4 The Exponential Distribution and the Poisson Process

4.1 Properties of the Exponential Distribution

Exercises

Exercise 4.1.1. Let X_1 and X_2 be independent random variables where $X_1 \sim \text{EXP}(\lambda_1)$ and $X_2 \sim \text{EXP}(\lambda_2)$. Find the pdf of $Z = \max\{X_1, X_2\}$, the largest order statistic of X_1 and X_2. Is the rv Z exponentially distributed?

Solution: For $z \geq 0$,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(\max\{X_1, X_2\} \leq z) \\
&= P(X_1 \leq z, X_2 \leq z) \\
&= P(X_1 \leq z)P(X_2 \leq z) \;\; \text{by independence of } X_1 \text{ and } X_2 \\
&= (1 - e^{-\lambda_1 z})(1 - e^{-\lambda_2 z}) \\
&= 1 - e^{-\lambda_1 z} - e^{-\lambda_2 z} + e^{-(\lambda_1+\lambda_2)z}.
\end{aligned}$$

Therefore, it immediately follows that

$$f_Z(z) = F_Z'(z) = \lambda_1 e^{-\lambda_1 z} - \lambda_2 e^{-\lambda_2 z} - (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)z}, z > 0.$$

Looking at the form of the above pdf, it is clear that $Z = \max\{X_1, X_2\}$ is not exponentially distributed.

Exercise 4.1.2. Let X_1 and X_2 be independent random variables where $X_1 \sim \text{EXP}(\lambda_1)$ and $X_2 \sim \text{EXP}(\lambda_2)$. For $0 \leq y \leq x$, show that

$$P(X_2 - y \leq X_1 \leq \min\{x, X_2\}) = \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(1 - e^{-\lambda_2 y} - e^{-(\lambda_1+\lambda_2)x} + e^{-\lambda_2 y}e^{-(\lambda_1+\lambda_2)x}\right).$$

Solution: For $0 \leq y \leq x$, we have

$$\begin{aligned}
&P(X_2 - y \leq X_1 \leq \min\{x, X_2\}) \\
&= \int_0^\infty P(X_2 - y \leq X_1 \leq \min\{x, X_2\}|X_2 = w) f_{X_2}(w)dw \\
&= \int_0^\infty P(w - y \leq X_1 \leq \min\{x, w\}) f_{X_2}(w)dw \text{ since } X_1 \text{ and } X_2 \text{ are independent} \\
&= \int_0^y P(\underbrace{w-y}_{<0} \leq X_1 \leq \underbrace{\min\{x,w\}}_{=w})\lambda_2 e^{-\lambda_2 w}dw + \int_y^x P(\underbrace{w-y}_{>0} \leq X_1 \leq \underbrace{\min\{x,w\}}_{=w})\lambda_2 e^{-\lambda_2 w}dw \\
&+ \int_x^{x+y} P(\underbrace{w-y}_{>0} \leq X_1 \leq \underbrace{\min\{x,w\}}_{=x})\lambda_2 e^{-\lambda_2 w}dw + \int_{x+y}^\infty P(\underbrace{w-y}_{>x} \leq X_1 \leq \underbrace{\min\{x,w\}}_{=x})\lambda_2 e^{-\lambda_2 w}dw \\
&= \int_0^y P(X_1 \leq w)\lambda_2 e^{-\lambda_2 w}dw + \int_y^x [P(X_1 > w-y) - P(X_1 > w)]\lambda_2 e^{-\lambda_2 w}dw \\
&+ \int_x^{x+y} [P(X_1 > w-y) - P(X_1 > x)]\lambda_2 e^{-\lambda_2 w}dw \\
&= \int_0^y (1 - e^{-\lambda_1 w})\lambda_2 e^{-\lambda_2 w}dw + \int_y^x (e^{-\lambda_1(w-y)} - e^{-\lambda_1 w})\lambda_2 e^{-\lambda_2 w}dw \\
&+ \int_x^{x+y} (e^{-\lambda_1(w-y)} - e^{-\lambda_1 x})\lambda_2 e^{-\lambda_2 w}dw \\
&= 1 - e^{-\lambda_2 y} - \frac{\lambda_2}{\lambda_1+\lambda_2}(1 - e^{-(\lambda_1+\lambda_2)y}) + \frac{\lambda_2 e^{\lambda_1 y}}{\lambda_1+\lambda_2}(e^{-(\lambda_1+\lambda_2)y} - e^{-(\lambda_1+\lambda_2)x}) \\
&- \frac{\lambda_2}{\lambda_1+\lambda_2}(e^{-(\lambda_1+\lambda_2)y} - e^{-(\lambda_1+\lambda_2)x}) + \frac{\lambda_2 e^{\lambda_1 y}}{\lambda_1+\lambda_2}(e^{-(\lambda_1+\lambda_2)x} - e^{-(\lambda_1+\lambda_2)(x+y)}) - e^{-\lambda_1 x}(e^{-\lambda_2 x} - e^{-\lambda_2(x+y)}) \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2} - \frac{\lambda_1}{\lambda_1+\lambda_2}e^{-\lambda_2 y} - \frac{\lambda_1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)x} + \frac{\lambda_1}{\lambda_1+\lambda_2}e^{-\lambda_2 y}e^{-(\lambda_1+\lambda_2)x} \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}(1 - e^{-\lambda_2 y} - e^{-(\lambda_1+\lambda_2)x} + e^{-\lambda_2 y}e^{-(\lambda_1+\lambda_2)x}).
\end{aligned}$$

Exercise 4.1.3. Verify the formula for the pdf of an Erlang(n, λ) rv by differentiating the cdf given by (4.2).

Solution: Using the cdf given by (4.2), the corresponding pdf is given by

$$\begin{aligned}
f(x) &= \frac{d}{dx}\left(1 - e^{-\lambda x}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}\right) \\
&= -e^{-\lambda x}\sum_{j=1}^{n-1}\frac{j(\lambda x)^{j-1}\cdot\lambda}{j!} + \lambda e^{-\lambda x}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!} \\
&= -\lambda e^{-\lambda x}\sum_{j=1}^{n-1}\frac{(\lambda x)^{j-1}}{(j-1)!} + \lambda e^{-\lambda x}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!} \\
&= -\lambda e^{-\lambda x}\sum_{m=0}^{n-2}\frac{(\lambda x)^m}{m!} + \lambda e^{-\lambda x}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{j!} + \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!} \\
&= \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}\ ,\ x > 0,
\end{aligned}$$

which we recognize as the Erlang(n, λ) pdf.

Exercise 4.1.4. Player A is currently playing a game of chess. Player B will be starting its own chess game t minutes from now. If the length of chess game (measured in minutes) involving player i is exponentially distributed with rate λ_i, $i = 1, 2$, what is the probability that player 1 finishes their game first? Assume that games lengths are independent of each other.

Solution: For $i = 1, 2$, let $X_i \sim \text{EXP}(\lambda_i)$ be the rv representing the game length involving player i. We wish to find

$$\begin{aligned}
&P(\text{Player 1 finishes first}) \\
&= P(X_1 < t + X_2) \\
&= \underbrace{P(X_1 < t + X_2 | X_1 \le t)}_{=1} P(X_1 \le t) + P(X_1 < t + X_2 | X_1 > t) P(X_1 > t) \\
&= P(X_1 \le t) + [1 - P(X_1 \ge t + X_2 | X_1 > t)] P(X_1 > t) \\
&= P(X_1 \le t) + [1 - P(X_1 > t + X_2 | X_1 > t)] P(X_1 > t) \text{ since } X_1 \text{ is a continuous rv} \\
&= P(X_1 \le t) + [1 - P(X_1 > X_2)] P(X_1 > t) \text{ due to the generalized memoryless property} \\
&= 1 - e^{-\lambda_1 t} + \left(1 - \frac{\lambda_2}{\lambda_1 + \lambda_2}\right) e^{-\lambda_1 t} \\
&= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 t}.
\end{aligned}$$

Exercise 4.1.5. Suppose that a silicon transistor is made up of four independent components which all function to govern the transfer of electronic impulses. For $k = 1, 2, 3, 4$, assume that the lifetime X_k (measured in years) of component k has an exponential distribution with pdf denoted by $f_k(x) = \lambda_k e^{-\lambda_k x}$, $x > 0$. In addition, let $\lambda_1 = \lambda_2 = 1/9$ and $\lambda_3 = \lambda_4 = 1/6$.

(a) What is the probability that component 4 outlives component 1?

(b) What is the probability that component 2's lifetime lasts between 6 and 10 years?

(c) On average, how many years does it take for the first component breakdown to occur?

(d) Use the memoryless property of the exponential distribution to calculate $P(X_3 < X_1 + X_2)$.

Solution: (a) We wish to calculate

$$P(X_1 < X_4) = \frac{1/9}{1/9 + 1/6} = \frac{1/9}{5/18} = \frac{2}{5} \, .$$

(b) We wish to calculate

$$P(6 < X_2 < 10) = P(X_2 > 6) - P(X_2 > 10) = e^{-(1/9)(6)} - e^{-(1/9)(10)} = e^{-2/3} - e^{-10/9} \approx 0.184.$$

(c) We note that $T = \min\{X_1, X_2, X_3, X_4\} \sim \text{EXP}(\lambda)$, where T represents the time of the first component breakdown and

$$\lambda = \sum_{i=1}^{4} \lambda_i = \frac{1}{9} + \frac{1}{9} + \frac{1}{6} + \frac{1}{6} = \frac{5}{9} \, .$$

Therefore, we end up with $\text{E}[T] = 1/\lambda = 9/5 = 1.8$ years.

(d) Note that

$$\begin{aligned}
P(X_3 < X_1 + X_2) &= 1 - P(X_3 > X_1 + X_2) \\
&= 1 - P(X_3 > X_1 + X_2 | X_3 > X_1) P(X_3 > X_1) - \underbrace{P(X_3 > X_1 + X_2 | X_3 < X_1)}_{=0} P(X_3 < X_1) \\
&= 1 - P(X_3 > X_2) P(X_3 > X_1) \ \text{ by the generalized memoryless property of } X_3 \\
&= 1 - \left(\frac{1/9}{1/9 + 1/6} \right)^2 \\
&= 1 - \left(\frac{2}{5} \right)^2 \\
&= \frac{21}{25} \, .
\end{aligned}$$

Exercise 4.1.6. Suppose X_1, X_2, and X_3 are independent random variables where $X_i \sim \text{EXP}(\lambda_i)$, $i = 1, 2, 3$.

(a) Determine $P(X_1 < X_2 < X_3)$.

(b) Determine $P(X_1 < X_2 | X_3 = \max\{X_1, X_2, X_3\})$.

(c) Determine $\text{E}[\max\{X_1, X_2, X_3\} | X_1 < X_2 < X_3]$.

Solution: (a) Note that

$$\begin{aligned}
P(X_1 < X_2 < X_3) &= P(X_1 < X_2, X_2 < X_3) \\
&= \int_0^\infty P(X_1 < X_2, X_2 < X_3 | X_2 = y) \lambda_2 e^{-\lambda_2 y} dy \\
&= \int_0^\infty P(X_1 < y, X_3 > y) \lambda_2 e^{-\lambda_2 y} dy \quad \text{since } X_2 \text{ is independent of } X_1 \text{ and } X_3 \\
&= \int_0^\infty P(X_1 < y) P(X_3 > y) \lambda_2 e^{-\lambda_2 y} dy \quad \text{since } X_1 \text{ and } X_3 \text{ are independent} \\
&= \int_0^\infty (1 - e^{-\lambda_1 y}) e^{-\lambda_3 y} \lambda_2 e^{-\lambda_2 y} dy \\
&= \lambda_2 \int_0^\infty \left\{ e^{-(\lambda_2 + \lambda_3) y} - e^{-(\lambda_1 + \lambda_2 + \lambda_3) y} \right\} dy \\
&= \lambda_2 \left(\frac{1}{\lambda_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} \right) \\
&= \lambda_2 \left(\frac{\lambda_1}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)} \right) \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3} \ .
\end{aligned}$$

As an interesting sidenote, based on the above result obtained, we see that $P(X_1 < X_2 < X_3)$ is expressible as

$$P(X_1 < X_2 < X_3) = P(X_1 = \min\{X_1, X_2, X_3\}) P(X_2 = \min\{X_2, X_3\}).$$

(b) Using the result of part (a), we have

$$
\begin{aligned}
P(X_1 < X_2 | X_3 = \max\{X_1, X_2, X_2\}) &= \frac{P(X_1 < X_2, X_3 = \max\{X_1, X_2, X_3\})}{P(X_3 = \max\{X_1, X_2, X_3\})} \\
&= \frac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3)} \\
&= \frac{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}}{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3} + \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_1}{\lambda_1+\lambda_3}} \\
&= \frac{\frac{1}{\lambda_2+\lambda_3}}{\frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_1+\lambda_3}} \\
&= \frac{\lambda_1 + \lambda_3}{\lambda_1 + \lambda_2 + 2\lambda_3} .
\end{aligned}
$$

(c) We consider

$$
\begin{aligned}
& \mathrm{E}[\max\{X_1, X_2, X_3\} | X_1 < X_2 < X_3] \\
= \; & \mathrm{E}[X_3 | X_1 < X_2 < X_3] \\
= \; & \mathrm{E}[X_3 - X_2 + X_2 - X_1 + X_1 | X_1 < X_2 < X_3] \\
= \; & \mathrm{E}[(X_3 - X_2) + (X_2 - X_1) + X_1 | X_1 < X_2 < X_3] \\
= \; & \mathrm{E}[X_1 | X_1 < X_2 < X_3] + \mathrm{E}[X_2 - X_1 | X_1 < X_2 < X_3] + \mathrm{E}[X_3 - X_2 | X_1 < X_2 < X_3].
\end{aligned}
$$

If we now apply the result from Example 4.2 (i.e., $Y_1 | (Y_1 < Y_2) \sim \min\{Y_1, Y_2\}$ where $Y_i \sim \mathrm{EXP}(\alpha_i)$, $i = 1, 2$, and Y_1 and Y_2 are independent), we get

$$
\begin{aligned}
\mathrm{E}[X_1 | X_1 < X_2 < X_3] &= \mathrm{E}[X_1 | X_1 < X_2, X_2 < X_3] \\
&= \mathrm{E}[X_1 | X_1 < X_2, X_2 = \min\{X_2, X_3\}] \\
&= \mathrm{E}[X_1 | X_1 < \min\{X_2, X_3\}] \\
&= \mathrm{E}[\min\{X_1, \min\{X_2, X_3\}\}] \\
&= \mathrm{E}[\min\{X_1, X_2, X_3\}].
\end{aligned}
$$

Using the generalized memoryless property, we next obtain

$$
\begin{aligned}
\mathrm{E}[X_2 - X_1 | X_1 < X_2 < X_3] &= \mathrm{E}[X_2 - X_1 | X_1 < X_2, X_2 = \min\{X_2, X_3\}] \\
&= \mathrm{E}[\min\{X_2, X_3\} - X_1 | X_1 < X_2, X_2 = \min\{X_2, X_3\}] \\
&= \mathrm{E}[\min\{X_2, X_3\} - X_1 | \min\{X_2, X_3\} > X_1] \\
&= \mathrm{E}[\min\{X_2, X_3\}].
\end{aligned}
$$

Finally, applying the generalized memoryless property again, we get

$$
\begin{aligned}
\mathrm{E}[X_3 - X_2 | X_1 < X_2 < X_3] &= \mathrm{E}[X_3 - X_2 | X_2 = \max\{X_1, X_2\}, X_2 < X_3] \\
&= \mathrm{E}[X_3 - \max\{X_1, X_2\} | X_2 = \max\{X_1, X_2\}, X_2 < X_3] \\
&= \mathrm{E}[X_3 - \max\{X_1, X_2\} | X_3 > \max\{X_1, X_2\}] \\
&= \mathrm{E}[X_3].
\end{aligned}
$$

Putting these pieces together, we ultimately get

$$\begin{aligned}\mathrm{E}[\max\{X_1, X_2, X_3\}|X_1 < X_2 < X_3] &= \mathrm{E}[\min\{X_1, X_2, X_3\}] + \mathrm{E}[\min\{X_2, X_3\}] + \mathrm{E}[X_3] \\ &= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3}.\end{aligned}$$

Exercise 4.1.7. For $a > 0$, consider the following statements for a rv $X \sim \mathrm{EXP}(\lambda)$:

(i) $\mathrm{E}[X^2|X > a] = \mathrm{E}[X^2] + a$,

(ii) $\mathrm{E}[X^2|X > a] = (a + \mathrm{E}[X])^2$,

(iii) $\mathrm{E}[X^2|X > a] = \mathrm{E}[(X + a)^2]$.

Only one of these statements is actually true. Identify which one is the correct statement.

Solution: Let $Y = X|(X > a)$. For $y \geq a$, the cdf of Y is given by

$$F_Y(y) = P(Y \leq y) = P(X \leq y|X > a) = \frac{P(X > a) - P(X > y)}{P(X > a)} = 1 - \frac{e^{-\lambda y}}{e^{-\lambda a}} = 1 - e^{-\lambda(y-a)}.$$

So, $f_Y(y) = F_Y'(y) = \lambda e^{-\lambda(y-a)}$, $y > a$, and

$$\mathrm{E}[Y^2] = \mathrm{E}[X^2|X > a] = \int_a^\infty y^2 \lambda e^{-\lambda(y-a)} dy = \int_0^\infty (z + a)^2 \lambda e^{-\lambda z} dz = \mathrm{E}[(X + a)^2].$$

Thus, the conditional rv $X|(X > a)$ has the <u>same</u> distribution as $X + a$ (essentially due to the memoryless property). Thus, statement (iii) is the correct one.

Exercise 4.1.8. Suppose that X and Y are independent exponentially distributed random variables with respective rates λ and μ. Let $M = \min\{X, Y\}$.

(a) Determine $\mathrm{E}[MX|M = X]$.

(b) Show that the conditional joint pdf of $(X, Y)|(X > Y)$ is given by

$$g(x, y) = \lambda(\lambda + \mu)e^{-\lambda x - \mu y}, \; 0 < y < x < \infty.$$

(c) Use the result in part (b) to determine $\mathrm{E}[MX|M = Y]$.

(d) Determine $\mathrm{Cov}(X, M)$.

Solution: (a) We have

$$\begin{aligned}
\mathrm{E}[MX|M=X] &= \mathrm{E}[X^2|M=X] \\
&= \mathrm{E}[X^2|X<Y] \\
&= \mathrm{E}[M^2] \text{ since } X|(X<Y) \sim M \text{ from Example 4.2} \\
&= \mathrm{E}[M]^2 + \mathrm{Var}(M) \\
&= \left(\frac{1}{\lambda+\mu}\right)^2 + \frac{1}{(\lambda+\mu)^2} \text{ since } M \sim \mathrm{EXP}(\lambda+\mu) \\
&= \frac{2}{(\lambda+\mu)^2}\ .
\end{aligned}$$

(b) Let $G(x,y)$ represent the conditional joint cdf of $(X,Y)|(X>Y)$. For $0 \le y \le x < \infty$, it follows that

$$\begin{aligned}
G(x,y) &= P(X \le x, Y \le y | X > Y) \\
&= \frac{P(X \le x, Y \le y, X > Y)}{P(X > Y)} \\
&= \frac{P(Y < X \le x, Y \le y)}{\frac{\mu}{\lambda+\mu}}\ . \qquad (4.1)
\end{aligned}$$

Looking at the numerator, we have

$$\begin{aligned}
P(Y < X \le x, Y \le y) &= \int_0^y P(Y < X \le x, Y \le y | Y = w) f_Y(w) dw \\
&= \int_0^y P(w < X \le x, w \le y) \mu e^{-\mu w} dw \text{ since } X \text{ and } Y \text{ are independent} \\
&= \int_0^y P(w < X \le x) \mu e^{-\mu w} dw \\
&= \int_0^y \left[P(X > w) - P(X > x)\right] \mu e^{-\mu w} dw \\
&= \int_0^y e^{-\lambda w} \mu e^{-\mu w} dw - e^{-\lambda x} \int_0^y \mu e^{-\mu w} dw \\
&= \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)y}\right) - e^{-\lambda x}\left(1 - e^{-\mu y}\right) \\
&= \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)y}\right) - e^{-\lambda x} + e^{-\lambda x} e^{-\mu y}.
\end{aligned}$$

Substituting the above result into (4.1), we obtain

$$\begin{aligned}
G(x,y) &= \frac{\frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)y}\right) - e^{-\lambda x} + e^{-\lambda x} e^{-\mu y}}{\frac{\mu}{\lambda+\mu}} \\
&= 1 - e^{-(\lambda+\mu)y} - \frac{\lambda+\mu}{\mu} e^{-\lambda x} + \frac{\lambda+\mu}{\mu} e^{-\lambda x} e^{-\mu y},\ 0 \le y \le x < \infty,
\end{aligned}$$

and so the corresponding conditional joint pdf is given by

$$g(x,y) = \frac{\partial^2 G(x,y)}{\partial x \partial y} = \lambda(\lambda+\mu)e^{-\lambda x}e^{-\mu y},\ 0 < y < x < \infty.$$

(c) Using the result in part (b), we wish to obtain

$$\begin{aligned}
\mathrm{E}[XY|X>Y] &= \int_0^\infty \int_y^\infty xy\, g(x,y)dxdy \\
&= \int_0^\infty \int_y^\infty xy\lambda(\lambda+\mu)e^{-\lambda x}e^{-\mu y}dxdy \\
&= \frac{\lambda(\lambda+\mu)}{\lambda^2}\int_0^\infty ye^{-\mu y} \underbrace{\int_y^\infty \frac{\lambda^2 x^{2-1}e^{-\lambda x}}{(2-1)!}dx}_{\text{Erlang}(2,\lambda)\text{ tail probability}}\ dy \\
&= \frac{\lambda+\mu}{\lambda}\int_0^\infty ye^{-\mu y}e^{-\lambda y}(1+\lambda y)dy \\
&= \frac{\lambda+\mu}{\lambda}\left\{\int_0^\infty ye^{-(\lambda+\mu)y}dy + \lambda\int_0^\infty y^2 e^{-(\lambda+\mu)y}dy\right\} \\
&= \frac{1}{\lambda}\cdot\frac{1}{\lambda+\mu} + \mathrm{E}[V^2] \text{ where } V \sim \mathrm{EXP}(\lambda+\mu) \\
&= \frac{1}{\lambda(\lambda+\mu)} + \mathrm{Var}(V) + \mathrm{E}[V]^2 \\
&= \frac{1}{\lambda(\lambda+\mu)} + \frac{1}{(\lambda+\mu)^2} + \frac{1}{(\lambda+\mu)^2} \\
&= \frac{1}{\lambda(\lambda+\mu)} + \frac{2}{(\lambda+\mu)^2}\ .
\end{aligned}$$

(d) First of all, we have that

$$\mathrm{Cov}(X,M) = \mathrm{E}[MX] - \mathrm{E}[M]\mathrm{E}[X] = \mathrm{E}[MX] - \left(\frac{1}{\lambda+\mu}\right)\left(\frac{1}{\lambda}\right).$$

Therefore, we simply need to determine $\mathrm{E}[MX]$. Conditioning on whether $M = X$ or

$M = Y$ and using the results from parts (a) and (c), we obtain

$$\begin{aligned}
\mathrm{E}[MX] &= \mathrm{E}[MX|M = X]P(X < Y) + \mathrm{E}[MX|M = Y]P(Y < X) \\
&= \mathrm{E}[MX|M = X] \cdot \frac{\lambda}{\lambda + \mu} + \mathrm{E}[MX|M = Y] \cdot \frac{\mu}{\lambda + \mu} \\
&= \frac{2}{(\lambda + \mu)^2} \cdot \frac{\lambda}{\lambda + \mu} + \left(\frac{2}{(\lambda + \mu)^2} + \frac{1}{\lambda(\lambda + \mu)} \right) \cdot \frac{\mu}{\lambda + \mu} \\
&= \frac{2\lambda^2}{\lambda(\lambda + \mu)^3} + \frac{2\lambda\mu}{\lambda(\lambda + \mu)^3} + \frac{\mu}{\lambda(\lambda + \mu)^2} \\
&= \frac{2\lambda + \mu}{\lambda(\lambda + \mu)^2} \, .
\end{aligned}$$

Hence,

$$\mathrm{Cov}(X, M) = \frac{2\lambda + \mu}{\lambda(\lambda + \mu)^2} - \left(\frac{1}{\lambda + \mu} \right) \left(\frac{1}{\lambda} \right) = \frac{1}{(\lambda + \mu)^2} \, .$$

Exercise 4.1.9. Consider a hair salon with three stylists who work independently of each other. Four customers, A, B, C, and D, enter the salon simultaneously to find no other customers inside. A, B, and C go directly to the stylists, and D waits until either A, B, or C finishes before beginning its own service. What is the probability that A is still in the salon after the other three customers have left when

(a) the service time for each stylist is *exactly* (i.e., non-random) m minutes, $m > 0$?

(b) the service time for each stylist is either $5, 9$, or 30 minutes with equal probability?

(c) the service time for each stylist is exponentially distributed with mean $1/\mu$?

Solution: (a) If the service time for each stylist is *exactly* m minutes (i.e., constant), then A, B, and C will leave together – thus making it impossible for A to still be in the salon after B, C, and D have left. Therefore,

$$P(A \text{ is still in the salon after the other 3 have left}) = 0.$$

(b) Let X_i denote customer i's service time, $i = A, B, C, D$. Since service times are independent of each other,

$$\begin{aligned}
&P(A \text{ is still in the salon after the other 3 have left}) \\
&= P(X_A = 30, X_B = 5 \text{ or } 9, X_C = 5 \text{ or } 9, X_D = 5 \text{ or } 9) \\
&= P(X_A = 30)P(X_B = 5 \text{ or } 9)P(X_C = 5 \text{ or } 9)P(X_D = 5 \text{ or } 9) \\
&= \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \\
&= \frac{8}{81} \, .
\end{aligned}$$

(c) Applying the memoryless property of the exponential distribution, we obtain

$$\begin{aligned}
&P(A \text{ is still in the salon after the other 3 have left})\\
&= P(B \text{ finishes first}, C \text{ finishes second}, D \text{ finishes third})\\
&\quad + P(B \text{ finishes first}, D \text{ finishes second}, C \text{ finishes third})\\
&\quad + P(C \text{ finishes first}, B \text{ finishes second}, D \text{ finishes third})\\
&\quad + P(C \text{ finishes first}, D \text{ finishes second}, B \text{ finishes third})\\
&= 4\left(\frac{\mu}{\mu+\mu+\mu}\right)\left(\frac{\mu}{\mu+\mu+\mu}\right)\left(\frac{\mu}{\mu+\mu}\right)\\
&= 4(1/3)(1/3)(1/2)\\
&= 2/9.
\end{aligned}$$

Exercise 4.1.10. Consider an inventory of $m \geq 2$ batteries with independent lifetimes. It is known that battery i functions for an exponentially distributed amount of time with rate α_i, $i = 1, 2, \ldots, m$. A remote control requires two batteries to be operational and batteries 1 and 2 are initially installed. Whenever a battery fails, it is immediately replaced by the lowest numbered functional battery that has not yet been put in use. At a random time denoted by T, a battery will fail and our inventory will be empty. At that moment, exactly one of the batteries (which we call battery W) will not yet have failed.

(a) Assuming that $\alpha_i = \alpha$, $i = 1, 2, \ldots, m$, determine $P(W = i)$ for each $i = 1, 2, \ldots, m$.

(b) Assuming that $\alpha_i = \alpha$, $i = 1, 2, \ldots, m$, determine $\mathrm{E}[T]$.

(c) Assuming that $\alpha_i = \alpha$, $i = 1, 2, \ldots, m$, determine $P(T > \mathrm{E}[T])$. Calculate its value for $m = 2, 3, 4, 5$.

(d) Suppose instead that an electronic switchboard that employs all m batteries is to be turned on. All m batteries from the inventory are initially placed into the switchboard. Determine the probability that battery i, $i = 1, 2, \ldots, m$, is the second battery to fail.

(e) Under the same conditions as in part (d), determine the expected time for the second battery failure to occur.

Solution: (a) Let us assume that $i \geq 2$. When battery i is first put into use, $i - 2$ batteries have already failed. As a result, battery i has to outlive the other battery currently in use as well as the upcoming $m-i$ batteries which will be put in use one at a time as failures occur. Since any remaining lifetimes are exponentially distributed with rate α (due to the memoryless property), whenever battery i competes with another battery and a failure does occur, the probability is $1/2$ that it is not battery i that has failed. Therefore, it follows that

$$P(W = i) = (1/2)^{m-i+1}, \ i = 2, 3, \ldots, m.$$

In the case when $i = 1$, we see that battery 1 must outlive all the other $m-1$ batteries. Thus, $P(W = 1) = (1/2)^{m-1}$.

(b) Due to the memoryless property of the exponential distribution, note that T is the sum of $m-1$ independent EXP(2α) random variables (since each time a failure occurs, the time until the next failure is exponentially distributed with rate 2α). In other words, $T \sim \text{Erlang}(m-1, 2\alpha)$, and so $\text{E}[T] = \frac{m-1}{2\alpha}$.

(c) Since $T \sim \text{Erlang}(m-1, 2\alpha)$ from part (b), its tail probability is given by

$$P(T > t) = e^{-2\alpha t} \sum_{j=0}^{m-2} \frac{(2\alpha t)^j}{j!}, \ t \geq 0.$$

Therefore,

$$P(T > \text{E}[T]) = P\left(T > \frac{m-1}{2\alpha}\right) = e^{-(m-1)} \sum_{j=0}^{m-2} \frac{(m-1)^j}{j!}, \ t \geq 0.$$

Plugging into our above formula, we obtain

$$\begin{aligned}
m = 2 &\implies P(T > \text{E}[T]) = P(T > 0.5\alpha^{-1}) = e^{-1} \approx 0.368, \\
m = 3 &\implies P(T > \text{E}[T]) = P(T > \alpha^{-1}) = 3e^{-2} \approx 0.406, \\
m = 4 &\implies P(T > \text{E}[T]) = P(T > 1.5\alpha^{-1}) = 17e^{-3}/2 \approx 0.423, \\
m = 5 &\implies P(T > \text{E}[T]) = P(T > 2\alpha^{-1}) = 71e^{-4}/3 \approx 0.440.
\end{aligned}$$

(d) Conditioning on the first battery to fail (and noting that we do not want battery i to be the first one that fails), we obtain

$$\begin{aligned}
&P(\text{battery } i \text{ is the second to fail}) \\
= &\sum_{j=1; j\neq i}^{m} P(\text{battery } j \text{ fails first}) \\
&\times P(\text{battery } i \text{ fails next among } m-1 \text{ batteries, of which battery } j \text{ is not one of them}) \\
= &\sum_{j=1; j\neq i}^{m} \frac{\alpha_j}{\sum_{k=1}^{m} \alpha_k} \cdot \frac{\alpha_i}{\sum_{k=1; k\neq j}^{m} \alpha_k}.
\end{aligned}$$

(e) Let $X_i \sim \mathrm{EXP}(\alpha_i)$ be the lifetime of battery i, $i = 1, 2, \ldots, m$. Moreover, let Y represent the time when the second battery failure occurs. Conditioning on which battery fails first, we obtain

$$\begin{aligned}
\mathrm{E}[Y] &= \sum_{i=1}^{m} \mathrm{E}[Y|X_i = \min\{X_1, X_2, \ldots, X_m\}] P(X_i = \min\{X_1, X_2, \ldots, X_m\}) \\
&= \sum_{i=1}^{m} \mathrm{E}[Y|X_i < \underbrace{\min\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m\}}_{=Z \sim \mathrm{EXP}(\sum_{j=1;j\neq i}^{m} \alpha_j)}] \cdot \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \\
&= \sum_{i=1}^{m} \mathrm{E}[X_i + \min\{X_1 - X_i, \ldots, X_{i-1} - X_i, X_{i+1} - X_i, \ldots, X_m - X_i\}|Z > X_i] \cdot \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \\
&= \sum_{i=1}^{m} \mathrm{E}[X_i + (Z - X_i)|Z > X_i] \cdot \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \\
&= \sum_{i=1}^{m} (\mathrm{E}[X_i|X_i < Z] + \mathrm{E}[Z - X_i|Z > X_i]) \cdot \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \\
&= \sum_{i=1}^{m} (\mathrm{E}[\min\{X_1, X_2, \ldots, X_m\}] + \mathrm{E}[Z]) \cdot \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \\
&= \frac{1}{\sum_{j=1}^{m} \alpha_j} + \sum_{i=1}^{m} \frac{1}{\sum_{j=1;j\neq i}^{m} \alpha_j} \cdot \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \ ,
\end{aligned}$$

where we applied the result from Example 4.2 and the generalized memoryless property in the second last equality.

4.2 The Poisson Process

Exercises

Exercise 4.2.1. Vehicles cross a certain point on Highway 401 in accordance with a Poisson process at rate $\lambda = 10$ per minute. If a deer blindly runs across the highway, then what is the probability that the deer will be unhurt if the amount of time that it takes to cross the highway is c seconds? Assume that if the deer is on the highway when a vehicle drives by, then the deer will be hurt. Calculate this probability for $c = 2, 4, 8$, and 12.

Solution: Let $N(t)$ be the number of vehicles crossing a certain point on Highway 401, so that $\{N(t), t \geq 0\}$ is a Poisson process at rate $\lambda = 10$ per minute. If we assume that the deer starts to cross the highway at time t, then it follows that

$$\begin{aligned}
P(\text{deer is unhurt}) &= P\left(N\left(t + \frac{c}{60}\right) - N(t) = 0\right) \\
&= P\left(N\left(\frac{c}{60}\right) = 0\right) \text{ due to stationary increments} \\
&= \frac{e^{-10c/60}(10c/60)^0}{0!} \\
&= e^{-c/6}.
\end{aligned}$$

If $c = 2$, then $P(\text{deer is unhurt}) = e^{-2/6} = e^{-1/3} \approx 0.717$.
If $c = 4$, then $P(\text{deer is unhurt}) = e^{-4/6} = e^{-2/3} \approx 0.513$.
If $c = 8$, then $P(\text{deer is unhurt}) = e^{-8/6} = e^{-4/3} \approx 0.264$.
If $c = 12$, then $P(\text{deer is unhurt}) = e^{-12/6} = e^{-2} \approx 0.135$.

Exercise 4.2.2. Patients arrive to a walk-in clinic according to a Poisson process with a rate of 6 patients per hour. However, the doctor only begins to examine patients when the third patient has arrived.

(a) Calculate the mean and variance of the time from the opening of the clinic until the first patient starts to be examined.

(b) In the opening first hour, what is the probability that the doctor does not start examining at all?

Solution: (a) Since the doctor only begins to examine patients when the third patient has arrived to the clinic, we simply wish to calculate $\mathrm{E}[S_3]$ and $\mathrm{Var}(S_3)$. Since $S_3 \sim \mathrm{Erlang}(3, 6)$, it immediately follows that $\mathrm{E}[S_3] = 3/6 = 1/2$ and $\mathrm{Var}(S_3) = 3/6^2 = 1/12$.

(b) We wish to calculate

$$\begin{aligned} P(\text{no examinations take place in the opening first hour}) &= P(S_3 > 1) \\ &= e^{-6(1)} \sum_{i=0}^{3-1} \frac{[6(1)]^i}{i!} \\ &= 25e^{-6} \\ &\approx 0.0620. \end{aligned}$$

Exercise 4.2.3. Suppose that events occur according to a Poisson process at rate $\lambda = 0.75$.

(a) What is the probability that the 5^{th} event occurs more than 2 time units after the 4^{th} event occurs?

(b) What is the probability that the 5^{th} event occurs more than 3 time units after the 3^{rd} event occurs?

(c) Calculate the joint probability that exactly 1 event occurs in the time interval $[1, 4]$ and exactly 3 events occur in the time interval $[3, 5]$.

Solution: (a) For $i \geq 2$, let the time between the $(i-1)^{\text{th}}$ and i^{th} event occurrences be denoted by T_i. Since events occur according to a Poisson process, we know that $T_5 \sim \text{EXP}(0.75)$. Therefore, it follows that

$$P(T_5 > 2) = e^{-0.75(2)} = e^{-1.5} \approx 0.223.$$

(b) Let the time between the 3^{rd} and 5^{th} event occurrences be given by $T = T_4 + T_5$, where T_4 and T_5 are independent EXP(0.75) random variables. Therefore, $T \sim \text{Erlang}(2, 0.75)$, and so

$$P(T > 3) = e^{-0.75(3)} \sum_{i=0}^{2-1} \frac{[(0.75)3]^i}{i!} = 3.25e^{-2.25} \approx 0.343.$$

(c) Let $\{N(t), t \geq 0\}$ represent the Poisson process under consideration. We wish to calculate

$$
\begin{aligned}
& P(N(4)-N(1)=1, N(5)-N(3)=3) \\
= \; & P(N(4)-N(1)=1, N(4)-N(3)=0, N(5)-N(3)=3) \\
& + P(N(4)-N(1)=1, N(4)-N(3)=1, N(5)-N(3)=3) \\
= \; & P(N(3)-N(1)=1, N(4)-N(3)=0, N(5)-N(4)=3) \\
& + P(N(3)-N(1)=0, N(4)-N(3)=1, N(5)-N(4)=2) \\
= \; & P(N(3)-N(1)=1)P(N(4)-N(3)=0)P(N(5)-N(4)=3) \\
& + P(N(3)-N(1)=0)P(N(4)-N(3)=1)P(N(5)-N(4)=2) \text{ by independent increments} \\
= \; & \frac{e^{-0.75(2)}[0.75(2)]^1}{1!} \cdot \frac{e^{-0.75(1)}[0.75(1)]^0}{0!} \cdot \frac{e^{-0.75(1)}[0.75(1)]^3}{3!} \\
& + \frac{e^{-0.75(2)}[0.75(2)]^0}{0!} \cdot \frac{e^{-0.75(1)}[0.75(1)]^1}{1!} \cdot \frac{e^{-0.75(1)}[0.75(1)]^2}{2!} \\
= \; & \frac{(0.75)^4 e^{-3}}{3} + \frac{(0.75)^3 e^{-3}}{2} \\
= \; & (0.75)^4 e^{-3} \\
\approx \; & 0.0158.
\end{aligned}
$$

Exercise 4.2.4. Two individuals, A and B, both require liver transplants. If a new liver is not received, then A will die following an exponentially distributed time with rate μ_A, and B after an exponentially distributed time with rate μ_B. New livers arrive in accordance with a Poisson process at rate λ. It has been decided that the first liver will go to A (or to B if B is alive and A is not at that time) and the next one to B (if B is still alive).

(a) What is the probability that A obtains a new liver?

(b) What is the probability that B obtains a new liver?

(c) What is the probability that neither A nor B obtains a new liver?

(d) What is the probability that both A and B obtain new livers?

Solution: (a) In terms of notation, we define the following random variables:

K_1 is the time to arrival of the first liver,
K_2 is the time from the arrival of the first liver to the second liver arrival,
X_A is the lifetime of individual A,
X_B is the lifetime of individual B.

Since livers arrive according to a Poisson process, it follows that K_1 and K_2 are iid EXP(λ) random variables. Moreover, we know that $X_A \sim$ EXP(μ_A) and $X_B \sim$

EXP(μ_B). Now, we realize that A will get a new liver if the first liver arrives before A dies, implying simply that

$$P(A \text{ obtains a new liver}) = P(K_1 < X_A) = \frac{\lambda}{\lambda + \mu_A}.$$

(b) We remark that individual B will get a liver under two circumstances: (i) the first liver arrives before A dies and the second liver arrives before B dies, and (ii) if A dies first and the first liver arrives before B dies. Note that at time 0, there are three competing exponential timers to consider, namely K_1, X_A, and X_B. To determine the desired probability, we condition on which timer completes first (i.e., which of K_1, X_A, or X_B has the smallest value).

First of all, let us consider the situation if $K_1 = \min\{K_1, X_A, X_B\}$. In this case, upon observing the arrival of the first liver, we stop observing the timer X_A (since A receives this first liver) and we replace the completed K_1 timer by the K_2 timer (i.e., the time until the arrival of the second liver). Additionally, upon observing the arrival of the first liver, the remaining timer distribution for X_B probabilistically resets itself due to the memoryless property. Therefore, the conditional probability that B obtains a new liver, given that $K_1 = \min\{X_A, X_B, K_1\}$, is

$$\begin{aligned}
& P(B \text{ obtains a new liver} | K_1 = \min\{K_1, X_A, X_B\}) \\
= \;& P(B \text{ obtains a new liver} | \text{liver 1 arrives first}) \\
= \;& P(K_2 < X_B) \text{ due to the memoryless property} \\
= \;& \frac{\lambda}{\lambda + \mu_B}\;.
\end{aligned}$$

Next, suppose that $X_A = \min\{K_1, X_A, X_B\}$. As before, by the memoryless property, the remaining timer distributions for X_B and K_1 probabilistically reset themselves after A dies, and thus the conditional probability that B obtains a new liver, given that $X_A = \min\{K_1, X_A, X_B\}$, is

$$\begin{aligned}
& P(B \text{ obtains a new liver} | X_A = \min\{K_1, X_A, X_B\}) \\
= \;& P(B \text{ obtains a new liver} | A \text{ dies first}) \\
= \;& P(K_1 < X_B) \text{ due to the memoryless property} \\
= \;& \frac{\lambda}{\lambda + \mu_B}\;.
\end{aligned}$$

Finally, we remark that if $X_B = \min\{K_1, X_A, X_B\}$, then the first observed event is individual B's death, and hence B will clearly not obtain a new liver. Putting the pieces together, we ultimately obtain

$$
\begin{aligned}
& P(B \text{ obtains a new liver}) \\
= \; & P(B \text{ obtains a new liver} | K_1 = \min\{K_1, X_A, X_B\}) P(K_1 = \min\{K_1, X_A, X_B\}) \\
& + P(B \text{ obtains a new liver} | X_A = \min\{K_1, X_A, X_B\}) P(X_A = \min\{K_1, X_A, X_B\}) \\
& + P(B \text{ obtains a new liver} | X_B = \min\{K_1, X_A, X_B\}) P(X_B = \min\{K_1, X_A, X_B\}) \\
= \; & \frac{\lambda}{\lambda + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_A + \mu_B} + \frac{\lambda}{\lambda + \mu_B} \cdot \frac{\mu_A}{\lambda + \mu_A + \mu_B} + 0 \cdot \frac{\mu_B}{\lambda + \mu_A + \mu_B} \\
= \; & \frac{\lambda}{\lambda + \mu_B} \left(\frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B} \right).
\end{aligned}
$$

(c) To determine the probability that neither individual A nor B obtains a new liver, we look at part (b) and consider only the cases where $X_A = \min\{K_1, X_A, X_B\}$ or $X_B = \min\{K_1, X_A, X_B\}$, since in the case when K_1 is the minimum, individual A is sure to receive a new liver. Moreover, once the first event to occur is either A or B dying, the remaining timer distributions for K_1 and the individual who is still alive probabilistically reset themselves (due to the memoryless property). Therefore, we obtain

$$
\begin{aligned}
& P(\text{neither } A \text{ nor } B \text{ obtains a new liver}) \\
= \; & P(\text{neither } A \text{ nor } B \text{ obtains a new liver} | X_A = \min\{K_1, X_A, X_B\}) P(X_A = \min\{K_1, X_A, X_B\}) \\
& + P(\text{neither } A \text{ nor } B \text{ obtains a new liver} | X_B = \min\{K_1, X_A, X_B\}) P(X_B = \min\{K_1, X_A, X_B\}) \\
= \; & P(X_B < K_1) P(X_A = \min\{K_1, X_A, X_B\}) + P(X_A < K_1) P(X_B = \min\{K_1, X_A, X_B\}) \\
& \hfill \text{due to the memoryless property} \\
= \; & \frac{\mu_B}{\lambda + \mu_B} \cdot \frac{\mu_A}{\lambda + \mu_A + \mu_B} + \frac{\mu_A}{\lambda + \mu_A} \cdot \frac{\mu_B}{\lambda + \mu_A + \mu_B} \\
= \; & \frac{\mu_A \mu_B}{\lambda + \mu_A + \mu_B} \left(\frac{1}{\lambda + \mu_B} + \frac{1}{\lambda + \mu_A} \right).
\end{aligned}
$$

(d) To determine the probability that both individuals obtain new livers, we again look at part (b) and consider only the case where $K_1 = \min\{K_1, X_A, X_B\}$, since in the other two cases when X_A or X_B is the minimum, someone dies prior to obtaining a new liver (and hence we cannot observe both A and B obtaining new livers). Recall that $K_1 = \min\{K_1, X_A, X_B\}$ results in A obtaining a new liver, and so both will obtain new livers if after this observation, we also eventually observe B obtain a new liver. Therefore, this leads to

$$
\begin{aligned}
& P(\text{both } A \text{ and } B \text{ obtain new livers}) \\
= \; & P(\text{both } A \text{ and } B \text{ obtain new livers} | K_1 = \min\{K_1, X_A, X_B\}) P(K_1 = \min\{K_1, X_A, X_B\}) \\
= \; & P(K_2 < X_B) P(K_1 = \min\{K_1, X_A, X_B\}) \text{ due to the memoryless property} \\
= \; & \frac{\lambda}{\lambda + \mu_B} \left(\frac{\lambda}{\lambda + \mu_A + \mu_B} \right).
\end{aligned}
$$

Exercise 4.2.5. Let $S(t)$ denote the price of a security at time t. A commonly-used model for the process $\{S(t), t \geq 0\}$ assumes that the price remains unchanged until a "shock" occurs, at which time the price is multiplied by a *random factor*. Specifically, if $N(t)$ represents the number of shocks by time t and X_i represents the i^{th} multiplicative factor, then the model under consideration is given by

$$S(t) = S(0) \prod_{i=1}^{N(t)} X_i,$$

where $S(0) = s > 0$ and $\prod_{i=1}^{N(t)} X_i$ is equal to 1 when $N(t) = 0$. Suppose that $\{X_i\}_{i=1}^{\infty}$ are iid EXP(μ) random variables and $\{N(t), t \geq 0\}$ is a Poisson process with rate λ. Assume that $\{N(t), t \geq 0\}$ is independent of each rv X_i.

(a) Determine E[$S(t)$].

(b) Determine Var($S(t)$).

Solution: (a) Note that

$$\begin{aligned}
\mathrm{E}[S(t)|N(t) = n] &= s\mathrm{E}\left[\prod_{i=1}^{N(t)} X_i \middle| N(t) = n\right] \\
&= s\mathrm{E}\left[\prod_{i=1}^{n} X_i \middle| N(t) = n\right] \\
&= s\mathrm{E}\left[\prod_{i=1}^{n} X_i\right] \quad \text{since each } X_i \text{ and } N(t) \text{ are independent} \\
&= s(\mathrm{E}[X])^n \text{ since } \{X_i\}_{i=1}^{\infty} \text{ are iid} \\
&= s\left(\frac{1}{\mu}\right)^n.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathrm{E}[S(t)] &= \sum_{n=0}^{\infty} \mathrm{E}[S(t)|N(t) = n] P(N(t) = n) \\
&= s\sum_{n=0}^{\infty} \left(\frac{1}{\mu}\right)^n \cdot \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= se^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t/\mu)^n}{n!} \\
&= se^{-\lambda t + \lambda t/\mu} \\
&= se^{-\lambda t(1-\mu^{-1})}.
\end{aligned}$$

(b) Following the same approach used in part (a), we have

$$\begin{aligned}
\mathrm{E}[S(t)^2|N(t)=n] &= s^2\mathrm{E}\left[\left(\prod_{i=1}^{N(t)} X_i\right)^2 \middle| N(t)=n\right] \\
&= s^2\mathrm{E}\left[\prod_{i=1}^{n} X_i^2 \middle| N(t)=n\right] \\
&= s^2\mathrm{E}\left[\prod_{i=1}^{n} X_i^2\right] \text{ since each } X_i \text{ and } N(t) \text{ are independent} \\
&= s^2(\mathrm{E}[X^2])^n \text{ since } \{X_i\}_{i=1}^{\infty} \text{ are iid} \\
&= s^2\left(\frac{2}{\mu^2}\right)^n.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathrm{E}[S(t)^2] &= \sum_{n=0}^{\infty} \mathrm{E}[S(t)^2|N(t)=n]P(N(t)=n) \\
&= s^2\sum_{n=0}^{\infty}\left(\frac{2}{\mu^2}\right)^n \cdot \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= s^2 e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(2\lambda t/\mu^2)^n}{n!} \\
&= s^2 e^{-\lambda t + 2\lambda t/\mu^2}.
\end{aligned}$$

As a result, we can now determine

$$\begin{aligned}
\mathrm{Var}(S(t)) &= \mathrm{E}[S(t)^2] - \mathrm{E}[S(t)]^2 \\
&= s^2 e^{-\lambda t + 2\lambda t/\mu^2} - \left(s e^{-\lambda t + \lambda t/\mu}\right)^2 \\
&= s^2\left(e^{-\lambda t + 2\lambda t/\mu^2} - e^{-2\lambda t + 2\lambda t/\mu}\right) \\
&= s^2 e^{-\lambda t}\left(e^{2\lambda t\mu^{-2}} - e^{\lambda t(2\mu^{-1}-1)}\right).
\end{aligned}$$

Exercise 4.2.6. Customers arrive at a service facility according to a Poisson process at rate $\lambda = 2$ customers per hour. Let $N(t)$ be the number of customers that have arrived up to time t.

(a) What is the probability that at least 3 customers arrive between the morning hours of 8:00 am and 10:00 am (of the same day)?

(b) Suppose that after 15 minutes, the first customer has yet to arrive at the service facility. What is the probability that at most an additional 10 minutes goes by without the arrival of the first customer?

(c) Calculate $E[N(3)|N(1) = 2]$.

(d) Calculate $P(N(1) = 2|N(3) = 6)$.

(e) What is the probability that the 7^{th} customer arrival occurs more than 45 minutes after the 5^{th} customer arrival?

Solution: (a) We simply wish to calculate

$$\begin{aligned} P(N(2) \geq 3) &= 1 - P(N(2) \leq 2) \\ &= 1 - \sum_{i=0}^{2} \frac{e^{-2(2)}[2(2)]^i}{i!} \\ &= 1 - e^{-4} - 4e^{-4} - \frac{16}{2!}e^{-4} \\ &= 1 - 13e^{-4} \\ &\approx 0.762. \end{aligned}$$

(b) Let T_1 be the time of the first customer arrival, where $T_1 \sim \text{EXP}(2)$. Note that 15 minutes is equal to 1/4 hour, whereas 10 minutes is equal to 1/6 hour. Thus, we wish to calculate

$$\begin{aligned} P(T_1 \leq 1/4 + 1/6|T_1 > 1/4) &= 1 - P(T_1 > 1/4 + 1/6|T_1 > 1/4) \\ &= 1 - P(T_1 > 1/6) \text{ by the memoryless property} \\ &= 1 - e^{-2(1/6)} \\ &= 1 - e^{-1/3} \\ &\approx 0.284. \end{aligned}$$

(c) First of all, note that $N(3) - N(1) \sim \text{POI}(4)$. Therefore, we obtain

$$\begin{aligned} \text{E}[N(3)|N(1) = 2] &= \text{E}[N(3) - N(1) + N(1)|N(1) = 2] \\ &= \text{E}[N(3) - N(1)|N(1) = 2] + 2 \\ &= \text{E}[N(3) - N(1)] + 2 \;\; \text{due to independent increments} \\ &= 4 + 2 \\ &= 6. \end{aligned}$$

(d) Note that

$$
\begin{aligned}
P(N(1)=2|N(3)=6) &= \frac{P(N(1)=2, N(3)=6)}{P(N(3)=6)} \\
&= \frac{P(N(1)=2, N(3)-N(1)+N(1)=6)}{P(N(3)=6)} \\
&= \frac{P(N(1)=2, N(3)-N(1)=4)}{P(N(3)=6)} \\
&= \frac{P(N(1)=2)P(N(3)-N(1)=4)}{P(N(3)=6)} \quad \text{due to independent increments} \\
&= \frac{e^{-2}2^2}{2!} \cdot \frac{e^{-4}4^4}{4!} \cdot \frac{6!}{e^{-6}6^6} \\
&= \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 \\
&= \frac{80}{243} \approx 0.329.
\end{aligned}
$$

(e) Let X be the time between the 5^{th} and 7^{th} customer arrivals. If T_i denotes the i^{th} interarrival time, then $X = T_6 + T_7$ where T_6 and T_7 are iid EXP(2) random variables. Thus, $X \sim \text{Erlang}(2,2)$. Since 45 minutes is equivalent to 3/4 hour, it follows that

$$P(X > 3/4) = e^{-2(3/4)} \sum_{i=0}^{2-1} \frac{[2(3/4)]^i}{i!} = e^{-3/2}\left(1 + \frac{3}{2}\right) = \frac{5}{2}e^{-3/2} \approx 0.558.$$

.