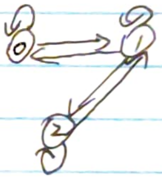
1. I have carefully read the integrity statement
Muchi Wang

2.(a) $\begin{cases} \frac{1}{2}\pi(0) + \frac{1}{4}\pi(1) = \pi(0) \\ \frac{1}{2}\pi(0) + \frac{2}{3}\pi(1) + \frac{1}{6}\pi(2) = \pi(1) \\ \frac{1}{12}\pi(1) + \frac{5}{6}\pi(2) = \pi(2) \\ \pi(0) + \pi(1) + \pi(2) = 1 \end{cases}$

$\Rightarrow \pi = [\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4}]$

Since π is the only solution, it is unique.

(b) Since $i \to j \ \forall i,j \in S$, the chain is irreducible



Since $P_{ii} > 0 \ \forall i \in S$, the chain is aperiodic

By part (a), stationary distribution exists

By theorem,

$\lim_{n\to\infty} P(X_n = j \mid X_0 = i) = \pi(j) \quad \forall j \in S$

$= \begin{cases} \frac{1}{4} & j=0 \\ \frac{1}{2} & j=1 \\ \frac{1}{4} & j=2 \end{cases}$

(c) Since I (irreducible), S (stationary distribution exists)

$E_y(T_y) = \frac{1}{\pi(y)}$

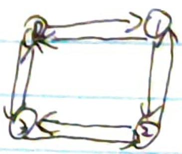
$E_0(T_0) = \frac{1}{\pi(0)} = 4$

3. (a) $P(X_0=0, X_3=1) = P(X_3=1 \mid X_0=0)P(X_0=0) = P_{01}^3 \mu(0) = \frac{77}{120} \times \frac{1}{2} = \frac{77}{240}$

(b) $P(X_{11}=0 \mid X_9=2, X_0=3) = P(X_{11}=0 \mid X_9=2) = P_{20}^2 = \frac{11}{20}$

Markov property

(c) Assume this MC is at state X_n,



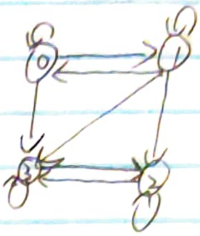
X_{n+1} is even if X_n is odd

X_{n+1} is odd if X_n is even

$\sum_{n=0}^{9} X_n$ has odd number of odd terms if $X_0 = 3$

Hence $\sum_{n=0}^{9} X_n$ is odd

$P(\sum_{n=0}^{9} X_n \text{ is even} \mid X_0=3) = 0$

4. (a) $\{0,1\}$, $\{2,3\}$ 

~~[illegible] is transient~~

Since $\{2,3\}$ is closed and $P_{03}>0$

$P_{00}<1$ ($0\to 3$ and never exits $\{2,3\}$)

So $\{0,1\}$ is transient

Since $\{2,3\}$ is closed an finite,

$\{2,3\}$ is recurrent by corollary, $\{2,3\}$ is positive recurrent

(b) $A=\{2\}$, $B=\{3\}$, $C=\{0,1\}$

$$\begin{cases} h(0)=\frac{1}{2}h(0)+\frac{1}{3}h(1) \\ h(1)=\frac{1}{4}h(0)+\frac{1}{2}h(1)+\frac{1}{6} \end{cases}$$ where $h(2)=1$ and $h(3)=0$

$h=$ $[\frac{1}{3} \quad \frac{1}{2}]$

The prob. is $\frac{1}{3}$

(c) $A=\{2,3\}$ $C=\{0,1\}$

$$\begin{cases} g(0)=1+\frac{1}{2}g(0)+\frac{1}{3}g(1) \\ g(1)=1+\frac{1}{4}g(0)+\frac{1}{2}g(1) \end{cases}$$

$$g=\begin{array}{cc} 0 & 1 \end{array}$$
$$g=[5 \quad \frac{9}{2}]$$

The expected time is $\frac{9}{2}$

(d) ~~Let [illegible]~~ $A=\{3\}$ $C=\{0,1,2\}$

$$\begin{cases} g(0)=1+\frac{1}{2}g(0)+\frac{1}{3}g(1) \\ g(1)=1+\frac{1}{4}g(0)+\frac{1}{2}g(1)+\frac{1}{6}g(2) \\ g(2)=1+\frac{2}{3}g(2) \end{cases}$$

$$\begin{array}{ccc} 0 & 1 & 2 \end{array}$$
$$g=[6 \quad 6 \quad 3]$$

Let T_3^i be the i-th time that it visits state 3.

~~$E_1(T_3^3)=$ [illegible]~~

~~$=6+\frac{2}{3}E_2(T_3^2)+\frac{1}{3}E(T_3^2)$ Markov property~~

~~$+\frac{2}{3}(3+1+\frac{3}{2}E_2(T_3^1))$~~

~~$=7+\frac{2}{3}(4+\frac{3}{2}\times 3)$~~

~~$=\frac{?}{6}$~~

$E_1(T_3^3)=6+\frac{1}{3}E_2(T_3^2)+\frac{2}{3}E_3(T_3^2)+1$

$=7+\frac{1}{3}(3+1+\frac{1}{3}E_2(T_3))+\frac{2}{3}(1+\frac{1}{3}E_2(T_3))$

$=10$

5.(a) Assume $\rho_{ij} \neq 0$

since $\rho_{ij} > 0$ (ρ_{ij} is a probability, $\rho_{ij} \in [0,1]$)

and i is recurrent

$\rho_{ji} = 1$ by theorem

Since $\rho_{ij} > 0$ and $\rho_{ji} = 1 > 0$

i and j should be in the same class

But i is null recurrent and j is positive recurrent

$\Rightarrow$ i and j are not in the same class

By contradiction $\rho_{ij} = 0$

(b) $\rho_{jj} = P(T_j < \infty \mid X_0 = j)$

$= \sum_{i \in S \setminus \{j\}} P(T_j < \infty \mid X_0 = j, X_1 = i)\, P(X_1 = i \mid X_0 = j) + P(T_j < \infty \mid X_0 = j, X_1 = j)\, P(X_1 = j \mid X_0 = j)$

$= \sum_{i \in S \setminus \{j\}} P(T_j < \infty \mid X_1 = i)\, P_{ji} + P(T_j < \infty \mid X_0 = j, X_1 = j)\, P_{jj}$

If $P_{jj} = 0$

$\rho_{jj} = \sum_{i \in S \setminus \{j\}} \rho_{ij} P_{ji}$ Markov property

$= \sum_{i \in S \setminus \{j\}} P_{ji}$ ($\rho_{ij} = 1$)

$= 1$ by defn and $P_{jj} = 0$

If $P_{jj} > 0$

$\rho_{jj} = \sum_{i \in S \setminus \{j\}} \rho_{ij} P_{ji} + P_{jj}$ ($P(T_j < \infty \mid X_0 = j, X_1 = j) = 1$ if $P_{jj} > 0$)

$= \sum_{i \in S \setminus \{j\}} P_{ji} + P_{jj}$

$= \sum_{i \in S} P_{ji}$

$= 1$ by defn

Hence j is recurrent

6.(a) $P(X_1=0) \overset{\vee}{=} p^3$ (since independence)

(b) $E(Y) = 1\cdot r + 2\cdot q = r+2q$

~~[illegible]~~

extinction happens for sure $\Longrightarrow E(Y) \leq 1$

$E(Y) = r+2q = 1-p-q+2q = 1-p+q$

$1-p+q \leq 1$

$p \geq q$

$\varphi(s) = p+rs+qs^2$

$p+rs+qs^2 = s$

$qs^2+(r-1)s+p=0$

$qs^2-(q+p)s+p=0$

$$s = \frac{q+p \pm \sqrt{(q+p)^2-4qp}}{2q}$$

$$= \frac{q+p \pm \sqrt{(q-p)^2}}{2q}$$

$$= \frac{q+p \pm (q-p)}{2q}$$ since $q-p>0$

$s = 1$ or $\frac{p}{q}$

since $s \in (0,1)$

$s = \frac{p}{q}$

(c) $S = \{0,1,2,4,6,\cdots\}$

$\{0\}$ is the obsorbing state

$\{0\}$ is recurrent

Since $P_{11}=0$ (The # of offsprings is even)

$\{1\}$ is transient

Let $i \in S \setminus \{0,1\}$,

Since $P_{i0}>0$ and 0 is the obsorbing state

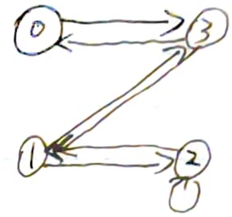
$P_{ii}<1$

so $\{2,4,6,\cdots\}$ is transient

7. (a) Let X_n be the # of umbrellas at current location

$S = \{0,1,2,3\}$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1-p & p \\ 0 & 1-p & p & 0 \\ 1-p & p & 0 & 0 \end{bmatrix} \end{array}$$



$$\begin{cases} \pi(0) = (1-p)\pi(3) \\ \pi(1) = (1-p)\pi(2) + p\pi(3) \\ \pi(2) = (1-p)\pi(1) + p\pi(2) \\ \pi(3) = \pi(0) + p\pi(1) \\ \pi(0)+\pi(1)+\pi(2)+\pi(3) = 1 \end{cases}$$

$\pi = [\frac{1-p}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}]$

Since $i \to j \ \forall i,j \in S$,

this MC is irreducible

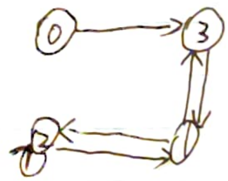
$I, S \Rightarrow R$

By theorem $\frac{N_n(0)}{n} = \pi(0) = \frac{1-p}{4-p}$

long-run percentage of time that the professor gets wet is

$p\frac{1-p}{4-p} = \frac{p-p^2}{4-p}$

$$(c)\ P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1-p & p \\ 0 & 1-p & p & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$



$$\begin{cases} \pi(0) = 0 \\ \pi(1) = (1-p)\pi(2) + \pi(3) \\ \pi(2) = (1-p)\pi(1) + p\pi(2) \\ \pi(3) = \pi(0) + p\pi(1) \\ \pi(0)+\pi(1)+\pi(2)+\pi(3) = 1 \end{cases}$$

$\pi(3) + p\pi(1) + p\pi(2)$

$= \frac{p}{2+p} + \frac{p}{2+p} + \frac{p}{2+p}$

$= \frac{3p}{2+p}$

$\pi = [0 \ \ \frac{1}{2+p} \ \ \frac{1}{2+p} \ \ \frac{p}{2+p}]$

Since the state 0 is only visited initially and it does not affect the long-run percentage, we can just remove it

$\pi = [\frac{1}{2+p} \ \ \frac{1}{2+p} \ \ \frac{p}{2+p}]$

since $i \to j \ \forall i,j \in \{1,2,3\}$

this MC is irreducible

$I, S \Rightarrow R$

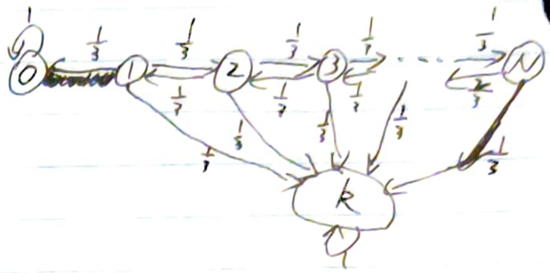
By theorem $\frac{N_n(i)}{n} = \pi(i)$

The long-run percentage of time that the professor ~~gets we~~ brings an umbrella is

8. (a) $\{0\}$ is positive recurrent since $P_{00}=1$ and $E_0(T_0)=1$

$\{k\}$ is positive recurrent since $P_{kk}=1$ and $E_k(T_k)=1$

$\{1,2,3,\cdots,N\}$ is transient since $\forall i\in\{1,2,\cdots,N\}$ $P_{ik}>0$ and k is absorbing so $P_{ii}<1$



(b) $P(X_4=1|X_0=1)=P_{11}^4=\sum_{i\in S} P_{1i}^2 P_{i1}^2$

$P_{1i}^2=[\frac{1}{3}\ \frac{1}{9}\ 0\ \frac{1}{9}\ 0\cdots 0\ \frac{4}{9}]$ if $N>3$ $\qquad P^2_{1i}=[\frac{1}{3}\ \frac{1}{9}\ 0\ \frac{1}{9}\ \frac{4}{9}]$

$P_{i1}^2=[0\ \frac{1}{9}\ 0\ \frac{1}{9}\ 0\cdots 0]^T$ if $N>3$ $\qquad P^2_{i1}=[0\ \frac{1}{9}\ 0\ \frac{2}{9}\ 0]$

$P_{11}^4=\frac{1}{81}+\frac{1}{81}$ (if $N>3$) $\qquad P_{11}^4=\frac{1}{81}+\frac{2}{81}=\frac{1}{27}$

$=\frac{2}{81}$

(c)
$$P=\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & k \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 2 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 3 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ k & 0 & 0 & 0 & 0 & 1 \end{array}$$

$A=\{0\}$, $B=\{k\}$, $C=\{1,2,3\}$

$$\begin{cases} h(1)=\frac{1}{3}+\frac{1}{3}h(2) \\ h(2)=\frac{1}{3}h(1)+\frac{1}{3}h(3) \\ h(3)=\frac{2}{3}h(2) \end{cases} \qquad (h(0)=1,\ h(k)=0)$$

$$h=\begin{array}{ccc} 1 & 2 & 3 \\ \left[\frac{7}{18}\right. & \frac{1}{6} & \left.\frac{1}{9}\right] \end{array}$$

The probability is $\frac{1}{6}$

9. (a) $P(X_3 \neq 0, X_2 = 0 | X_0 = 1)$

$= P(X_3 = 1, X_2 = 0 | X_0 = 1) + P(X_3 = 2, X_2 = 0 | X_0 = 1)$

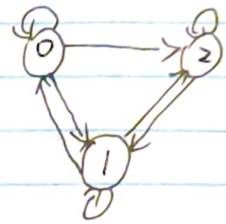
$= \dfrac{P(X_3 = 1, X_2 = 0, X_0 = 1) + P(X_3 = 2, X_2 = 0, X_0 = 1)}{P(X_0 = 1)}$

$= \dfrac{P(X_3 = 1 | X_2 = 0) P(X_2 = 0 | X_0 = 1) P(X_0 = 1) + P(X_3 = 2 | X_2 = 0) P(X_2 = 0 | X_0 = 1) P(X_0 = 1)}{P(X_0 = 1)}$ Markov Property

$= P_{01} P_{10}^2 + P_{02} P_{10}^2$

$= \frac{1}{4} \times \frac{3}{16} + \frac{1}{4} \times \frac{3}{16}$

$= \frac{3}{32}$



(b) $\begin{cases} \pi(0) = \frac{1}{2}\pi(0) + \frac{1}{4}\pi(1) \\ \pi(1) = \frac{1}{4}\pi(0) + \frac{1}{4}\pi(1) + \frac{5}{8}\pi(2) \\ \pi(2) = \frac{1}{4}\pi(0) + \frac{1}{2}\pi(1) + \frac{3}{8}\pi(2) \end{cases}$

$\pi = \begin{bmatrix} \pi(0) \\ \pi(1) \\ \pi(2) \end{bmatrix}^T = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} x \right)^T$ where $x \in \mathbb{R}$ (This gives all the stationary measure of P)

since this chain is irreducible ($i \to j \ \forall i, j \in S$)

and recurrent (finite and closed)

By theorem, $\mu_0(1) = E(\sum_{n=0}^{T_0 - 1} \mathbb{1}_{\{X_n = 1\}})$,

$x = 1$ $\quad \mu_0(0) = 1$ $\quad \pi(0) = 1 = \mu_0(0)$

since is still a stationary measure and

$E(\sum_{n=0}^{T_0 - 1} \mathbb{1}_{\{X_n = 1\}}) = 2$

(c) $P(X_n \neq 1, X_{n-1} \neq 1, \cdots, X_1 \neq 1 | X_0 = 0)$

$= (\frac{1}{2})^n + \sum_{k=0}^{n-1} (\frac{1}{2})^k \frac{1}{4} (\frac{3}{8})^{n-k-1}$ (the chain stays at 0 all the time or stays at 0 k times, move to 2 and stays at 2 for the rest of the times)

$= 2^{-n} + \frac{1}{4} (\frac{3}{8})^{n-1} \sum_{k=0}^{n-1} (\frac{4}{3})^k$

$= 2^{-n} + \frac{1}{4} (\frac{3}{8})^{n-1} \dfrac{1 - (\frac{4}{3})^n}{-\frac{1}{3}}$

$= 2^{-n} - \frac{3}{4} (\frac{3}{8})^{n-1} + 2^{-n+1}$

$= (\frac{1}{2})^n \cdot 3 - (\frac{3}{4})^n \cdot \frac{2}{2^n}$

$= (\frac{1}{2})^n (3 - 2(\frac{3}{4})^n)$

Hence $a = \frac{1}{2}$ $\quad b = \frac{3}{4}$

10.(a) The chain is ~~ined~~ irreducible

Since ~~the~~ S is finite

By theorem, at least one of the states is ~~re~~ positive recurrent

~~Hence~~ Since it is irreducible

~~All of~~ S is ~~null~~ positive recurrent

By observation, all the cycles have ~~even~~ lengths even

So the # of steps it takes to go back to the same state is even

Hence the period of all states is 2

It is not aperiodic

(b) $A=\{7\}$ $C=\{0,1,2,3,4,5,6\}$

$$\begin{cases} g(0) = 1+\frac{1}{3}g(1)+\frac{1}{3}g(3)+\frac{1}{3}g(5) \\ g(1) = 1+\frac{1}{3}g(0)+\frac{1}{3}g(2)+\frac{1}{3}g(4) \\ g(2) = 1+\frac{1}{3}g(1)+\frac{1}{3}g(3) \\ g(3) = 1+\frac{1}{3}g(0)+\frac{1}{3}g(2)+\frac{1}{3}g(6) \\ g(4) = 1+\frac{1}{3}g(1)+\frac{1}{3}g(5) \\ g(5) = 1+\frac{1}{3}g(0)+\frac{1}{3}g(4)+\frac{1}{3}g(6) \\ g(6) = 1+\frac{1}{3}g(3)+\frac{1}{3}g(5) \\ g(7) = 1 \end{cases}$$

$g(0) = 10$ ← starting from 0

The expected number of steps until the chain reaches state 7 for the first time is 10

11.(a) Define a distribution π where $\pi(i)=P_{ai}$ $(a\in S)$

$\sum_{i\in S}\pi(i)P_{ij}$, $j\in S$

$=\sum_{i\in S}P_{ai}P_{ij}$

$=P^2_{aj}$

$=P_{aj}$ $(P=P^2)$

$=\pi(j)$

$\sum_{i\in S}\pi(i)=\sum_{i\in S}P_{ai}=1$ by defn of DTMC

Hence π is a stationary distribution

Since it is irreducible

By theorem, $\{X_n\}_{n=0,1,2,\cdots}$ is positive recurrent

(b) Since $P=P^2$

$\Rightarrow P^n=P$ $\forall n\in\mathbb{R}$

By part (a), all states are positive recurrent

$\exists n, s.t.\ P^n_{ii}>0$ $\forall i\in S$

Hence $P_{ii}>0$ $\forall i\in S$

$\Rightarrow$ it is aperiodic

Since A (aperiodic), I (irreducible), S (stationary distribution exists from part (a))

$\lim_{n\to\infty}P^n_{xy}=\pi(y)$

$\pi(x)P_{xy}=\pi(x)\lim_{n\to\infty}P^n_{xy}=\pi(y)\pi(x)=\pi(y)\lim_{n\to\infty}P^n_{yx}=\pi(y)P_{yx}$

By theorem, it is time reversable

(c) By theorem, the chain can be partitioned into $T\cup R_1\cup R_2\cup\cdots$ where T are all the transient states and R_i, $i=1,2,\cdots$ are closed recurrent classes

① Assume $i,j\in S$ are not positive recurrent

$\Rightarrow \pi(i)=0=\pi(j)$ by theorem

$\Rightarrow \pi(i)P_{ij}=\pi(j)P_{ji}$

Which statisfies detail balance condition

② Assume i is positive recurrent and j is not

$\Rightarrow \pi(i)P_{ij}=\pi(i)0=0$ since the class that i is in is closed $\Rightarrow i\not\to j$

$\Rightarrow \pi(j)P_{ji}=0P_{ji}=0$ by theorem

③ Assume i and j are positive recurrent

If i and j are not in the same class, $\pi(i)P_{ij}=0=\pi(j)P_{ji}$ (closed)

else, since the class is irreducible, by part (b) it satisfies the detail balance condition

Hence, the chain is time reversable

12 (a) $\pi(i) = \sum_{j\in S} \pi(j) P_{ji}$

$$= \sum_{j=0}^{\infty} \pi(j) P_{ji}$$

Since $P_{ji} = P_{i-j}, 0 < i \le j+1$, and $i > 0$

$\Rightarrow P_{ji} = \begin{cases} P_{i-j} & -1 < i-1 \le j \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow \pi(i) = \sum_{j=i-1}^{\infty} \pi(j) P_{i-j}$

(b) $\Phi(\alpha) = q_0 + q_1\alpha + q_2\alpha^2 + \cdots$ is the generating function of $\{q_n\}$

$= P_1 + P_0\alpha + P_{-1}\alpha^2 + \cdots$

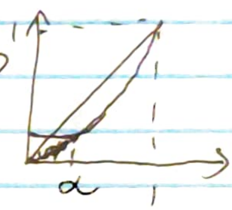
$\Rightarrow \Phi'(\alpha) = q_1 + 2q_2\alpha + 3q_3\alpha^2 + \cdots > 0$ since all terms are non-negative and at least one term is positive, otherwise, $P_1, P_0, P_{-1}, \cdots$ wouldn't be a distribution

$\Phi''(\alpha) = 2q_2 + 6q_3\alpha + 12q_4\alpha^2 + \cdots > 0$ for the same reason as above

$\Rightarrow \Phi(\alpha)$ is increasing and convex $\swarrow$ $\Phi(1) = \sum_{k=0}^{\infty} q_k = 1$

$\Phi'(1) = P_0 + 2P_{-1} + 3P_{-2} + \cdots = \sum_{k=0}^{\infty} kq_k > 1$ since $k \in [0, \infty)$, $q_k > 0$

Hence there exists an $\alpha \in (0,1)$, s.t. $\alpha = q_0 + q_1\alpha + q_2\alpha^2 + \cdots$



(c) Let $\pi \sim Geo(1-\alpha)$

By part (a) $\pi(i) = \sum_{j=1}^{\infty} \pi(j) P_{i-j}$, for $i > 0$

$\pi(i) = \pi(i-1)P_1 + \pi(i)P_0 + \pi(i+1)P_{-1} + \cdots$

$(1-\alpha)\alpha^{i} = (1-\alpha)\alpha^{i-1}q_0 + (1-\alpha)(1-\alpha)^{i}q_1 + \cdots$

$\alpha = q_0 + \alpha q_1 + \alpha^2 q_2 + \cdots$

Since α is the solution, $\pi(i) = (\pi P)(i)$ for $i > 0$

For $i = 0$, $\pi(0) = \pi(0)\sum_{j\le 0} P_j + \pi(1)\sum_{j\le 0} P_{j-1} + \cdots$

$(1-\alpha)\sum_{j\le 0} P_j + (1-\alpha)\alpha\sum_{j\le 0} P_{j-1} + (1-\alpha)\alpha^2\sum_{j\le 0} P_{j-2}$

$= \sum_{j\le 0}\sum_{i=0}^{\infty} P_{j-i}\alpha^i \cdot (1-\alpha)$

$= \begin{bmatrix} P_0 + P_{-1}\alpha + P_{-2}\alpha^2 + P_{-3}\alpha^3 + \cdots \\ + P_{-1} + P_{-2}\alpha + P_{-3}\alpha^2 + \cdots \\ + P_{-2} + P_{-3}\alpha + \cdots \end{bmatrix} \cdot (1-\alpha)$

$= (1-\alpha)\sum_{i=1}^{\infty} q_i \sum_{v=0}^{i-1} \alpha^v$

$= (1-\alpha)\sum_{i=1}^{\infty} q_i \frac{1-\alpha^i}{1-\alpha}$

$= \sum_{i=1}^{\infty} q_i - \sum_{i=1}^{\infty} q_i\alpha^i$

$= 1 - q_0 - (\alpha - q_0)$ by part (b)

$= 1-\alpha$

$= \pi(0)$

Hence $\pi = \pi P$

Since π is already a distribution

$\sum_{i\in S} \pi(i) = 1$

Hence $\pi \sim Geo(1-\alpha)$ is a stationary distribution

13. (a) $P(Y_{n+1}=j \mid Y_n=i)$

$= P(X_{\varepsilon_1+\cdots+\varepsilon_n+\varepsilon_{n+1}}=j \mid X_{\varepsilon_1+\cdots+\varepsilon_n}=i)$

$=\sum_{k=1}^{\infty} P(X_{\varepsilon_1+\cdots+\varepsilon_n+\varepsilon_{n+1}}=j \mid X_{\varepsilon_1+\cdots+\varepsilon_n}=i, \varepsilon_{n+1}=k)\, P(\varepsilon_{n+1}=k)$

$=\sum_{k=1}^{\infty} P_{ij}^{k} P_k$

(b) Since $\{X_n\}_{n=0,1,\cdots}$ is irreducible

$\exists\, n$, s.t. $P_{ij}^{n}>0,\ \forall i,j\in S$

Let P' be the transition matrix of $\{Y_n\}_{n=0,1,\cdots}$

$P'_{ij}=\sum_{k=1}^{\infty} P_{ij}^{k} P_k \geq P_{ij}^{n} P_n>0 \quad (P_n>0)$

$i\to j,\ \forall i,j\in S \Longrightarrow \{Y_n\}_{n=0,1,\cdots}$ is irreducible

By the proof from part (c) and the chain is irreducible

$\Longrightarrow \{Y_n\}_{n=0,1,\cdots}$ is positive recurrent

(c) The stationary distribution of $\{Y_n\}_{n=0,1,\cdots}$ is also π

~~[illegible]~~ $\sum_{i\in S}\pi(i)P'_{ij}$

$=\sum_{i\in S}\pi(i)\sum_{k=1}^{\infty}P_{ij}^{k}P_k$

$=\sum_{k=1}^{\infty}P_k\sum_{i\in S}\pi(i)P_{ij}^{k}$

$=\sum_{k=1}^{\infty}P_k\,\pi(j) \qquad (\pi=\pi P^n\ \forall n\in\mathbb{N})$

$=\pi(j) \qquad (\sum_{k=1}^{\infty}P_k=1)$

Clearly $\sum_{i\in S}\pi(i)=1$

Hence, the stationary distribution of $\{Y_n\}_{n=0,1,\cdots}$ is π