4.(a)
Since $\frac{d}{dx}h'(x)\mu''(x) = h''(x)\mu''(x) + \mu'''(x)h'(x)$, $h''(x)\mu''(x) = \frac{d}{dx}h'(x)\mu''(x) - \mu'''(x)h'(x)$

$$\begin{aligned}
\int_a^b \mu''(x)h''(x)dx =& \int_a^b \frac{d}{dx}h'(x)\mu''(x) - \mu'''(x)h'(x)dx \\
=& \int_a^b \frac{d}{dx}h'(x)\mu''(x)dx - \int_a^b \mu'''(x)h'(x)dx \\
=& h'(b)\mu''(b) - h'(a)\mu''(a) - \int_a^b \mu'''(x)h'(x)dx \\
& \text{ by fundamental theorem of calculus} \\
=& -\int_a^b \mu'''(x)h'(x)dx \\
& \text{ since } \mu(x) \text{ is linear at } a \text{ and } b,\ \mu''(a) = \mu''(b) = 0 \\
=& -\int_{x_1}^{x_n} \mu'''(x)h'(x)dx \\
& \text{ since } \mu(x) \text{ is linear in } [a, x_1) \text{ and } (x_n, b],\ \mu'''(x) = 0 \\
=& -\sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \mu'''(x)h'(x)dx \\
& \text{ by the properties of integral}
\end{aligned}$$

4.(b)

Since $\mu(x)$ is cubic in $[x_1, x_n]$. $\mu'''(x)$ is constant. Hence $\mu'''(x) = c_i$ where $x \in [x_i, x_{i+1}]$ and c_i is the coefficient of x^3 in the cubic function times 6.

4.(c)

From part (b), let $\mu'''(x) = c_i$ where $x \in [x_i, x_{i+1}]$.

$$\int_{x_i}^{x_{i+1}} \mu'''(x)h'(x)dx$$
$$= \int_{x_i}^{x_{i+1}} c_i h'(x)dx$$
$$= c_i \int_{x_i}^{x_{i+1}} h'(x)dx$$

by the properties of integral

$$= c_i(h(x_{i+1}) - h(x_i))$$

by fundamental theorem of calculus

4.(d)

Since both $g(x)$ and $\mu(x)$ pass through points (x_i, y_i) for all i, $h(x_i) = g(x_i) - \mu(x_i) = 0$ for all i.

$$\begin{aligned}
\int_a^b \mu''(x)h''(x)dx &= -\sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \mu'''(x)h'(x)dx \quad \text{by part (a)} \\
&= -\sum_{i=1}^{n-1} c_i(h(x_{i+1}) - h(x_i)) \quad \text{by part (c)} \\
&= 0
\end{aligned}$$

4.(e)
Since,

$$\begin{aligned}
\int_a^b \mu''(x)h''(x)dx &= \int_a^b \mu''(x)(g''(x)-\mu''(x))dx \\
&= \int_a^b \mu''(x)g''(x)dx - \int_a^b (\mu''(x))^2 dx \\
&= 0
\end{aligned}$$

So, $\int_a^b \mu''(x)g''(x)dx = \int_a^b (\mu''(x))^2 dx$
Also,

$$\begin{aligned}
\int_a^b \mu''(x)h''(x)dx &= \int_a^b (\mu''(x)+g''(x)-g''(x))h''(x)dx \\
&= \int_a^b (g''(x)-h''(x))h''(x)dx \\
&= \int_a^b g''(x)h''(x)dx - \int_a^b (h''(x))^2 dx \\
&= \int_a^b g''(x)(g''(x)-\mu''(x))dx - \int_a^b (h''(x))^2 dx \\
&= \int_a^b (g''(x))^2 dx - \int_a^b \mu''(x)g''(x)dx - \int_a^b (h''(x))^2 dx \\
&= \int_a^b (g''(x))^2 dx - \int_a^b (\mu''(x))^2 dx - \int_a^b (h''(x))^2 dx \\
&= 0
\end{aligned}$$

Hence, $\int_a^b (g''(x))^2 dx - \int_a^b (\mu''(x))^2 dx = \int_a^b (h''(x))^2 dx \geq 0$ since, $(h''(x))^2 \geq 0$.
Thus, $\int_a^b (g''(x))^2 dx \geq \int_a^b (\mu''(x))^2 dx$
$\int_a^b (h''(x))^2 dx = 0$ if and only if $(h''(x))^2 = 0$
$\int_a^b (g''(x))^2 dx = \int_a^b (\mu''(x))^2 dx$ if and only if $(h''(x))^2 = 0$ which is equivalent to $g(x) = \mu(x)$

4.(f)
Let $\mu(x)$ be a natural cubic spline with knots at each value x_i, $i = 1, \ldots, n$.
By part (e), $\int_a^b (g''(x))^2 dx \geq \int_a^b (\mu''(x))^2 dx$,
so $\lambda \int_a^b (g''(x))^2 dx \geq \lambda \int_a^b (\mu''(x))^2 dx$ for $\lambda \geq 0$.
By assumption $\mu(x_i) = y_i$ for all i, $\sum_{i=1}^n (y_i - \mu(x_i))^2 = 0$.
Since $(y_i - g(x_i))^2 \geq 0$, $\sum_{i=1}^n (y_i - g(x_i))^2 \geq \sum_{i=1}^n (y_i - \mu(x_i))^2$.

$$
\begin{aligned}
&\sum_{i=1}^n (y_i - g(x_i))^2 + \lambda \int_a^b (g''(x))^2 dx \\
\geq &\sum_{i=1}^n (y_i - \mu(x_i))^2 + \lambda \int_a^b (\mu''(x))^2 dx \\
= &\lambda \int_a^b (\mu''(x))^2 dx
\end{aligned}
$$

$\implies \min_g [\sum_{i=1}^n (y_i - g(x_i))^2 + \lambda \int_a^b (g''(x))^2 dx] = \lambda \int_a^b (\mu''(x))^2 dx$
Therefore the minimizer must be a natural cubic spline with knots at each value x_i, $i = 1, \ldots, n$.