Applications of spectral graph theory in probing quantum entanglement

Endterm Evaluation

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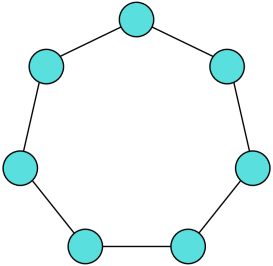
What is Spectral Graph Theory?

Spectral graph theory is the study of the properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvectors of matrices associated with the graph, such as its adjacency matrix or Laplacian matrix. (Spielman, 2019)

Terms

A simple graph G is an ordered pair (V, E), consisting of a nonempty set V of vertices and a set E of edges, each edge a two-tuple of V with no edge having identical ends.



Terms

Adjacency Matrix is a square matrix used to represent a graph. The adjacency matrix A has elements,

$$A_{ij} = \begin{cases} 1 & (i,j) \in E \\ 0 & o.w. \end{cases} \tag{1}$$

In case of no self-loops (aka Simple Graph), $A_{ii} = 0 \; \forall \; i \in V$.

Terms

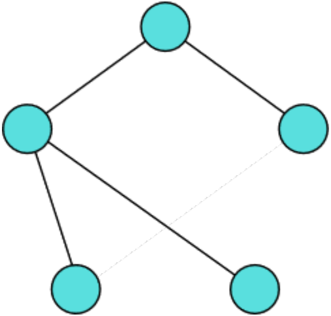
The **degree** of a node in an undirected network is the number of edges connected to it. We denote the degree of node $i \in V$ by k_i.

We sometimes use D to represent the **degree matrix** which is a diagonal matrix with the values k_i where $i = j$.

Terms

A connected graph G is one in which there exists a path between vertices a and b, $\forall a, b \in V$.

A **tree** T is a poorly connected graph as removing any one edge will render it disconnected.



Terms

The **Graph Laplacian** for a simple undirected, unweighted network is an $n \times n$ symmetric matrix L with elements,

$$L_{ij} = \begin{cases} k_i, & i = j \\ -1, & i \neq j \\ 0, & \text{o.w.} \end{cases} \tag{2}$$

$$L_{ij} = k_i \delta_{ij} - A_{ij}$$

where A_{ij} is an element of the adjacency matrix and δ_{ij} is the Kronecker delta, which is 1 if $i = j$ and 0 otherwise.

Another way to find the laplacian matrix is,

$$L = D - A$$

where D is the diagonal matrix with the node degrees along its diagonal.

Connecting classical graphs with quantum graph states

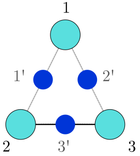
In this project we aim to find a classical metric using spectral graph theory and relate it with its quantum counterpart in order to find a good relation through which if we know one metric, we can guess the other with good certainty.

We will explore the metrics which we have come up with for now and also discuss the future metrics which can be valid for comparision.

1. Introduction
2. Experiments
3. Expander Graphs
4. Error Correction
5. Observations

Entanglement Value

For every corresponding edge pair, we fill $\log(n)$ if separating the system along the midpoints of the respective edges leads to two separated components unless when the edge is being compared to itself in which case we see if the edge is actually present in the graph state and thus assign it a value of $\log(n)$ and the rest of the edge pairs are assigned a value of zero.



(a) Graph representation

	1'	2'	3'
1'	0	1	0
2'	1	1	1
3'	0	1	0

(b) Entanglement Matrix

Properties of the Graph Laplacian

We consider only connected graph which means the number of connected components is one. Thus the number of zero eigenvalues of L is also one. (Li, 2022)

As the laplacian is a PSD, the smallest eigenvalue is zero.

The second smallest eigenvalue $\alpha(G)$ is also known as the algebraic connectivity of a graph and is greater than 0 in our case. (Marsden, 2013)

For a given value of n which is the number of vertices for the graphs generated, we find the laplacian and subsequently $\alpha(G)$. Our aim is then to compare the entanglement values obtained with the second smallest eigenvalue of the respective graphs.

Inverse Distance Determinant Value

We denote the shortest path between two nodes i and j with s_{ij} and define a matrix T where,

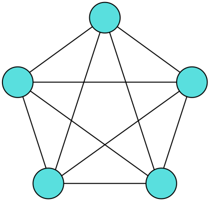
$$T_{ij} = \begin{cases} \frac{1}{s_{ij}}, & \text{if } s_{ij} \neq 0 \\ 0, & \text{o.w.} \end{cases} \tag{3}$$

We record our observations with values of $n = 4, 5, 6, \ldots$

Definition

An Expander Graph is a *sparsely populated graph* that is *well connected*.

- A *sparse graph* is a graph in which the total number of edges is few compared to the maximal number of edges.
- A graph G is connected if there exists a path between vertices a and b $\forall\, a, b \in G$.



Cheeger Constant

The **Cheeger constant** or the **edge expansion** of a finite graph G, (Siegel, 2014)

$$c(G) = \min_{0<|S|\leq \frac{n}{2}} \frac{|\partial(S)|}{|S|} \tag{4}$$

where the boundary of S is δS.

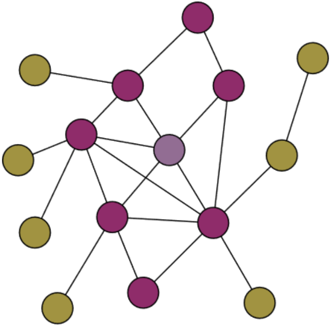
The larger the Cheeger constant, the better connected the graph is. $c(G) > 0$ iff G is a connected graph.

Vertex Expansion

The vertex expansion or the vertex isoperimetric number $h_{\text{out}}(G)$ of a graph G is defined as,

$$h_{\text{out}}(G) = \min_{0 \leq |S| \leq \frac{n}{2}} \frac{|\partial_{\text{out}}(S)|}{|S|}$$

$$h_{\text{in}}(G) = \min_{0 \leq |S| \leq \frac{n}{2}} \frac{|\partial_{\text{in}}(S)|}{|S|}$$



Spectral Expansion

The spectral gap of G, $s(G)$ is defined as $s(G) = \lambda_1 - \lambda_2$. (Goldreich, 2011)

Cheeger's and Buser's inequalities

$$\frac{s(G)}{2} \leq c(G) \leq \sqrt{2\lambda_1 s(G)}$$

Families of Expanders

- The Iterated Zig-Zag Construction
- Margulis-Gabber-Galil
- Chordal Cycle Graph
- Paley Graph

Some Interesting Properties of Expanders

- The second largest eigenvalue, λ_2 is bound by $2\sqrt{d-1}$ for a d regular graph.
- If the graph is bipartite, the eigenvalues are symmetric about zero.
- $\lambda_1 = d$ so $s(G) = d - \lambda_2$.
- $\lambda_2(G) \leq \max_{G_1,G_2} \min\{\lambda_1(G_1), \lambda_2(G_1)\}$.

Pearson correlation coefficient values for various n and methods used

The correlation is obtained between the method used vs second largest eigenvalue.

n	Entanglement	Inverse Distance	Cheeger Constant
4	0.91874	0.94776	0.92042
5	-	0.90590	0.90685
6	-	0.55140	0.81209

Introduction

- The message sent from Alice is not hte same as that received by Bob.
- It is a mapping $f : \{0, 1\}^k \to \{0, 1\}^n$.
- The *original message* is encoded into *codewords*.
- The *minimum relative distance* δ should be large enough.
- The amount of information transmitted by the *rate* which is k/n.
- We'll deal in linear codes which are linear combinations over $GF(2)$.

Expanders : Are they good enough?

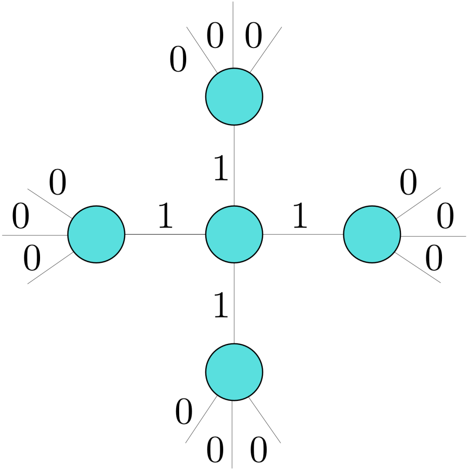
- For a graph $G \in \mathcal{G}$ with n vertices, construct a code of length $dn/2$.
- Since C has minimum distance of at least δd, it is possible to correct $\delta d/2$ errors.

Algorithm for Decoding Expander Codes

Linear time decoding is possible with Expander Graphs. (Spielman, 1999)

Flip the edges suggested by a vertex only within distance $\delta d/4$.

Edge Cases



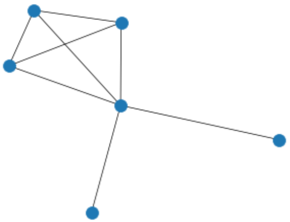
Parity Check Matrices for Expander Graphs

For a graph with n vertices and k edges, we construct an $n \times k$ matrix where the entry corresponding to a certain row i and certain column j signifies whether the node i and edge j are connected. This is done by enemerating the edges in any order starting from 0 to $k - 1$. After that for every node we fill the entries corresponding to each of the columns.

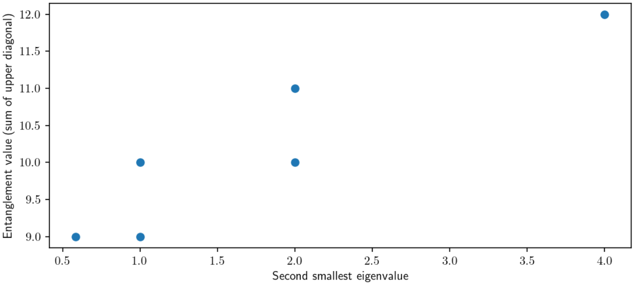
The Appendix displays all the graphs which satisfy the above constraint. The parity matrix for the graph with $n = 6$ and $si = 012$ is,

Parity Check Matrices for Expander Graphs

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

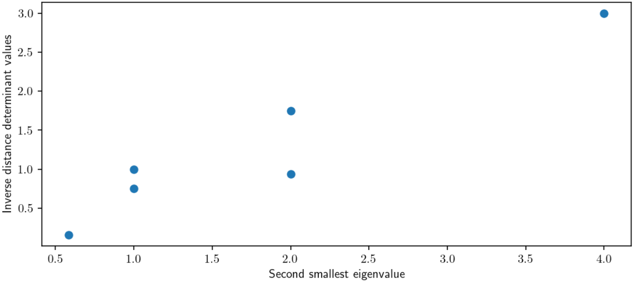


Entanglement value vs Second smallest eigenvalue for graphs with $n = 4$



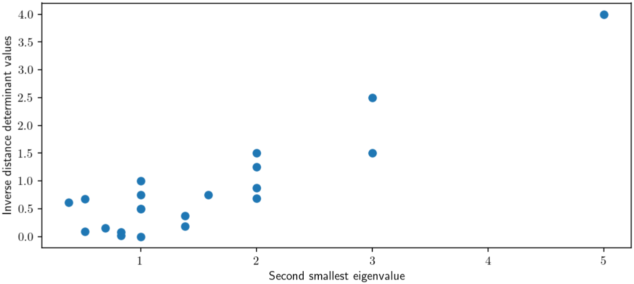
For $n = 3$, there are only 2 graphs possible (2 and 3 edges) therefore no clear observation is possible. In case of $n = 5$, the graph can no longer be spectrally represented on a plane in case of high connections, hence quantifying the values is hard.

Inverse Distance Determinant Value vs Second smallest eigenvalue for graphs with $n = 4$

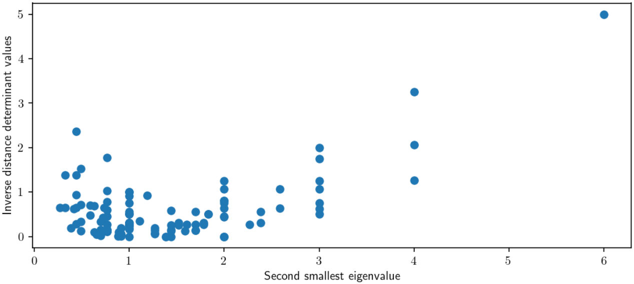


For $n = 3$, there are only 2 graphs possible (2 and 3 edges) therefore no clear observation is possible.

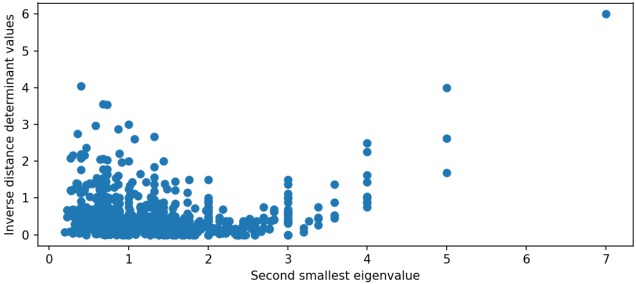
Inverse Distance Determinant Value vs Second smallest eigenvalue for graphs with $n = 5$



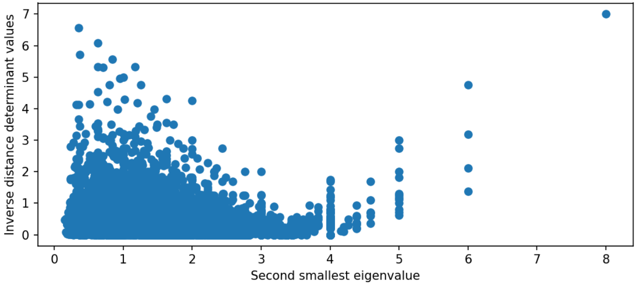
Inverse Distance Determinant Value vs Second smallest eigenvalue for graphs with $n = 6$



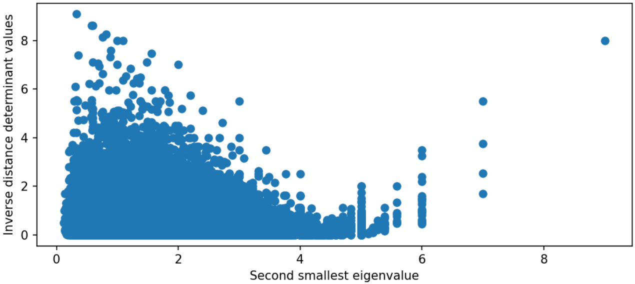
Inverse Distance Determinant Value vs Second smallest eigenvalue for graphs with $n = 7$



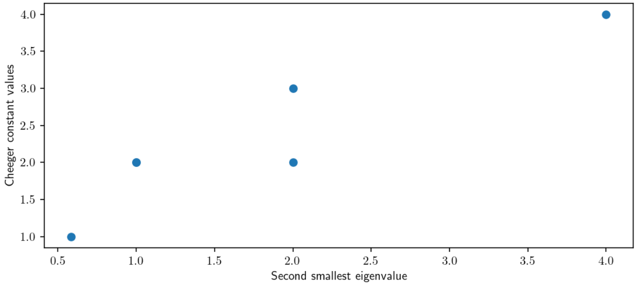
Inverse Distance Determinant Value vs Second smallest eigenvalue for graphs with $n = 8$



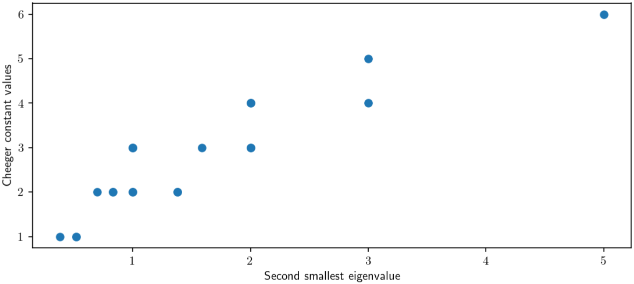
Inverse Distance Determinant Value vs Second smallest eigenvalue for graphs with $n = 9$



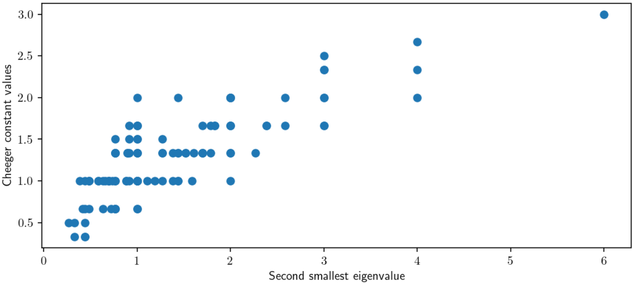
Cheeger constant value vs Second smallest eigenvalue for graphs with $n = 4$



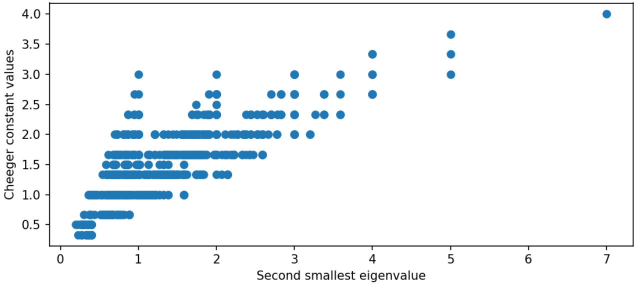
Cheeger constant value vs Second smallest eigenvalue for graphs with $n = 5$



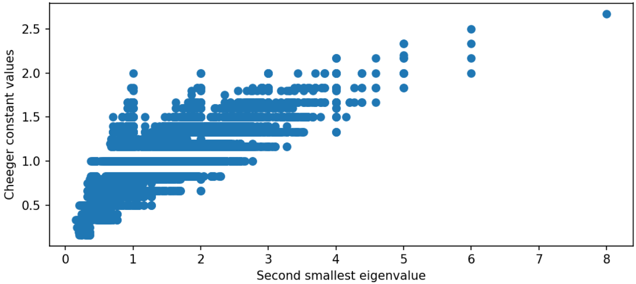
Cheeger constant value vs Second smallest eigenvalue for graphs with $n = 6$



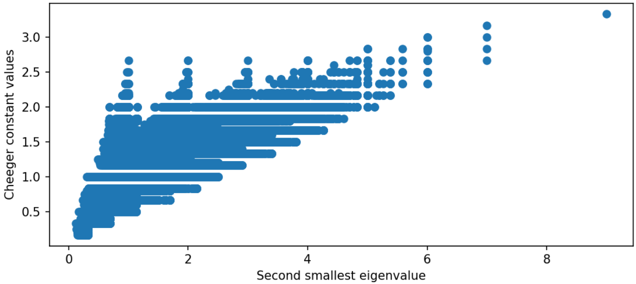
Cheeger constant value vs Second smallest eigenvalue for graphs with $n = 7$



Cheeger constant value vs Second smallest eigenvalue for graphs with $n = 8$



Cheeger constant value vs Second smallest eigenvalue for graphs with $n = 9$



Deficiency or nullity of a matrix

We calculate the d value which is simply the minimum number of columns in a matrix which can perform row-wise modular addition to give a column with all zeros. The d value is to be calculated over the parity check matrices of a graph which satisfy the constraint that is $n < k$. The method for obtaining the parity check matrices has been explained before.

Correlation values between different paramters

Why correlation instead of covariance?

Covariance represents change while *correlation* represents connection. Coefficient of correlation lies between $(-1, 1)$ while covariance can range between $(-\infty, \infty)$. The correlation coefficient is calculated as follows:

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2}\sqrt{\sum(y-\bar{y})^2}}$$

n	C(G) vs D vals	C(G) vs $\alpha(G)$	$\alpha(G)$ vs D vals
4	nan	1.00000	nan
5	0.75289	0.00003	0.98950
6	0.03303	0.00000	0.83271
7	0.87909	0.00000	0.00635

Neural Network

The model has three dense layers of 64, 32 and, 1 output sizes each. The first two layers use the ReLU activation function. I have used the *adam* optimizer and Mean Square Error as the loss function. For the actual training and testing part, I proceeded with taking sets of graphs with different sets of vertices for both the training and testing. For the set with smaller n we rescale the matrix by first flattening and then filling with zeros to match the size. For disjoint sets we get a suboptimal performace with an accuracy of around 21% which is akin to random guessing.

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Thank You!