_{m=0}^{\infty} z^m \alpha_m &= z \\ 1 - q + (1-q)qz + q^2z^2 &= z \\ q^2z^2 - q^2z + qz - z + 1 - q &= 0 \\ q^2z(z-1) - (1-q)z + (1-q) &= 0 \\ q^2z(z-1) - (1-q)(z-1) &= 0 \\ (z-1)[q^2z - (1-q)] &= 0. \end{aligned}$$

The two roots are $z = (1-q)/q^2$ and $z = 1$. Thus, we must have that $\pi_0 = (1-q)/q^2$.

Exercise 3.4.4. For a *Galton-Watson Branching Process* having $X_0 = n$, $n \in \mathbb{Z}^+$, calculate the probability that the population will eventually die out when the number of offspring (produced by any one individual) has pmf given by $\alpha_m = (1-\beta)^m \beta$, $m = 0, 1, 2, \dots$ and $0 < \beta < 1$.

Solution: First of all, let π_0 represent the probability that the population will eventually die out, given $X_0 = n$. Note that the offspring distribution is identical to that of the random variable $Y - 1$ where $Y \sim \text{GEO}(\beta)$. As a result, we have that

$$\mu = \mathrm{E}[Y-1] = \frac{1}{\beta} - 1 = \frac{1-\beta}{\beta} .$$

If $\mu \le 1$ (or equivalently, $\beta \ge 1/2$), then $\pi_0 = 1$.

On the other hand, if $\mu > 1$ (i.e., $\beta < 1/2$), then we must first find the roots of the equation

$$\begin{aligned}
\sum_{m=0}^{\infty} z^m \alpha_m &= z \\
\sum_{m=0}^{\infty} z^m (1-\beta)^m \beta &= z \\
\beta \sum_{m=0}^{\infty} [(1-\beta)z]^m &= z \\
\frac{\beta}{1-(1-\beta)z} &= z \quad \text{provided that } |z| < (1-\beta)^{-1} < 2 \\
z - (1-\beta)z^2 &= \beta \\
(1-\beta)z^2 - z + \beta &= 0 \\
(1-\beta)z^2 - (1-\beta)z - \beta z + \beta &= 0 \\
(1-\beta)z(z-1) - \beta(z-1) &= 0 \\
(z-1)[(1-\beta)z - \beta] &= 0.
\end{aligned}$$

The two roots are $z = \beta/(1-\beta)$ and $z = 1$. Thus, we must have that

$$\pi_0 = \left(\frac{\beta}{1-\beta}\right)^n .$$

Exercise 3.4.5. Consider a *Galton-Watson Branching Process* with $X_0 = 1$ and $\mu < 1$.

(a) How many individuals, on average, ever existed in this population?

(b) What would the answer to part (a) become if $X_0 = n$, $n \in \mathbb{Z}^+$?

Solution: (a) Let W represent the number of individuals that ever existed in this population. As such, $W = \sum_{j=0}^{\infty} X_j$. With $X_0 = 1$ and $\mu < 1$, recall that $\mathrm{E}[X_j] = \mu^j$ for $j \in \mathbb{N}$. Therefore,

$$\mathrm{E}[W] = \mathrm{E}\left[\sum_{j=0}^{\infty} X_j\right] = \sum_{j=0}^{\infty} \mathrm{E}[X_j] = \sum_{j=0}^{\infty} \mu^j = \frac{1}{1-\mu} .$$

(b) For a *Galton-Watson Branching Process*, we know that $\mathrm{E}[X_j] = \mu \mathrm{E}[X_{j-1}]$, $j \in \mathbb{Z}^+$. Under the assumption that $X_0 = n$, let us consider $\mathrm{E}[X_j]$ for several values of j:

$$\begin{aligned}
&\text{Take } j = 1 \Rightarrow \mathrm{E}[X_1] = \mu \mathrm{E}[X_0] = n\mu, \\
&\text{Take } j = 2 \Rightarrow \mathrm{E}[X_2] = \mu \mathrm{E}[X_1] = n\mu^2, \\
&\text{Take } j = 3 \Rightarrow \mathrm{E}[X_3] = \mu \mathrm{E}[X_2] = n\mu^3.
\end{aligned}$$

Based on the above observations, we deduce that $\mathrm{E}[X_j] = n\mu^j$, $j \in \mathbb{N}$. Therefore,

$$\mathrm{E}[W] = \mathrm{E}\left[\sum_{j=0}^{\infty} X_j\right] = \sum_{j=0}^{\infty} \mathrm{E}[X_j] = \sum_{j=0}^{\infty} n\mu^j = \frac{n}{1-\mu} .$$

Exercise 3.4.6. For the *Gambler's Ruin Problem* in Example 3.13, prove that $\{1, 2, \ldots, N-1\}$ is a transient communication class.

Solution: Let us assume that $\{1, 2, \ldots, N-1\}$ is a recurrent communication class and try to get a contradiction. Under this assumption, state 1 is recurrent. Moreover, since since 0 is an absorbing state, it is clear that state 1 does not communicate with state 0. By Theorem 3.5, it must be that $P_{1,0} = 0$. However, for this DTMC, we actually have $P_{1,0} = 1 - p > 0$. This contradicts the recurrence of state 1. Hence, we must have that state 1 (as well as states $2, 3, \ldots, N-1$) is transient.

Exercise 3.4.7. Consider the following modification to the *Gambler's Ruin Problem*. Instead of tossing one coin to determine whether the gambler wins or loses, two iid coins (each with probability p of landing heads) are tossed. If both coins come up heads, the gambler wins the game and receives \$1. If both coins come up tails, he or she loses the game and pays out \$1. Otherwise, the gambler's wealth remains the same. Let X_n denote the wealth of the gambler after the n^{th} play of the game. Assume that the gambler starts with $\$i$ and only quits playing when he or she reaches either \$0 or $\$N$. Let $U_i = P(\text{gambler reaches } \$N \mid X_0 = i)$.

(a) For $i = 1, 2, \ldots, N-1$, show that

$$U_i = aU_{i-1} + bU_{i+1}$$

and find the values of a and b.

(b) Assuming that $p \neq 1/2$, show that $U_i = (1 - r^i)/(1 - r^N)$, where $r = [(1-p)/p]^2$.

Solution: (a) Clearly, $U_0 = 0$ and $U_N = 1$. Thus, let us consider $i = 1, 2, \ldots, N-1$. If we condition on the outcome of the first game (i.e., $Y = -1$ if the gambler loses \$1, $Y = 0$ if the wealth remains the same, or $Y = 1$ if the gambler wins \$1), then we obtain

$$\begin{aligned}
U_i &= P(\text{gambler reaches } \$N|X_0 = i) \\
&= P(Y=-1|X_0=i)P(\text{gambler reaches } \$N|X_0 = i, Y=-1) \\
&+ P(Y=0|X_0=i)P(\text{gambler reaches } \$N|X_0 = i, Y=0) \\
&+ P(Y=1|X_0=i)P(\text{gambler reaches } \$N|X_0 = i, Y=1) \\
&= q^2U_{i-1} + 2pqU_i + p^2U_{i+1}.
\end{aligned}$$

Isolating the above equation for U_i, we get

$$(1-2pq)U_i = q^2U_{i-1} + p^2U_{i+1} \Longrightarrow U_i = \frac{q^2}{1-2pq}U_{i-1} + \frac{p^2}{1-2pq}U_{i+1},$$

and we see that $a = \frac{q^2}{1-2pq}$ and $b = \frac{p^2}{1-2pq}$.

(b) We note that the equations for $U_1, U_2, \ldots, U_{N-1}$ are identical in structure to those of the original *Gambler's Ruin Problem*, with the role of p being played by b and the role of q being played by a. Note that $b = \frac{p^2}{1-2pq} \neq 1/2$ is equivalent to $p \neq 1/2$. Therefore, applying the result for $p \neq 1/2$ in the original *Gambler's Ruin Problem*, we immediately obtain

$$U_i = \frac{1-(a/b)^i}{1-(a/b)^N} = \frac{1 - \left(\frac{q^2}{1-2pq}\cdot\frac{1-2pq}{p^2}\right)^i}{1 - \left(\frac{q^2}{1-2pq}\cdot\frac{1-2pq}{p^2}\right)^N} = \frac{1-r^i}{1-r^N}\ ,$$

where $r = (q/p)^2 = [(1-p)/p]^2$.

Exercise 3.4.8. For the *Gambler's Ruin Problem* in Example 3.13, let M_i denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N units, given that he or she has an initial fortune of i units, $i = 0, 1, \ldots, N$.

(a) Show that M_i satisfies

$$M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, 2, \ldots, N-1.$$

(b) Using $M_0 = M_N = 0$, solve the equations in part (a) to obtain

$$M_i = \begin{cases} i(N-i) & , \text{ if } p = \frac{1}{2}, \\ \frac{i}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^i}{1-(q/p)^N} & , \text{ if } p \neq \frac{1}{2}. \end{cases}$$

Solution: (a) Let us consider $i = 1, 2, \ldots, N-1$. If we condition on the outcome of the first game (i.e., $Y = 0$ if game is lost or $Y = 1$ if game is won), then we obtain

$$\begin{aligned} M_i &= \mathrm{E}[\text{number of games played until broke or gets fortune}|X_0 = i] \\ &= P(Y=0|X_0=i)\mathrm{E}[\text{number of games played until broke or gets fortune}|X_0 = i, Y = 0] \\ &+ P(Y=1|X_0=i)\mathrm{E}[\text{number of games played until broke or gets fortune}|X_0 = i, Y = 1] \\ &= q(M_{i-1}+1) + p(M_{i+1}+1) \\ &= 1 + pM_{i+1} + qM_{i-1}. \end{aligned}$$

(b) Considering $M_i = 1 + pM_{i+1} + qM_{i-1}$, $i = 1, 2, \ldots, N-1$, we first observe:

$$\begin{aligned} pM_i + qM_i &= 1 + pM_{i+1} + qM_{i-1} \\ p(M_{i+1} - M_i) &= q(M_i - M_{i-1}) - 1 \\ M_{i+1} - M_i &= \frac{q}{p}(M_i - M_{i-1}) - \frac{1}{p}. \end{aligned}$$

$$\begin{aligned} \text{Take } i = 1 \Longrightarrow \quad & M_2 - M_1 = \frac{q}{p}M_1 - \frac{1}{p}, \\ \text{Take } i = 2 \Longrightarrow \quad & M_3 - M_2 = \frac{q}{p}\left(\frac{q}{p}M_1 - \frac{1}{p}\right) - \frac{1}{p} = \left(\frac{q}{p}\right)^2 M_1 - \frac{q}{p^2} - \frac{1}{p}, \\ \text{Take } i = 3 \Longrightarrow \quad & M_4 - M_3 = \frac{q}{p}\left(\left(\frac{q}{p}\right)^2 M_1 - \frac{q}{p^2} - \frac{1}{p}\right) - \frac{1}{p} = \left(\frac{q}{p}\right)^3 M_1 - \frac{q^2}{p^3} - \frac{q}{p^2} - \frac{1}{p}, \\ & \vdots \\ \text{Take } i = k \Longrightarrow \quad & M_{k+1} - M_k = \left(\frac{q}{p}\right)^k M_1 - \frac{1}{p}\sum_{j=0}^{k-1}\left(\frac{q}{p}\right)^j. \end{aligned}$$

Adding these first k equations together yields

$$M_{k+1} - M_1 = M_1 \sum_{l=1}^{k} \left(\frac{q}{p}\right)^l - \frac{1}{p} \sum_{l=1}^{k} \sum_{j=0}^{l-1} \left(\frac{q}{p}\right)^j, \quad k = 1, 2, \ldots, N-1.$$

<u>Consider first the case of $p = 1/2$ (i.e., $q/p = 1$)</u>:

$$\begin{aligned}
M_{k+1} - M_1 &= kM_1 - \frac{1}{p} \sum_{l=1}^{k} l \\
M_{k+1} - M_1 &= kM_1 - 2\frac{k(k+1)}{2} \\
\text{Let } k = N-1 \Longrightarrow M_N - M_1 &= (N-1)M_1 - 2\frac{N(N-1)}{2} \quad \text{(note that } M_N = 0) \\
NM_1 &= N(N-1) \\
M_1 &= N-1.
\end{aligned}$$

Thus, $M_{k+1} = (k+1)(N-1) - k(k+1) = (k+1)(N-k-1)$, implying that $M_k = k(N-k)$, which holds true for $k = 0, 1, \ldots, N$.

<u>Consider next the case $p \neq 1/2$ (i.e., $q/p \neq 1$)</u>:

$$\begin{aligned}
M_{k+1} &= M_1 \sum_{l=0}^{k} \left(\frac{q}{p}\right)^l - \frac{1}{p} \sum_{l=1}^{k} \sum_{j=0}^{l-1} \left(\frac{q}{p}\right)^j \\
&= M_1 \frac{1-(q/p)^{k+1}}{1-(q/p)} - \frac{1}{p} \sum_{l=1}^{k} \frac{1-(q/p)^l}{1-(q/p)} \\
&= M_1 \frac{1-(q/p)^{k+1}}{1-(q/p)} - \frac{1}{p-q} \left\{ k - \frac{(q/p)(1-(q/p)^k)}{1-(q/p)} \right\} \\
&= M_1 \frac{1-(q/p)^{k+1}}{1-(q/p)} - \frac{1}{p-q} \left\{ \frac{(k+1-1)(1-q/p) - q/p + (q/p)^{k+1}}{1-(q/p)} \right\} \\
&= M_1 \frac{1-(q/p)^{k+1}}{1-(q/p)} - \frac{1}{p-q} \left\{ (k+1) - \frac{1-(q/p)^{k+1}}{1-(q/p)} \right\} \\
&= \frac{1-(q/p)^{k+1}}{1-(q/p)} \left(M_1 + \frac{1}{p-q} \right) - \frac{k+1}{p-q} \\
&= \frac{1-(q/p)^{k+1}}{1-(q/p)} \left(M_1 - \frac{1}{q-p} \right) + \frac{k+1}{q-p} \ .
\end{aligned}$$

Let $k = N-1 \Longrightarrow M_1 - \dfrac{1}{q-p} = -\dfrac{N}{q-p} \cdot \dfrac{1-(q/p)}{1-(q/p)^N}$.

Thus,

$$M_{k+1} = \frac{1-(q/p)^{k+1}}{1-(q/p)} \cdot \frac{-N}{q-p} \cdot \frac{1-(q/p)}{1-(q/p)^N} + \frac{k+1}{q-p} = \frac{k+1}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^{k+1}}{1-(q/p)^N}$$

$$\Longrightarrow M_k = \frac{k}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^k}{1-(q/p)^N}\ ,\ \ k = 0, 1, \ldots, N.$$