EECS475
Introduction to Cryptography

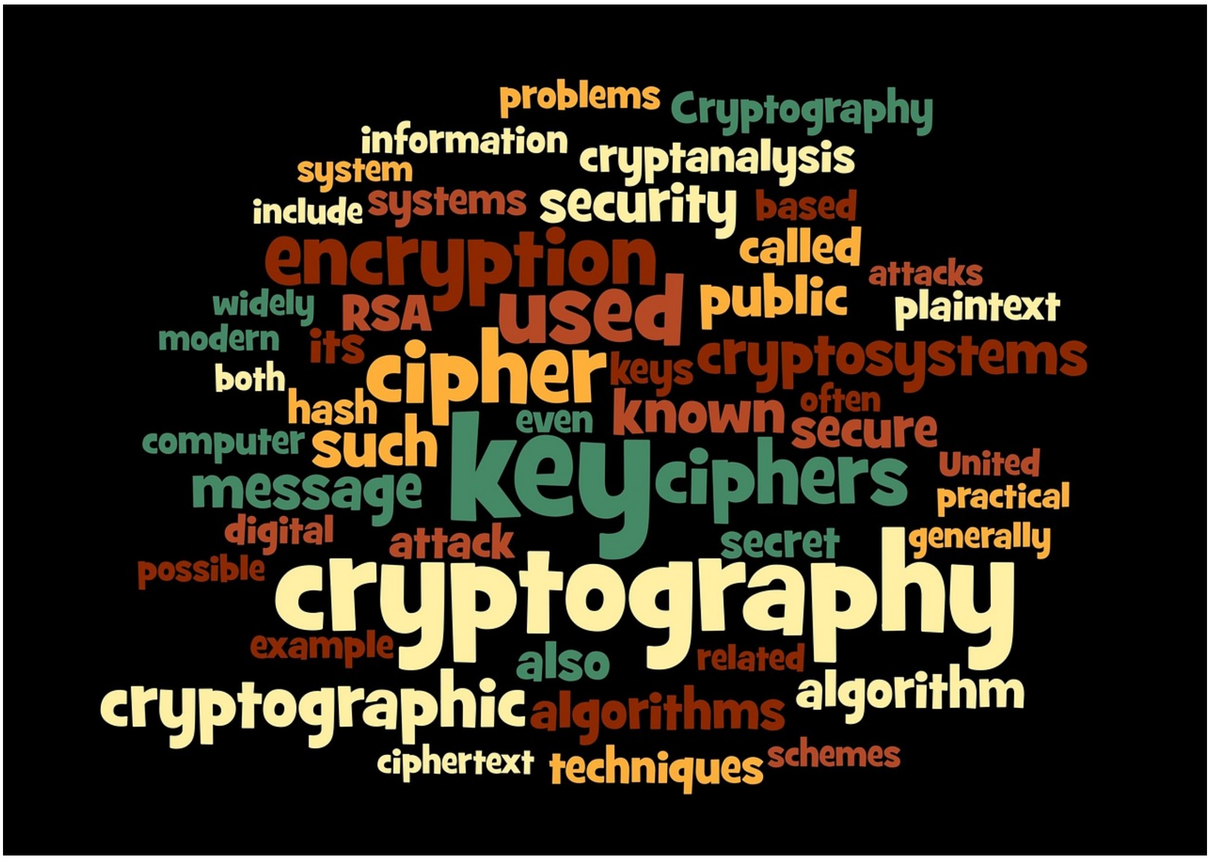
Winter 2023 All Course Members

January 21, 2024

Abstract

This is an accumulated scribe notes taken by the course members[1] of EECS475, an upper-level course taught at University of Michigan by Mahdi Cheraghchi. Topics include various historic ciphers, perfect secrecy, symmetric encryption (including pseudorandom generators, stream ciphers, pseudorandom functions/permutations), message authentication, cryptographic hash functions, and public key encryption.

We'll use *Introduction to Modern Cryptography* [KL20] as our main references.



This course is offered in Winter 2023, and the date on the cover page is the last updated time.

[1]Maintain by course staff via selecting/rearranging the submitted scribe notes. For a complete list of contribution, please see Appendix A.

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Chapter 1

Introduction to Cryptography

Lecture 1: Introduction

Here are some important links: 4 Jan. 10:30

- Course information and syllabus.
- Piazza.
- Slack channel.
- Or just see here!

If you have anything want to say, please don't hesitate to email us via eecs475-staff@umich.edu.

Lecture 2: The Cryptographic Methodology: Modeling encryption

We first see the general picture of cryptography, i.e., the cryptographic methodology: 9 Jan. 10:30

(a) Form a realistic *model* of the scenario, adjusting as necessary to allow for possibility of solution.

(b) *Precisely define* the desired functionality and security properties of a potential solution.

(c) Constructing and analyze a solution, *ideally* proving that it satisfies the desired properties.

1.1 Language of Cryptography

Let's first define some formal notions to help us *speak*:

Definition 1.1.1 (Plaintext). Given a space of message $\mathcal{M}$, $m \in \mathcal{M}$ is called a *plaintext*.

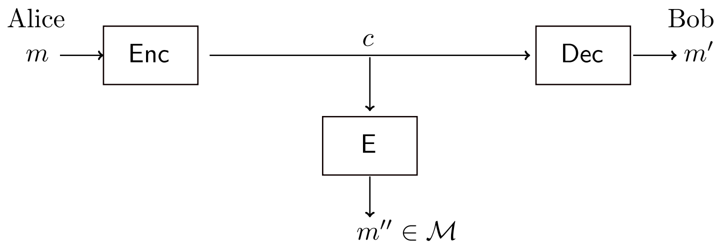
Definition 1.1.2 (Ciphertext). Given a space of encrypted message $\mathcal{C}$ (also called *ciphertext space*), $c \in \mathcal{C}$ is called a *ciphertext*.

Then, our model will be

(a) A sender uses an algorithm $\mathsf{Enc}(\cdot)$ which takes a plaintext $m \in \mathcal{M}$ to a ciphertext $c \in \mathcal{C}$.

(b) A receiver uses an algorithm $\mathsf{Dec}(\cdot)$ which takes some ciphertext $c \in \mathcal{C}$ to a plaintext $m' \in \mathcal{M}$.

(c) An eavesdropper is represented by an algorithm $\mathsf{E}(\cdot)$ that takes $c \in \mathcal{C}$ and output m''.



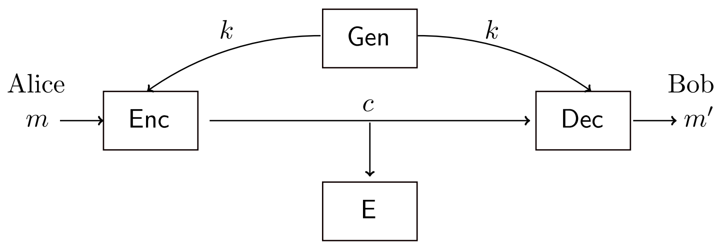
Remark. I will use hyperlinks extensively to help you follow the document, but for plaintext and ciphertext, this is the last time I explicitly reference it: it's used all over the place!

Often time, we will have some sorts of private key k in the key space $\mathcal{K}$ which is assumed to be unknown to the eavesdropper, and the key is generated by $\mathsf{Gen}(\cdot)$.[1] We summarize the above discussion to the following.

Definition 1.1.3 (Encryption scheme)**.** An *encryption scheme* is a tuple $\Pi = (\mathsf{Gen}(\cdot), \mathsf{Enc}(\cdot), \mathsf{Dec}(\cdot))$.

There are several common schemes, one of which is called symmetric key encryption.

Definition 1.1.4 (Symmetric key encryption)**.** *Symmetric key encryption* is the scheme that a secret key is available to the sender and the receiver in advance, but not the eavesdropper.



1.2 Correctness and Security

Ideally, the scheme we just introduced should have the functionality that we expect, specifically, the correctness and the security.

1.2.1 Correctness

Consider the following.

Definition 1.2.1 (Correctness)**.** An encryption scheme $\Pi = (\mathsf{Gen}(\cdot), \mathsf{Enc}(\cdot), \mathsf{Dec}(\cdot))$ satisfies *correctness* if for all $m \in \mathcal{M}$,

$$\mathsf{Dec}(\mathsf{Enc}(m)) = m.$$

Remark. In the case of symmetric key encryption, we incorporate with the key $k \in \mathcal{K}$ for correctness, i.e., we now require that for all $k \in \mathcal{K}$ and $m \in \mathcal{M}$,

$$\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = m.$$

1.2.2 Shannon Secrecy

Definition 1.2.1 is natural, but what about security? At minimum, $\mathsf{E}(\cdot)$ should not be able to recover m from c. Or more generally, we have the following.

[1] We often just sample a uniform random key k from $\mathcal{K}$ by invoking Gen, hence often time there is no explicit input to Gen.

Definition 1.2.2 (Kerckhoff's principle). The *Kerckhoff's principle* states that an encryption scheme should remain secure even if all its algorithm are known to the public or attacker.

Remark. Kerckhoff's principle might be awkward at first, but this is an essential reason of the success of the modern cryptography.

But this description is vague, and we want a precise, mathematical definition of security!

Lecture 3: Shannon Secrecy and Perfect Secrecy

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As previously seen. We have seen one crucial property of a desired scheme, i.e., the correctness.

We now want to discuss the security aspect. The idea is quite simple.

Intuition. Seeing the ciphertext should give the eavesdropper no information about the probability distribution of the message space.

This leads to the following.

Definition 1.2.3 (Shannon secrecy*). An encryption scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is *Shannon secret* if for any message distribution $\mathcal{D}$ over $\mathcal{M}$, and any fixed $\overline{m} \in \mathcal{M}$ and $\overline{c} \in \mathcal{C}$,

$$\Pr_{\substack{m \leftarrow \mathcal{D} \\ k \leftarrow \mathsf{Gen}}}(m = \overline{m} \mid \mathsf{Enc}_k(m) = \overline{c}) = \Pr_{\substack{m \leftarrow \mathcal{D} \\ k \leftarrow \mathsf{Gen}}}(m = \overline{m}).$$

Notation. For a variable with an overline, e.g., $\overline{m}$, it means that it's fixed.

Remark. To interpret Definition 1.2.3, we see that the left-hand side is the *posterior* probability after seeing the ciphertext $\overline{c}$, while the right-hand side is a *priori* distribution.

In fact, Definition 1.2.3 can be difficult to work with. Therefore, we will derive an equivalent but more convenient definition. Starting with Shannon secrecy*, by the definition of conditional probability we have that

$$\begin{aligned}\Pr_{m,k}(m = \overline{m} \mid \mathsf{Enc}_k(m) = \overline{c}) &= \frac{\Pr_{m,k}(m = \overline{m} \wedge \mathsf{Enc}_k(\overline{m}) = \overline{c})}{\Pr_{m,k}(\mathsf{Enc}_k(m) = \overline{c})} \\ &= \frac{\Pr_{m,k}(m = \overline{m} \wedge \mathsf{Enc}_k(\overline{m}) = \overline{c})}{\Pr_{m,k}(\mathsf{Enc}_k(\overline{m}) = \overline{c})} = \Pr_m(m = \overline{m}) \times \frac{\Pr_k(\mathsf{Enc}_k(\overline{m}) = \overline{c})}{\Pr_{m,k}(\mathsf{Enc}_k(m) = \overline{c})},\end{aligned}$$

where the last equality is from the independence of m and k.[2] We know the ratio of the fraction should equal 1 if we assume Definition 1.2.3, leading to an equivalent definition of Shannon secrecy*:

Definition 1.2.4 (Shannon secrecy). An encryption scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is *Shannon secret* if for any message distribution $\mathcal{D}$ over $\mathcal{M}$, and any fixed $\overline{m} \in \mathcal{M}$ and $\overline{c} \in \mathcal{C}$,

$$\Pr_{k \leftarrow \mathsf{Gen}}(\mathsf{Enc}_k(\overline{m}) = \overline{c}) = \Pr_{\substack{m \leftarrow \mathcal{D} \\ k \leftarrow \mathsf{Gen}}}(\mathsf{Enc}_k(m) = \overline{c}). \tag{1.1}$$

Note. This is trivial when $\Pr(m = \overline{m}) = 0$, so $\overline{m} \in \operatorname{supp}(\mathcal{D})$ should hold true.

1.2.3 Perfect Secrecy

If we take a particular distribution $\mathcal{D}$, such as the uniform distribution (where all outcomes are equally likely), then Shannon secrecy means that the probability that we get a particular outcome is the same for all possible messages $\overline{m}$.

[2] The key is naturally independent of the message.

Remark. The right-hand side of Equation 1.1 does not rely on $\overline{m}$ at all.

This observation leads to the following.

Definition 1.2.5 (Perfect secrecy). An encryption scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is *perfect secret* if for all $m_0, m_1 \in \mathcal{M}$ and for all $\overline{c} \in \mathcal{C}$,

$$\Pr_{k \leftarrow \mathsf{Gen}}(\mathsf{Enc}_k(m_0) = \overline{c}) = \Pr_{k \leftarrow \mathsf{Gen}}(\mathsf{Enc}_k(m_1) = \overline{c}).$$

Intuition. Basically, the probability distributions of $\mathsf{Enc}_k(m_0)$ and $\mathsf{Enc}_k(m_1)$ should be the same.

Lemma 1.2.1. Perfect secrecy is equivalent to Shannon secrecy.

Proof. The way we define perfect secrecy suggests that Shannon secrecy implies perfect secrecy. Conversely, we can show that perfect secrecy implies Shannon secrecy. Using the law of total probability,

$$\Pr_{m,k}(\mathsf{Enc}_k(m) = \overline{c}) = \sum_{m' \in \mathcal{M}} \Pr(m = m') \times \Pr(\mathsf{Enc}_k(m') = \overline{c}).$$

From the definition of perfect secrecy, we know that the probabilities for any two messages are the same, so we can replace m' with $\overline{m}$, i.e., we further have

$$\Pr_{m,k}(\mathsf{Enc}_k(m) = \overline{c}) = \underbrace{\sum_{m' \in \mathcal{M}} \Pr(m = m')}_{1} \times \Pr_k(\mathsf{Enc}_k(\overline{m}) = \overline{c}) = \Pr_{m,k}(\mathsf{Enc}_k(\overline{m}) = \overline{c})$$

which is the definition of Shannon secrecy. ■

1.3 One-Time Pad

It's all great, if we can actually design an scheme which achieves Shannon secrecy.

Problem. Can we achieve Shannon secrecy?

Answer. Yes, we can! The one-time pad, also known as the Vernam Cipher, achieves Shannon secrecy. ⊛

Definition 1.3.1 (One-time pad). The *one-time pad* scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ defined over $\mathcal{M} = \mathcal{C} = \mathcal{K} = \{0,1\}^\ell$ is given by

- $\mathsf{Gen}(\cdot)$: chooses a key $k \in \{0,1\}^\ell$ uniformly at random.
- $\mathsf{Enc}_k(m)$: outputs $c = m \oplus k \in \{0,1\}^\ell$.[a]
- $\mathsf{Dec}_k(c)$: outputs $\overline{m} = c \oplus k \in \{0,1\}^\ell$.

[a] $\oplus$ is the bit-wise XOR.

Lemma 1.3.1. One-time pad is correct.

Proof. Let $k, m \in \{0,1\}^\ell$ be arbitrary. Then,

$$\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = (m \oplus k) \oplus k = m \oplus \underbrace{(k \oplus k)}_{0^\ell} = m.$$

■

Lecture 4: One-time Pad, Limitations of Perfect Secrecy

We now show that one-time pad is secure. 18 Jan. 10:30

Theorem 1.3.1. One-time pad is perfectly secret.

Proof. Let $\overline{m} \in \{0,1\}^\ell$ be any fixed plaintext, k be a randomly generated key by $\mathsf{Gen}(\cdot)$. For any $\overline{c} \in \{0,1\}^\ell$

$$\Pr_k(\mathsf{Enc}_k(\overline{m}) = \overline{c}) = \Pr_k(\overline{m} \oplus k = \overline{c}) = \Pr_k(k = \overline{m} \oplus \overline{c}) = 2^{-\ell},$$

where second equality comes from the identity $\overline{m} \oplus \overline{m} \oplus k = \overline{m} \oplus \overline{c}$ and $a \oplus a = 0$, and the last equality follows from the fact that k is random and the $\overline{m} \oplus \overline{c}$ is fixed. In all, this means that for any fixed m_0, m_1, the probability is always equal to $2^{-\ell}$, hence it's perfectly secret. ∎

1.3.1 Problems with the One-Time Pad

The one-time pad seems to be the perfect code as it satisfies perfect secrecy. However, one-time pad is rarely used in real life, and neither is the perfect secrecy criterion. In fact, there are several problems that come with the one-time pad.

(a) The length of the key must be equal to the message length. This means that if we have to encode a long message, we will need to transmit a key that is as long as this message, which is a waste of space.

(b) The key has to be truly randomly generated, otherwise the proof wouldn't work.

(c) The key couldn't be reused, which is why the one-time pad is *one time*. For instance, if we have key k, message m_1 and m_2, let

$$c_1 = \mathsf{Enc}_k(m_1) = m_1 \oplus k, \qquad c_2 = \mathsf{Enc}_k(m_2) = m_2 \oplus k,$$

then if we calculate $c_1 \oplus c_2$, the k's cancel, and we'll figure out $m_1 \oplus m_2$, which is bad. The figure below illustrates this scenario:

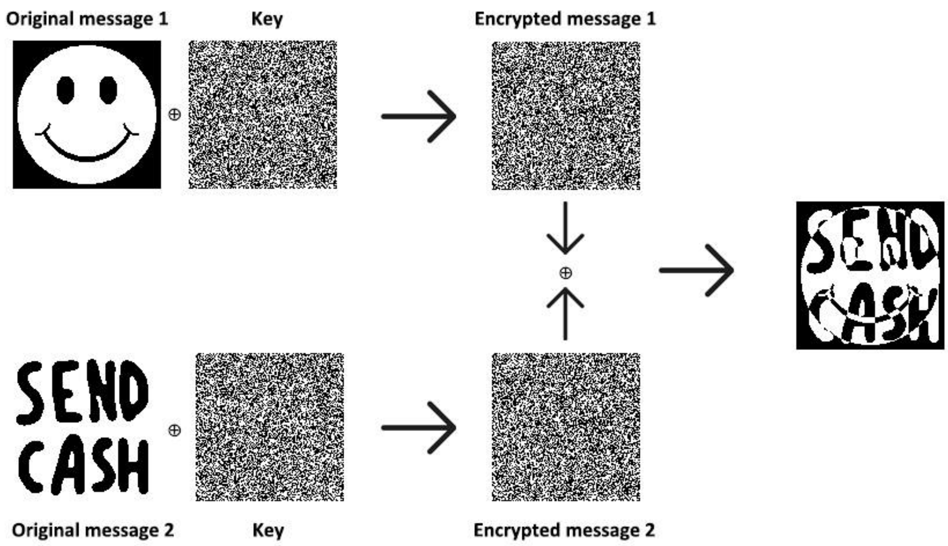


Figure 1.1: Recover $m_1 \oplus m_2$.

Can we do better than one-time pad, i.e., satisfy the perfectly secret requirement without the above limitations? For the first bullet point, sadly, the answer is no.

Theorem 1.3.2. For any perfect encoding scheme, we must have $|\mathcal{K}| \geq |\mathcal{M}|$, i.e., to achieve perfect secrecy, we have to have at least as many potential keys as messages.

Proof. For any fixed message $m_0 \in M$ and any fixed key k_0, consider the set of all possible decryption outcomes of m_0's encoding under k_0, i.e.,

$$D := \{\mathsf{Dec}_k(\mathsf{Enc}_{k_0}(m_0)) \colon k \in \mathcal{K}\}.$$

Since we could have at most k outcomes, $|D| \leq |\mathcal{K}|$. Suppose $|\mathcal{K}| < |\mathcal{M}|$, then $|D| \leq |\mathcal{K}| < |\mathcal{M}|$, which implies that $\exists m_1 \in \mathcal{M} \backslash D$. But then $\Pr_k(\mathsf{Enc}_k(m_0) = \overline{c}) > 0$, which is not equal to $\Pr_k(\mathsf{Enc}_k(m_1) = \overline{c}) = 0$ by definition of m_1 and correctness. ■

1.4 Computational Security

Rather than requiring the encryption scheme to be perfectly secret, we wish to look for some kind of encryption schemes that are secure enough – more specifically, secure against eavesdroppers that are efficient.

Intuition. Recall problem (a), there we said that the key is too long. The upside is that it gives us more possible keys when the eavesdropper is making a random guess – but since computational power is limited, we wouldn't need that big a key space for it to be infeasible for an eavesdropper.

Let's first have a look at the sizes of key spaces and their corresponding computing time to give us an idea about which attacks are *feasible*:

- 2^{30} (1 billion): totally feasible;
- 2^{40}: a few minutes on a PC:
- 2^{60}: a few minutes on a supercomputer;
- 2^{80}: 1 to 2 years on a supercomputer;
- 2^{100}: 1 million years on a supercomputer;
- 2^{128}: 1 trillion years after 200 more years of Moore's law. This is really infeasible;
- 2^{256}: more than the number of atoms ($\approx 2^{240}$) in the universe.

Lecture 5: Computational Security and PRGs

1.4.1 Concrete Security

23 Jan. 10:30

As previously seen. The perfect secrecy is stronger than what we actually need. What is sufficient security in practice, and how to formally define it?

One approach is to use *concrete security*, which deals with the exact setup of the notion of feasibility (number of operations, architecture, etc.).

Intuition. An adversary can have a *tiny* chance of security violation, where this tiny chance is much smaller than the chance that something goes wrong. And since the adversary can boost chance of success by repeating or doing more work, so we really measure the runtime over the probability of success as the *cost* of the attack, i.e.,

$$\text{cost} = \frac{\text{run time}}{\Pr(\text{success})}.$$

In other words, we wish to maximize the cost of an adversary's attack.

> **Remark.** These ideas of thinking about feasibility and costs of an attacker diverges from Shannon's in that he did not concern himself with understanding the computational power of attackers, rather his theory only looked at mathematical foundation.

1.4.2 Asymptotic Security

However, defining the equation above requires us to quantify the duration of an attack algorithm. Since numbers can get messy and involve knowledge of OS/computer architecture, we abstract it away in this course and focus on asymptotic behavior instead.

> **As previously seen.** An algorithm efficient if it runs in polynomial time rather than exponential.

We introduce a *security parameter* n which quantifies our level of security. This parameter is selected by the user for different use cases.

> **Intuition.** Logically, it wouldn't make sense to use the same security parameter for TV streaming services as nuclear launch codes.

> **Example.** Selecting 128 vs 256 bits for the key length of OTP.

However, with higher security we also increase the computation time of the legitimate communicating parties. Thus, the problem becomes finding a security level which makes legitimate communication reasonable while making adversary attacks difficult.

> **Example.** Legitimate entity runs fast, so rapid communication is possible, efficient in terms of n
>
> - $O(n)$, $O(n \log n)$, $O(n^2)$, etc.
>
> Attacker can afford to take longer, but still feasible:
>
> - $O(n^c)$ where c is some large constant.

In essence, we want the scenario where the attack spends a feasible amount of time but can only gain a *tiny advantage*.

Chapter 2

Message Security

In this chapter, we're going to focus on schemes that consider asymptotic security, where we naturally have some kinds of *adversaries* who is trying to attack the system.

2.1 Pseudorandom Generators

Pseudorandom generators, noted as *the most important notion in cryptography*, are used in virtually every cryptosystem either explicitly or under the hood. The motivation is that, recalling that a major headache of OTP requires the key being the same length with messages, and we don't have access to this kind of random source, so we want to generate a "random" key with small true randomness.

Moreover, the interesting thing is that, the "advantage" we care about relates directly to the quality of the randomness we use!

2.1.1 Negligible Functions

We keep mentioning the word *tiny advantage*, and we now give a formal way to model it by using the notion of negligible functions.

Definition 2.1.1 (Negligible). A function $\epsilon(n)$ is *negligible*, written as $\epsilon(n) = \operatorname{negl}(n)$, if

$$\epsilon(n) = o(n^{-c})$$

for all constant $c > 0$.

Equivalently,

$$\lim_{n\to\infty} n^c \cdot \epsilon(n) = 0.$$

As previously seen (Little-o). Recall that $f(n) = o(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Example. $\epsilon(n) = \frac{1}{2^{n/4}}$ is negligible.

Proof. Since $\lim_{n\to\infty} n^c \times \epsilon(n) = 0$ for any positive constant c by L'Hôpital's rule. ⊛

Example. $\epsilon(n) = \frac{1}{n^5}$ is not negligible.

Proof. Since $\lim_{n\to\infty} n^c \times \epsilon(n) = \infty$ for $c > 5$. ⊛

We then use the notion of negligible to model *tiny advantage* and construct a generalized template for computational security.

Definition 2.1.2 (Computational security). Every randomized polynomial-time attacker has only negligible "advantage" when attacking our cryptographic system.

Now, there's one fundamental question to ask: how should we define "advantage" formally? This leads to the next big topic, pseudorandom generators.

Lecture 6: PRGs, Distinguishing Games, and Stream Ciphers

2.1.2 Pseudorandom Generators

25 Jan. 10:30

We're now ready to define randomness and pseudorandomness, and dive deep into the difference between these two concepts. We must also quantify pseudorandomness:

(a) How random is pseudorandom?

(b) Is there a way to predict a pseudorandom outcome better than a blind guess?

(c) Is there a feasible algorithm that can improve our prediction of the outcome?

To answer these questions, we considered methods of "differentiating" pseudorandom and random, as well as the efficiency of this operation. A pseudorandom generator is an efficient, deterministic algorithm G that takes in a seed $s \in \{0,1\}^n$ for some seed length n and generates a pseudorandom output $G(s) \in \{0,1\}^{\ell(n)}$, where $\ell(n) > n$ is some extension of the initial seed.

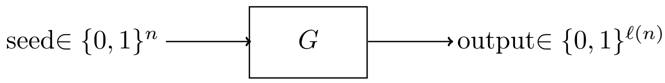


Figure 2.1: G generates a pseudorandom bit-string of length $\ell(n)$ given a truly random seed of length n.

Let's first give the definition of PRGs.

Definition 2.1.3 (Pseudorandom generator). A *pseudorandom generator (PRG) G* with expansion $\ell(n)$ is a deterministic, polynomial time algorithm satisfying pseudorandom property.

Notation (Expansion). The *expansion* of a PRG G is the length of its output, i.e., $|G(s)| = \ell(n) > n$ for all $s \in \{0,1\}^n$.

We see that we haven't defined the notion of pseudorandom, since to define it we need some additional machinery. Before doing so, we see the following.

Intuition. Informally, pseudorandom means that for uniform random $s \in \{0,1\}^n$, $G(s)$ looks random (like a uniform $\ell(n)$-bit string) to all feasible attackers.

From Definition 2.1.3, new questions arise: What does it mean to look random? What defines a feasible attacker? We can answer these questions by defining a distinguisher that attempts to distinguish between a pseudorandom and random input, and the resulting "advantage" from this distinguisher's output.

2.1.3 Distinguishers

A distinguisher takes in $y \in \{0,1\}^{\ell(n)}$ and outputs a decision bit that determines whether y was randomly or pseudorandomly generated.

Definition 2.1.4 (Distinguisher). A *distinguisher D* is a polynomial time (potentially probabilistic) algorithm such that

- takes in inputs 1^n and $y = G(s)$ for some uniformly random $s \leftarrow \{0,1\}^n$ ("real world"), or some uniformly random $y \leftarrow \{0,1\}^{\ell(n)}$ ("ideal world");

- outputs a decision bit (0/1) for whether y was pseudorandom or random.

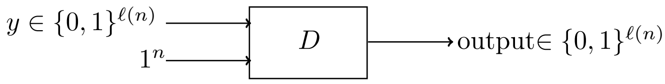


Figure 2.2: D distinguishes a bit-string of length $\ell(n)$ given the length n (by 1^n).

Note. The 1^n input is often also implicit, since it tells D the length of the seed (which is publicly known). n is also known as the security parameter because it determines the length of the random seed, and therefore, also the randomness of the generated result.

A distinguisher can potentially be probabilistic, meaning it can make random choices. In this way, the output of the distinguisher has a distribution based on the randomness of the inputs, as well as the internal randomness of the algorithm. We want to know the probability that the distinguisher outputs 1 or accepts when given random and pseudorandom inputs. Considering these probabilities, we can then define advantage, a way to quantify pseudorandomness.

Definition 2.1.5 (Advantage)**.** Given a distinguisher D, its *advantage* $\mathrm{Adv}_G(D)$ in distinguishing the "real world" ($y \leftarrow G(s)$ for uniformly random s) and the "ideal world" ($y \leftarrow \{0,1\}^{\ell(n)}$ is uniformly random) is given by

$$\mathrm{Adv}_G(D) := \left| \Pr_{s \leftarrow \{0,1\}^n}(D(1^n, G(s)) = 1) - \Pr_{y \leftarrow \{0,1\}^{\ell(n)}}(D(1^n, y) = 1) \right|.$$

Intuition. The first probability is over the "real world" and the second probability is over the "ideal world," and both probabilities are also over the randomness of the distinguisher D, since D can be probabilistic algorithm.

For a pseudorandom generator, we want our advantage to be small and as close to 0 as possible, which can be characterize by the notion of negligible.

Definition 2.1.6 (Fool)**.** If the advantage of a distinguisher D against a PRG G is negligible, then D is *fooled* by G, or G *fools* D.

When we have a generator G, we want it to fool all distinguishers D. In other words, to be secure, the generator G must fool all distinguishers D.

Definition 2.1.7 (Pseudorandom)**.** A PRG G is *pseudorandom* (or *secure*) if for all (randomized) polynomial time[a] distinguishers D, we have $\mathrm{Adv}_G(D) = \mathrm{negl}(n)$.

[a] Or probabilistic polynomial time.

A natural question from Definition 2.1.7 arises.

Problem. Why must D be efficient?

Answer. Otherwise, we can construct an attack that essentially checks if y is the same as $G(s)$ for every possible seed s. Note that according to the Kerckhoff principle (open source assumption), the length of the seed n and the generator algorithm G are both public knowledge. ⊛

Consider any generator G that maps $s \in \{0,1\}^n$ to some subset of $\{0,1\}^{\ell(n)}$:

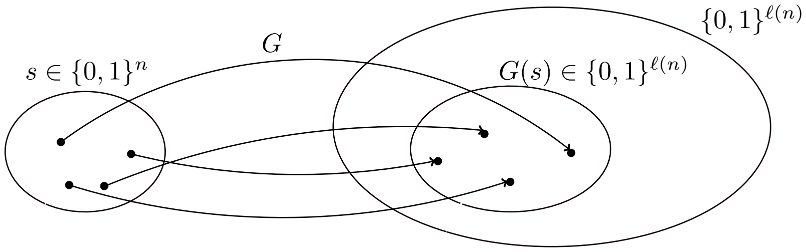


Figure 2.3: Venn diagram of the domain and the image of G.

Then, let's define a distinguisher D that accepts $y \in \{0,1\}^{\ell(n)}$ if and only if there exists $s \in \{0,1\}^n$ such that $G(s) = y$, i.e., the image of G, $\operatorname{Im} G$.

We know that $\ell(n) > n$, so $\ell(n) \geq (n+1)$ and therefore $2^{\ell(n)} \geq 2^{n+1} > 2^n$. So, the size of $\{0,1\}^{\ell(n)}$ is at least two times the size of what's mapped to by $s \in \{0,1\}^n$. From here, we see that

$$\Pr_{s \leftarrow \{0,1\}^n}(D(1^n, G(s)) = 1) = 1$$

since D accepts if G maps some s to $G(s)$, and also

$$\Pr_{y \leftarrow \{0,1\}^{\ell(n)}}(D(1^n, y) = 1) = \frac{|\operatorname{Im} G|}{\left|\{0,1\}^{\ell(n)}\right|} = \frac{2^n}{2^{\ell(n)}} \leq \frac{2^n}{2^{n+1}} = \frac{1}{2}.$$

So, we can determine a lower-bound for the advantage of D, which is

$$\mathrm{Adv}_G(D) = \left| \Pr_{s \leftarrow \{0,1\}^n}(D(1^n, G(s)) = 1) - \Pr_{y \leftarrow \{0,1\}^{\ell(n)}}(D(1^n, y) = 1) \right| \geq \left| 1 - \frac{1}{2} \right| = \frac{1}{2},$$

which is non-negligible. However, the problem here is that D is not efficient. For this procedure, D must check every seed $s \in \{0,1\}^n$, which includes 2^n possible bit strings. This wouldn't be efficient since it's exponential with respect to the bit string length. So, for practicality, we want to consider feasible algorithms since this would be a trivial, infeasible algorithm that produces a non-negligible advantage for any generator G.

2.1.4 Existence of PRGs

Continuing the discussion of complexity, another question is whether PRGs exist. Although it seems trivial, we actually don't know whether PRGs exist! This boils down to whether P equals NP.

Proposition 2.1.1. If PRGs exist, then $\mathsf{P} \neq \mathsf{NP}$.

Proof. Firstly, observe the following.

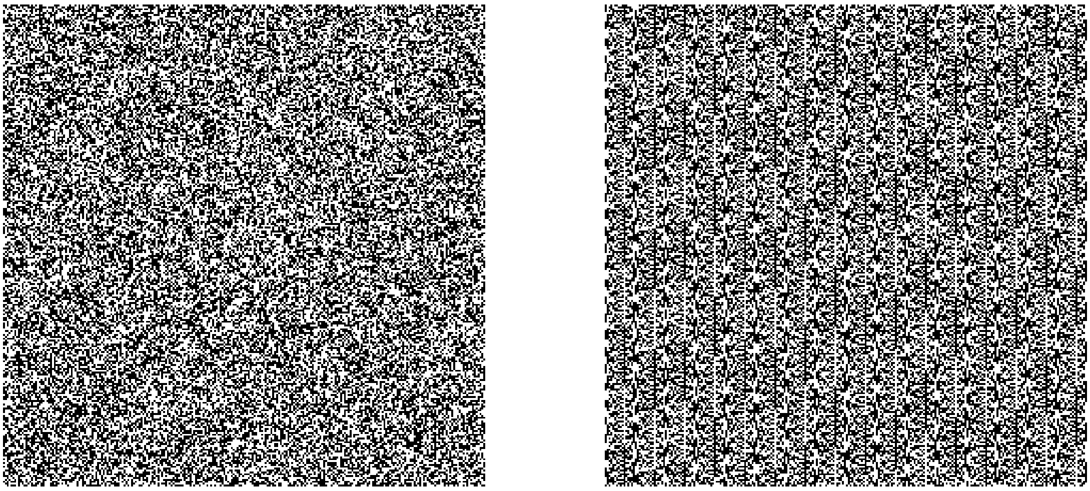
> **Claim.** PRGs are efficiently verifiable, i.e., in NP.
>
> **Proof.** Since G is open source, we can determine whether $y \in \{0,1\}^{\ell(n)}$ was generated by a seed $s \in \{0,1\}^n$ (the certificate) by simply running $G(s)$, hence PRGs must be in NP. ⊛

Therefore, if PRGs exist, i.e., there is no efficient algorithm to find s given y, so $\mathsf{P} \neq \mathsf{NP}$. ■

Solving $\mathsf{P} = \mathsf{NP}$ or not is out of our reach currently, and this problem is hunting us for centuries! However, even though we don't know whether PRGs really exist or not, there are heuristics for pseudorandom generators. There are high quality implementations of pseudorandom generators.

Example. `/dev/random` in Unix, `CryptGenRandom()` in Windows, and `SecureRandom()` in Java are all high quality implementations of PRGs.

There is a distinction between the above implementation and one like `rand()`, since `rand()` is easy to break even though it is sufficiently random for algorithms. The repetition in `rand()` is illustrated below.



`RANDOM.ORG` PHP `rand()` on Microsoft Windows

Figure 2.4: Comparison between the patterns of generated pseudo-randomness.[1]

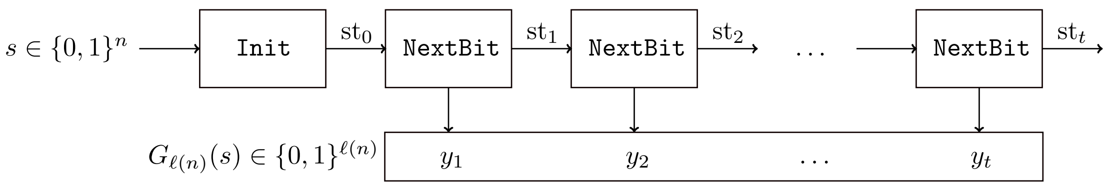
2.1.5 Stream Cipher

Stream ciphers are equivalent to PRGs, and can essentially be seen as a PRG on demand. Suppose we have a PRG that takes n bits and produces $\ell(n)$ bits, what if we don't know what $\ell(n)$ is? Stream cipher can be abstracted as a button you push, where each time you push the button, you get one more pseudorandom bit. You can then continue to push the button until you have the length you want.

Definition 2.1.8 (Stream cipher). A *stream cipher* is PRG-like procedure that generates pseudorandom bits with two functions, `Init` and `NextBit`.

- `Init`(s): takes seed s and outputs some state data st_0.
- `NextBit`(st_i): takes in state data and outputs the next state st_{i+1} and a bit y_{i+1}.

Compiling all bits will be a pseudorandom bit-string of length $\ell(n)$ (essentially $G_{\ell(n)}(s)$).



Lecture 7: Stream Ciphers and Eavesdropping Security

Intuition. The purpose of having a stream cipher is to have a PRG-like procedure that generates pseudorandom bits as needed, or a pseudorandom generator *on demand*. 30 Jan. 10:30

If implemented in `C++`, the st_i would be set as a global variable to keep track of what the seed has evolved to and additionally use this variable in our changing functions. Note this process is stateful.[2]. The stream cipher is pseudorandom if for every $\ell(n) = \mathsf{poly}(n)$, the function G_ℓ defined above is pseudorandom.

[1]Source: random.org
[2]One that uses the last updated state to produce a new state, i.e., one that remembers what was done last

> **Note.** To get a stream cipher from a PRG, take PRG and compose with itself for as many times as needed.

2.2 Eavesdropping Security

In this section, we aim to address the key length problem in perfect secrecy, as depicts by Theorem 1.3.2.

> **As previously seen.** Perfect secrecy dictates that
>
> (a) Given a ciphertext, you cannot determine the plaintext.
>
> (b) c_0 and c_1 will have the exact same statistical distribution.
>
> (c) If a shorter key than the message length is used, perfect secrecy is impossible.

Now, we try to relax to only require that given c_0 or c_1, it's hard to distinguish between the two cases.

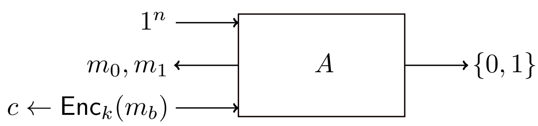
2.2.1 Eavesdropping Game

To formulate the above idea, we design the so-called EAV game, which requires that any adversary A is unable to distinguish between $\mathsf{Enc}_k(m_0)$ and $\mathsf{Enc}_k(m_1)$ given two plaintexts m_0 and m_1.

Definition 2.2.1 (Eavesdropping game)**.** The *eavesdropping game* for an adversary A against an encryption scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ proceeds as follows:

1. A is given the security parameter;
2. A outputs two messages, m_0, m_1 with $|m_0| = |m_1|$;
3. the game generates $c \leftarrow \mathsf{Enc}_k(m_b)$ for $b \in \{0, 1\}$[a] with $k \leftarrow \mathsf{Gen}$;
4. A is given c;
5. A outputs a decision bit.

[a] Indicating which "world" A is in.



2.2.2 Eavesdropping Secrecy

Consider the new notion of advantage against the EAV game.

Definition 2.2.2 (Advantage)**.** Given an adversary A in an EAV game, the *advantage* $\mathrm{Adv}_{\Pi}^{\mathrm{EAV}}(A)$ in distinguishing "world 0" and the "world 1" is given by

$$\mathrm{Adv}_{\Pi}^{\mathrm{EAV}}(A) := \left|\Pr(A \text{ in "world 1" outputs } 1) - \Pr(A \text{ in "world 0" outputs } 1)\right|.$$

Naturally, we have the following.

Definition 2.2.3 (Eavesdropping secrecy)**.** An encryption scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is *eavesdropping secure* if for every probabilistic polynomial time adversary A, the advantage is negligible.

> **Intuition.** Probabilities of adversary accepting on both $c_0 = \mathsf{Enc}_k(m_0)$ and $c_1 = \mathsf{Enc}_k(m_1)$ should

be similar or the same (the adversary should not know what is going on).[a]

[a] We have seen the similar definition in the assignment: A is trying to get some information about m_b from c_b!

Remark. We assume the length of both messages are the same, i.e., $|m_0| = |m_1|$ because otherwise A can distinguish messages by length.

Note. If we change $\mathrm{negl}(n)$ to 0 and remove the polynomial time requirement from Definition 2.2.3, then we get back the perfect secrecy.

2.2.3 Eavesdropping Secure Schemes

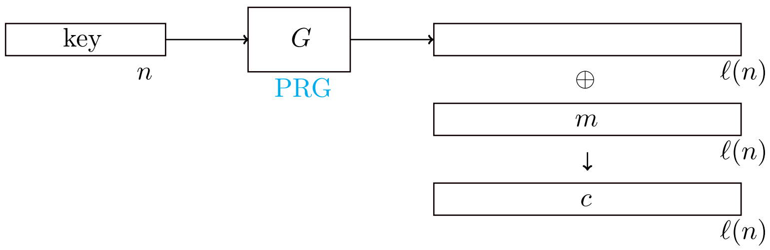
We are now interested in constructing an EAV-secure scheme. The idea is simple, we start from one-time pad:

(a) use pseudorandom generators to create a short key;

(b) stretch the short key to be an effective key and run one-time pad.

With $|m| = \ell(n)$, and a PRG G, we define our scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ by

- $\mathsf{Gen}(1^n)$: outputs a uniformly random $k \leftarrow \{0,1\}^n$;
- $\mathsf{Enc}_k(m)$: given $m \in \{0,1\}^{\ell(n)}$, outputs $c = m \oplus G(k)$;
- $\mathsf{Dec}_k(c)$: given $c \in \{0,1\}^{\ell(n)}$, outputs $c \oplus G(k)$.



Claim. Π is correct.

Proof. On a broad scale, this is just one-time pad ⊛

As for EAV secrecy, consider proving the contrapositive, i.e., if Π is not EAV secure, an adversary can win the eavesdropping game. By using this adversary to break the security of the PRG since the security of Π relies on the security of the PRG.

Intuition. We recognize that this is just a proof by reduction! Assume some adversary can win this game. Incorporate this adversary into a game that can break the PRG.

Lecture 8: EAV Construction Analysis and CPA Security

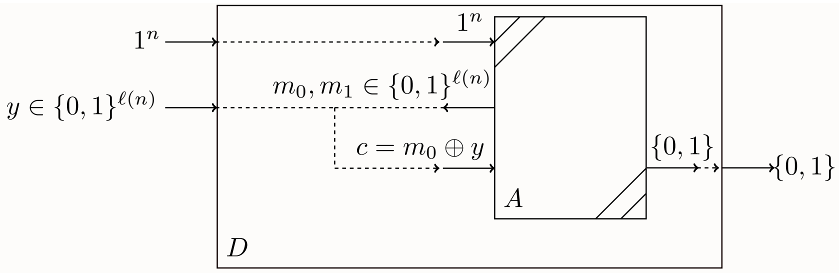
Theorem 2.2.1. If G is a (secure) PRG, then Π is EAV secure. 1 Feb. 10:30

Proof. Let A be any probabilistic polynomial time EAV-attacker against Π. We use A to build a probabilistic polynomial time distinguisher $D(1^n, y)$ for $y \in \{0,1\}^{\ell(n)}$ against G by

(a) run $A(1^n)$ and receive messages $m_0, m_1, \in \{0,1\}^{\ell(n)}$;

(b) give $c = m_0 \oplus y$ to A;[a]

(c) output A's verdict.



We see that D's advantage is

$$\mathrm{Adv}_G^{\mathrm{PRG}}(D) = \left| \Pr_{k \leftarrow \{0,1\}^n}(D(1^n, G(k)) = 1) - \Pr_{y \leftarrow \{0,1\}^{\ell(n)}}(D(1^n, y) = 1) \right|,$$

which is just

$$\mathrm{Adv}_G^{\mathrm{PRG}}(D) = \left| \Pr_k(A \text{ in "world 0" accepts}) - \Pr(A \text{ in random } c \text{ accepts}) \right|.$$

Recall that we want to bound the advantage of A in the EAV game, which is

$$\mathrm{Adv}_\Pi^{\mathrm{EAV}}(A) = \left| \Pr(A \text{ in "world 0" accepts}) - \Pr(A \text{ in "world 1" accepts}) \right|.$$

By adding 0, we have

$$\begin{aligned} \mathrm{Adv}_\Pi^{\mathrm{EAV}}(A) = |&\Pr(A \text{ in "world 0" accepts}) - \Pr(A \text{ in "world 1" accepts}) \\ &+ \Pr(A \text{ in random } c \text{ accepts}) - \Pr(A \text{ in random } c \text{ accepts})|. \end{aligned}$$

By using the hybrid argument, i.e., the triangle inequality,[b] we have

$$\begin{aligned} \mathrm{Adv}_\Pi^{\mathrm{EAV}}(A) &= |\Pr(A \text{ in "world 0" accepts}) - \Pr(A \text{ in "world 1" accepts}) \\ &\qquad + \Pr(A \text{ in random } c \text{ accepts}) - \Pr(A \text{ in random } c \text{ accepts})| \\ &\leq |\Pr(A \text{ in "world 0" accepts}) - \Pr(A \text{ in random } c \text{ accepts})| \\ &\qquad + |\Pr(A \text{ in "world 1" accepts}) - \Pr(A \text{ in random } c \text{ accepts})|. \end{aligned}$$

We can replace step two of D to use $c = m_1 \oplus y$, which we name it D'. And now we can substitute $\mathrm{Adv}_G^{\mathrm{PRG}}(D)$ and $\mathrm{Adv}_G^{\mathrm{PRG}}(D')$ into the above inequality and get

$$\mathrm{Adv}_\Pi^{\mathrm{EAV}}(A) \leq \mathrm{Adv}_G^{\mathrm{PRG}}(D) + \mathrm{Adv}_G^{\mathrm{PRG}}(D') = \mathrm{negl}(n)$$

since we know that G is PRG, hence both $\mathrm{Adv}_G^{\mathrm{PRG}}(D)$ and $\mathrm{Adv}_G^{\mathrm{PRG}}(D')$ are negligible. ■

[a] We will deal with m_1 later.
[b] $|a+b| \leq |a| + |b|$, or $|a-b| = |a-b+c-c| \leq |a-c| + |c-b|$.

2.3 Chosen Plaintext Attack Security

Although the constructed EAV secure scheme sounds promising, it still suffers from the key-reuse problem: suppose we have $c_1 = m_1 \oplus G(k)$ and $c_2 = m_2 \oplus G(k)$, then when an adversary gets c_1 and c_2,

$$c_1 \oplus c_2 = m_1 \oplus m_2,$$

then we fall into the same situation as depicted in Figure 1.1.

Note (Potential solution). We might use a stream cipher to generate new keys, but this is inconvenient for the decrypting user if the ciphertext are not ordered.

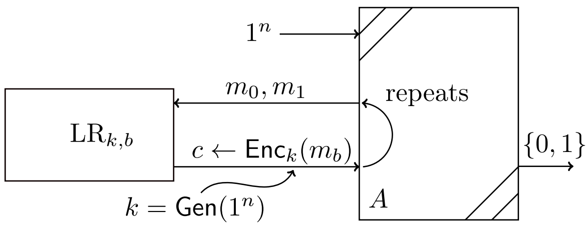
2.3.1 Chosen Plaintext Attack Game

To enhance the notion of EAV security, we can address the key-reuse issue, i.e., consider a repeated EAV game. In order to simplify things a bit, we abstract the "game mechanism" in EAV games, i.e., the generation of c, into a call-able function, called left-right oracle.

Definition 2.3.1 (Left-right oracle). The *left-right oracle* $\mathrm{LR}_{k,b}(\cdot,\cdot)$ with parameters k, b on input (m_0, m_1) is defined as

$$\mathrm{LR}_{k,b}(m_0, m_1) = \begin{cases} \mathsf{Enc}_k(m_b), & \text{if } |m_0| = |m_1|; \\ \varnothing, & \text{if } |m_0| \neq |m_1|. \end{cases}$$

Definition 2.3.2 (Chosen plaintext attack game). The *chosen plaintext attack game* is a model of a repeated EAV game for a probabilistic polynomial time adversary A against $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$. A can only make $\mathsf{poly}(n)$ number of queries (m_0, m_1) to a Left-right oracle $\mathrm{LR}_{k,b}(\cdot,\cdot)$ to get a ciphertext for each query, and output a decision bit in the end.



Intuition. The CPA game is a model for EAV security that supports key-reuse. That way, if a key-reuse problem exists, the adversary will be able to use it to their advantage.

Remark. The decision bit b and the encryption key k are only chosen *once* at the start. For example, if the key is 7 and b is 1, then the LR oracle will return $\mathsf{Enc}_7(m_1)$ for every query.

Lecture 9: CPA Secure Scheme and the Construction

To decipher Definition 2.3.2, a CPA game is conducted as follows. 6 Feb. 10:30

1. Adversary A takes the security parameter 1^n and a key k will be generated according to the protocol.
2. Adversary A will then make adaptive queries by inputting two messages m_0, m_1 into the left-right oracle $\mathrm{LR}_{k,b}$. The output of the left-right oracle will be sent to the adversary A. These processes of the oracle will be repeated polynomial (in n) many times.
3. Finally, adversary A will output the final decision.

Example. In WWII, American forces managed to decipher part of Japanese communication codes and found out their next target was a location called "AF." American forces suspected that "AF" was Midway, so they sent a false message saying "Midway is running out of water." Another piece of codes was deciphered afterwards, saying "AF is running out of water" and it was clear that "AF" was Midway. In this example, the adversary is American forces and the false message is the chosen plaintext.

2.3.2 Chosen Plaintext Attack Secrecy

Consider the new notion of advantage against the CPA game.

Definition 2.3.3 (Advantage). Given an adversary A in a CPA game, the *advantage* $\mathrm{Adv}_{\Pi}^{\mathrm{CPA}}(A)$ in distinguishing "world 0" and "world 1" is given by

$$\mathrm{Adv}_{\Pi}^{\mathrm{CPA}}(A) := \left|\Pr\left(A^{\mathrm{LR}_{k,0}(\cdot,\cdot)} \text{ accepts}\right) - \Pr\left(A^{\mathrm{LR}_{k,1}(\cdot,\cdot)} \text{ accepts}\right)\right|.$$

Notation. $A^{\mathrm{LR}_{k,b}(\cdot,\cdot)}$ means the adversary A can interact with the oracle.

Then, we have the following.

Definition 2.3.4 (Chosen Plaintext attack secrecy). An encryption scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is *chosen plaintext attack secure* if for every probabilistic polynomial time adversary A, the advantage is negligible.

Remark. Definition 2.3.2 is equivalent as saying that A have access to Enc and a single call to LR.

2.3.3 Chosen Plaintext Attack Secure Schemes

Now we ask that whether a CPA secure scheme exists. Unfortunately, we see the following.

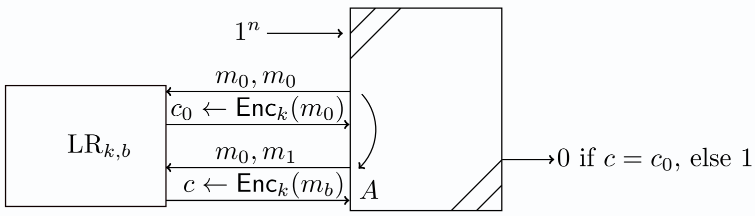
Claim. A CPA secure encryption scheme does not exist.

Proof. Let $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ be CPA secure and A be an efficient attacker and consider the following.

(a) Firstly, take any m_0 and query $\mathrm{LR}_{k,b}(m_0, m_0)$ to receive $c_0 = \mathsf{Enc}_k(m_0)$.

(b) Then take $\mathrm{LR}_{k,b}(m_0, m_1)$ where $m_1 \neq m_0$ but $|m_0| = |m_1|$ and receive $c = \mathrm{LR}_{k,1}(m_0, m_1)$.

(c) If $c = c_0$, outputs 0, else output 1.



If A is in the "left world" ($b = 0$), then it will always output 0; if A is in the "right world" ($b = 1$), A will always output 1 by correctness. Therefore,

$$\mathrm{Adv}_{\Pi}^{\mathrm{CPA}}(A) = \left|\Pr\left(A^{\mathrm{LR}_{k,0}(\cdot,\cdot)} \text{ accepts}\right) - \Pr\left(A^{\mathrm{LR}_{k,1}(\cdot,\cdot)} \text{ accepts}\right)\right| = 1 \neq \mathrm{negl}(n).$$

⊛

Problem. There's a bug in the above explanation!

Answer. The attacker A assumes that each time the ciphertext for m_0 is the same, which is not the case in the reality since it is dangerous to do so (and there's no reason to do so since Enc can be probabilistic). ⊛

Intuition. From the encryption side, the information of whether the same message is encrypted should be protected, so the cipthertext space should be much larger than the message space and cannot be solved efficiently by brute-force.

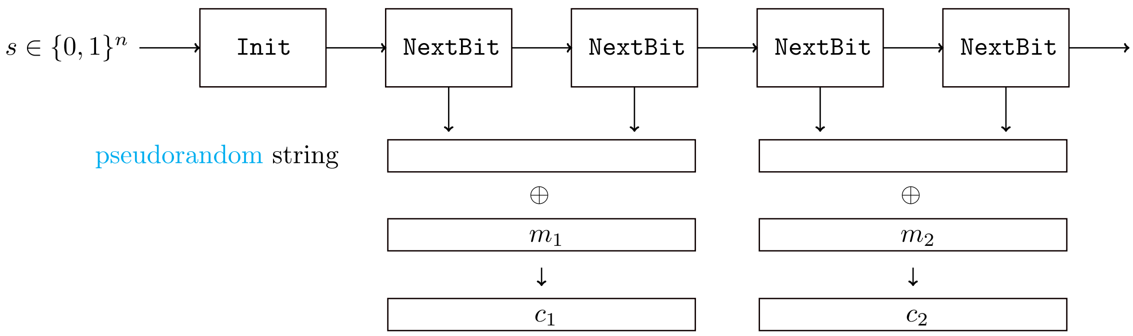
Hence, the correct way to put this is the following.

Theorem 2.3.1. There does not exist a CPA secure encryption scheme with deterministic and stateless[a] $\mathsf{Enc}_k(\cdot)$.

[a]This means the ciphertext for the same message in different times should be the same.

This suggests that we still have hope! Theorem 2.3.1 suggests that we can try to construct a CPA secure scheme by using stream cipher.

- Firstly, initialize the stream cipher with the key.[3]
- Then encrypt the message by XOR-ing it with part of the string, and we get ciphertext c_1.
- Each time we encrypt the message, the string used in last time will be discarded and next bits of string will be used.
- Note that the sequence of the message is also sent in the ciphertexts.



Remark. This encryption scheme will survive in the chosen plaintext attack.

Proof. Because each time a same message m_0 will be encrypted into different ciphertexts. The sequence does not affect the security because even if the attacker knew it, it cannot know where to start deciphering the text. ⊛

Note. However, if the number of messages is very large, the decryption will take a great deal of time since we have to look through the history and locate the piece of message we want.

Lecture 10: Pseudorandom Functions

As previously seen. For a deterministic Enc in a CPA game, adversary could send in two identical messages (m_0, m_0) to find the encryption of an individual message. Therefore, we need an encryption strategy that appears randoms and provides a different output for the same message if ran multiple times. 8 Feb. 10:30

Stream cipher seemed promising, but fell short in being tedious and inefficient in ensuring that the users who are supposed to know the message are in the same state. The following problem illustrates this issue.

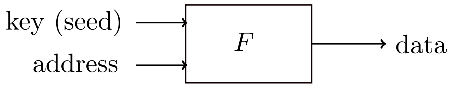
Example. If a user wants to know the 20-th bit they need to use the key to get the initial state and go through each of the previous bits.

So, our next goal is to find a methodology that allows stateless access.

2.4 Pseudorandom Functions

To address the aforementioned issue, we want something like this:

[3]Note that the string generated is pseudorandom.



This suggests the notion of keyed function.

Definition 2.4.1 (Keyed function)**.** A *keyed function* $F: \mathcal{K} \times X \to Y$ is a function with domain being the product between the key space and the input space (address).

We see that a keyed function is just any kind of function with domain in the form of $\mathcal{K} \times X$ and with an appropriate interpretation.

Example. $\mathcal{K} = X = Y = \{0,1\}^n$.

Notation. We usually write $F(k,x) =: F_k(x)$ for a keyed function F.

Now, we simply want that given a key $k \in \mathcal{K}$, we can access an arbitrary address $x \in X$ such that the output y looks random, i.e., we want a random function.

Notation (Random function)**.** The *random function* $\mathcal{U}: X \to Y$ is a deterministic (consistent) function such that the lookup table is a uniformly random.

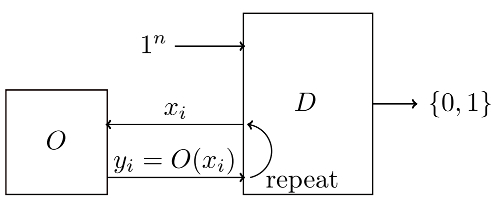
Input	Output
00	100
01	111
10	010
11	100

Table 2.1: A random function with $X = \{0,1\}^2, Y = \{0,1\}^3$.

2.4.1 Pseudorandom Functions Game

And naturally, we want a *pseudo* version of a random functions, which we called pseudorandom functions. Before we formally introduce the definition, as always, we should define the advantage under some adversary-style games set-up. To do this, we design a PRF game similar to the game of PRG.

Definition 2.4.2 (Pseudorandom function game)**.** The *pseudorandom function game* contains a distinguisher D and an oracle (black box) O which is either $O(\cdot) = F_k(\cdot)$ for secret random key $k \leftarrow \{0,1\}^n$ (the "real world") or $O(\cdot) = \mathcal{U}(\cdot)$ (the "ideal world"), where D can query O with various inputs and get the corresponding outputs, then output a decision bit.



Then, we naturally define the advantage in this context as follows.

Definition 2.4.3 (Advantage)**.** Given a distinguisher D in a PRF game, the *advantage* $\mathrm{Adv}_F(D)$ in distinguishing the "real world" ($O = F_k$) and the "ideal world" ($O = \mathcal{U}$) is given by

$$\mathrm{Adv}_F(D) := \left| \Pr_{k \leftarrow \mathcal{K}}(D^{F_k(\cdot)}(1^n) = 1) - \Pr_{\mathcal{U} \leftarrow \mathrm{RF}}(D^{\mathcal{U}(\cdot)}(1^n) = 1) \right|.$$

Intuition. The first probability represents the "real world" $O = F_k$ for a uniform $k \in \mathcal{K}$, and the second probability represents the "ideal world" $O = \mathcal{U}$.

2.4.2 Pseudorandom Functions

Finally, we have the following definition.

Definition 2.4.4 (Pseudorandom function)**.** A keyed function F is a *pseudorandom function* if every probabilistic polynomial time distinguisher D has negligible advantage in the PRF game against F.

Intuition. You can think of stream ciphers as cassette tapes, and PRFs as CDs.

Remark. PRFs exist if and only if PRGs exist. You can think of the key as similar to the seed in the PRG.

Example (Poor example)**.** Let $F_k(x) = k \oplus x$, given O.

Proof. Query $m_0 = 0 \dots 0, m_1 = 1 \dots 1$ to get $y_0 = O(0 \dots 0)$ and $y_1 = O(1 \dots 1)$. If $y_0 \oplus y_1 = 1 \dots 1$ accept, else reject. We see that in the real world, we have

$$(k \oplus x_0) \oplus (k \oplus x_1) = x_0 \oplus x_1 = 1 \dots 1,$$

hence $\Pr(\text{accept}) = 1$. While in the ideal world, $\Pr(\text{accept}) = 2^{-n}$, hence

$$\mathrm{Adv}_F(D) = |\Pr(\text{accept in real world}) - \Pr(\text{accept in ideal world})| = 1 - 2^{-n} \approx 1.$$

⊛

2.4.3 PRF-Based CPA-Secure Schemes

Now we're interested in utilizing a PRF $F_k(\cdot) \colon \mathcal{K} \times X \to Y$ to build a CPA secure scheme, where we let $\mathcal{K} = X = Y = \{0,1\}^n$. Consider the following scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ by

- $\mathsf{Gen}(1^n)$: outputs a uniformly random $k \leftarrow \{0,1\}^n$;
- $\mathsf{Enc}_k(m)$ given $m \in \{0,1\}^n$: choose a random $x \leftarrow \{0,1\}^n$, outputs $(x, m \oplus F_k(x))$;
- $\mathsf{Dec}_k(c)$ given $c = (x, c')$: outputs $c' \oplus F_k(x)$.

In real world applications, it is important to construct it with a key such that the key space $\mathcal{K}$ is fairly large since if $\mathcal{K}$ is small, PRF is broken because with brute force you can guess a key or try all the keys.

Example. DES (data encryption standard, 1976-1977) fell fault to this for $\mathcal{K} = \{0,1\}^{56}$, $X = Y = \{0,1\}^{64}$.

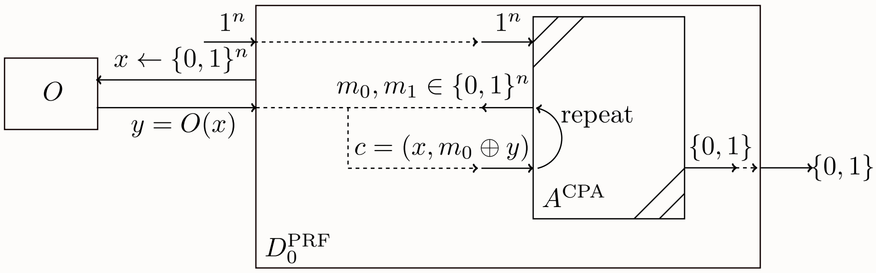
Example. Currently AES (2001-2002) is using $X = Y = \{0,1\}^{128}$, while $\mathcal{K} = \{0,1\}^n$ with $n = 128, 192, 256$. Even for $n = 128$ is considered secure.

Lecture 11: PRF-Based and Arbitrary Length Encryption

Now, we show that Π is actually CPA-secure. 13 Feb. 10:30

Theorem 2.4.1. If F is a (secure) PRF, then the above scheme Π is CPA-secure for messages of length n.

Proof. We will conduct a proof by reduction. Let A be any CPA adversary against the system. If A has non-negligible advantage over the system, then we can write a distinguisher D for the PRF backing the system. In diagram, we have the following.



In this diagram, we are utilizing the adversary A within distinguisher D_0 to help output a decision bit. A outputs some messages m_0 and $m_1 \leftarrow \{0,1\}^n$, out of which we arbitrarily choose m_0. D_0 can then be defined as such:

(a) choose random x;

(b) query the oracle O to obtain y, which is either $F_k(x)$ or a uniform random string;

(c) encode the previously chosen m_0 and pass $c = (x, m_0 \oplus y)$ to the adversary A;

(d) output A's decision bit.

There are two "worlds" for D_0:

- real world: $O = F_k(x)$ for random k. D_0 perfectly simulates the "left world" in the CPA game for adversary A because it evaluates $c = (x, m_0 \oplus y) = (x, m_0 \oplus F_k(x))$. This is exactly the input c that A is expecting as the encryption of m_0;
- ideal world: $O = \mathcal{U}$ (a random function). D_0 perfectly simulates the following "hybrid," "random ciphertext" world. In this world, our adversary receives nonsense because $(x, m_0 \oplus y)$ is complete randomness, and A does not expect this! A is expecting either an encryption of m_0 or m_1, which are the true "left" and "right worlds" for A.

Problem. We note that O is **consistent**, i.e., for a fixed a, $O(a)$ will always be the same. Specifically, when $O = \mathcal{U}$, we can't treat it as truly random, since independence of c' holds if and only if all queries x to the oracle O are distinct.

Answer. This is not a concern because of the birthday paradox.

> **Note** (Birthday paradox). If we pick q random objects from a population of a size N, then $\Pr(\text{two or more are the same}) \approx q^2/N$, where in our scenario q is the number of queries to the oracle and N is the size of the sample space $X = \{0,1\}^n$.

From here we can show that the probability of the "ideal world" accepting for D_0 in the PRF game is negligibly different from the probability of the "hybrid world" accepting in the CPA game, specifically,

$$\begin{aligned}\Pr\left(D_0^{\mathcal{U}(\cdot)} = 1\right) &= \Pr(A = 1 \text{ in "hybrid"}) \pm \text{negl}(n) \\ &\leq \frac{\#\text{ queries}^2}{2^{n+1}} \pm \text{negl}(n) = \frac{\mathsf{poly}(n)}{2^{n+1}} \pm \text{negl}(n) = \text{negl}(n).\end{aligned} \tag{2.1}$$

⊛

Now we can write the advantage of A in the CPA game against Π as

$$\begin{aligned}\mathrm{Adv}_{\Pi}^{\mathrm{CPA}}(A) &= \left|\begin{aligned}&\Pr(A = 1 \text{ in "left"}) - \Pr(A = 1 \text{ in "right"}) \\ &\qquad\qquad + \Pr(A = 1 \text{ in "hybrid"}) - \Pr(A = 1 \text{ in "hybrid"})\end{aligned}\right| \\ &\leq \begin{aligned}&|\Pr(A = 1 \text{ in "left"}) - \Pr(A = 1 \text{ in "hybrid"})| \\ &\qquad\qquad + |\Pr(A = 1 \text{ in "right"}) - \Pr(A = 1 \text{ in "hybrid"})|\end{aligned} \\ &\leq (\mathrm{Adv}_F^{\mathrm{PRF}}(D_0) + \mathrm{negl}(n)) + (\mathrm{Adv}_F^{\mathrm{PRF}}(D_1) + \mathrm{negl}(n)),\end{aligned}$$

where the last line follows from Equation 2.1.[a] Finally, notice that the advantage on both D_0 and D_1 is negligible because we are assuming that F is a PRF for which all polynomial time distinguishers have a negligible advantage in the PRF game against F.

Hence, in all, the overall advantage of $\mathrm{Adv}^{\mathrm{CPA}}(A)$ is

$$\mathrm{Adv}_{\Pi}^{\mathrm{CPA}}(A) \leq \mathrm{negl}(n) + \mathrm{negl}(n) + \mathrm{negl}(n) + \mathrm{negl}(n) = \mathrm{negl}(n),$$

which is negligible for all possible A, hence Π is CPA-secure. ■

[a] Note that we implicitly use the advantage where we build D_1 as D_0 by passing the adversary m_1 instead of m_0.

Theorem 2.4.1 shows that given a valid PRF, we can build a valid CPA-secure scheme that prevents the original drawbacks such as key reuse.

2.4.4 Remaining Problems

However, new issues arise:

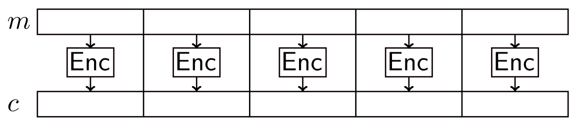
(a) Messages must be of fixed-length by the definition of Π at hand (length n and **only** length n).

(b) CPA security does not address **active adversaries**, i.e., every adversary so far has been a passive observer of behaviors of these encryption schemes, whereas an active adversary would be able to affect what is decrypted as well. This type of attack is called *chosen ciphertext attack*.

> **Example.** Enigma machine was also only for passive adversaries.

> **Example.** Suppose an adversary wants to mess with your salary. They may not know what it is, but they want to mess with the ciphertext of your salary to make it higher or lower by some factor, so that the decryption is incorrect.

Let's first focus on the long message issue. To address it, a naive approach will be reusing the key! We could chop up the message into parts that our CPA secure scheme can take.



This is CPA secure, but we have an efficiency issue because our ciphertext $c = (x, c')$ where $x \in \{0,1\}^n$ and $c' \in \{0,1\}^n$ is with size $2 \times n$. This is too large, and we need to strive for smaller overhead.

2.5 Modes of Operations and Encryption in Practice

To address the problems mentioned, we discuss modes of operation for encrypting arbitrary-length messages using a stream cipher.

2.5.1 Counter Mode

Let F be a PRF. To encrypt $m \in \{0,1\}^*$, we first break up m into blocks of length n each as

$$m = m_1 \| m_2 \| \ldots \| m_t$$

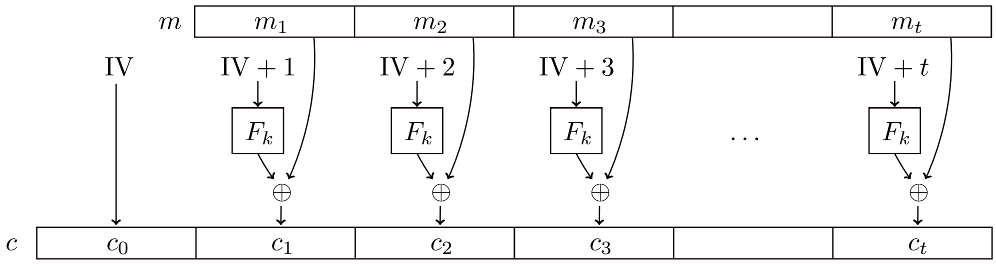
for $|m_i| = n$ and $|m_t| \leq n$. Then, we proceed as follows.

(a) Choose a random address as an initialization vector $\text{IV} \leftarrow \{0,1\}^n$.

(b) Initiate the ciphertext c with $c_0 \leftarrow \text{IV}$.

(c) For m_i, encode with $F_k(\text{IV}+i)$ such that $c_i \leftarrow m_i \oplus F_k(\text{IV}+i)$.

(d) Concatenate them together to get $c \leftarrow c_0 \| c_1 \| \dots \| c_t$.



Notation (Counter mode). We call such a scheme *counter mode*, or *CTR*.

Lecture 12: Modes of Operations

By using CTR, we can now encrypt messages of various lengths. Specifically, counter mode gets around the issue by having a single initialization vector (IV) that gets incremented with each block. Subsequent blocks of the message will be padded with the result of the PRF given the input of $\text{IV}+i$, where i is the block number. 15 Feb. 10:30

Proposition 2.5.1. If F is PRF, then CTR is CPA secure.

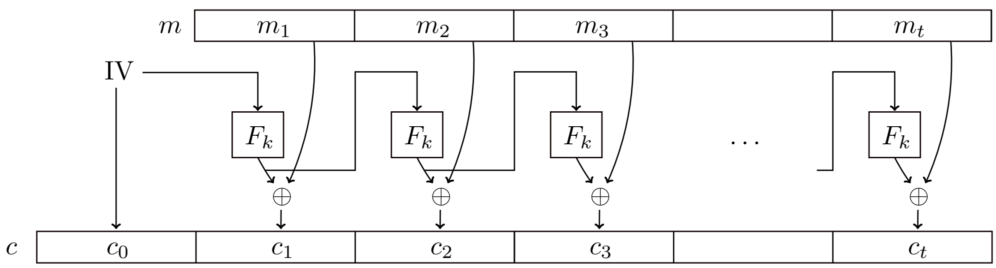
Proof. Every Input to F across the entire CPA game is distinct, with a very negligible probability to be the same. Therefore, all output of F will "look like" truly random and independent. ■

Remark. Advantages of using CTR:

- simple and satisfies CPA secure;
- does not require a message length that is a multiple of n;
- fast and efficient because it can be computed in parallel;
- no need for padding (we can just trim the output of F to fit the last message block size $|m_t|$).

2.5.2 Output Feedback Mode

One potential issue with counter mode is that the input for each PRF is easy for an adversary to calculate for each block if they are able to guess the initialization vector. This is a concern because if the function that generates the IV does not have a good random distribution, the system will not be secure. Consider the following scheme.



Notation (Output feedback mode)**.** We call such a scheme *output feedback mode*, or *OFB*.

OFB mode gets around this issue by feeding the output from the previous block as an input to each block, starting with the IV for the first block. This makes the index of each block practically random which is much closer to the original CPA game model.

Proposition 2.5.2. If F is PRF, then OFB is CPA secure.

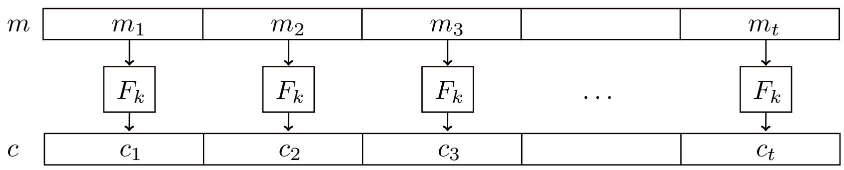
Proof. Since OFB does not have repeated input to F (happen with negligible probability). ■

Remark. Some remarks of using OFB:

- OFB is random by the chain, which is independent of message, not by counter;
- OFB can increase security: because IV is not random, if IV is attacked, every block in CTR can in danger. In contrast, in OFB, because IV is executed with F before XOR with message block, OFB can still be secure even if IV is guessed.
- OFB **cannot** be computed in parallel.

2.5.3 Electronic Codebook Mode

Consider the following scheme.



Notation (Electronic Codebook mode)**.** We call such a scheme *electronic codebook mode*, or *ECB*.

We see that this is not similar to CTR or OFB. Since for ECB, we use the message as input to generate ciphertext.

Proposition 2.5.3. Even if F is PRF, ECB is **not** CPA secure.

Proof. Since same message block is encrypted to the same ciphertext block, implying that ECB is not CPA secure because it is stateless and deterministic from Theorem 2.3.1. ■

Clearly, we have a clear problem when trying to decrypt.

Remark (Decryption)**.** We see the following.

- If F is PRF, we are not able to decrypt the ciphertext because it is not guaranteed that the inverse function of F exists.
- If F is PRP (will be introduced soon!), we are able to decrypt the ciphertext, but it is still not CPA secure because the same message still shares the same ciphertext.

2.5.4 Pseudorandom Permutation

To introduce the next mode, we need some preliminaries.

Definition 2.5.1 (Block cipher)**.** A bijective keyed function $F\colon \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ such that both F_k and F_k^{-1} can be computed efficiently given the key k is called a *block cipher*.

Intuition. Block cipher is an invertible version of a PRF.

We see that we need invertibility, which proposes the following notion.

Definition 2.5.2 (Pseudorandom permutation)**.** A keyed function F is a *pseudorandom permutation* if every probabilistic polynomial time adversary A has negligible advantage in distinguishing between F_k and a random bijection with a random key $k \leftarrow \mathcal{K}$.

Where here, the advantage is defined as follows.[4]

Note. Notice that in A can call the oracle multiple times.

Definition 2.5.3 (Advantage)**.** Given a adversary A which tries to distinguish a PRP from a random bijection, the *advantage* $\mathrm{Adv}_F(A)$ in distinguishing the "real world" (F_k) and the "ideal world" $(P \leftarrow P_n)$[a] is given by

$$\mathrm{Adv}_F(A) \coloneqq \left| \Pr_{k \leftarrow \mathcal{K}}(A^{F_k(\cdot)} = 1) - \Pr_{P \leftarrow P_n}(A^{P(\cdot)} = 1) \right|.$$

[a]We use P_n to denote the set of all bijection on $\{0,1\}^n$.

Note. If we can give A access to $F_k^{-1}(\cdot)/P^{-1}(\cdot)$ as well, then F_k is a strong PRP.

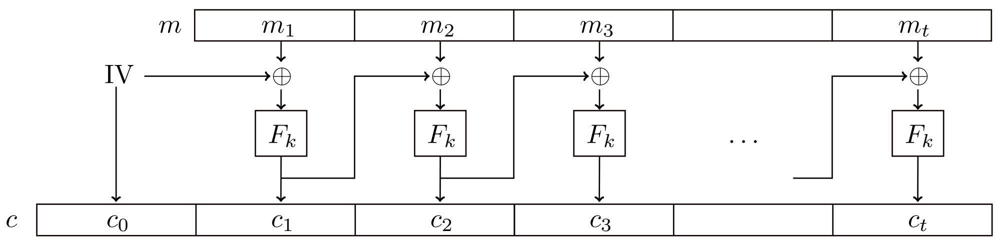
Theorem 2.5.1. If F is a PRP, F is also a PRF.

Proof idea. Given oracle access, a random permutation is identical to a random function as long as distinct input queries to the random function don't return the same value (because if $c_1 = c_2$, function F is not invertible), which implies "birthday collision" on outputs. However, collision happens with negligible probability: $\frac{\mathsf{poly}(n)}{2^n}$. Thus, under the efficient setting, if F is a PRP, it is also a PRF. ■

With all these, we now introduce the final mode of operation combines ideas from OFB and ECB.

2.5.5 Cipher Block Chaining

Like OFB, we now use the result of the previous block to help encrypt each block, starting with IV for the first block. However, instead of feeding a pseudorandom string into a PRF, the pad of the message and the original block is used as an input. To decrypt, the operations are reversed for each block using the inverse of F when necessary. In diagram, we see that for encryption, we have



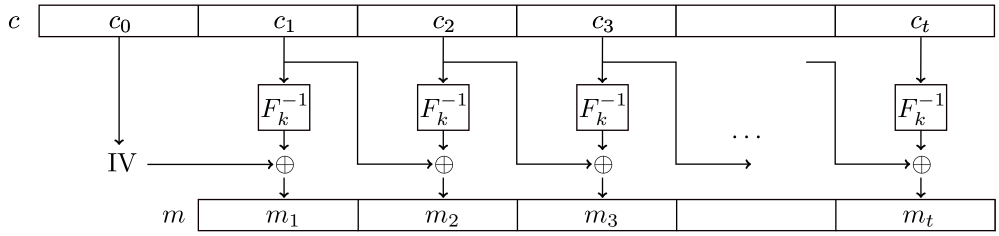
In other words, we have

$$\begin{cases} c_0 = \mathrm{IV} \\ c_i = F_k(m_i \oplus c_{i-1}). \end{cases}$$

Note. We need to find a way to increase the length of m_t to n-bits.

And to decrypt, we have

[4]Formally, we should define the "PRP game," which is just a variation of PRF game!



In other words, we have

$$m_i = c_{i-1} \oplus F_k^{-1}(c_i).$$

Note. We need to un-pad "m_t" to the right size.

Notation (Cipher block chaining)**.** We call such a scheme *cipher block chaining*, or *CBC*.

Theorem 2.5.2. If F is a PRP (which is also a PRF), then CBC is CPA secure.

Proof idea. All ciphertexts look like random independent strings as long as no input to $F_k(\cdot)$ is ever repeated. Based on the birthday paradox, repetitions happen with only negligible ($\mathsf{poly}(n)/2^n$) probability by the choice of IV and pseudorandom outputs of prior blocks. ■

Remark. Some remarks of using CBC:

- encryption is sequential, i.e., we cannot compute ciphertext without computing all prior blocks;[a]
- provides more security than CTR if IV is predictable;
- requires padding the last message block since we cannot trim the output of F to fit the last message block.[b]

[a] Although decryption can be done in parallel.
[b] Which will also increase the execution time.

The last point means that the plaintext must be extended or padded to the correct length, i.e., multiple of n. This leads to a potential vulnerability we next discuss.

Chapter 3

Message Authentication

Lecture 13: Integrity and Authentication

The fact that CBC requires padding the last block of the message leads to a potential vulnerability called "authenticity": if attacker can change the ciphertexts and send it to the receiver in the CBC mode, the attacker can decrypt the entire message. More broadly, with the previous methodology, we can't know whether the message is sent by a particular user, which is a huge problem. But firstly, let's see what's wrong if the authenticity can't be ensured in the CBC mode. 20 Feb. 10:30

3.1 Security vs. Authenticity

3.1.1 Cipher Block Chaining Padding

This padding must be done in an unambiguous way, and one possible standard to accomplish this is the PKCS standard, and we are interested in the following specific version.

Definition 3.1.1 (PKCS #5). Let L be the length of a message block in bytes. On message m, let $b \in \{1, \dots, L\}$[a] be the number of bytes that when added to the message m, makes the total message length equal to a multiple of L as required in CBC. The *PKCS #5* standard requires that the new message $\hat{m}$ after the padding to be

$$\hat{m} := m \| \underbrace{bb \dots b}_{b \text{ times}},$$

i.e., we add exactly b b's at the end of m.

[a] b can't be 0 implies that if the length of m is equal to a multiple of L already, we must add another L bytes.

Note. We're now thinking about characters, not bits (otherwise the above doesn't make sense).

Example. If $b = 5$, we have $\hat{m} = m \| 55555$.

However, this can create some vulnerabilities in CBC mode. This is based on the following fact.

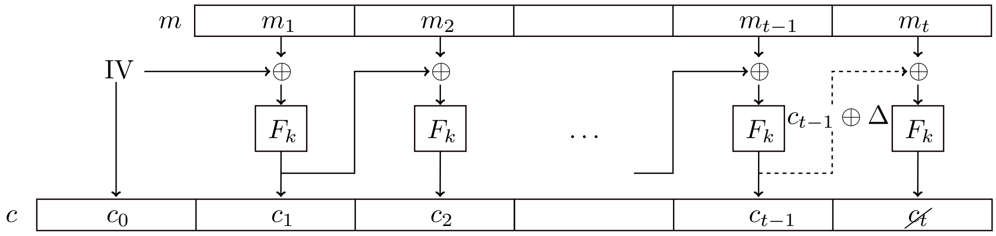
Remark (Practical implementation). If the last message block is formatted incorrectly w.r.t. PKCS #5, it's common that a public error will be announced.

Example. `Cat55555` is a well formatted message block, which is easy to decrypt.

Example. `Cat54555` is a poorly formatted message block is, which will error.

3.1.2 Padded Oracle Attacks

All these may seem irrelevant: what is this all about? Don't we already prove that CBC is CPA secure in Theorem 2.5.2? In reality is that, whenever we do something outside the scheme (in this case, using PKCS #5 for padding and allowing other things as we will see) that is not considered in the standard CPA game, there's no guarantee anymore. To be more specific, there's a potential "integrity" issue: what if the adversary is able to change c_{t-1} and throw it back to the sender?



Formally, if the adversary is able to do this, then the output of the last block, i.e., c_t, becomes

$$c_t' = F_k(m_t \oplus (c_{t-1} \oplus \Delta)) = F_k((m_t \oplus \Delta) \oplus c_{t-1}).$$

Problem. What does this suggest?

Answer. By letting $c_{t-1}' = c_{t-1} \oplus \Delta$, we're able to deceive the user (or the "format checker") that "m_t is $m_t \oplus \Delta$"! ⊛

We see that the same thing can be applied in any block, hence, we are able to "shift" the message any way we want.

Intuition. An active adversary can recover the plaintext by shifting each bit in the ciphertext and observing whether an error gets thrown.

Now, we claim the following.

Claim. Just knowing that an error occurs is enough for the adversary to recover the entire plaintext!

Proof. Firstly, we claim that we can recover b. It's simple: we start by checking whether $b = L$. If it is, then the last block of the message should be $m_t = L\ L\ \ldots\ L$. By "shifting" the left most character[a] by δ, i.e., we make the "format checker" to think that m_t looks like

$$m_t' = (L + \delta)\ L\ \ldots\ L.$$

Now the formatter thinks there's an error since by looking at the last $L - 1$ L's, it thinks that the final (left most) character should also be an L, not $L + \delta$! So we got an error prompt. This only happens when $b = L$, hence we can determine whether $b = L$. In the same way, we can keep ruling out the possibility by doing the same thing from left to right, until we found out b.

Once the adversary recovers b, the message is basically exposed in the same way: Suppose

$$m_t = \ldots\ x\ \underbrace{b\ b\ \ldots\ b}_{b \text{ times}},$$

where x is the last character of the data. Observe that if we replace b by $b + 1$, the formatter now will complain if x is not $b + 1$! This implies that we can use the formatter to determine x is $b + 1$ or not. Since we also shift x by δ', so we can actually determine whether $x + \delta'$ is $b + 1$ or not:

$$m_t' = \ldots\ (x + \delta')\ \underbrace{(b+1)\ (b+1)\ \ldots\ (b+1)}_{b \text{ times}},$$

and if $x + \delta' \neq b + 1$, then we get an error prompt. We can simply try out different δ' until there's no error. In that case, $x = b + 1 - \delta'$! By recovering the last $b + 1$ characters, we can replace them by $b+2$, and keep doing the same thing until every character in this block is decrypted. ⊛

[a]Notice that I didn't use the work "bit" due to the string representation.

Notation (Padded oracle attack). The above attack is called the *padded oracle attack*.

We learn that we need to be very careful with padding and consider attacks outside the CPA model. In this case, the integrity of the cyphertexts used in the next block.

3.2 Message Authentication Codes

The padded oracle attack raises the problem of authenticity of the message, i.e., does this message really comes from a particular user? Obviously, this is very important from the above example. But there are many others obvious reasons suggesting that this is important.

Example. Alice is trying to send the message "Please give Eve \$20" to Bob. If Eve is able to read it and change the message to "Please give Eve \$200", then that would clearly be advantageous to Eve.

So how do we ensure that an active adversary can't modify the data being transmitted? Following our previous approach, we should define something similar to the encryption scheme, i.e., we formalize how to generate string that we're going to sent. Now, instead of considering ciphertext, what we want is authenticity, so naturally, we define something called "tag", which can be thought of to be used to verify the identity of a user. Then, we define the following.

Definition 3.2.1 (Message authentication code). The *message authentication code* or *MAC* is a tuple $\Pi = (\mathsf{Gen}, \mathsf{Tag}, \mathsf{Ver})$ defined as follows.

- $\mathsf{Gen}(1^n)$ outputs a key $k \in \mathcal{K}$ given the security parameter n.
- $\mathsf{Tag}_k(m)$ outputs some short string called a "tag" given key k and message m.
- $\mathsf{Ver}_k(m', t')$ accepts or rejects depending on whether m' and t' is a valid message-tag pair.

Same as before, we want "correctness" and "security".

Definition 3.2.2 (Correctness). A MAC $\Pi = (\mathsf{Gen}, \mathsf{Tag}, \mathsf{Ver})$ satisfies *correctness* if for all $m \in \mathcal{M}$ and $k \in \mathcal{K}$,

$$\mathsf{Ver}_k(m, \mathsf{Tag}_k(m)) = 1.$$

As for "security", we want that the adversary can't produce a valid tag for a different $m' \neq m$.

Intuition. Seeing (m, t), the adversary can't produce (m', t') with $m' \neq m$ and $\mathsf{Ver}_k(m', t') = 1$.

But since the adversary can wait for a long time, i.e., observing lots of valid (m, t) pairs, hence we should not just consider a one-time game.

3.3 Chosen Message Attack Security

3.3.1 Chosen Message Attack

Consider the following game.

Definition 3.3.1 (Chosen message attack). The *chosen message attack* or *CMA* game given an adversary (called a *forger*) F[a] and a MAC $\Pi = (\mathsf{Gen}, \mathsf{Tag}, \mathsf{Ver})$ is conducted as follows.

1. $k \leftarrow \mathsf{Gen}(1^n)$ is kept secret from F.
2. F receives polynomially many tags for messages of their choices by querying the $\mathsf{Tag}_k(\cdot)$ oracle.

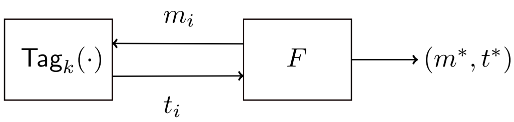
3. F outputs an attempted forgery (m^*, t^*).

[a] F gets access to the security parameter 1^n.

Definition 3.3.2 (Weak forgery). In the CMA, F *weakly forges* if $\mathsf{Ver}_k(m^*, t^*) = 1$ and m^* is not a query from A to the tag oracle.

Notation. It's canonical to drop the word "weak" in Definition 3.3.2.

If you like a picture:



Remark. Notice that we didn't consider the case that F can resend a seen (m, t) pair. That is, CMA game doesn't consider the repeated case.

Lecture 14: Message Authentication via PRFs

3.3.2 Chosen Message Attack Secrecy 22 Feb. 10:30

Then, we have the following security notion.

Definition 3.3.3 (Advantage). Given a forger F in an CMA, the *advantage* $\mathrm{Adv}_{\Pi}^{\mathrm{CMA}}(F)$ in generating a fake message/tag pair is given by

$$\mathrm{Adv}_{\Pi}^{\mathrm{CMA}}(F) \coloneqq \Pr_{k \leftarrow \mathsf{Gen}}(F^{\mathsf{Tag}_k(\cdot)} \text{ forges}).$$

Definition 3.3.4 (Unforgeable). A MAC is *unforgeable* under CMA if for every probabilistic polynomial time forger F, the advantage is negligible.

This has a more common name.

Notation (UFCMA). An unforgeable MAC under CMA is abbreviated as *UFCMA*.

Remark (Canonical MAC). If $\mathsf{Tag}_k(m)$ and $\mathsf{Ver}_k(m, t)$ are deterministic, then this MAC is called *canonical*. In this case, $\mathsf{Ver}_k(m, t)$ is automatic defined, i.e., it accepts if and only if $t = \mathsf{Tag}_k(m)$.

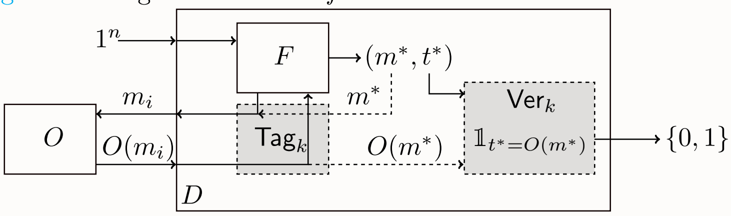
3.3.3 Chosen Message Attack Message Authentication Codes

We can use a PRF to construct a MAC that is UFCMA. For a fixed-length message $m \in \{0, 1\}^\ell$, suppose f is a PRF such that $f \colon \mathcal{K} \times \{0, 1\}^\ell \to \{0, 1\}^n$. We define a MAC as

- $\mathsf{Gen}(1^n)$: Choose a random $k \leftarrow \mathcal{K}$.
- $\mathsf{Tag}_k(m)$: Output $\mathsf{Tag}_k(m) = f_k(m)$.
- $\mathsf{Ver}_k\,(m, t)$: Accept if and only if $t = \mathsf{Tag}_k(m)$.

Theorem 3.3.1. This MAC is UFCMA if f is a PRF.

Proof. We prove the theorem by reduction. Let F be a probabilistic polynomial time forger, and we construct a distinguisher D against the PRF f as follows.



In words:

- D runs a query m_i by F to $O(\cdot)$ and relays the answer $t_i = O(m_i)$ back to F.
- When F outputs (m^*, t^*), query O again to get $O(m^*)$.
- Output 1 if and only if $O(m^*) = t^*$ and $m^* \notin \{m_i\}_i$.

For F, the "real world" is when $O = f_k$ and the "ideal world" is when $O = \mathcal{U}$ where $\mathcal{U}$ is a random function. The advantage of D is defined as

$$\mathrm{Adv}^{\mathrm{PRF}}(D) = |\Pr(D \text{ in "real world" accepts}) - \Pr(D \text{ in "ideal world" accepts})|\,.$$

Observe the following two cases:

(a) If $O = f_k$: We are in the "real world" and D is a perfect simulation of the MAC. Thus,

$$\Pr(D \text{ in "real world" accepts}) = \Pr(F \text{ forges}) = \mathrm{Adv}^{\mathrm{CMA}}(F).$$

(b) If $O = \mathcal{U}$: We are in the "ideal world" so we keep giving random values t to F. Since m^* is new, $O(m^*)$ will be uniformly random and independent. Thus,

$$\Pr(D \text{ in "ideal world" accepts}) = 2^{-n}.$$

This suggests that $\mathrm{Adv}^{\mathrm{PRF}}(D) = |\,\mathrm{Adv}^{\mathrm{CMA}}(F) - \mathrm{negl}(n)|$, which is also negligible since f is a PRF and D runs in polynomial time, we have

$$\mathrm{Adv}^{\mathrm{CMA}}(F) = \mathrm{Adv}^{\mathrm{PRF}}(D) \pm \mathrm{negl}(n) = \mathrm{negl}(n).$$

■

Remark. This MAC has a deterministic $\mathsf{Tag}_k(m)$ after specifying k.

3.3.4 Strong Unforgeability

You may wonder why we use the word "weak" in Definition 3.3.2 but never actually mention it again. This is because while it's satisfactory for many applications, CMA doesn't necessarily rule out the ability to find a new tag t^*, for an old (queried) message m, but in Definition 3.3.2 we rule this possibility out. In response, we define the following.

Definition 3.3.5 (Strong forgery)**.** In the CMA, F *strongly forges* if $\mathsf{Ver}_k(m^*, t^*) = 1$ and (m^*, t^*) is different from all (m, t)'s.

Note. For deterministic $\mathsf{Tag}_k(m)$, strong forgery is equivalent to weak forgery. Also, for deterministic MACs, strong forgery is the same if we give the forger access to $\mathsf{Ver}_k(\cdot)$ in addition.

3.3.5 Domain Expansions

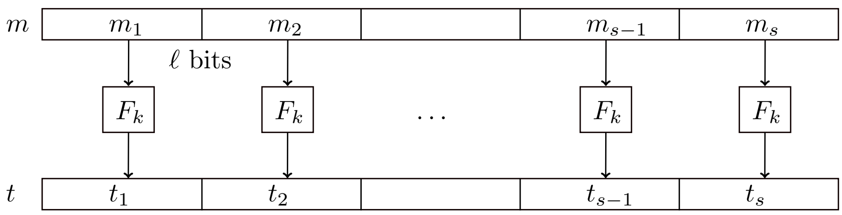
A natural question to ask next is whether there exists a secure MAC for messages of arbitrary length. Consider the previously constructed secure MAC for messages of length n, and let m be an arbitrary message of arbitrary length, we need to produce a method of tagging m without leaking any information to an adversary.

Intuition. An attractive naive implementation would be to simply parse the message into blocks of fixed-length and tag each block directly.

However, this implementation is not secure, and the final MAC for arbitrary length messages is not trivial.

Attempt 1

Firstly, we formalize the above idea. Suppose s is the number of blocks and $m = m_1 \| \ldots \| m_s$ where $|m_i| = \ell$ and $m \in \{0,1\}^*$. To tag m, we tag each part of the message.



Claim. This **not** UFCMA.

Proof. Consider the "reordering attack":

(a) Query $m_1 \| m_2$ and get $t_1 \| t_2$.

(b) Output (m^*, t^*) where $m^* = m_2 \| m_1$ and $t^* = t_2 \| t_1$.

This adversary wins the CMA game every time since this is a valid message/tag pair for messages with length 2ℓ. ⊛

Attempt 2

We now try to include indices when tagging m to fix the problem. First, break the message into $m = m_1 \| m_2 \| \ldots \| m_s$ in blocks of length $\ell/2$. Suppose $t_i = F_k(m_i \| \langle i \rangle)$ with $\langle i \rangle$ being length $\ell/2$.

Notation. Given an integer $i \in \mathbb{Z}$, $\langle i \rangle$ is its binary representation in bits.[a]

[a] The length can vary and should be specified.

Additionally, we demand that $\mathsf{Ver}_k(\cdot)$ to verify individual blocks and make sure indices are in the correct order.

Claim. This is still **not** UFCMA.

Proof. Consider the "truncation attack":

(a) Query $m = m_1 \| m_2$, get $t = t_1 \| t_2$.

(b) Output $m^* = m_1, t^* = t_1$.

This adversary wins the CMA game again, obviously. ⊛

Attempt 3

To prevent "truncation attack", we try to also include the length (number of blocks) when tagging the message m. Again, break the message into $m = m_1 \| m_2 \| \ldots \| m_s$ in blocks of length $\ell/3$, and set $t_i = F_k(m_i \| \langle i \rangle \| \langle s \rangle)$, where both $\langle i \rangle$ and $\langle s \rangle$ are in $\ell/3$ bits.

Additionally, we demand that $\mathsf{Ver}_k(\cdot)$ to verify individual blocks, make sure indices are in the correct order, and also the number of the total number of blocks matches with the entire m.

Claim. This is still **not** UFCMA.

Proof. Consider the "mix-match attack":

(a) Query $m = m_1 \| m_2$ and get $t_1 \| t_2$.

(b) Query $m' = m'_1 \| m'_2$ and get $t'_1 \| t'_2$.

(c) Output $m^* = m_1 \| m'_2$ and $t^* = t_1 \| t'_2$.

This adversary wins the CMA game again since not only the order is preserved, the length is fixed to be the same too (compared to the previous attempt). ⊛

Attempt 4

Finally, we give each message a random identifier. Again, we break the message into $m = m_1 \| m_2 \| \ldots \| m_s$ in blocks of length $\ell/4$, and let $r \leftarrow \{0,1\}^{\ell/4}$ be uniform. Set $t_i = F_k(m_i \| \langle i \rangle \| \langle s \rangle \| r)$ for each i and let $t = r \| t_1 \| t_2 \| \ldots \| t_s$.

Additionally, we demand that $\mathsf{Ver}_k(\cdot)$ to verify i and s as before, and also verify r for all block.

Theorem 3.3.2. This is a unforgeable MAC if F is a PRF.

Proof idea. The extra information we include in each block prevents the previously described attacks, and no more attacks are possible, i.e., a forgery m^*, t^* must include a block $m_i \| r \| \langle i \rangle \| \langle l \rangle$. For the full proof, see [KL20, Page 117-120] ■

Remark. Although this is a solution, but now the tag is 4 times longer than the message, so we seek for a better solution.

Lecture 15: CBC-MAC and Authenticated Encryption

3.3.6 Cipher Block Chaining-Message Authentication Codes

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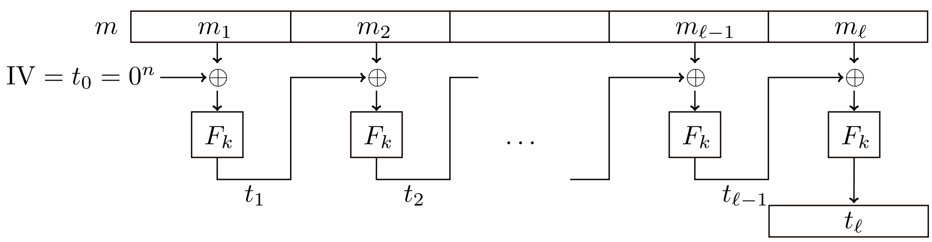
One potential solution is the following, which is used more in practice. Let $m \in \{0,1\}^{\ell \cdot n}$ such that $m = m_1 \| m_2 \| \ldots \| m_\ell$ with $|m_i| = n$, i.e., consider the block length being n and the message length being $\ell \cdot n$, and

- $F \colon \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ be a PRF;
- $\mathsf{Gen}(1^n)$: chooses $k \leftarrow \mathcal{K}$ uniformly at random for the PRF;
- $\mathsf{Tag}_k(m)$: set $t_0 = 0^n$ and $t_i = F_k(t_{i-1} \oplus m_i)$ where $i \in [\ell]$, output at the end t_ℓ;
- $\mathsf{Ver}_k(m, t)$ is canonical that it could just run the whole thing again and verify that we have the same tag since F_k is deterministic.



Notation (Cipher block chaining-message authentication codes)**.** The above MAC is called *Cipher block chaining-message authentication codes*, or *CBC-MAC* for short.

Remark. It is necessary to have a fixed t_0, otherwise the system is totally insecure.

Proof. If we do allow a non-fixed t_0, then consider now our tag is defined as (t_0, t_ℓ) for m. Now, if we change t_0 to $t_0 + \Delta$, it would be valid for $(m_1 + \Delta)\|m_2\| \ldots \|m_\ell$ since

$$t_1 = F_k(t_0 \oplus m_1) = F_k((t_0 \oplus \Delta) \oplus (m_1 \oplus \Delta)) = F_k(t_0' \oplus m_1'),$$

and everything else in the chain would be the same, hence it's valid, i.e., the adversary wins. ⊛

With this, we see that it is also necessary to **not** output all t_i's, since we can consider replacing t_i and m_i to $t_i + \Delta$ and $m_i + \Delta$.

So, with these caveats, we have the following.

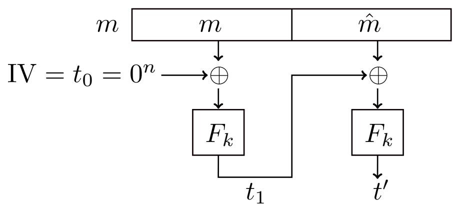
Theorem 3.3.3. If F is a PRF, then CBC-MAC is UFCMA for message of length exactly $\ell \cdot n$ for a fix ℓ.

The proof is tedious, but it's quite obvious to see why this shouldn't work for arbitrary length by considering the "length extension attacks":

(a) query m (one block) and get tag $t = F_k(m)$;

(b) query $m' = \hat{m} \oplus t$ and get tag $t' = F_k(t \oplus \hat{m})$;

(c) Now consider the forge $m^* = m\|\hat{m}$, $t^* = t'$, which is valid since:



Remark. This is the only attack that an adversary can do for CBC-MAC.

Hence, if we can resolve the above issue, we can support arbitrary length message.

Notation (Prefix-free encoding)**.** The *prefix-free encoding* is an encoding of a message such that no valid message is a prefix of another.

For example, given m, we can encode it into $\langle |m| \rangle \| m \| 0 \ldots 0$, where $\langle |m| \rangle$ denotes the length of m in binary and $0 \cdots 0$ is used for padding to block length.

Example. Suppose $n = 4$ and $m = 001$, so $\langle |m| \rangle = \langle 3 \rangle = 0011$ and the encoding of m is $0011\|0010$. To decode, we just read the first block to obtain the information of $|m|$ and read the second block.

It is easy to see that the above encoding is prefix-free.

Claim. The encoding $m \mapsto \langle |m| \rangle \| m \| 0 \ldots 0$ is prefix-free.

Proof. Suppose $\hat{m}$ (the encoding of m) is a prefix of $\hat{m}'$ (the encoding of m'), then $\hat{m}_1 = \hat{m}_1'$,[a] and hence $|m| = |m'|$. But $\hat{m}, \hat{m}'$ have the same number of block, meaning $\hat{m} = \hat{m}'$. ⊛

[a] This denotes the first block.

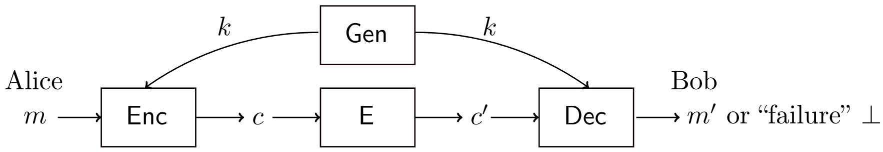
Now we see an improvement of Theorem 3.3.3 based on the above finding.

Theorem 3.3.4. If we use a prefix-free encoding before tagging, and remove encoding after verifying, then CBC-MAC is UFCMA for arbitrary length message.

Remark. In reality, there are a better option, but theoretically this works and the above encoding is not that bad since we only need to add $\log |m|$ bits.

3.4 Authenticated Encryption Scheme

After seeing MAC, which is all about integrity, we now ask whether we can combine it with encryption, i.e., we want both confidentiality and integrity. What we want is the following.



This leads to the so-called cryptosystem.

Definition 3.4.1 (Cryptosystem)**.** A *cryptosystem* is defined as a tuple $\Pi = (\mathsf{Gen}(\cdot), \mathsf{Enc}(\cdot), \mathsf{Dec}(\cdot))$ where

- $\mathsf{Gen}(1^n)$ output a secret key k;
- $\mathsf{Enc}_k(m)$ output a ciphertext c;
- $\mathsf{Dec}_k(c')$ output either a message m' or a special failure symbol $\perp$.[a]

[a]Indicating that c' is inauthentic.

For "correctness", we can still use Definition 1.2.1, i.e., we require that for all k and m,

$$\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = m.$$

Note. We don't allow $\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = \perp$ to be specific, since $\mathsf{Enc}_k(m)$ should always be a valid ciphertext.

As for "security", we now combine what we discussed so far:

(a) Confidentiality: Same as CPA-security.

(b) Authenticity: Attacker shouldn't be able to produce an "authentic-looking" ciphertext on its own, even after seeing many ciphertexts from the sender (on messages of its choice).

We see that there's no "tag" here, i.e., we need to define a similar game as CMA game in this context.

3.4.1 Ciphertext Forgery Game

Formally, we have the following.

Definition 3.4.2 (Ciphertext forgery game)**.** The *ciphertext forgery game* for an adversary A let A has access to $\mathsf{Enc}_k(\cdot)$ oracle, and A wins if it can produce a new and authentic-looking ciphertext.

Then, naturally, we have the following definition.

Definition 3.4.3 (Advantage)**.** Given an adversary in a ciphertext forgery game, the *advantage* $\mathrm{Adv}_{\Pi}^{\mathrm{UNF}}(A)$ is given by

$$\mathrm{Adv}_{\Pi}^{\mathrm{UNF}}(A) := \Pr_{k \leftarrow \mathsf{Gen}(1^n)}(A^{\mathsf{Enc}_k(\cdot)} \text{ “forges”}),$$

where "forges" means that the output c^* satisfies $\mathsf{Dec}_k(c^*) \neq \perp$ and c^* was not a reply for the $\mathsf{Enc}_k(\cdot)$ oracle.

Definition 3.4.4 (Unforgeable). A cryptosystem $\Pi = (\mathsf{Gen}(\cdot), \mathsf{Enc}(\cdot), \mathsf{Dec}(\cdot))$ is *unforgeable* if for all probabilistic polynomial time adversary A, the advantage is negligible.

3.4.2 Authenticated Encryption Scheme

Bringing everything together, we have the following.

Definition 3.4.5 (Authenticated encryption scheme). A cryptosystem $\Pi = (\mathsf{Gen}(\cdot), \mathsf{Enc}(\cdot), \mathsf{Dec}(\cdot))$ is an *authenticated encryption (AE) scheme* if it is CPA-secure and unforgeable.

Now we would like to construct AE schemes. We want to combine the existing construction of CPA-secure schemes and UFCMA MACs in a *modular way*.

Remark. This is not what people do in reality, since we might want everything in one-shot in a more efficient way.

Lecture 16: AE Scheme Construction and Cryptographic Hashing

Now, we explore different approaches to AE constructions. Assume we have an encryption scheme $\Pi^E = (\mathsf{Gen}^E, \mathsf{Enc}^E, \mathsf{Dec}^E)$ and a MAC $\Pi^M = (\mathsf{Gen}^M, \mathsf{Tag}^M, \mathsf{Ver}^M)$, we want to use these to construct $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ which is AE. 8 Mar. 10:30

Note. It is important to use different randomness for the encryption scheme and MAC.

Candidate 1 – Encrypt and Tag

Consider the following.

- $\mathsf{Gen}(1^n)$: $k_E \leftarrow \mathsf{Gen}^E(1^n), k_M \leftarrow \mathsf{Gen}^M(1^n), k = (k_E, k_M)$.
- $\mathsf{Enc}_k(m)$: compute $c = \mathsf{Enc}^E_{k_E}(m)$ and $t = \mathsf{Tag}^M_{k_M}(m)$, output the ciphertext (c, t).
- $\mathsf{Dec}_k(c, t)$: compute $m = \mathsf{Dec}^E_{k_E}(c)$. If $\mathsf{Ver}^M_{k_M}(m, t) = 1$, output m, else output $\perp$.

Claim. This construction approach of AE is not CPA secure nor unforgeable.

Proof. We see the following.

- Although the encryption scheme we use is CPA secure, the MAC we use does not guarantee anything about confidentiality, e.g., it may contain a part of the message. If Tag is deterministic, then it also directly breaks CPA security from the same proof as Theorem 2.3.1.
- This construction is also not necessarily unforgeable since similarly, the encryption scheme is not guaranteed to be unforgeable. For example, ciphertext can have extra parts (junk) that one can "tweak" without causing failure, hence we get a new ciphertext for the same message.

⊛

Remark. If we can somehow enforce the tag to be the same length (as we did before), then we might be okay. But this rule out of the generic theorem we want, i.e., we want to allow any kind of UFCMA MAC to be used.

So, we might we to do things sequentially.

Candidate 2 – Tag, then Encrypt

This was a popular choice a few decades ago. Consider the following.

- $\mathsf{Gen}(1^n)$: $k_E \leftarrow \mathsf{Gen}^E(1^n), k_M \leftarrow \mathsf{Gen}^M(1^n), k = (k_E, k_M)$.
- $\mathsf{Enc}_k(m)$: compute $t = \mathsf{Tag}^M_{k_M}(m)$, then output ciphertext $c = \mathsf{Enc}^E_{k_E}(m \| t)$.
- $\mathsf{Dec}_k(c, t)$: compute $\mathsf{Dec}^E_{k_E}(c)$, parse as $m \| t$. If it is well formatted, check if $\mathsf{Ver}^M_{k_M}(m, t) = 1$. If so, output m, else output $\perp$.

Claim. This construction approach of AE is not CPA secure nor unforgeable.

Proof. We see the following.

- This AE construction is not necessarily CPA secure because the tag can be of different lengths for different messages. The concatenation of the message and tag then might be of different lengths. This could break CPA security. In addition, this AE construction might provide different types of error messages, which can be a potential vulnerability.
- This AE construction is not necessarily unforgeable for the same reason as candidate 1. The ciphertext can be tweaked, and MAC still might accept a tweaked ciphertext into a valid one.

⊛

Now, we try a different order of doing encryption and tagging.

Candidate 3 – Encrypt, then Tag

Consider the following.

- $\mathsf{Gen}(1^n)$: $k_E \leftarrow \mathsf{Gen}^E(1^n), k_M \leftarrow \mathsf{Gen}^M(1^n), k = (k_E, k_M)$.
- $\mathsf{Enc}_k(m)$: compute $c = \mathsf{Enc}^E_{k_E}(m)$, and $t = \mathsf{Tag}^M_{k_M}(c)$, output ciphertext and the tag pair (c, t).
- $\mathsf{Dec}_k(c, t)$: if $\mathsf{Ver}^M_{k_M}(c, t) = 1$, output $\mathsf{Dec}^E_{k_E}(c)$, else output $\perp$.

Theorem 3.4.1. If Π^E is CPA secure and Π^M is strongly unforgeable, then this cryptosystem (encrypt then tag) is a valid AE scheme.

Proof sketch. We see that this AE construction is

- CPA secure because the MAC function is only dependent on c and independent of the encryption key k_E.[a] Hence, confidentiality guarantee by the encryption scheme is preserved;
- unforgeable because is indicated directly from the strong unforgeability of MAC.

■

[a]This is why independent is important.

Remark. Some disadvantages of the "encrypt, then tag" AE scheme are

- the two encryption keys must be independent, which is tricky in practice;
- the tag and encryption have to be done sequentially, i.e., the AE needs two passes on data which is inefficient.

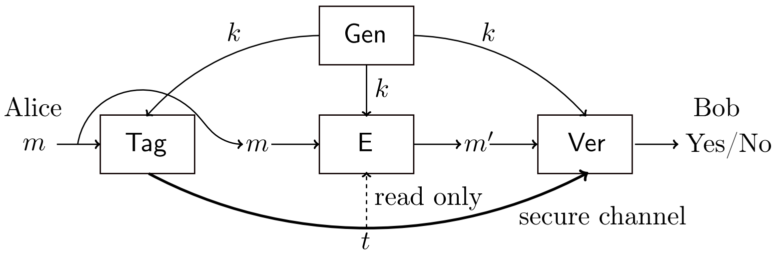
Note. There are other AE proposals, we name a few here.

- Galois counter mode (GCM).
- Offset codebook mode (OCB).
- Integrity aware parallelizable mode (IAPM).

Chapter 4

Symmetric Public Key Message Authentication

The encryption scheme we have talked about relies heavily on secret keys. However, key exchanges can be difficult in real life, and we want to construct an AE scheme that relies less on secret keys. To do this, we first focus on the integrity, i.e., we want a UFCMA MAC that does not rely on secret key. Consider the following new public key model that shares tags through a secure channel that is read only to the adversaries:



Getting integrity first already has some implication.

Example. Downloading a free software over P2P networks.

Proof. Everyone can see the data (message, ciphertext, whatever), but it's crucial to know that the data we're downloading is from the authority. ⊛

In this new public key MAC, instead of using PRF to generate a tag for the message, we use the so-called cryptographic Hash functions. These Hash functions are different from Hash functions (tables) used in data structures and algorithms: while collisions happened in both cases, sometimes we're able to compute what data will collide with each other in a Hash table, this is prohibited in the former case.[1]

Intuition. The adversary can't generate a different message such that it with the original tag is a valid pair.

4.1 Cryptographic Hash Family

We now define the main object we are going to focus on in this chapter formally.

Definition 4.1.1 (Cryptographic Hash function). A *cryptographic hash function* is a deterministic polynomial time function $H : \mathcal{K} \times X \to Y$ with $|X| > |Y|$ such that H_k satisfies collision resistance for every k.

[1] We see that this is another hardness assumption!

Intuition (Collision resistance)**.** Given any probabilistic polynomial time algorithm A, a function f is *collision resistance* if A can't find $x \neq x'$ such that $f(x) = f(x')$.

Later, we will formally define collision resistance in Definition 4.1.2.

Notation (Cryptographic Hash family)**.** We sometimes call Definition 4.1.1 a *Hash family* since it's actually a family of functions we're going to use over keys $k \in \mathcal{K}$.

Notice that H_k are compression functions since $|X| > |Y|$, so by the pigeon-hole principle, collisions will happen after the mapping, and hence it's reasonable to ask for collision resistance.

Intuition. Our goal is to make the adversary can't generate the same tag with a different message, and this is done by collision resistance.

Note. k, X, Y can depend on the security parameter.

The key generation algorithm $\mathsf{Gen}(1^n)$ outputs $k \in \mathcal{K}$ where k is not necessarily uniformly random. Usually we have $Y = \{0,1\}^n$ and there are two common choices of X spaces:

- $X = \{0,1\}^\infty$: $|X| = \infty, |Y| = 2^n$;
- $X = \{0,1\}^{2n}$: $|X| = 4^n, |Y| = 2^n$.

Lecture 17: Cryptographic Hash Functions & Merkle-Damgård Construction

4.1.1 Collision Resistance and Second Pre-Image Resistance

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We now discuss the effectiveness of hash families through the concepts of collision resistance and second pre-image resistance.

Definition 4.1.2 (Collision resistance)**.** A hash family (Gen, H) is said to be *collision resistant* if for all probabilistic polynomial time adversaries A,

$$\mathrm{Adv}_H^{\mathrm{CR}}(A) = \Pr_{k \leftarrow \mathsf{Gen}(1^n)}(A(k) \text{ outputs a collision}) = \mathrm{negl}(n),$$

where a collision is some x and x' where $H_k(x) = H_k(x')$ but $x \neq x'$.

Remark. It's critical to remember here that k is public. This means that the adversary A has access to k.

Definition 4.1.3 (Second pre-image resistance)**.** A hash family (Gen, H) is said to be *second pre-image resistant* (or target collision resistant) if for all probabilistic polynomial time adversaries A,

$$\mathrm{Adv}_H^{\mathrm{SPR}}(A) = \Pr_{k \leftarrow \mathsf{Gen}(1^n), x \leftarrow X}(A(x, k) \text{ outputs a collision involving } x) = \mathrm{negl}(n),$$

where a collision involving x is some x' such that $H_k(x) = H_k(x')$ but $x \neq x'$.[a]

[a] Note that x is given, and is chosen at random from some distribution X.

Remark. Once again, k is public.

Problem. What's the difference between these two definitions?

Answer. Against collision resistance, an adversary has to produce a collision of any two values in the domain. Against second pre-image resistance however, an adversary has to find a collision for some x chosen at random.

In fact, it's not hard to see that any hash family that is collision resistant is also second pre-image resistant, i.e., second pre-image resistance has a weaker level of security. ⊛

4.2 Forming Attacks

Now that we have definitions in place, we want to ask the following question: what sort of attacks could break hash families which are collision resistant and/or second pre-image resistant? Given a hash function, how long might it take to discover a collision?

4.2.1 Pigeon-Hole Attack on Collision Resistant Families

As we have seen before, suppose we have some hash functions $H_k\colon X \to \{0,1\}^\ell$. In this attack, an adversary tries $2^\ell + 1$ possible inputs where every input is a member of the domain space X.[2] By the pigeon-hole principle, we know that we must have at least one collision.

Remark. This attack takes $O(2^\ell)$ attempts. Is it possible to generate a more efficient attack?

4.2.2 Birthday Paradox Attack on Collision Resistant Families

Once again, suppose that we have some hash functions $H_k\colon X \to \{0,1\}^\ell$. If we choose q inputs from the domain space X, then we'll have $\binom{q}{2}$ possible pairs, which is approximately $\frac{q^2}{2}$ from the birthday paradox. Each one of these pairs represents a chance of colliding with a probability of $\frac{1}{2^\ell}$.

Suppose that we choose $q = \sqrt{2^{\ell+1}} \approx 2^{\ell/2}$, then the probability of a collision is now approximately

$$\frac{q^2}{2} \cdot \frac{1}{2^\ell} = \frac{(\sqrt{2^{\ell+1}})^2}{2} \cdot \frac{1}{2^\ell} = \frac{2^\ell}{2^\ell} = 1,$$

hence when running a birthday paradox attack against a collision resistant hash family, we know that we're likely to have a collision when the attacker attempts $\sqrt{2^{\ell+1}}$ queries.

Example. If $\ell = 128$, an attacker would need 2^{64} queries. This number of queries is computationally feasible, which means that collision resistant hash functions should not use key spaces this small.

Example (SHA-1). If we have $\ell = 166$, then we need approximately 2^{80} queries. This key space size is used by SHA-1 and is considered to be on the edge of feasible.

Example (SHA-256). $\ell = 256$ is computationally infeasible and is used in systems like SHA-256.

4.2.3 Attacks on Second Pre-Image Resistant Families

Both of the attacks described above discuss collision resistant families.

Problem. What about attacks on second pre-image resistant families?

Answer. For these hash families, pigeon-hole attacks are effective, but birthday paradox attacks are not effective. ⊛

For collision resistant systems, adversaries only need to find an arbitrary collision. Every x_n drawn from the domain space potentially conflicts with $x_1, x_2, x_3, \ldots, x_{n-1}$ which were drawn previously. For second pre-image resistant families however, $x_1, x_2, x_3, \ldots, x_{n-1}$ are irrelevant, and we only care if x_n

[2] Remember also that $|X|\rangle 2^\ell$ is true for a hash function.

collides with the original x which was provided as input to A. Therefore, $\ell = 128$ is considered to be on the lower end of acceptable key space sizes for hash families that are second pre-image resistant.

4.3 Merkle-Damgård Construction

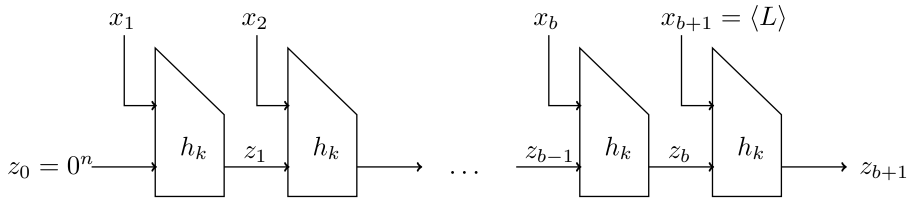
Sometimes we may be interested in hashing messages with variable sizes.

> **Example.** Suppose that we are sending a package over the internet and want to create a check sum by hashing the contents of package. This check sum will allow the receiver of the package to verify that the package they received wasn't corrupted by hashing its contents and comparing the result to the check sum. In this use case, the size of the package can vary.

> **Problem.** Is there a way that we can construct a hash function which can accept variable length messages using a hash function which accepts messages of a fixed-length?

> **Answer.** The solution here is the so-called *Merkle-Damgård construction*. ⊛ cite

Suppose we have some hash function $h\colon \mathcal{K} \times \{0,1\}^{2n} \to \{0,1\}^n$ which is collision resistant. If x is our message, suppose then that we split x into blocks $x_1 \| x_2 \| x_3 \| \ldots \| x_b$, where we let L represent the length of (un-padded) x such that $b = \lceil L/n \rceil$. We can then implement the following algorithm to "chain" the blocks together



Let this new construction be the function H, and note the following.

> **Note.** We append 0's to x to make it of length bn so $|x_i| = n$ indeed. Let the "initialization value" $z_0 = 0^n$ and compute $z_i = h(z_{i-1} \| x_i)$ for all $1 \leq i \leq b+1$, where $x_{b+1} = \langle L \rangle \in \{0,1\}^n$ is the n-bit binary representation of L. Finally, output $H_k(x) \coloneqq z_{b+1}$ as the hash value.

Theorem 4.3.1. If (Gen, h) is collision resistant, then (Gen, H) is also collision resistant.

Proof. Let A be an arbitrary probabilistic polynomial time attacker against (Gen, H), we build a new attacker A' against (Gen, h) such that $A'(k)$ receives a hash key k and runs $A(k)$, which outputs two distinct strings $x \neq x'$.

We show that whenever H succeeds, our constructed attacker A' against h also succeeds. Then from the fact that (Gen, h) is collision resistant, so is (Gen, H), i.e., if $H_k(x) = H_k(x')$, we must describe A' which obtains (from x, x') two distinct $2n$-bit strings $w \neq w'$ such that $h_k(w) = h_k(w')$.

Let $L = |x|$, $L' = |x'|$ be the length of the messages, and let z_i, z'_i be the intermediate values for input x and x', respectively. Recall that $z_{b+1} = H_k(x) = H_k(x') = z'_{b'+1}$.

- If $L \neq L'$, then we know that $z_b \| \langle L \rangle \neq z'_{b'} \| \langle L' \rangle$ and $h(z_b \| \langle L \rangle) = z_{b+1} = z'_{b'+1} = h(z'_{b'} \| \langle L' \rangle)$, so $w = z_b \| \langle L \rangle$ and $w' = z'_{b'} \| \langle L' \rangle$ is a collision in h_k.
- If $L = L'$, then x and x' have the same number of blocks ($b = b'$). We check whether $z_b = z'_b$:
  - If not, then by the same logic above, $w = z_b \| \langle L \rangle$ and $w' = z'_b \| \langle L' \rangle$ form a collision in h_k.
  - Otherwise, we have $z_b = z'_b$, and we "work backwards" from there: we have $h(z_{b-1} \| x_b) = h(z'_{b-1} \| x'_b)$, so we check whether $z_{b-1} \| x_b = z'_{b-1} \| x'_b$. If not, we have found a collision in h_k, and if so, we have $z_{b-1} = z'_{b-1}$ and continue working backwards.

> **Claim.** There is some i such that $h(z_{i-1} \| x_i) = h(z'_{i-1} \| x'_i)$ but $z_{i-1} \| x_i \neq z'_{i-1} \| x'_i$.

Proof. If not, then all blocks would satisfy $z_{i-1} \| x_i = z'_{i-1} \| x'_i$, i.e., $x_i = x'_i$ for $1 \leq i \leq b$, hence $x = x'$, a contradiction. ⊛

■

Lecture 18: Hash-and-Mac, HMAC, Public Key Cryptography

With the Merkle-Damgård construction, we see *the power of hash functions*: we can now transform arbitrary-length data to a fixed-length (very versatile). 15 Mar. 10:30

Remark. Hence, we can now use hash functions to turn a primitive that can only handle a fixed number of bits into one that can take arbitrary lengths.

One such example is MACs.

4.4 Arbitrary Length UFCMA MAC

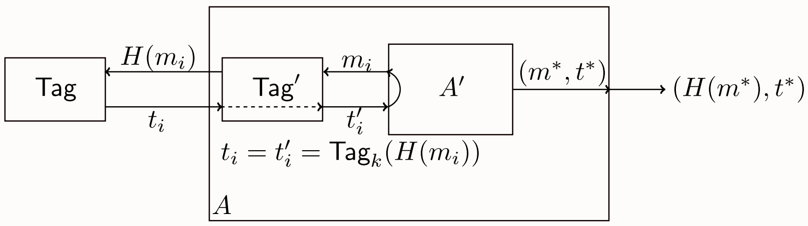
As previously seen. Recall the CBC-MAC.

Suppose we now have a hash function $H \colon X \to \{0,1\}^n$[3] which is a collision resistant, and a fixed-length MAC $\Pi = (\mathsf{Gen}, \mathsf{Tag}, \mathsf{Ver})$. Naturally, to tag an arbitrary length $m \in \{0,1\}^*$, simply output $\mathsf{Tag}_k(H(x))$.

Theorem 4.4.1. This composition gives a UFCMA MAC.

Proof. We prove this by reduction, where we must prove security of both components. First, from $\Pi = (\mathsf{Gen}, \mathsf{Tag}, \mathsf{Ver})$ (fixed-length), we define $\Pi' = (\mathsf{Gen}', \mathsf{Tag}', \mathsf{Ver}')$ were $\mathsf{Tag}'_k(m) = \mathsf{Tag}_k(H(m))$.

Then, given any probabilistic polynomial time adversary A against Π', we built an adversary A' against Π.



To show that A' has only negligible advantage, we look at the collision event. Let Q be the query set of A', i.e., the set of all m_i, and suppose $m^* \notin Q$.[a] Then, for any collision event, $\exists m \in Q$ such that $H(m^*) = H(m)$.

There are two cases, i.e., a collision either happen or doesn't happen, where

- if we have this collision, then we have found a collision in the hash function;
- otherwise, winning the game for A' and for A would be related.

Formally, we have

$$\begin{aligned}
\Pr(A' \text{ forges}) &= \Pr(A' \text{ forges} \land \text{collisions}) + \Pr(A' \text{ forges} \land \neg\text{collisions}) \\
&\leq \Pr(\text{collision}) + \Pr(A' \text{ forges} \land \neg\text{collisions}) \\
&= \operatorname{negl}(n) + \operatorname{negl}(n) \\
&= \operatorname{negl}(n),
\end{aligned}$$

where

- $\Pr(\text{collision}) = \operatorname{negl}(n)$ because H is collision resistant, and

[3] We no longer specify the key while defining the Hash as we discussed before.

- $\Pr(A' \text{ forges} \wedge \neg\text{collisions}) = \text{negl}(n)$ because Π is UFCMA.

∎

[a]Since forged message must be new, so if $m^* \in Q$, we don't need to include this into our probability.

Remark. We see that a fixed-length MAC with a hash function is an arbitrary length MAC!

Problem. How can we do this same type of proof for PRFs?

4.4.1 HMAC

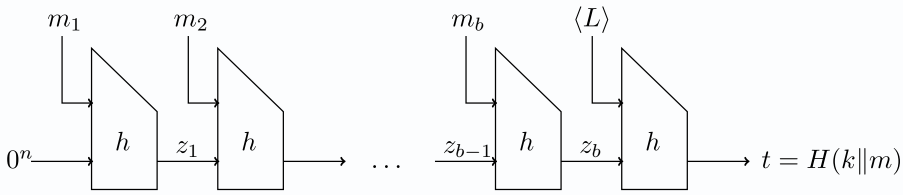
We see that the above construction starts with a fixed-length MAC and an arbitrary length hash function. Is that possible to use only a hash function to directly construct MACs/PRFs? In other words, are hash functions stronger than MACs?

Surprisingly, the answer is yes. Given an arbitrary length hash function H, it's possible to construct an arbitrary length MAC directly.

However, one might naively try the following.

Example (Amateur construction). To use an arbitrary length hash function H to construct an arbitrary length MAC, consider $\mathsf{Tag}_k(m) = H(k\|m)$. This is not UFCMA in general!

Proof. We see that it is easily forgeable if H uses Merkle-Damgård.[a] Consider the following.



We see that the length extension attack is trivial, we simply continue the chain to get the correct tag, i.e., $m^* = m\| \langle L \rangle \|$anything with $t^* = H(k\|m^*)$. ⊛

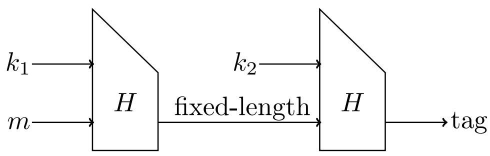
[a]As most mainstreams hashes do.

To fix this, someone proposed the so-called *HMAC*, and the idea is that given key $k = (k_1, k_2)$, generate $\mathsf{Tag}_k(m) = H(k_2\|H(k_1\|m))$.

Note. $H(k_1\|m)$ is fixed-length.

Theorem 4.4.2. If $H(k_1\|m)$ is a secure fixed-length MAC, and H has appropriate "pseudorandom properties", then the HMAC is unforgeable for arbitrary lengths.

Remark. This is informal, and it's beyond our scope. But the intuition should be clear, i.e.,



Now, the length extension attack doesn't work anymore.

To avoid 2 keys k_1 and k_2, people use heuristics.

Example. Take one key k and let $k_1 = k \oplus \mathsf{ipad}$ and $k_2 = k \oplus \mathsf{opad}$, where ipad (inner-pad) and opad (outer-pad) are two fixed "randomly looking" string.

Chapter 5

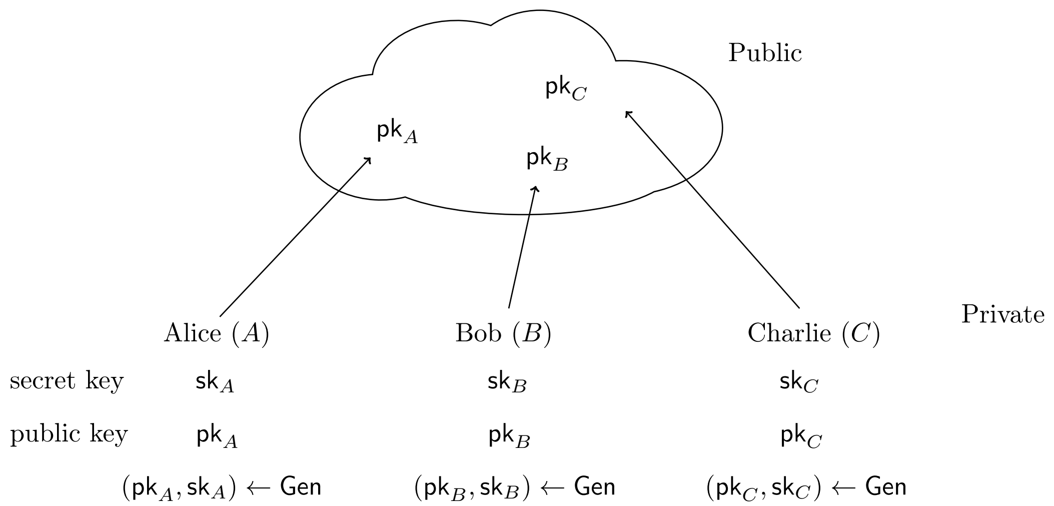
Asymmetric Public Key Message Security

What we have considered so far (i.e., CPA and MAC) is the so-called **symmetric** key cryptography, i.e., same key for encryption, decryption, or tagging. In this chapter, we again, first focus on the message security. However, symmetric key encryption is not realistic since it isn't scalable, e.g., Internet. So, the following question arose.

Problem. Do we really need the same key for encryption/decryption? In other words, what if we used different keys for encryption and decryption?

Answer. Actually, we can achieve quite a lot! ⊛

The idea is that we use 2 correlated (i.e. not independently generated) keys, the *public key* pk and *secret (private) key* sk.



Example. If Bob wants to send m to Alice, he encrypts it using pk_A. Only Alice who has sk_A can decrypt.

Remark. In the real world, we *combine* public key and secret key.

Even better, in 1976, Diffie and Hellman come up with the idea of exchanging a secret key over a public communication channel, which makes the whole thing works. The theory behind this is *number theory*!

Lecture 19: Elementary Number Theory

5.1 Number Theory

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Let's start by developing some basic number theory.

Definition 5.1.1 (Integer). The set of *integers* is denoted by $\mathbb{Z}$,

$$\mathbb{Z} := \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

Definition 5.1.2 (Natural number). The set of *natural numbers* is denoted by $\mathbb{N}$ or $\mathbb{Z}^+$,

$$\mathbb{N} = \mathbb{Z}^+ = \{1, 2, 3, \cdots\}.$$

Note. We do not include 0 in $\mathbb{N}$!

5.1.1 Primes

The following is the most important object in number theory.

Definition 5.1.3 (Prime). A natural number $p > 1$ is *prime* if it has no divisors other than 1 and p.

Example. $2, 3, 5, 7, 11, 13, 101, \cdots$ are all primes.

Definition 5.1.4 (Composite). A natural number $p > 1$ is *composite* if it is not prime.

Example. $4, 6, 8, 9, 15, 102, \cdots$ are all composites.

Remark. 1 is neither prime nor composite.

Perhaps one of the most important facts in basic number theory is the following.

Theorem 5.1.1 (Prime factorization). Every integer $N > 1$ can be written uniquely (up to ordering) as a product of (powers of) primes, i.e.,

$$N = \prod_i p_i^{e_i}$$

where p_i are prime with $e_i \geq 1$.

Example. $6 = 2 \cdot 3$, $102 = 2 \cdot 3 \cdot 17$, and $72 = 2^3 \cdot 3^2$.

Lemma 5.1.1 (Division with remainder). Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. There exist unique integers q, r such that

$$a = bq + r$$

for some $0 \leq r < b$.

Remark. These integers can be efficiently computed, i.e. we can find q, r in time polynomial in bit-length, e.g., $\log_2 a + \log_2 b + O(1)$.

Definition 5.1.5 (Greatest common divisor). The *greatest common divisor* $\gcd(a, b)$ of integers a, b is the largest integer g such that $g \mid a$ and $g \mid b$.

Example. $\gcd(24, 36) = 12$, $\gcd(15, 55) = 5$, $\gcd(12, 35) = 1$.

Definition 5.1.6 (Co-prime). Two integers a and b are said to be *co-prime* (or *relatively prime*) if $\gcd(a, b) = 1$.

It's important to note that the naive algorithm for computing GCD is inefficient.

Example. Consider $24 = 2^3 \cdot 3$ or $36 = 2^2 \cdot 3^2$, where gcd is found by taking the least power of primes common in both factorization. In this case $\gcd(24, 36) = 2^2 \cdot 3 = 12$. This is inefficient as factoring is inefficient.

Next we try to build a more efficient method of computing GCD's. It's based on the following.

Theorem 5.1.2 (Bezout theorem). Let $a, b \in \mathbb{Z}^+$. Then there exist $x, y \in \mathbb{Z}^+$ such that

$$\gcd(a, b) = ax + by.$$

Moreover, $\gcd(a, b)$ is the smallest positive integer that can be written this way.[a]

[a] That is, GCD is the smallest positive linear combination of a, b.

Proof. Let $I := \{a\hat{x} + b\hat{y} \mid \hat{x}, \hat{y} \in \mathbb{Z}\}$, in particular, $a, b \in I$, so I has positive integers.

Let $d = ax + by \in I$ for some $x, y \in \mathbb{Z}$ be the smallest positive integer in I. We must show that $d \mid a$, $d \mid b$ and if $d' \mid a$ and $d' \mid b$, then $d' \mid d$ (or alternatively $d' \leq d$, both conditions are equivalent).

Claim. d divides every element of I.

Proof. Say $c = a\hat{x} + b\hat{y} \in I$. Dividing c by d, we have $c = qd + r$ for some $0 \leq r < d$. Hence,

$$r = c - qd = a\hat{x} + b\hat{x} - q(ax + by) = a(\hat{x} - qx) + b(\hat{x} - qy) \in I.$$

But since d is the smallest positive integer in I we must have $r = 0$, therefore $d \mid c$. ⊛

Claim. d is the largest common divisor of a, b.

Proof. Suppose we have some d' such that $d' \mid a$ and $d' \mid b$. Then $d' \mid ax$ and $d' \mid by$. So, $d' \mid ax + by = d$, and $d' \leq d$. ⊛

■

5.1.2 Extended Euclid's Algorithm

We now want an efficient algorithm to compute not only $\gcd(a, b)$, but also the x, y coefficients as described in Bezout theorem. This is done using extended Euclid's algorithm.

Notation. $a \bmod b$ means remainder of division of a by b ($b \neq 0$).

Theorem 5.1.3 (Extended Euclid's algorithm). Let $a, b > 1$ be some integers. If $b \mid a$ then $\gcd(a, b) = b$. Now suppose $b \nmid a$, then

$$\gcd(a, b) = \gcd(b, a \bmod b).$$

Proof. We first note the following.

Claim. (a, b) and $(b, a \bmod b)$ have the same common divisors.

Proof. We need to show that $d \mid a, d \mid b \Leftrightarrow d \mid b, d \mid (a \bmod b)$. First, $d \mid b \Leftrightarrow d \mid b$ is trivial. Also, we know that $(a \bmod b) = a - qb$ for some $q \in \mathbb{Z}$. Now

- if $d \mid a$ and $d \mid b$, then $d \mid a - qb = (a \bmod b)$;
- if $d \mid b$ (so $d \mid qb$) and $d \mid (a \bmod b) = a - qb$, then $d \mid (a - qb) + qb = a$.

⊛

So (a, b) and $(b, a \bmod b)$ have the same common divisors, and so in particular, the same greatest common divisor. ∎

This naturally leads to the following (recursive) algorithm for computing GCD.

Algorithm 5.1: Extended Euclidean Algorithm

Data: $a, b \in \mathbb{Z}^+$ with $a \geq b > 0$.
Result: x, y such that $ax + by = \gcd(a, b)$

```
if b | a then
    return (0, 1)
else
    a = qb + r with 0 < r < b            // From Lemma 5.1.1
    (x', y') ← ExtendedEuclid(b, r)
    return (y', x' − y'q)
```

Theorem 5.1.4. The extended Euclid theorem is correct.

Proof. Since

$$bx' + ry' = \gcd(b, r) = \gcd(a, b)$$

from Theorem 5.1.3, and

$$bx' + (a - qb)y' = ay' + b(x' - qy') = ax + by,$$

so the extended Euclid theorem outputs $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$. ∎

Remark. The extended Euclid theorem makes a linear number (in the input length) of recursive calls, hence is efficient.

5.1.3 Group Theoretic View of Numbers

Consider the following.

Definition 5.1.7 (Integers modulo n)**.** The set $\mathbb{Z}_n$ of *integers modulo* n is defined as

$$\mathbb{Z}_n \coloneqq \{0, 1, 2, \cdots, n - 1\},$$

which is the set of all possible remainders of division by n.

Definition 5.1.8 (Integers co-prime to n)**.** The set $\mathbb{Z}_n^*$ of *integers co-prime to* n is defined as

$$\mathbb{Z}_n^* \coloneqq \{x \in \mathbb{Z}_n : \gcd(x, n) = 1\},$$

which is the set of all possible remainders of division by n that are co-prime to n.

Example. $\mathbb{Z}_6^* = \{1,5\}$, $\mathbb{Z}_{10}^* = \{1,3,7,9\}$, $\mathbb{Z}_7^* = \{1,2,3,4,5,6\}$.

Remark. For any prime p we have $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$.

Proof. Since p is prime all numbers less than p (except 0) are not divisible by p, and so must be co-prime to p.[a] ⊛

[a] The only factors of p are $1, p$.

5.1.4 Modular Arithmetic

Consider the following.

Definition 5.1.9 (Equivalence modulo n)**.** We define

$$a = b \pmod n \Leftrightarrow n \mid a - b,$$

i.e., elements are identified if their remainders of division of a and b by n are the same.

Notation. $a \equiv b \pmod n$ and $a \equiv^n b$ mean the same as $a = b \pmod n$.

Remark (Equivalence Relation)**.** Congruence modulo n is an equivalence relation.

Proof. We see that

(a) for all $a \in \mathbb{Z}$, $n \mid 0 = a - a$ so $a \equiv a \pmod n$;

(b) for all $a, b \in \mathbb{Z}$ if $a \equiv b \pmod n$ that is, $n \mid a - b$ then $n \mid b - a$, i.e., $b \equiv a \pmod n$;

(c) for all $a, b, c \in \mathbb{Z}$ if we have $a \equiv b \pmod n$ and $b \equiv c \pmod n$. Then $n \mid a - b$ and $n \mid b - c$. So $n \mid (a-b) + (b-c) = a - c$, hence $a \equiv c \pmod n$.

⊛

Lecture 20: Group Theory

Remark (Operation of mod)**.** For $a = a' \pmod n$ and $b = b' \pmod n$, we have 22 Mar. 10:30

- $a + b = a' + b' \pmod n$;
- $a - b = a' - b' \pmod n$;
- $a \cdot b = a' \cdot b' \pmod n$.

We don't have such a definition for division.

Example. Observe the counter-example that $3 \cdot 2 = 15 \cdot 2 \pmod{24}$ however, $3 \neq 15 \pmod{24}$.

Definition 5.1.10 (Invertible)**.** b is *invertible* if there is some c such that $b \cdot c = 1 \pmod n$.

Notation (Moldular inverse)**.** In this case, we say that c is the *modular inverse* of b.

Note. In $\mathbf{R}$, we don't see inverses of integers being integers; however, with modular arithmetic we can actually define integer inverses.

The following lemma tells us when do we have inverses for integers.

Lemma 5.1.2. Given $a \geq 1$ and $N > 1$, a is invertible if and only if $\gcd(a, N) = 1$.

Proof. The forward direction is easy. Let $a \cdot c = 1 \pmod{N}$, then $a \cdot c - 1 = N \cdot q$, implying $ac - Nq = 1$, which is $\gcd(a, N)$ since GCD is the smallest positive integer expressible in this way.

For the backward direction, since $\gcd(a, N) = 1$, we know that there exists x, y such that $x \cdot a + y \cdot N = 1$, hence $x \cdot a = 1 \pmod{N}$, i.e., x is the modular inverse. ∎

Corollary 5.1.1. We can calculate the modular inverse of integers.

Proof. We use the extended Euclid theorem with Lemma 5.1.2. ∎

Theorem 5.1.5 (Uniqueness of modular inverses). If c, c' are both inverses of a, then $c = c' \pmod{N}$.

Proof. Since

$$\begin{cases} a \cdot c = 1 & \Rightarrow N \mid a \cdot c - 1; \\ a \cdot c' = 1 & \Rightarrow N \mid a \cdot c' - 1 \end{cases} \Rightarrow N \mid a \cdot (c - c') \Rightarrow \gcd(N, a) = 1$$

since the inverse exist. Hence, $N \mid c - c'$, so $c = c' \pmod{N}$. ∎

Example. Let $a = 11, N = 17$, then

$$(-3) \cdot 11 + \underbrace{2 \cdot 17} = 1 \Rightarrow -3 = 14 \pmod{17}.$$

Verifying, we see that indeed $11 \cdot 14 = 1 \pmod{17}$.

Note. We take modulo 17 on both sides of the first equation.

5.2 Group Theory

Let's introduce group formally.

Definition 5.2.1 (Group). A *group* is a set G along with a binary operation $\circ \colon G \times G$ for which the following conditions hold.

- Closure: for all $g, h \in G$, $g \circ h \in G$.
- Existence of an identity: there exists $e \in G$ such that for all $g \in G$, $e \circ g = g = g \circ e$.
- Existence of inverses: for all $g \in G$, there exists an inverse $h \in G$ such that $g \circ h = e = h \circ g$.
- Associativity: for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

Notation. We sometimes denote e as 1.

Definition 5.2.2 (Abelian group). A group G with operation $\circ$ is *Abelian* if $\circ$ is commutative, i.e., for all $g, h \in G$, $g \circ h = h \circ g$.

Note. When the binary operation is understood, we simply call the set G a group.

Definition 5.2.3 (Finite group). When G has a finite number of elements, we say G is a *finite group*.

Definition 5.2.4 (Infinite group). If G has infinitely many elements, we say G is an *infinite group*.

Definition 5.2.5 (Order). The *order* of a group G is defined as $|G|$.[a]

[a] I.e., the number of elements in G.

Example. $(\mathbb{Z}_N, +)$ is a groups.

Proof. $(\mathbb{Z}_N, +)$ with order N satisfies:

- ✓ $a + b = b + a \pmod N$;
- ✓ $a + 0 = 0 + a = a \pmod N$;
- ✓ $a + (-a) = 0 \pmod N$;
- ✓ $a + (b + c) = (a + b) + c \pmod N$.

⊛

Example. $(\mathbb{Z}_N^*, \cdot)$ is a group.

Proof. $(\mathbb{Z}_N^*, \cdot)$, where p is prime and has order $p - 1$ satisfies:

- ✓ $a \cdot b = b \cdot a \pmod N$;
- ✓ $a \cdot 1 = 1 \cdot a = a \pmod N$;
- ✓ $a \cdot (-a) = 1 \pmod N$;
- ✓ $a \cdot (b \cdot c) = (a \cdot b) \cdot c \pmod N$.

⊛

Definition 5.2.6 (Cyclic group). A group G is *cyclic* if $\exists g \in G$ such that

$$G = \{1 = g^0, g^1, g^2, \dots, g^{n-1}\}.$$

Notation (Generated). We say that a cyclic group G is *generated* by g, and written as $G = \langle g \rangle$.

Note. In the above notation, we assume that $|G| = n$. However, we can also define a cyclic group as an infinite group generated by a single element g and its inverse g^{-1}[a] and still write $G = \langle g \rangle$.

[a] Think about why.

5.2.1 Lagrange's Theorem

The first important theorem in group theory is considering the size (order) of the subgroup and its parent.

Definition 5.2.7 (Subgroup). A group $G' \subseteq G$ is a *subgroup* if $\leq (G', \circ)$ is a group.

Theorem 5.2.1 (Lagrange's theorem). If $G' \subseteq G$ is a subgroup, then $|G'| \mid |G|$.[a]

[a] I.e., the order of subgroup divides the order of the original group.

Note. Lagrange's theorem tell us that the orders of sub- and parent groups cannot be too close.

Example. Consider $\mathbb{Z}_p^*$ for prime p with $p = 7$, i.e., $\mathbb{Z}_7^* = \{1, \dots, 6\}$.

Proof. Consider the set of powers of 3: $\{1, 3, 9 = 2, 6, 18 = 4, 12 = 5, 15 = 1\} = \{1, 3, 2, 6, 4, 5\}$. We see that 3 generate $\mathbb{Z}_7^*$.

However, consider the set of powers of 2: $\{1, 2, 4, 8 = 1, \dots\} = \{1, 2, 4\}$. Clearly, 2 is not a generator since $3 \not| 7$. ⊛

5.2.2 Group Exponentiation

In Cryptography, it is often useful for us to be able to describe a group operation applied, say m, number of times to a group element $g \in G$. We will now state and prove a very useful theorem.

Theorem 5.2.2. Let G be a finite group with $m = |G|$. Then for any element $g \in G$, $g^m = 1$.

Proof. For simplicity, we prove for when G is Abelian.[a] Fix arbitrary $g \in G$, and let $g_1, \dots, g_m$ be the elements of G. We claim that

$$g_1 \cdot g_2 \cdots g_m = (gg_1) \cdot (gg_2) \cdots (gg_m).$$

Observe that $gg_i = gg_j \Rightarrow g_i = g_j$ since we can multiply both sides by g^{-1}, so each of the m elements in parentheses on the right-hand side of the displayed equation is distinct.

Also notice how there are exactly m elements in G. We see that the m elements being multiplied together on the right-hand side are just the elements of G, just in some permuted order.

Now as, G is Abelian, we have commutativity and the order of multiplication doesn't matter. We can also "pull out" all occurrences of g to obtain

$$g_1 \cdot g_2 \cdots g_m = (gg_1) \cdot (gg_2) \cdots (gg_m) = g^m \cdot (g_1 \cdot g_2 \cdots g_m),$$

giving us $g^m = 1$. ■

[a] Though it holds for any finite group.

Theorem 5.2.2 gives us several very useful corollaries that we'll state below whose proofs could be found in [KL20, §7].

Corollary 5.2.1 (Fermat's little theroem)**.** For all prime p, $\gcd(a, p) = 1$ implies $a^{p-1} = 1 \bmod p$.

Definition 5.2.8 (Euler's totient function)**.** The *Euler's totient function* $\varphi(N)$ is defined as $\varphi(N) = |\{a \mid 1 \leq a \leq N, \gcd(a, N) = 1\}|$.

Example. We see that $|\mathbb{Z}_N^*| = \varphi(N)$.

Corollary 5.2.2 (Euler's theorem)**.** If $\gcd(g, N) = 1$, then $g^{\varphi(N)} = 1 \bmod N$.

Corollary 5.2.3. For $m = |G|$, for all $g \in G$ and all $x \in \mathbb{Z}$, $g^x = g^{x \bmod m}$.

Proof. Since we know that $g^m = 1$ from Theorem 5.2.2. ■

Corollary 5.2.4. For $m = |G| > 1$, $e \in \mathbb{Z}$, and $\gcd(e, m) = 1$. Let $d = e^{-1} \bmod m$, then the function $f_e \colon G \to G$ defined as $f_e(g) = g^e$ is a bijection with f_d being the inverse.

Proof. We see that $f_d(f_e(g)) = f_d(g^e) = (g^e)^d = g^{e \cdot d} = g^{e \cdot d \bmod m} = g^1 = g$. ■

5.2.3 Fast Exponentiation

Say we have a group element g, and we want to compute g^M.

Example (Naive method). We simply calculate g^M as

$$g^m = \underbrace{g \cdot g \cdots g}_{M \text{ times}} = \underbrace{(((g \cdot g) \cdot g) \cdot g) \cdots}_{M \text{ times}}.$$

This clearly not-efficient (polynomial runtime) in the bit-length of M, so we want something faster. The observation below forms the foundation for fast exponentiation, that'd allow us to do this computation in $O(\log |M|)$ time.

Note. When $M = 2^m$,

$$g^{2^m} = g^{2^{m-1}} \cdot g^{2^{m-1}} = \left(g^{2^{m-1}}\right)^2$$

That is we can get the following terms recursively:

$$g, g^2, g^4, g^8, g^{16}, \cdots.$$

Problem. But how many operations do we end up performing in this case?

Answer. We see that $T(M) = T(M/2) + 1$, hence $T(M) = \log M = m$. ⊛

We can generalize this as follows. Firstly, write M as

$$M = \sum_{i=0}^{\ell} m_i \cdot 2^i$$

for $m_i \in \{0, 1\}$.

Intuition. Think about the binary representation of M.

Then, for g^M, we see that

$$g^M = \prod_{i=0}^{\ell} g^{m_i \cdot 2^i},$$

hence by applying the above trick for each g^{2^i} for those $m_i = 1$, we only need $O(\ell^2)$ multiplications altogether if $\log M = \ell$.

Corollary 5.2.5. Fast exponentiation allows us to compute inverses very fast as $g^{-1} = g^{|G|-1}$.

Proof. Since $g^{|G|} = 1$. ■

Remark. For $\mathbb{Z}_p^*$, we have an even faster method, i.e., the extended Euclid algorithm.

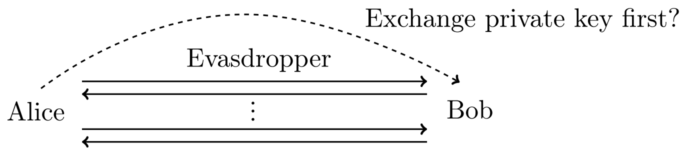
5.3 Diffie-Hellman Key Exchange

We now know that how to compute g^m from g efficiently, but do we know how to compute m from g^m? This is known as the discrete logarithm function.[1] This problem is conjectured to be extremely difficult, hence the main idea is that, can we utilize the hardness of computing m from g^m to build a strong cryptographic protocol?

[1]Think of it as "$\log_g$"$g^m = m$.

Lecture 21: Diffie-Hellman Key Exchange

The answer is yes! In fact, imagine Alice and Bob want to communicate on the public channel, and they want to exchange keys first. 27 Mar. 10:30



The up shot is that, we can use the hardness of computing discrete logarithmto build such a secure key exchange protocol! To do this, we start by fixing a large cyclic group G of known order q, where the length of q is approximately the security parameter.[2]

Usually, we let $G = \mathbb{Z}_{q+1}^*$ for $q+1$ being a prime number.[3] Since q corresponds to the security parameter, so one stand-alone question which is worth thinking about is the following.

Exercise. How to generate any large prime number (with some certain number of bits)?

Answer. Gauss has solved this for us. Specifically, he showed a fundamental question in number theory: there are $\pi(n) \approx n/\ln n$ many prime numbers between 1 to n? By interpreting this as some sorts of "density" $(\pi(n)/n \approx \ln^{-1} n)$ of primes, modern computer tries different random numbers w.r.t. this density and test whether they are prime to generate one. ⊛

Hence, we're able to generate a large prime $q+1$, hence generate $\mathbb{Z}_{q+1}^*$. Then we define the following key exchange procedure, called Diffie-Hellman protocol.

Definition 5.3.1 (Diffie-Hellman protocol)**.** The *Diffie-Hellman key exchange protocol* works as cite

(a) Alice sends Bob the group G, the generator g, and $A = g^a$ where a is randomly picked in G;

(b) Bob sends back $B = g^b$ where b is randomly picked in G;

(c) both of them calculate the key $K = g^{ab}$.

$$\begin{array}{ccc} \text{Alice} & \xrightarrow{\langle G\rangle,\ g,\ A = g^a \in G} & \text{Bob} \\ a \leftarrow \mathbb{Z}_q & \xleftarrow[B = g^b \in G]{} & b \leftarrow \mathbb{Z}_q \end{array}$$

Note. Given G, we can obtain a generator g for G.

Proof. There's an efficient non-trivial trial-and-error approach [KL20, Appendix B.3]. ⊛

We see that an Eavesdropper can obtain A and B, but not a, b. Moreover, getting a, b from A, B requires solving the *discrete logarithmproblem.* There is no known polynomial time algorithm. But on the other hand, Alice and Bob now shares the key K where

$$K = A^b = (g^a)^b = (g^b)^a = B^a.$$

Informally, Eavesdropper will have to take discrete logarithm to break this, i.e., after getting $\langle G\rangle, g, A = g^a$, and $B = g^b$, to get $K = g^{ab}$, the only way is to obtain either a or b. Formally, the security primitive is based on the decisional Diffie-Hellman assumption.

Definition 5.3.2 (Decisional Diffie-Hellman assumption)**.** Given a group $G = \langle g\rangle$ with $q = |G|$, the *decisional Diffie-Hellman assumption (DDH)* holds if $(g, g^a, g^b, g^{ab}) \in G^4$ where $a, b \leftarrow \mathbb{Z}_q$ is indistinguishable from (g, g^a, g^b, g^c) where $c \leftarrow \mathbb{Z}_q$.

[2] Since the length of q is the security parameter (1^n), i.e., $q = \log(1^n)$.
[3] The reason to use $q+1$ is, if $q+1$ is a large prime, then q is always an even number.

Intuition. The decisional Diffie-Hellman assumption is trying to say that if g^a, g^b are uniform random from the group, then the key K generated as g^{ab} will also be a uniform random element in the group.

Conjecture 5.3.1. DDH holds for $\mathbb{Z}^\star_{q+1}$ with $q+1$ prime.

An immediate consequence of Conjecture 5.3.1 is that Diffie-Hellman protocol is secure.

Note. Since there are so many sophisticated algorithms for solving the discrete logarithmproblem for groups like $\mathbb{Z}^\star_q$, we normally require $q > 2^{>1000}$ to make the security parameter larger than 1000.

However, we note that our network system works with bit-strings, not group elements (not even $\mathbb{Z}^*_{p+1}$), i.e., we want a key to be a uniform random bit-string.

Note (Key derivation). There are many *key derivation algorithms* which turn a random group element g^{ab} into a random bit-string, some involving hashing. This is used in practice heavily for instance inside VPN protocols.

Remark. The Diffie-Hellman protocol might not work alone if the Eavesdropper is active in the real world, but it works fine in our passive setup.

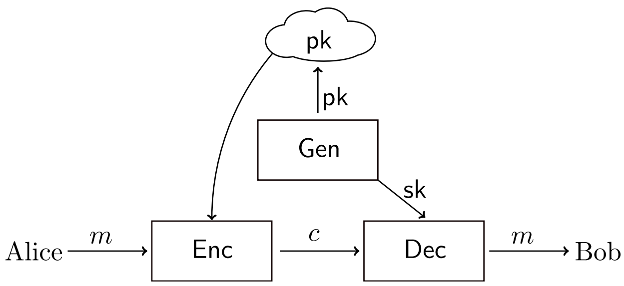
5.4 Public Key Message Encryption

By using Diffie-Hellman protocol, we can first exchange the key at the beginning of the communication. But what if we want to have a protocol $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ that can *directly* handle public key encryption? How to model this? Analogs of EAV/CPA security in public key setting? Consider the following.

Definition 5.4.1 (Public key scheme). The *public key scheme* is a cryptosystem $\Pi = (\mathsf{Gen}, \mathsf{Dec}, \mathsf{Enc})$ where

- $\mathsf{Gen}(1^n)$: outputs $(\mathsf{pk}, \mathsf{sk})$;
- $\mathsf{Enc}(\mathsf{pk}, m)$: outputs ciphertext c where $m \in \mathcal{M}$;
- $\mathsf{Dec}(\mathsf{sk}, c)$: outputs $m \in \mathcal{M}$ (or fail "$\perp$"[a]).

[a] With some protocols happen with negligibles probability.



Naturally, we should have the following.

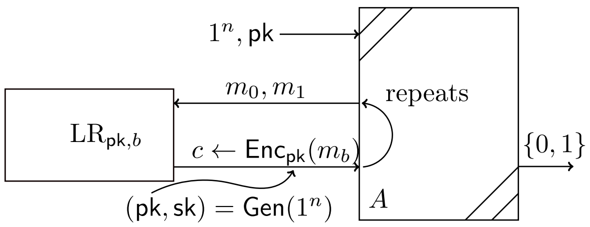
Definition 5.4.2 (Correctness). A public key scheme is *correct* if for $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^n)$ and all $m \in \mathcal{M}$,

$$\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, m)) = m.$$

5.4.1 Chosen Plaintext Attack Secrecy

How about the security definition? Analogous to how we define the CPA secrecy, we again use a left-right oracle to define the CPA game, but this time, we replace k by pk.

Definition 5.4.3 (Chosen plaintext attack game)**.** The *public key version chosen plaintext attack game* for a probabilistic polynomial time adversary A against a public key scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ allows A to make $\mathsf{poly}(n)$ number of queries (m_0, m_1) to a Left-right oracle $\mathrm{LR}_{\mathsf{pk},b}(\cdot,\cdot)$ to get a ciphertext for each query, and output a decision bit in the end.



Correspondingly, the advantage is defined as follows.

Definition 5.4.4 (Advantage)**.** Given an adversary A in a CPA game, the *advantage* $\mathrm{Adv}_{\Pi}^{\mathrm{CPA}}(A)$ in distinguishing "world 0" and "world 1" is given by

$$\mathrm{Adv}_{\Pi}^{\mathrm{CPA}}(A) := \left| \Pr_{(\mathsf{pk},\mathsf{sk})\leftarrow\mathsf{Gen}(1^n)} (A^{\mathrm{LR}_{\mathsf{pk},0}(\cdot,\cdot)} \text{ accepts}) - \Pr_{(\mathsf{pk},\mathsf{sk})\leftarrow\mathsf{Gen}(1^n)} (A^{\mathrm{LR}_{\mathsf{pk},1}(\cdot,\cdot)} \text{ accepts}) \right|.$$

Then, the public key version CPA secrecy is defined as follows.

Definition 5.4.5 (Chosen Plaintext attack secrecy)**.** A public key scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is *chosen plaintext attack secure* if for every probabilistic polynomial time adversary A, the advantage is negligible.

5.4.2 EAV Secrecy and CPA Secrecy

One might ask, why do we directly allow $\mathsf{poly}(n)$ queries, rather than start by only one query as in the EAV game? The reason is that in fact, the number of queries to the left-right oracle doesn't matter! I.e., if we have "EAV"-secrecy, then we also have security under multiple queries.

Intuition. The intuition behind the proof is, imagine a many-query attacker A that makes up to $q = \mathsf{poly}(n)$ queries. Consider the following worlds:

- Hybrid 0 (left world): all queries (m_0, m_1) to the LR oracle are answered by $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$.
- Hybrid 1: First query (m_0, m_1) is answered by $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$, then $\mathsf{Enc}_{\mathsf{pk}}(m_0)$ thereafter.
- Hybrid 2: Similar for the first 2 queries

  $\vdots$

- Hybrid q (right world): All queries are answered by $\mathsf{Enc}_{\mathsf{pk}}(m_1)$.

Then, by sequentially bound the advantage, we're done.

Lecture 22: CPA Security and ElGamal Cryptosystem

Formally, we have the following. 29 Mar. 10:30

Theorem 5.4.1. For public key encryption schemes Π, EAV secrecy is equivalent to CPA secrecy.

Proof. Since EAV is weaker than CPA in terms of secrecy, we only need to show that EAV implies CPA. Imagine a many-query attacker A that makes up to $q = \mathsf{poly}(n)$ queries, and consider the following worlds:

- Left world (H_0): all queries (m_0, m_1) to the LR oracle are answered by $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$.
- Hybrid 1 (H_1): First query (m_0, m_1) is answered by $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$, then $\mathsf{Enc}_{\mathsf{pk}}(m_0)$ thereafter.
- Hybrid 2 (H_2): Similarly to H_1, but answers $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$ for the first 2 queries.

  $\vdots$

- Hybrid q (H_q): All queries answered by $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$ (i.e., right world).

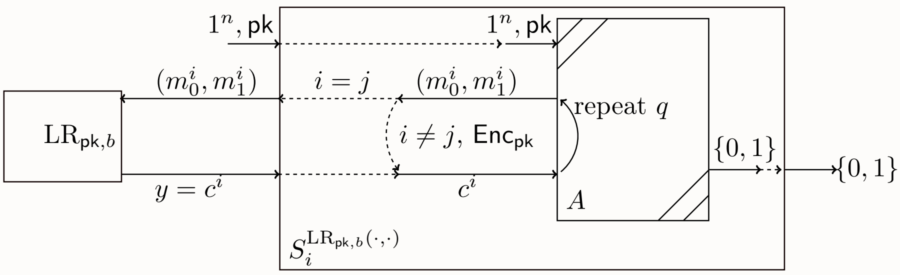
Intuition. The difference between H_{i-1} and H_i is only how the i-th query is answered.

Now, we build a "simulator" $S_i^{\mathrm{LR}_{\mathsf{pk},b}(\cdot,\cdot)}(\mathsf{pk})$ that gets *one query* and simulates either H_{i-1} or H_i depending on b. Specifically, on the j-th query (m_0^j, m_1^j) of A:

- If $j < i$, S_i runs $c^j \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1^j)$.
- If $j > i$, S_i runs $c^j \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0^j)$.
- If $j = i$, S_i queries its LR oracle and gives the result to A.

We see that

- if S_i is in the left world ($b = 0$), then we perfectly simulate H_{i-1};
- if S_i is in the right world ($b = 1$), then we perfectly simulate H_i.

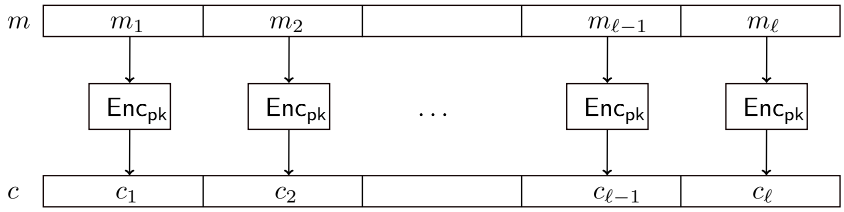


By the triangle inequality, we have

$$
\begin{aligned}
\mathrm{Adv}_{\Pi}^{\mathrm{CPA}} &= |\Pr(A = 1 \text{ in } H_0) - \Pr(A = 1 \text{ in } H_q)| \\
&= \left| \sum_{i=1}^{q} (\Pr(A = 1 \text{ in } H_{i-1}) - \Pr(A = 1 \text{ in } H_i)) \right| \\
&\leq \sum_{i=1}^{q} |\Pr(A = 1 \text{ in } H_{i-1}) - \Pr(A = 1 \text{ in } H_i)| \leq \sum_{i=1}^{q} \underbrace{\mathrm{Adv}_{\Pi}^{\mathrm{EAV}}(S_i)}_{\mathrm{negl}(n)} = \mathrm{negl}(n)
\end{aligned}
$$

where $\mathrm{Adv}_{\Pi}^{\mathrm{EAV}}(S_i) = \mathrm{negl}(n)$ by assumption and $q \cdot \mathrm{negl}(n) = \mathsf{poly}(n) \cdot \mathrm{negl}(n) = \mathrm{negl}(n)$. ■

Theorem 5.4.1 implies that we can encrypt long messages bit-by-bit, block-by-block, or broken up in any other reasonable way. One call to Enc on "long" messages translates to many calls on "short" messages, which is allowed by Theorem 5.4.1.



Theorem 5.4.2. Any public key encryption scheme with deterministic $\mathsf{Enc}_{\mathsf{pk}}(\cdot)$ algorithm can't be CPA secure.[a]

[a] Not even for one query due to Theorem 5.4.1.

Proof. Query $c \leftarrow \mathsf{LR}_{\mathsf{pk},b}(m_0, m_1)$ for any $m_0 \neq m_1$. Then run $c' = \mathsf{Enc}_{\mathsf{pk}}(m_0)$.[a] If $c = c'$, output 0, else 1. This will produce a perfect advantage. ■

[a] Notice that both pk and the encryption function are public.

5.4.3 ElGamal Encryption

Let's construct a CPA secure public key encryption scheme! We *kind of* already saw this in Diffie-Hellman, which is formalized by ElGamal — "the public key encryption version of Diffie-Hellman". cite

Intuition. ElGamal converted Diffie-Hellman into $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$. Basically message is $m \in G$, the "one-time-pad effect" would involve multiplying m with something random (e.g., K).

Formally, consider the following.

Definition 5.4.6 (ElGamal encryption scheme)**.** The *ElGamal encryption scheme* $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is defined as

- $\mathsf{Gen}(1^n)$: choose random $a \leftarrow \mathbb{Z}_q$, output $(\mathsf{pk}, \mathsf{sk}) = (g^a, a)$;[a]
- $\mathsf{Enc}(\mathsf{pk}, m)$ for and $m \in G$: choose random $b \leftarrow \mathbb{Z}_q$, output ciphertext $(B, c) = (g^b, m \cdot \mathsf{pk}^b)$;[b]
- $\mathsf{Dec}(\mathsf{sk}, (B, c))$: compute $K = B^{\mathsf{sk}}$, output $c \cdot K^{-1} \in G$.

[a] This is what Alice does in Diffie-Hellman.
[b] We see that B is random and pk^b is essentially the "key": this is what Bob does in Diffie-Hellman.

Intuition. Consider substituting $\mathsf{pk} = A = g^a$ and $\mathsf{sk} = a$.

Note. Recall that given K, we can compute K^{-1} efficiently.

We now show that the ElGamal scheme is really what we want.

Claim. The ElGamal scheme is correct.

Proof. Since for all $m \in G$, $(\mathsf{pk}, \mathsf{sk}) = (g^a, a)$, hence

$$\mathsf{Enc}(\mathsf{pk}, A) = (B, c) = (g^b, m \cdot (g^a)^b),$$

so

$$\mathsf{Dec}(B, c) = c \cdot (B^a)^{-1} = m \cdot g^{ab} \cdot (g^{ab})^{-1} = m.$$

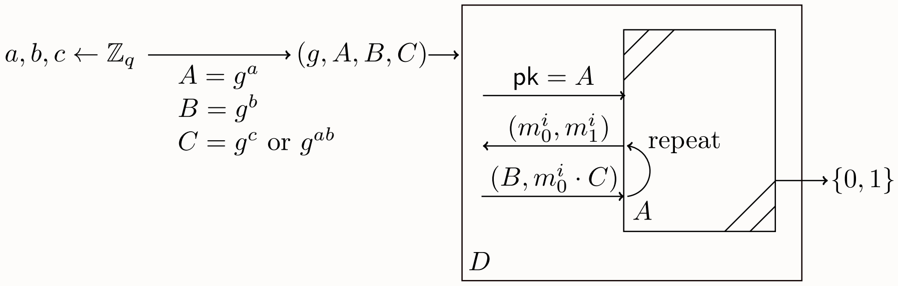
⊛

As for CPA security, we again consider the decisional Diffie-Hellman assumption over G, i.e.,

$$(g, g^a, g^b, g^{ab}) \in G^4 \text{ for } a, b \leftarrow \mathbb{Z}_q \text{ is indistinguishable from } (g, g^a, g^b, g^c) \in G^4 \text{ for } a, b, c \leftarrow \mathbb{Z}_q.$$

Theorem 5.4.3. Assuming decisional Diffie-Hellman assumption for G, then the ElGamal scheme Π is CPA secure.

Proof. Let A be any feasible probabilistic polynomial time attacker against Π, we use A to build a distinguisher D which breaks the decisional Diffie-Hellman assumption.



There are two worlds:

- "real" world: if (g, A, B, C) is a Diffie-Hellman tuple, D perfectly simulates the left CPA world since $C = g^{ab}$, so the ciphertext is $\mathsf{Enc}_{\mathsf{pk}}(m_0^i) = (g^b, m_0^i \cdot \mathsf{pk}^b) = (g^b, m_0^i \cdot g^{ab}) = (g^b, m_0^i \cdot C)$;
- "ideal" world: if (g, A, B, C) is random, then D simulates a "hybrid" CPA world where the ciphertext is two independent random group elements (regardless of the message).

Symmetrically, we can construct D' against decisional Diffie-Hellman assumption that replies to A with $(B, m_1^i \cdot C)$. Then, we have

$$\mathrm{Adv}^{\mathrm{CPA}}(A) \leq \mathrm{Adv}^{\mathrm{DH}}(D) + \mathrm{Adv}^{\mathrm{DH}}(D') = \mathrm{negl}(n) + \mathrm{negl}(n) = \mathrm{negl}(n),$$

which is a contradiction, hence we're done. ∎

Lecture 23: RSA Function and RSA Encryption

5.5 RSA Cryptosystem

3 Apr. 10:30

The hardness assumption of Diffie-Hellman relies on the hardness of the discrete logarithm problem (finding an unknown exponent, i.e., given g^a) in a group of known order. We now ask the following.

Problem. Is there other hardness assumption we can use?

Answer. Yes! As we will see, we can utilize the RSA function, which relies on the hardness of "factoring",[a] and finding the base (unknown but known exponent) in a group of unknown order. ⊛

[a] I.e., the prime factorization.

By utilizing this new hardness, beyond public key encryption (as provided by ElGamal), we can also sign messages, i.e., we get a cryptosystem. We first focus on encrypting.

5.5.1 RSA Function

Let $N = pq$ be the product of two large distinct primes p and q. Then, $\mathbb{Z}_N^* = \{a \in \mathbb{Z}_N \mid \gcd(a, N) = 1\}$, i.e., we start with $\mathbb{Z}_N$ and throw out all multiples of p and q.

Remark. This means $\varphi(N) = (p-1)(q-1)$.

As previously seen. In any group G, $a^{|G|} \equiv 1 \pmod{N}$ for all $a \in G$. Furthermore, Euler's theorem states that for any $a \in \mathbb{Z}_N^*$, we have $a^{\varphi(N)} \equiv 1 \pmod{N}$.

Intuition. If any element is raised to a multiple of $\varphi(N) \pmod{N}$, we get back the original element!

This motivates the following: given N, e, and d, we defined the so-called RSA function and its inverse.

Definition 5.5.1 (RSA function)**.** The *RSA function* $\mathrm{RSA}_{N,e}(x)$ is defined as

$$\mathrm{RSA}_{N,e}(x) = x^e \bmod N.$$

A particularly useful property of RSA is that by defining $d \equiv e^{-1} \pmod{\varphi(N)}$, the RSA function is a bijection from $\mathbb{Z}_N^*$ to itself with the inverse function being

$$\mathrm{RSA}_{N,d}(y) = y^d \bmod N.$$

Proposition 5.5.1. RSA is a bijection with $\mathrm{RSA}_{N,d}(y)$ for $d \equiv e^{-1} \pmod{\varphi(N)}$ being the inverse of $\mathrm{RSA}_{N,e}(x)$.

Proof. We need to show that $\mathrm{RSA}_{N,d}(\mathrm{RSA}_{N,e}(x)) = x$ for all $x \in \mathbb{Z}_N^*$. We have

$$\mathrm{RSA}_{N,d}(\mathrm{RSA}_{N,e}(x)) = (x^e)^d \bmod N.$$

Using the property that $e \cdot d \equiv 1 \pmod{\varphi(N)}$, we know that there exists an integer k such that $e \cdot d = 1 + k \cdot \varphi(N)$. Therefore,

$$(x^e)^d = x^{ed} = x^{1+k\cdot\varphi(N)} = x \cdot (x^{\varphi(N)})^k \bmod N.$$

By Euler's theorem, we have $x^{\varphi(N)} \equiv 1 \pmod N$ for all $x \in \mathbb{Z}_N^*$. Thus,

$$x \cdot (x^{\varphi(N)})^k \equiv x \cdot 1^k \equiv x \bmod N,$$

hence $\mathrm{RSA}_{N,d}(\mathrm{RSA}_{N,e}(x)) = x$ for all $x \in \mathbb{Z}_N^*$. This also proves that the RSA is a bijection, and we can efficiently evaluate and invert it using the trapdoor information d. ■

5.5.2 RSA Construction

From Proposition 5.5.1, it's possible to construct a public key scheme such that given a message m, we encrypt it as $c := \mathrm{RSA}_{N,e}(m) = m^e \pmod N$; to decrypt, we use $\mathrm{RSA}_{N,d}(c) = c^d \pmod N$, which is proved to be m, i.e., we consider Figure 5.1.

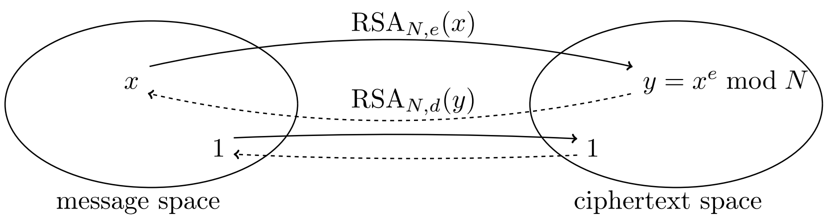


Figure 5.1: RSA is a bijection

Formally, consider the following key generation mechanism for RSA.

Definition 5.5.2 (RSA key)**.** The *RSA key* $(\mathsf{pk}, \mathsf{sk})$ is generated as follows:

(a) choose random, independent, large primes p and q having bit lengths approximately related to n, and compute $N = pq$;

(b) choose $e \in \mathbb{Z}_{\varphi(N)}^*$ such that $\gcd(e, \varphi(N)) = 1$, and compute $d \equiv e^{-1} \pmod{\varphi(N)}$;

(c) output $(\mathsf{pk}, \mathsf{sk})$ where the public key $\mathsf{pk} = (N, e)$ and the private key $\mathsf{sk} = (N, d)$.

As previously seen (Compute d)**.** Given a random e, to compute d, we note that since $\gcd(e, \varphi(N)) = 1$, by running extended Euclid algorithm, we can find A, B such that $1 = Ae + B\varphi(n) \bmod \varphi(N)$. Then, we have $Ae = 1 - B\varphi(N) = 1 \bmod \varphi(N)$, i.e., we have $d = A = e^{-1} \bmod \varphi(N)$.

Example. Common choices for e are $e = 3$ or $e = 2^{16} + 1$.

Proof. They are both primes, hence it's always the case that $\gcd(e, \varphi(N)) = 1$. It's particular nice since $\langle e \rangle$ only have two bits that are 1, so computing modular exponentiation is fast. ⊛

Then what we just saw is the following public key scheme.

Definition 5.5.3 (Textbook RSA cryptosystem)**.** The *RSA cryptosystem* given in [KL20] is a public key scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ where

- $\mathsf{Gen}(1^n)$: output $(\mathsf{pk}, \mathsf{sk}) = ((N, e), (N, d))$ from RSA key generation;
- $\mathsf{Enc}(\mathsf{pk}, m)$ for $m \in \mathbb{Z}_N^*$: output $c = \mathrm{RSA}_{N,e}(m) = m^e \bmod N$;
- $\mathsf{Dec}(\mathsf{sk}, c)$ for $c \in \mathbb{Z}_N^*$: output $m = \mathrm{RSA}_{N,d}(c) = c^d \bmod N$.

The text RSA cryptosystem is certainly correct due to Proposition 5.5.1; however, it is not secure: since the encryption is deterministic, hence Theorem 5.4.2 applies! To fix it,[4] we need to understand what hardness assumption does RSA rely on exactly.

5.5.3 Security of RSA

From the previous example, we see that what we really want is that for any probabilistic polynomial time adversary A, the probability that A can find the pre-image $x = m$ of $y = c$ under $\mathrm{RSA}_{N,e}$ for some random $y \in \mathbb{Z}_N^*$ is negligible in n. Formally, we have the following.

Conjecture 5.5.1 (RSA hardness)**.** The *RSA hardness* assume that for all probabilistic polynomial time A,

$$\Pr_{\substack{(\mathsf{pk},\mathsf{sk}) \leftarrow \mathsf{Gen}(1^n) \\ y \leftarrow \mathbb{Z}_N^*}} (A(1^n, \mathsf{pk}, y) \text{ outputs } x = \mathrm{RSA}_{N,e}^{-1}(y)) = \mathrm{negl}(n)$$

for $\mathsf{pk} = (N, e)$.

Intuition. It's hard to find the pre-image $x = y^d \pmod N$ given (N, e) and $y (= x^e \pmod N)$.

Remark. The hardness assumption is believed to be true.

Note. The RSA hardness assumption is specific:

- it holds only when y is sampled randomly: for particular values of y, it might be easy to find the pre-image of y;[a]
- the pre-image of a random y cannot be "completely" recovered: perhaps we can recover partial information about it.

[a] The inverse of some ciphertexts, e.g., 1, are easy to compute since $1^e = 1$; see Figure 5.1.

The problem of finding the pre-image of RSA is related to the factoring, and indeed, we can compare the hardness of them.

Claim. RSA $\leq$ factoring, i.e., if there's an efficient algorithm for prime factorization, then there is one for solving RSA.

[4] I.e., to make Enc not deterministic.

Proof. Given $\mathsf{pk} = (N, e)$, $y \in \mathbb{Z}_N^*$, by factoring $N = p \cdot q$, we obtain $\varphi(N) = (p-1)(q-1)$. Then, it's easy to compute $d = e^{-1} \pmod{\varphi(N)}$ hence $x = y^d \pmod N$. ⊛

Claim. Computing $\varphi(N) \Leftrightarrow$ factoring.

Proof. If we know $\varphi(N) = p \cdot q - p - q + 1 = (N+1) - (p+q)$, then we can solve for $p+q = (N+1) - \varphi(N)$ and $p \cdot q = N$. This is enough to have a factoring of N. ⊛

Claim. Factoring $\Leftrightarrow$ finding d from (N, e).

Proof. We know $e \cdot d - 1 = k \cdot \varphi(N)$ for some k. Through fancy math (see book!), it is multiple of $\varphi(N)$ to recover enough information to have a factoring of N. ⊛

Remark. It's still an open problem that whether RSA $\equiv$ factoring.[a]

[a] We suspect that factoring is easier than taking roots.

5.5.4 An Even Better RSA Cryptosystem

Since the textbook construction of RSA is not CPA secure because it has deterministic encryption, so an adversary can easily tell when the same message is sent twice.

Note. It is also not necessarily EAV secure.

Proof. Since messages don't usually follow a uniform random distribution, so the ciphertexts y's won't be uniformly random either. This doesn't fit the RSA hardness assumption directly, so we cannot say it is EAV secure. ⊛

To construct a secure public key scheme using RSA, we modify the above textbook construction and incorporate a cryptographic hash function H, and fix it by applying $\mathrm{RSA}_{N,e}$ on a random $x \leftarrow \mathbb{Z}_N^*$.[5]

Intuition. We don't have a guarantee that the encryption is random, but we do know it is hard to compute so if we hash x and pad that to the message, only someone with the private key will be able to recover the message.

So the ciphertext would be $c = (\mathrm{RSA}_{N,e}(x), H(x) \oplus m)$.

Lecture 24: Digital and RSA Signatures

However, in this case, we require something stronger than collision resistance for H. 5 Apr. 10:30

Intuition. A good hash function "practically behaves" like a uniform random function.

Notation (Random oracle)**.** Such an H is also known as a *random oracle.*

Example. SHA-3 is quite "random-like".

Remark. If x is not completely known, then $H(x)$ is completely random.

Formally, consider the following.

[5] To take advantage of the assumption.

Definition 5.5.4 (Randomized RSA cryptosystem). Given a random oracle Hash $H\colon \{0,1\}^n \to \mathbb{Z}_N^*$, the *randomized RSA cryptosystem* is a public key scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ where

- $\mathsf{Gen}(1^n)$: output $(\mathsf{pk}, \mathsf{sk}) = ((N, e), (N, d))$ from RSA key generation;
- $\mathsf{Enc}(\mathsf{pk}, m)$ for $m \in \mathbb{Z}_n^*$: output $(y, z) = (\mathrm{RSA}_{N,e}(x), H(x) \oplus m)$ for $x \leftarrow \mathbb{Z}_n^*$;
- $\mathsf{Dec}(\mathsf{sk}, c)$ for $c = (y, z)$: output $H(\mathrm{RSA}_{N,d}(y)) \oplus z$.

The correctness follows directly from Proposition 5.5.1 and computation, since

$$H(\mathrm{RSA}_{N,d}(y)) \oplus z = H(\mathrm{RSA}_{N,d}(\mathrm{RSA}_{N,e}(x))) \oplus (H(x) \oplus m) = H(x) \oplus H(x) \oplus m = m.$$

We claim that this construction is CPA secure.

Theorem 5.5.1. Randomized RSA cryptosystem is CPA secure.

Proof Idea. As noted, H behaves like a random function/oracle. Thus, without fully knowing x, $H(x)$ looks completely random, meaning $H(x) \oplus m$ looks completely random. By the RSA hardness assumption, an adversary cannot fully know x. So, an adversary cannot distinguish the encryption of a message from randomness. This ensures CPA secure. ■

Chapter 6

Asymmetric Public Key Message Authentication

We have studied MAC, where a sender and receiver share *symmetric keys*. Only senders who know the secret key can tag messages, and only receivers who know the secret key can verify them.

Now, we want to generalize this idea, i.e., we want to extend it to the setup of *asymmetric*, i.e., now a sender and receiver share asymmetric keys. Only senders who know the private key can sign messages, and only receivers who know the corresponding public key can verify them. This is called *signatures*.

6.1 Digital Signatures

Consider the following.

Definition 6.1.1 (Signature scheme)**.** A *signature scheme* is a tuple $\Pi = (\mathsf{Gen}, \mathsf{Sign}, \mathsf{Ver})$ where

- $\mathsf{Gen}(1^n)$: outputs $(\mathsf{vk}, \mathsf{sk})$;
- $\mathsf{Sign}(\mathsf{sk}, m)$ for $\mathsf{sk} \leftarrow \mathsf{Gen}(1^n)$ and $m \in \mathcal{M}$: outputs σ;
- $\mathsf{Ver}(\mathsf{vk}, m, \sigma)$ for $\mathsf{vk} \leftarrow \mathsf{Gen}, m \in \mathcal{M}, \sigma$: outputs Accept or Reject.

Notation (Digital signature)**.** We sometimes call signature scheme the *digital signature*.

To decrypt Definition 6.1.1 a bit, we see that

- Gen is given the security parameter and outputs a key-pair, where vk, sometimes denoted pk, is the *verification key* or *public key*; and sk is the *secret key* or *signing key*.
- Sign is given a verification key from Gen as well as a message m. It outputs a *signature* σ.
- Ver is given a public key from Gen, a message m, and a signature σ. It accepts or rejects.

Naturally, we want the following.

Definition 6.1.2 (Correctness)**.** A signature scheme Π is *correct* if for all $(\mathsf{vk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^n)$ and for all $m \in \mathcal{M}$,

$$\mathsf{Ver}(\mathsf{vk}, m, \mathsf{Sign}(\mathsf{vk}, m)) = \mathsf{Accept}.$$

There's something fundamentally different from MACs we have seen.

As previously seen. With MACs, verification came for free (i.e., canonical verification) since the sender and receiver shared the secret key, the receiver could verify messages by running the tag algorithm themselves.

Compared to MACs, digital signatures have no analogue of canonical verification because the receiver doesn't know the signing key, hence can't run Sign.

6.1.1 Chosen Message Attack

Nevertheless, apart from the above difference, the security definitions for digital signatures and MACs are very similar: we first define the analog to chosen message attack for MAC.

Definition 6.1.3 (Chosen message attack). The *chosen message attack* game given an adversary A, a parameter n, and a signature scheme Π is conducted as follows.

1. Let $(\mathsf{vk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^n)$.
2. $A(1^n, \mathsf{vk})$ receives polynomially many signatures for messages by querying the signing oracle $\mathsf{Sign}(\mathsf{sk}, \cdot)$.
3. A eventually outputs (m^*, σ^*).

Definition 6.1.4 (Weak forgery). In the CMA, A *forges* if $\mathsf{Ver}(\mathsf{vk}, m^*, \sigma^*)$ accepts and m^* is not a query from A to the signing oracle.

Definition 6.1.5 (Advantage). Given an adversary A in an CMA, the *advantage* $\mathrm{Adv}_{\Pi}^{\mathrm{CMA}}(A)$ in generating a fake message/signature pair is given by

$$\mathrm{Adv}_{\Pi}^{\mathrm{CMA}}(A) \coloneqq \Pr_{(\mathsf{vk},\mathsf{sk})\leftarrow\mathsf{Gen}(1^n)}(A^{\mathsf{Sign}(\mathsf{sk},\cdot)}(1^n, \mathsf{vk}) \text{ forges}).$$

In addition, we also consider the following as what we have done in MACs.

Definition 6.1.6 (Strong forgery). In the CMA, A *forges* if $\mathsf{Ver}(\mathsf{vk}, m^*, \sigma^*)$ accepts and (m^*, σ^*) is not a query-response pair from A to the signing oracle.

Note. $\mathrm{Adv}_{\Pi}^{\mathrm{SCMA}}(A)$ is defined in the natural way as in Definition 6.1.5.

6.1.2 Chosen Message Attack Security

Following the same vein as MACs, we define the following.

Definition 6.1.7 (Unforgeable). A signature scheme Π is *(existentially) unforgeable* under CMA if for every probabilistic polynomial time adversary A, the advantage is negligible.

Notation (UFCMA). An unforgeable signature scheme under CMA is abbreviated as *UFCMA*.

Similarly, we define the same notion for strong forgery.

Notation (Strongly UFCMA). A signature scheme Π is *strongly unforgeable* under CMA, or *strongly UFCMA*, if for every probabilistic polynomial time adversary A, the $\mathrm{Adv}_{\Pi}^{\mathrm{SCMA}}(A) = \mathrm{negl}(n)$.

When we discussed MACs, this caused some confusion: here again, we note the following.

Claim. strongly UFCMA is a strong security notion (i.e., more secure) than UFCMA.

Proof. We observe that

- to break strong UFCMA, an attacker must forge (m^*, σ^*) such that either m^* or σ^* is new;
- to break UFCMA, an attacker must forge (m^*, σ^*) such that m^* is new.

If you can do the latter, you can do the former. Thus, $\neg$UFCMA $\Rightarrow$ $\neg$strong UFCMA. By taking the contrapositive, strong UFCMA $\Rightarrow$ UFCMA. ⊛

Remark. UFCMA does not rule out, e.g., replay attacks (although time-stamping messages does).

6.2 RSA Signatures

In RSA encryption, e allows the public to encrypt messages and d allows one party to decrypt them. RSA's encryption and decryption functions are very similar, so with minimal changes, we can construct a scheme where e allows the public to decrypt messages and d allows one party to encrypt them. This leads to a digital signature scheme.

6.2.1 RSA Signature Scheme Construction

Definition 6.2.1 (Textbook RSA signature). The *RSA signature* given in [KL20] is a signature scheme $\Pi = (\mathsf{Gen}, \mathsf{Sign}, \mathsf{Ver})$ where

- $\mathsf{Gen}(1^n)$: output $(\mathsf{vk}, \mathsf{sk}) = ((N, e), (N, d))$ from RSA key generation;
- $\mathsf{Sign}(\mathsf{sk}, m)$ for $m \in \mathbb{Z}_N^*$: output $\sigma = \mathrm{RSA}_{N,d}(m)$.
- $\mathsf{Ver}(\mathsf{vk}, m, \sigma)$ for $m \in \mathbb{Z}_N^*$: output Accept if $\mathrm{RSA}_{N,e}(\sigma) = m$, otherwise output Reject.

The textbook RSA signature is correct from the Proposition 5.5.1. Moreover, given m, vk, and not d, it is hard to find $\sigma = \mathsf{Sign}(\mathsf{sk}, m)$ by the RSA hardness assumption. Thus, a forger cannot attack the textbook RSA signature by choosing a message and finding its signature. However, we note the following.

Remark. The RSA hardness assumption does not rule out choosing a signature and finding a message that produces it. In fact, given σ, vk, and not d, it is easy to find m such that $\sigma = \mathsf{Sign}(\mathsf{sk}, m)$.

This leads to the following.

Theorem 6.2.1. The textbook RSA signature is not UFCMA.

Proof. Consider a probabilistic polynomial time adversary A in the CMA game against the textbook RSA signature, where A first choose any $\sigma^* \in \mathbb{Z}_N^*$ and return $(m^*, \sigma^*) = (\mathrm{RSA}_{N,e}(\sigma^*), \sigma^*)$. We see that $\mathsf{Ver}(\mathsf{vk}, m^*, \sigma^*)$ always accepts because $\mathrm{RSA}_{N,e}(\sigma^*) = m^*$. Thus, A forges with probability 1 while making no queries. ∎

Another Proof. Consider a probabilistic polynomial time adversary A in the CMA game against the textbook RSA signature, where for arbitrary m and m', A queries to obtain (m, σ) and (m', σ'). Then, it outputs $(m \cdot m', \sigma \cdot \sigma')$.[a] We see that $\mathsf{Ver}(\mathsf{vk}, m^*, \sigma^*)$ always accepts because $\mathrm{RSA}_{N,e}(\sigma^*) \equiv (\sigma \cdot \sigma')^e \equiv \sigma^e \cdot {\sigma'}^e \equiv m \cdot m' \pmod N$. So, A always forges if mm' is neither m nor m'. ∎

[a]Remember that $m \in \mathbb{Z}_N^*$ in general, hence it makes to talk about multiplication.

6.2.2 An Even Better RSA Signature Scheme

One way to fix the textbook RSA signature is the following.

Definition 6.2.2 (Hash-and-Sign RSA signature). Given a random oracle Hash $H\colon \{0,1\}^n \to \mathbb{Z}_N^*$, the *Hash-and-Sign RSA signature* is a signature scheme $\Pi = (\mathsf{Gen}, \mathsf{Sign}, \mathsf{Ver})$ where

- $\mathsf{Gen}(1^n)$: output $(\mathsf{vk}, \mathsf{sk}) = ((N, e), (N, d))$ from RSA key generation;
- $\mathsf{Sign}(\mathsf{sk}, m)$ for $m \in \{0,1\}^n$: output $\sigma = \mathrm{RSA}_{N,d}(H(m))$;
- $\mathsf{Ver}(\mathsf{vk}, m, \sigma)$ for $m \in \{0,1\}^n$: output Accept if $\mathrm{RSA}_{N,e}(\sigma) = H(m)$, otherwise output Reject.

Again, the correct from the Proposition 5.5.1. We claim that this construction is UFCMA.

Theorem 6.2.2. The Hash-and-Sign RSA signature is UFCMA.

Proof Idea. To forge, an adversary can either

(i) choose a message and try to find a matching signature, or

(ii) choose a signature and try to find a matching message.

By the RSA hardness assumption, (i) is infeasible because the adversary doesn't know d; (ii) is also infeasible because it amounts to finding $m \in H^{-1}(\{y\})$ for some y of the adversary's choice while H is "secure". ■

Lecture 25: Schnorr's Identification and Fiat-Shamir

6.2.3 Comparison between RSA Function and Discrete Logarithm

10 Apr. 10:30

We showed that a simple application of the RSA function can create a digital signature.

Intuition. RSA function contains a "trapdoor" for calculating the inverse, making it easier for us to perform the verification step when ensuring that message was in fact signed by the authentic user.

Remark. Contrarily, it is not so easy to repurpose the Diffie-Hellman key exchange for this because it does not contain such a "trapdoor", where the hardness comes from the discrete logarithm problem.

But, as we will see, it can really be thought of as a one-way function, i.e., it's possible to construct digital signature based on the hardness of discrete logarithm.

6.3 Identification Schemes

An *identification scheme* is a digital signature utilizes the hardness of discrete logarithm problem, where it allows one to prove that you have the secret key without revealing it. In particular, we consider the Schnorr's identification, which (after some modifications) is a "zero-knowledge" protocols that utilizes the hardness of a discrete logarithm.

Note. The modifications are need, as we will see, since Schnorr's identification has a flaw that it can reveal x.

Specifically, the enhancement of Schnorr's identification is done by using the *Fiat-Shamir transform* that utilizes hashing to increase randomness and conceal the secret key. Let's first see the Schnorr's identification.

6.3.1 Schnorr's Identification

Consider the following discrete logarithm problem setup.

Problem 6.3.1 (Secret discrete logarithm problem). Let G be a large cyclic group of known prime order q,[a] generated by g, i.e., $|G| = q$ and $G = \{g^0, g^1, g^2, \cdots, g^{q-1}\}$. Let the secret key be a random number $x \in \mathbb{Z}_q$, and the public key be $y = g^x \in G$. The *secret discrete logarithm problem* asks for a prover $P(x)$ who knows the private key x to prove that it has the knowledge of x without revealing it to a verifier $V(y)$ who only has access to the public key y.

[a] For example, $G = \mathbb{Z}_q^*$.

Intuition. Everyone knows $y = g^x$, but if P want to "prove" to V that P has the secret key x, P need to solve the discrete logarithm problem. With the fact that P doesn't want to reveal x, we have the secret version of the discrete logarithm problem as defined above.

Claus-Peter Schnorr describes a four-step, *interactive protocol* between a prover (has x) and a verifier (has y) as described in the secret discrete logarithm setup. cite

Definition 6.3.1 (Schnorr's identification). The *Schnorr's identification* is a four-step interactive protocol between a prover $P(x)$ and a verifier $V(y)$ under the secret discrete logarithm problem setup, which works as follows.

1. The prover generates $k \leftarrow \mathbb{Z}_q$ and sends $c = g^k \in G$ to the verifier.[a]
2. The verifier generates $r \leftarrow \mathbb{Z}_q$ and sends it to the prover.
3. The prover $P(x)$ calculates $s = k + r \cdot x \pmod q \in \mathbb{Z}_q$ and sends it to the verifier.
4. The verifier $V(y)$ verifies whether $g^s = g^{k+r\cdot x \bmod q} = c \cdot y^r$, and accepts if this is correct.

[a]This commits the prover to some public key g^k.

6.3.2 Issues with Schnorr's Identification

Schnorr's identification is correct since if P, V run the protocol honestly, then V will accept; and if V accepts, then that means that P *knows* x. fAs for soundness,[1] we want to make sure that if V accepts with high probability, then P actually "knows" x.

Example (Thought experiment). Consider two challenges $r_1 \neq r_2$ sent by V to P for which P manages to make V accept. Then, say P's responses are s_1 and s_2, respectively, at the third step. We know that there are some r_1 and r_2, respectively, such that

$$g^{s_1} = c \cdot y^{r_1}, \quad g^{s_2} = c \cdot y^{r_2},$$

i.e., $g^{s_1 - s_2} = y^{r_1 - r_2} = g^{x(r_1 - r_2)}$.

From the above example, we see that since g is a generator, $s_1 - s_2 = x(r_1 - r_2) \bmod q$, i.e.,

$$x = (s_1 - s_2) \cdot (r_1 - r_2)^{-1} \bmod q,$$

implying that we can extract x via the extended Euclid algorithm by finding the inverse of $(r_1 - r_2)$.

Remark. Schnorr's identification may unknowingly reveal information about x.

This is what we wanted to keep secret according to the problem setup, hence Schnorr's identification doesn't really work.

6.3.3 Fiat-Shamir Transform

To make Schnorr's identification truly "zero knowledge", we want to create an *efficient* simulator that samples the distribution of the exchanged information *without knowing* x.

Intuition. The idea is to change the ordering of our random variable sampling to (c, r, s).

The only difference is that we first sample r, then s, then c. For now, we first choose $r \leftarrow \mathbb{Z}_q$, and $s \leftarrow \mathbb{Z}_q$, and finally set $c = g^s \cdot y^{-r} \in G$. This has exactly the same distribution as in Schnorr's identification, but knows nothing about x that V doesn't already know. So, V learns nothing new about x other than the fact that P knows it.

Intuition. Since x is in fact unused, we don't give any knowledge about x.

Remark. Simply by changing the order of the verification scheme, we don't give knowledge of x.

[1]I.e., you can't prove the wrong thing.

However, there are two issues: we didn't sign messages, and we had an interactive protocol. To get around this we do the normal hash function to reuse out OTP principles. This can be done by *Fiat-Shamir transform*.

cite

Fiat-Shamir transform is a technique for taking an interactive proof of knowledge and creating a digital signature based on it. This way, some fact (for example, knowledge of the secret key) can be publicly proven without revealing underlying information.

Definition 6.3.2 (Fiat-Shamir transformed Schnorr's identification)**.** Given a random oracle Hash $H\colon \{0,1\}^n \to \mathbb{Z}_N^*$, the *Fiat-Shamir transformed Schnorr's identification* is a signature scheme $\Pi = (\mathsf{Gen}, \mathsf{Sign}, \mathsf{Ver})$ where

- $\mathsf{Gen}(1^n)$: generate $x \leftarrow \mathbb{Z}_q$, and output $(\mathsf{vk}, \mathsf{sk}) = (g^x, x) = (y, x)$ where $y \in G$;
- $\mathsf{Sign}(\mathsf{sk}, m)$ for $m \in \{0,1\}^*$: generate $k \leftarrow \mathbb{Z}_q$, compute $c = g^k \in G$, and output $\sigma = (r, s) = (H(m, c), k + rx \bmod q)$;
- $\mathsf{Ver}(\mathsf{vk}, m, \sigma)$ for $m \in \{0,1\}^*$: output Accept if $H(m, c) = r$ for $c = g^s \cdot y^{-r}$, otherwise output Reject.

Remark. We replaced the interactive step by invoking H.

Theorem 6.3.1. Assuming Conjecture 5.3.1 on G and H is a random oracle, the Fiat-Shamir Schnorr identification is unforgeable.

Proof idea. Because H is a random oracle, the values $r = H(m, c)$ that a forger must deal with are like truly random challenges in Schnorr's identification protocol. A forger won't be able to answer such a challenge unless H gets extremely luck in receiving the one challenge it knows how to handle, and it's able to compute $x = \log_g y$ for the legit signer's public key y (a uniform random for the legit signer's element of G). ■

Chapter 7

Post-Quantum and Lattice-Based Cryptography

Lecture 26: Lattice-Based Cryptography

Finally, we give a *very* brief overview on the recent advances in cryptography, i.e., the post-quantum cryptography. Due to Shor, some hardness assumptions we rely on is already broken using quantum algorithms. Hence, people study the "post"-quantum cryptography, which aims to find new hardness assumptions against even quantum computers that we can rely on to build a secure cryptosystem. 12 Apr. 10:30 cite

A particular important subject is the lattice theory, which relies on the discrete nature of integers.

7.1 Post-Quantum Cryptography

7.1.1 Shor's Algorithm

Peter Shor had shocked the world with a quantum algorithm that can factorize integers in polynomial time. The idea was to use the quantum "weirdness", also known as "complex probabilities". This means that RSA could be broken. cite

Remark. Quantum search is also weird. You can search in an unstructured array of size N using $O(\sqrt{N})$ operations. This is Grove's algorithm. cite

In cryptography, quantum computers can brute force for a key of length N in time $2^{\frac{N}{2}}$ rather than 2^N.

Remark. A remedy to this is to double the key length.

Shor's algorithm also computes discrete logarithm in polynomial time. It breaks Diffie-Hellman, ElGamal, etc. cite

7.1.2 Post-Quantum Cryptography

Post-quantum cryptography can't rely on the hardness of factoring or discrete logarithm. An older proposal was to rely on hardness of things like subset sum or hash functions. Beyond this, more successful proposals rely on coding theory and lattice theory. cite cite

7.2 Lattice-Based Cryptography

Lattice-based cryptography is to build cryptography based on hardness of problems about lattices.

7.2.1 Lattice

Consider the following.

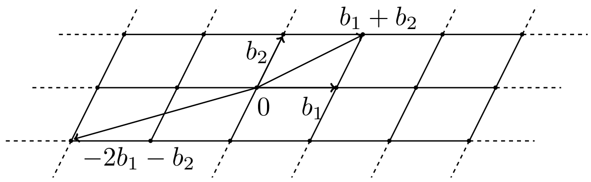
Definition 7.2.1 (Lattice). A *lattice* $\mathcal{L}(B)$ is a $\mathbb{Z}$-vector space spanned by B.

Note. We will mainly consider $B \subseteq \mathbb{R}^n$, i.e., $\mathcal{L}(B)$ is a $\mathbb{Z}$-vector space embedded in $\mathbb{R}^n$.

Basically, consider a set of basis vectors $B = \{b_1, b_2, \dots, b_n \mid b_i \in \mathbb{R}^n\}$, then

$$\mathcal{L}(B) = \{z_1 b_1 + \cdots + z_n b_n : z_i \in \mathbb{Z}\} \subseteq \mathbb{R}^n.$$

Intuition. It's a periodic, infinite grid in an n-dimensional space, generated by n basis vectors in $\mathbb{R}^n$.



Remark. In matrix form, $\mathcal{L}(B) = \{B \cdot Z : Z \subseteq \mathbb{Z}^n\}$.

7.2.2 Hardness on Lattices

One conjectured hard lattice problem is the shortest vector problem.

Problem 7.2.1 (Shortest vector problem). Given B, the *shortest vector problem* asks for finding the shortest (or a "very short") nonzero vector $v \in \mathcal{L}(B)$.

Another conjectured hard lattice problem is decoding, also known as closest vector problem.

Problem 7.2.2 (Closest vector problem). Given B and a target point $t \in \mathbb{R}^n$, the *closest vector problem* asks for finding the closest vector in $\mathcal{L}(B)$ to t.

Problem 7.2.3 (Learning with errors). Given n and a prime $q \approx n^2$, pick a secret $s \leftarrow \mathbb{Z}_q^n$. An instance of *learning with errors* problem is where a random matrix $A \leftarrow \mathbb{Z}_q^{m*n}$ is chosen and $b = A \cdot s + e$ is calculated for e being a short error vector (noise).[a] The task is to find s given n, q, A, and b.

[a] All operations are over $\mathbb{Z}_q$.

Remark. The problem of Learning with errors is conjectured to be hard.

Example. One example of e is that each entry is randomly chosen from $[-10, 10]$.

Learning with errors is closely related to closest vector problem, where $b \approx A \cdot s$ is a target vector close to the lattice point $v = A \cdot s$, where A is a lattice basis.

Problem 7.2.4 (Desicion learning with errors). The *decision learning with errors* is to, given (A, b), distinguish between $(A, b \approx A \cdot s)$ and (A, b) selected uniformly at random.

The relation between Learning with errors and decision learning with errors is illustrated by Theorem 7.2.1.

Theorem 7.2.1. Learning with errors and decision learning with errors are equivalent under a probabilistic polynomial time reduction, i.e., each can be solved efficiently if the other can.

7.2.3 Learning with Errors for Key Exchange

Consider a key exchange between Alice and Bob using learning with errors and the product $r^\top As$:

$$\begin{pmatrix} & r^\top & \end{pmatrix} \cdot \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \cdot \begin{pmatrix} \\ s \\ \\ \end{pmatrix}.$$

Specifically, it works as follows.

1. Alice first chooses a random $A \leftarrow \mathbb{Z}_q^{n*n}$ and sends it to Bob.
2. Bob chooses a "short" $s \leftarrow \mathbb{Z}_q^n$ and sends $u = A \cdot s + e \approx A \cdot s$ to Alice, where e is some errors.
3. Alice chooses a short $r \leftarrow \mathbb{Z}_q^n$ and sends $v^\top = r^\top A + d^\top \approx r^\top \cdot A$ to Bob, where d is some errors.
4. Bob calculates a key $k_B = v^\top \cdot s(r^\top A + d^\top) \cdot s = r^\top As + d^\top s \approx r^\top As$, where $d^\top s$ is some noise.
5. Alice calculates a key $k_A = r^\top \cdot u = r^\top(A \cdot s + e) = r^\top As + r^\top \cdot e \approx r^\top As$.

Finally,

- if k_A and k_B are both between 0 and $\frac{q}{2}$, then a 1 is transmitted;
- if k_A and k_B are both between $-\frac{q}{2}$ and 0, then a 0 is transmitted,

and a message 0 corresponds to 0, and a message 1 corresponds to $\frac{q}{2}$.

Note. Alice and Bob can agree on a bit in the end.

This can be turned into a public key encryption, just like how we turn Diffie-Hellman into ElGamal:

- to encrypt a bit $m \in \{0, 1\}$, Alice can compute $c \approx r^\top \cdot u + m \cdot (\frac{q}{2})$;
- to recover m, compute

$$p = c - v^\top \cdot s \approx r^\top A \cdot s + m\frac{q}{2} - r^\top As = m \cdot \frac{q}{2},$$

  where the $v^\top \cdot s$ corresponds to k_B, the first $r^\top A \cdot s$ corresponds to k_A, and the $r^\top As$ corresponds to k_B.

Everything works as guaranteed by the following.

Theorem 7.2.2. This public key encryption is CPA secure assuming decision learning with errors is hard.

Appendix

Appendix A

Acknowledgement

The following is a list of students, organized by the lecture scribe notes they are responsible for.[1]

A.1 Winter 2023

Lecture 1. Pingbang Hu.

Lecture 2. Pingbang Hu.

Lecture 3. Jason Zeng, Park Szachta, Meredith Benson.

Lecture 4. Nancy Liu, Nicklaus Sicilia.

Lecture 5. Bryan Nie, Andrew Marshall, Shuangyu Lei.

Lecture 6. Mathurin Gagnon, Aditya Sriram, Angelina Zhang, Adam Marakby.

Lecture 7. Edison Situ, Zhou Xinyue, Ashley Jeong, Samuel Costa.

Lecture 8. Matt Palazzolo, Mingye Chen.

Lecture 9. Nicholas Karns, Shufeng Chen, Jai Narayanan.

Lecture 10. Jason Obrycki.

Lecture 11. Trisha Dani, Nathan Curdo, Anthony Li.

Lecture 12. Yiwen Tseng, Erik Zhou, Lilly Phillips, Yoonsung Ji, Dylan Shelton.

Lecture 13. Benjamin Miller, Michael Hu, Enzo Metz.

Lecture 14. Jeremy Margolin, Katie Wakevainen, Jason Zheng, Jonathan Giha.

Lecture 15. Keming Ouyang, Chen Yuxiang, Haoyu Chen, Tao Zhu, Sean Chen.

Lecture 16. Dongqi Wu, Kevin Hua, Xun Wang, Benjamin Miller.

Lecture 17. Alex Young, Sohil Ramachandra.

[1] Noticeably, in the main document, the space of the header is limited, so I only list the main scribe notes I was referring to when organizing.

Lecture 18. Mei Lanting, Ava Banerjee, Lilly Phillips.

Lecture 19. Shaurya Gupta, Hussain Lokhandwala, David Yei.

Lecture 20. Madhav Shekhar Sharma, Ethan Kennedy, Yiwen Tseng, Noah Peters, Zhongqi Ma.

Lecture 21. Zeyu Chang, Meredith Benson, Ziyun Chi, Trisha Dani, Ethan Kennedy, Eric Leu, Yi Liu, Lilly Phillips.

Lecture 22. Trisha Dani, Aidan Gauthier, Ethan Kennedy, Yi Liu, Yiwen Tseng, Ashley Jeong, Vinamr Arya, Jai Narayanan.

Lecture 23. Ethan Kennedy, Yiwen Tseng, Aroosh Moulik, Justin Paul, Jeremy Roszko, Erik Zhou, Yi Liu, Nicklaus Sicilia.

Lecture 24. Ethan Kennedy, Luke Miga, Yi Liu, Nicklaus Sicilia, Yiwen Tseng, Zhiyuan Chen, Benjamin Miller, Vinamr Arya, Zhongqi Ma, Lauren Friedrich.

Lecture 25. Justin Paul, Park Szachta, Ji YoonSung, Sean Chen, Yi Liu, Lauren Friedrich, Julian Lane.

Lecture 26. Park Szachta, Lauren Friedrich.

More on Graph Theory Jason Thegn, Pingbang Hu.

Appendix B

More on Group Theory

Group theory provides a foundation for understanding the mathematical underpinnings of many cryptographic concepts and techniques. Understanding groups' structure often provides tools for analyzing the security of certain cryptographic algorithms. We have seen examples of these algorithms already, such as the factoring and the discrete logarithm problems, and there are many, many more. In this chapter, we provide more expository material regarding group theory.

Note. The notation is a bit different from what is used in lectures here and there, and also the way things are defined since I want to introduce everything in the most natural way. For example, the way I introduce quotient groups is different from most of the textbook.

In particular, we will use *lots of examples*, together with comments and motivations to provide a self-contained introduction to group theory. At the end of the chapter, we will see the *first isomorphism theorem*, which turns out to be natural if we follow everything.

B.1 Groups

Example (Protoypical example)**.** The additive group $(\mathbb{Z}, +)$ and the cyclic group $\mathbb{Z}_p = \mathbb{Z} / p\mathbb{Z}$. Just don't forget that most groups are non-commutative.

Generally speaking, groups encodes *symmetries*: given an element, there always exist a unique inverse of which w.r.t. an operation. Before written down the formal definition of a group, let us first see some examples.

Example (Additive integers)**.** Consider the pair $(\mathbb{Z}, +)$ where $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and $+$ is the associative *addition* operation. Observe that

- the element $0 \in \mathbb{Z}$ is an *identity*: $a + 0 = 0 + a = a$ for any $a \in \mathbb{Z}$;
- every element $a \in \mathbb{Z}$ has an additive *inverse*: $a + (-a) = (-a) + a = 0$.

Example (Nonzero rationals)**.** Consider the pair $(\mathbb{Q}^*, \cdot)$ where $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ be the set of *nonzero rational numbers*, and $\cdot$ is the associative *multiplication*. Again, we see that

- the element $1 \in \mathbb{Q}^*$ is an *identity*: $a \cdot 1 = 1 \cdot a = a$;
- for any rational number $x \in \mathbb{Q}^*$, there is an inverse x^{-1} such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

Now, you might want to make the following definition.

Definition B.1.1 (Group)**.** A *group* is a pair $(G, \star)$ consisting of a set of elements G, and a binary operator $\star \colon G \times G \to G$, such that

(a) G has an identity element 1_G (or just 1) such that for all $g \in G$, $1_G \star g = g \star 1_G = g$;

(b) $\star$ is associative, i.e., $(a \star b) \star c = a \star (b \star c)$ for any $a, b, c \in G$;

(c) every element $g \in G$ has an inverse $h \in G$ such that $g \star h = h \star g = 1_G$.

Note (Unimportant pendatic point). Some authors like to add a "closure" axiom, i.e., to say explicitly that $g \star h \in G$. This is implied already by the fact that $\star$ is a binary operation on G (and the fact that we already specified $\star \colon G \times G \to G$).

Remark. It is not required that $\star$ is commutative, i.e., $a \star b = b \star a$.

Definition B.1.2 (Abelian group). A group $(G, \star)$ is *Abelian* if $\star$ is commutative.

Notation. From now on, we will refer a group as just G if the operation is irrelevant or is canonically realized.

Definition B.1.3 (Order). The *order* of a group G is the number of elements of G.

B.1.1 Lots of Examples

It's almost impossible to understand algebraic objects without **lots of** examples, so here we see some.

Example (Non-examples). Let's see some non-examples.

- The pair $(\mathbb{Q}, \cdot)$ is not a group: While there is an identity element 1 the element $0 \in \mathbb{Q}$ does not have an inverse.
- The pair $(\mathbb{Z}, \cdot)$ is not a group: (Why?)
- Let $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ be the set of 2×2 real matrices. Then $(\mathrm{Mat}_{2\times 2}(\mathbb{R}), \cdot)$ (where $\cdot$ is the matrix multiplication) is not a group: Indeed, even though we have an identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we still run into the same issue as before: the zero matrix does not have a multiplicative inverse.

Example. Even if we delete the zero matrix from $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, the resulting structure is still not a group: those of you that know some linear algebra might recall that any matrix with determinant zero cannot have an inverse.

Let's resume writing down examples. Here are some Abelian examples of groups.

Example (Complex unit circle). Let S^1 denote the set of complex numbers z with absolute value one; that is, $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. Then $(S^1, \times)$ is a group since

- the complex number $1 \in S^1$ serves as the identity, and
- each complex number $z \in S^1$ has an inverse $1/z$ which is also in S^1, since $|z^{-1}| = |z|^{-1} = 1$.

There is one thing I ought to also check: that is, $z_1 \times z_2$ is actually still in S^1. But this follows from the fact that $|z_1 z_2| = |z_1||z_2| = 1$.

Here is an example from number theory as you might remember.

Example (Addition modulo (Cyclic group)). Let $n > 1$ be an integer, and consider the residues (remainders) modulo n. These form a group under addition. We call this the *cyclic group* of order n, and denote it as $\mathbb{Z}\,/\,n\mathbb{Z}$ or $\mathbb{Z}_n$, with elements $\overline{0}, \overline{1}, \ldots, \overline{n-1}$. The identity is $\overline{0}$.

Example (Multiplication mod$p$). Let p be a prime. Consider the nonzero residues modulo p, which we denote by $(\mathbb{Z}\,/\,p\mathbb{Z})^{\times}$ or $\mathbb{Z}_p^*$. Then, $(\mathbb{Z}_p^*, \times)$ is a group.

Exercise. Why do we need the fact that p is prime?

Here are some non-Abelian examples.

Example (General linear group). Let n be a positive integer. Then $\mathrm{GL}_n(\mathbb{R})$ is defined as the set of $n \times n$ real matrices which has nonzero determinant. It turns out that with this condition, every matrix does indeed have an inverse, so $(\mathrm{GL}_n(\mathbb{R}), \times)$ is a group, called the *general linear group.*[a]

[a] The fact that $\mathrm{GL}_n(\mathbb{R})$ is closed under $\times$ follows from the linear algebra fact that $\det(AB) = \det A \det B$.

Example (Special linear group). Recall $\mathrm{GL}_n(\mathbb{R})$, let $\mathrm{SL}_n(\mathbb{R})$ denote the set of $n \times n$ matrices whose determinant is actually 1. Again, for linear algebra reasons it turns out that $(\mathrm{SL}_n(\mathbb{R}), \times)$ is also a group, called the *special linear group.*

Example (Symmetric group). Let S_n be the set of permutations of $\{1, \ldots, n\}$. By viewing these permutations as functions from $\{1, \ldots, n\}$ to itself, we can consider *compositions* $\circ$ of permutations. Then the pair $(S_n \circ)$ is also a group, because

- there is an identity permutation, and
- each permutation has an inverse.

The group S_n is called the *symmetric group* on n elements.

If this is a bit confusing, consider S^n for $n = 3$.

Example (S^3). For example, (123) is the bijection on $\{1, 2, 3\}$ reads, from left to right, as

$$1 \mapsto 2, \quad 2 \mapsto 3, \quad 3 \mapsto 1.$$

Formally, the elements of S_3 are $(1), (12), (23), (13), (123), (132)$. As an example of composition, we obtain $(12)(23) = (123)$, while $(23)(12) = (132)$.

$\circ$	(1)	(12)	(23)	(13)	(123)	(132)
(1)	(1)	(12)	(23)	(13)	(123)	(132)
(12)	(12)	(1)	(123)	(132)	(23)	(13)
(23)	(23)	(132)	(1)	(123)	(13)	(12)
(13)	(13)	(123)	(132)	(1)	(12)	(23)
(123)	(123)	(13)	(12)	(23)	(132)	(1)
(132)	(132)	(23)	(13)	(12)	(1)	(123)

Table B.1: The complete binary operation compositions on S^3

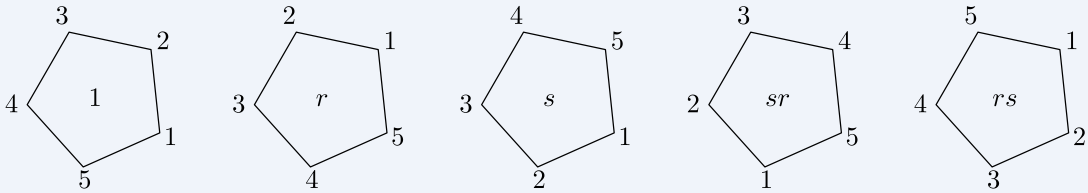
Example (Dihedral group). The *dihedral group* of order $2n$, denoted as D_{2n}, is the group of symmetries of a regular n-gon $A_1 A_2 \ldots A_n$, which includes rotations and reflections. It consists of the $2n$ elements

$$\{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}.$$

The element r corresponds to rotating the n-gon by $2\pi/n$, while s corresponds to reflecting it across the line OA_1 for O being the center of the polygon. So rs mean "reflect then rotate", just like function composition, we read from right to left.

In particular, $r^n = s^2 = 1$, and it's easy to see that $r^k s = sr^{-k}$.

Example (D_{10})**.** Here is a picture of some elements of D_{10}.



Remark. D_{12} is one of my favorite example of a non-Abelian group, and is the first group I try for any exam question of the form "find and example...".

We can also build a new group based on the exists groups.

Example (Product of groups)**.** Let $(G, \star)$ and $(H, *)$ be groups. We can define a *product group* $(G \times H, \circ)$, as follows. The elements of the group will be ordered pairs $(g, h) \in G \times H$, and we let

$$(g_1, h_1) \circ (g_2, h_2) \coloneqq (g_1 \star g_2, h_1 * h_2) \in G \times H$$

be the group operation.

Exercise. What are the identity and inverses of the product group?

Example (Trivial group)**.** The *trivial group*, often denoted 0 or 1, is the group with only an identity element. However, we use the notation $\{1\}$.

Exercise. Verify that the trivial group is actually a group.

B.1.2 Properties of Groups

Example (Prototypical example)**.** $\mathbb{Z}_p^*$ is possibly best.

Notation. From now on, given a group $(G, \star)$, we will abbreviate $a \star b$ to just ab, and since $\star$ is associative, we will omit unnecessary parentheses, e.g., $(ab)c = a(bc) = abc$.

Notation. For any $g \in G$ and $n \in \mathbb{N}$, we abbreviate

$$g^n \coloneqq \underbrace{g \star \cdots \star g}_{n \text{ times}}.$$

Moreover, we let g^{-1} denote the inverse of g, and $g^{-n} = (g^{-1})^n$.

In mathematics, a common theme is to require that objects satisfy certain minimalistic properties, with certain examples in mind, but then ignore the examples on paper and try to deduce as much as you can just from the properties alone.[1]

Let's try to do this here, and see what we can conclude just from knowing Definition B.1.1.

[1] For people who know something about functional equations, knowing a single property about a function is enough to determine the entire function.

Claim. Let G be a group.

(a) The identity of a group is unique.

(b) The inverse of any element is unique.

(c) For any $g \in G$, $(g^{-1})^{-1} = g$.

Proof. This mostly just some formal manipulations, and you needn't feel bad skipping it.

(a) If 1 and $1'$ are identities, then $1 = 1 \star 1' = 1'$.

(b) If h and h' are inverses to g, then $1_G = g \star h \Rightarrow h' = (h' \star g) \star h = 1_G \star h = h$.

(c) Trivial.

⊛

The following is slightly more useful.

Proposition B.1.1 (Inverse of products). Let G be a group, and $a, b \in G$. Then $(ab)^{-1} = b^{-1}a^{-1}$.

Proof. Since

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aa^{-1} = 1_G,$$

and $(b^{-1}a^{-1})(ab) = 1_G$ similarly, hence $(ab)^{-1} = b^{-1}a^{-1}$. ■

Finally, we state a very important lemma about groups, which highlights why having an inverse is so valuable.

Lemma B.1.1. Let G be a group, and pick a $g \in G$. Then the map $G \to G$ given by $x \mapsto gx$ is a bijection.

Proof. This can be done by showing injectivity and surjectivity directly. ■

Example ($\mathbb{Z}_7^*$). Let $G = \mathbb{Z}_7^*$ and pick $g = 3$. Lemma B.1.1 states that the map $x \mapsto 3 \cdot x \bmod 7$, and we can see this explicitly:

$$\begin{aligned} 1 &\mapsto 3 \pmod 7, & 4 &\mapsto 5 \pmod 7, \\ 2 &\mapsto 6 \pmod 7, & 5 &\mapsto 1 \pmod 7, \\ 3 &\mapsto 2 \pmod 7, & 6 &\mapsto 4 \pmod 7. \end{aligned}$$

Notation. Later on, sometimes the identity is denoted 0 instead of 1.

In most of our examples up until now, the operation $\star$ was thought of like multiplication of some sort, which is why $1 = 1_G$ was a natural notation for the identity element.

But there are groups like $\mathbb{Z} = (\mathbb{Z}, +)$ where the operation $\star$ is thought of as addition, in which case the notation $0 = 0_G$ might make more sense instead.

Remark. In general, whenever an operation is denoted $+$, it is almost certainly commutative.

B.1.3 Unimportant Long Digression

A common question is: why these axioms? For example, why associative but not commutative? One general heuristic is that, whenever you define a new type of general object, there's always a balancing act going on:

- on the one hand, you want to include enough constraints that your objects are "nice";
- on the other hand, if you include too many constraints, then the definition applies to too few objects.

So, for example, we include "associative" because that makes our lives easier and most operations we run into are associative. In particular, associativity is required to the inverse of an element to necessarily be unique.

However, we don't include "commutative", because examples below show that there are lots of non-Abelian groups we care about. Another comment is the following.

Intuition. A good motivation for the inverse axioms is that we get a large amount of *symmetry*. The set of positive integers with addition is not a group, for example, because we can't subtract 6 from 3: some elements are "larger" than others. By requiring an inverse element to exist, we get rid of this issue.

B.1.4 Lagrange's Theorem for Orders

Example (Prototypical example)**.** $\mathbb{Z}_p^*$.

Recall that we say the order of a group G is the number of elements of G. However, as is typical in math, we use the word "order" for way too many things. In groups, there is another definition.

Definition B.1.4 (Order)**.** The *order* $\operatorname{ord}(g)$ of an element $g \in G$ is the smallest positive integer n such that $g^n = 1_G$, or ∞ if no such n exists.

Example. The order of -1 in $\mathbb{Q}^\times$ is 2 while the order of 1 in $\mathbb{Z}$ is infinite.

It's easy to see the following.

Proposition B.1.2. If $g^n = 1_G$, then $\operatorname{ord}(g)$ divides n.

Remark. Any element of a group has a finite order.

Proof. Consider the infinite sequence $1_G, g, g^2, \ldots$, and find two elements that are the same. ⊛

One of the most first important result is the Lagrange's theorem.

Theorem B.1.1 (Lagrange's theorem for orders)**.** Let G be any finite group. Then $x^{|G|} = 1_G$ for any $x \in G$.

Proof. See Corollary B.4.1. You can assume it for now. ■

B.2 Isomorphisms

Example (Prototypical example)**.** $\mathbb{Z} \cong 10\mathbb{Z}$.

Next, we discuss the maps between groups. Consider the following two groups:

- $\mathbb{Z} = (\{\ldots, -2, -1, 0, 1, 2, \ldots\}, +)$;
- $10\mathbb{Z} = (\{\ldots, -20, -10, 0, 10, 20, \ldots\}, +)$.

These groups are "different", but only superficially so – you might even say they only differ in the names of the elements. Specifically, the map

$$\phi\colon \mathbb{Z} \to 10\mathbb{Z}, \quad x \mapsto 10x$$

is a bijection of the underlying sets which respects the group operation, i.e.,

$$\phi(x+y) = \phi(x) + \phi(y).$$

Intuition. ϕ is a way of re-assigning names of the elements without changing the structure of the group.

This is just formalism for capturing the obvious fact that $(\mathbb{Z}, +)$ and $(10\mathbb{Z}, +)$ are the same thing. Formally, we have the following.

Definition B.2.1 (Isomorphism)**.** Let $G = (G, \star)$ and $H = (H, *)$ be groups. A bijection $\phi\colon G \to H$ is called an *isomorphism* if

$$\phi(g_1 \star g_2) = \phi(g_1) * \phi(g_2)$$

for all $g_1, g_2 \in G$.

Notation (Isomorphic)**.** G and G are *isomorphic*, denoted as $G \cong H$, if there exists an isomorphism from G to H.

Note. The left-hand side $\phi(g_1 \star g_2)$ uses the operation of G, while the right-hand side $\phi(g_1) * \phi(g_2)$ uses the operation of H.

B.2.1 Lots of Examples

Let's see some examples.

Example. Let G and H be groups. We have the following isomorphisms.

- $\mathbb{Z} \cong 10\mathbb{Z}$.
- There is an isomorphism $G \times H \cong H \times G$ by the map $(g, h) \mapsto (h, g)$.
- The identity map $\mathrm{id}\colon G \to G$ is an isomorphism, hence $G \cong G$.
- There is another isomorphism of $\mathbb{Z}$ to itself: send every x to $-x$.

Example (Primitive roots modulo 7)**.** As a nontrivial example, we claim that $\mathbb{Z}_6 \cong \mathbb{Z}_7^*$, where the bijection is given by

$$\phi(a \bmod 6) = 3^a \bmod 7.$$

Proof. ϕ is a bijection by explicit calculation[a]

$$(3^0, 3^1, 3^2, 3^3, 3^4, 3^5) \equiv (1, 3, 2, 6, 4, 5) \pmod 7.$$

Moreover, ϕ respects the group actions, i.e., $\phi(a + b) = \phi(a) \times \phi(b)$ since the operation of $\mathbb{Z}_6$ and $\mathbb{Z}_7^*$ is $+$ and $\times$, respectively. But this is just saying

$$3^{a+b \bmod 6} \equiv 3^{a \bmod 6} 3^{b \bmod 6} \pmod 7,$$

which is true. ⊛

[a] To be pedantic, I should write $3^{0 \bmod 6} = 1$, etc.

More generally, we have the following.

Example (Primitive roots)**.** For any prime p, there exists an element $g \in \mathbb{Z}_p^*$ called a *primitive root* modulo p such that $1, g, g^2, \dots, g^{p-2}$ are all different modulo p. One can show by copying the above proof that $\mathbb{Z}_{p-1} \cong \mathbb{Z}_p^*$ for all primes p.

Exercise. Assuming the existence of primitive roots, give the isomorphism explicitly between $\mathbb{Z}_{p-1}$ and $\mathbb{Z}_p^*$.

Now, you might wonder why we care about isomorphism? The answer is that it's an equivalence relation.

Definition B.2.2 (Equivalence relation)**.** A relation is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence relations appear everywhere in mathematics.

Example. Here are some examples.

(a) The relation R on $\mathbb{Z} \times \mathbb{N}$ by $(n_1, m_1)R(n_2, m_2)$ provided that $n_1 m_2 = n_2 m_1$.

(b) The relation R on $\mathbb{C}$ by $z_1 R z_2$ if and only if $|z_1| = |z_2|$.

(c) For an $n \in \mathbb{Z}$, define a relation R on $\mathbb{Z}$ by $m_1 R m_2$ provided that $m_1 - m_2$ is a multiple of n.

Intuition. Given a equivalence relation, we can start by talking about things that should be viewed as "equal".

Often time, in mathematics, after you define something, you would want to classify them. In this case, isomorphisms help us classify groups since it's not hard to see that $\cong$ is an equivalence relation. Moreover, because we really only care about the structure of groups, we usually consider two groups to be the same when they are isomorphic.

B.2.2 Subgroups

Example (Prototypical example)**.** $\mathrm{SL}_n(\mathbb{R})$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Earlier we saw that $\mathrm{GL}_n(\mathbb{R})$, the $n \times n$ matrices with nonzero determinant, formed a group under matrix multiplication. But we also saw that a subset of $\mathrm{GL}_n(\mathbb{R})$, the $\mathrm{SL}_n(\mathbb{R})$, also formed a group with the same operation. This property should have its own name.

Definition B.2.3 (Subgroup)**.** Let $G = (G, \star)$ be a group. A *subgroup* of G is a group $H = (H, \star)$ where $H \subseteq G$.

Notation (Proper subgroup)**.** We say H is a *proper subgroup* of G if $H \subsetneq G$.

Remark. To specify a group G, I need to tell you both what the set G was and the operation $\star$ was. But to specify a subgroup H of G, I only need to tell you who its elements are since the operation of H is just inherited from G.

Example. For any group G, the trivial group $\{1_G\}$ and the entire group G are subgroup of G.

Example. $2\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, which is isomorphic to $\mathbb{Z}$ itself.

Example. Consider the symmetric group S_n on n elements. Let T be the set of permutations $\tau \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ for which $\tau(n) = n$. Then T is a subgroup of S_n, and is isomorphic to S_{n-1}.

Intuition. We're fixing the last element of the permutation.

Example. Consider the product group $G \times H$ between G and H and the set $\{(g, 1_H) \mid g \in G\}$. This is a subgroup of $G \times H$, and is isomorphic to G by the isomorphism $(g, 1_H) \mapsto g$.

Next is an especially important example.

Example (Generated subgroup)**.** Let x be an element of a group G. Consider the set

$$\langle x\rangle \coloneqq \{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\}.$$

This is a subgroup of G called the *subgroup generated* by x.

Finally, we present some non-examples of subgroups.

Example. Consider the group $\mathbb{Z} = (\mathbb{Z}, +)$.

(a) The set $\{0, 1, 2, \dots\}$ is not a subgroup of $\mathbb{Z}$ since it does not contain inverses.

(b) The set $\{n^3 \mid n \in \mathbb{Z}\} = \{\dots, -8, -1, 0, 1, 8, \dots\}$ is not a subgroup since it's not closed under addition: the sum of two cubes is not in general a cube.

(c) The empty set $\varnothing$ is not a subgroup of $\mathbb{Z}$ because it lacks an identity element.

B.3 Homomorphisms

B.3.1 Generators and Group Presentations

Example (Prototypical example)**.** $D_{2n} = \langle r, s \mid r^n = s^2 = 1\rangle$.

Let G be a group. Recall that for some element $x \in G$, we could consider the subgroup

$$\{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\}$$

of G. Here's a more pictorial version of what we did:

Intuition. Put x in a box, seal it tightly, and shake vigorously.

Using just the element x, we get a pretty explosion that produces the subgroup above. What happens if we put two elements x, y in the box? Among the elements that get produced are things like

$$xyxyx, \quad x^2y^9x^{-5}y^3, \quad y^{-2015}, \quad \dots .$$

Essentially, I can create any finite product of x, y, x^{-1}, y^{-1}. This leads us to define the following.

Definition B.3.1 (Generated subgroup)**.** Let S be a subset of G. The subgroup *generated* by S, denoted $\langle S\rangle$, is the set of elements which can be written as a finite product of elements in S (and their inverses).

Definition B.3.2 (Generator)**.** If $\langle S\rangle = G$ then we say S is a set of *generators* for G, as the elements of S together create all of G.

Exercise. Why is the condition "and their inverses" not necessary if G is a finite group (assume Lagrange's theorem)?

Example ($\mathbb{Z}$ is the infinite cyclic group)**.** Consider 1 as an element of $\mathbb{Z} = (\mathbb{Z}, +)$. We see $\langle 1\rangle = \mathbb{Z}$, meaning $\{1\}$ generates $\mathbb{Z}$. It's important that -1, the inverse of 1 is also allowed: we need it to write all integers as the sum of 1 and -1.

This gives us an idea for a way to try and express groups compactly.

Problem. Why not just write down a list of generators for the group? For example, we could write $\mathbb{Z} \cong \langle a \rangle$ meaning that $\mathbb{Z}$ is just the group generated by one element.

Answer. There's one issue: the generators usually satisfy certain properties. For example, consider $\mathbb{Z}_{100}$. It's also generated by a single element x, but this x has the additional property that $x^{100} = 1$. ⊛

This motivates us to write

$$\mathbb{Z}_{100} = \langle x \mid x^{100} = 1 \rangle.$$

I'm sure you can see where this is going. All we have to do is specify a set of generators and *relations* between the generators, and say that two elements are equal if and only if you can get from one to the other using relations.

Definition B.3.3 (Group presentation). Such an expression is appropriately called a *group presentation.*

Example (Dihedral group). The D_{2n} has a presentation

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

Thus, each element of D_{2n} can be written uniquely in the form r^α or sr^α, where $\alpha = 0, 1, \ldots, n-1$.

Example (Klein four group). The *Klein four group*, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, is given by the presentation

$$\langle a, b \mid a^2 = b^2 = 1, ab = ba \rangle.$$

Example (Free group). The *free group on n elements* is the group whose presentation has n generators and no relations at all. It is denoted F_n, so

$$F_n = \langle x_1, x_2, \ldots, x_n \rangle.$$

In other words, $F_2 = \langle a, b \rangle$ is the set of strings formed by appending finitely many copies of a, b, a^{-1}, b^{-1} together.

Remark. Notice that $F_1 \cong \mathbb{Z}$.

Notation. One might unfortunately notice that "subgroup generated by a and b" has exactly the same notation as the free group $\langle a, b \rangle$. We'll try to be clear based on context which one we mean.

Presentations are nice because they provide a compact way to write down groups. They do have some shortcomings, though.[2]

Example (Presentations can look very different). The same group can have very different presentations. For instance consider

$$D_{2n} = \langle x, y \mid x^2 = y^2 = 1, (xy)^n = 1. \rangle.$$

Proof. To see why this is equivalent, set $x = s$, $y = rs$. ⊛

B.3.2 Homomorphisms

Example (Prototypical example). The "mod out by 100" map, $\mathbb{Z} \to \mathbb{Z}_{100}$.

[2] Actually, determining whether two elements of a presentation are equal is undecidable. In fact, it is undecidable to even determine if a group is finite from its presentation.

How can groups talk to each other?

As previously seen. Two groups are "the same" if we can write an isomorphism between them.

If you know something about "metric space" (i.e., space with a distance function), two metric spaces are "the same" if we can write a "homeomorphism" (i.e., continuous and bijetive) between them. But what's the group analogy of a continuous map? We simply drop the bijection condition.

Definition B.3.4 (Group homomorphism)**.** Let $G = (G, \star)$ and $H = (H, *)$ be groups. A *group homomorphism* is a map $\phi\colon G \to H$ such that for any $g_1, g_2 \in G$,

$$\phi(g_1 \star g_2) = \phi(g_1) * \phi(g_2).$$

Intuition. A homomorphism between two groups G, H makes the groups "behave the same". In other words, we can "translate" any action performed in one of these groups into an action performed in the other.

Let's see some examples of homomorphisms: Let G and H be groups.

Example (Trivial homomorphism)**.** The *trivial homomorphism* $G \to H$ sends everything to 1_H.

Example. Any isomorphism $G \to H$ is a homomorphism. In particular, the identity map $G \to G$ is a homomorphism.

Example. There is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_{100}$ by sending each integer to its residue modulo 100.

Example. There is a homomorphism from $\mathbb{Z}$ to itself by $x \mapsto 10x$ which is injective but not surjective.

Example. There is a homomorphism from S_n to S_{n+1} by "embedding": every permutation on $\{1, \ldots, n\}$ can be thought of as a permutation on $\{1, \ldots, n+1\}$ if we simply let $n+1$ be a fixed point.

Example. A homomorphism $\phi : D_{12} \to D_6$ is given by $s_{12} \mapsto s_6$ and $r_{12} \mapsto r_6$.

Example. Specifying a homomorphism $\mathbb{Z} \to G$ is the same as specifying just the image of the element $1 \in \mathbb{Z}$. (Why?)

The last two examples illustrate something.

Remark. Suppose we have a presentation of G. To specify a homomorphism $G \to H$, we only have to specify where each generator of G goes, in such a way that the relations are all satisfied.

Remark. The right way to think about an isomorphism is as a "bijective homomorphism".

To be explicit,

Exercise. Show that $G \cong H$ if and only if there exist homomorphisms $\phi\colon G \to H$ and $\psi\colon H \to G$ such that $\phi \circ \psi = \mathrm{id}_H$ and $\psi \circ \phi = \mathrm{id}_G$.

Note. So the definitions of homeomorphism of metric spaces and isomorphism of groups are not too different.

Some obvious properties of homomorphisms follow.

Lemma B.3.1. Let $\phi\colon G \to H$ be a homomorphism. Then $\phi(1_G) = 1_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$.

Proof. Let $g \in G$. We know that $\phi(g) = \phi(g \star 1_G) = \phi(g) * \phi(1_G)$. Furthermore, $\phi(g) = \phi(g) * \phi(1_G) = \phi(g) * 1_H$ by definition of identity. Therefore, $\phi(1_G) = 1_H$.

Moreover, $\phi(1_G) = 1_H = \phi(g \star g^{-1}) = \phi(g) * \phi(g^{-1})$. Furthermore, $1_H = \phi(g) * \phi(g^{-1}) = \phi(g) * \phi(g)^{-1}$. Therefore, $\phi(g^{-1}) = \phi(g)^{-1}$. ∎

Lemma B.3.2. The composition of two homomorphisms is a homomorphism.

Proof. Let $(G, \star), (H, *), (I, \cdot)$ be groups, and let $\phi\colon G \to H$, $\psi\colon H \to I$ be two homomorphisms, we show that $\psi \circ \phi\colon g \to I$ is a homomorphism. Let $g_1, g_2 \in G$, we have

$$\psi \circ \phi(g_1 \star g_2) = \psi(\phi(g_1) * \phi(g_2)) = \psi(\phi(g_1)) \cdot \psi(\phi(g_2)) = \psi \circ \phi(g_1) \cdot \psi \circ \phi(g_2).$$

∎

B.3.3 Kernels

Now let me define a very important property of a homomorphism.

Definition B.3.5 (Kernel)**.** The *kernel* of a homomorphism $\phi\colon G \to H$ is defined by

$$\ker \phi := \{g \in G\colon \phi(g) = 1_H\}.$$

Remark. The kernel of a homomorphism $\phi\colon G \to H$ is a subgroup of G. In particular, $1_G \in \ker \phi$ for obvious reasons.

We also have the following important fact.

Theorem B.3.1 (Kernel determines injectivity)**.** Let $\phi : G \to H$ be a group homomorphism. Show that ϕ is injective if and only if $\ker(\phi) = \{1_G\}$.

Proof. Assume ϕ is injective. Since $\phi(1_G) = 1_H$, by injectivity, for all $x \in G$, $\phi(x) = 1_H \to x = 1_G$. Therefore, $\ker \phi = \{1_G\}$.

Now, assume $\ker \phi = \{1_G\}$. Fix $x, y \in G$ such that $\phi(x) = \phi(y)$. $\phi(x) \cdot \phi(y)^{-1} = 1_G$. We know that $\phi(x) \cdot \phi(y)^{-1} = \phi(x) \cdot \phi(y^{-1}) = \phi(x \cdot y^{-1})$. Furthermore, $x \cdot y^{-1} \in \ker \phi$, therefore $x \cdot y^{-1} = 1_G$, therefore $x = y$. We conclude that $\phi(x) = \phi(y) \to x = y$, and ϕ is injective. ∎

Remark. This extends naturally to vector spaces, i.e., any linear transformation $T\colon V \to W$ is injective if and only if $\ker(T) = \{0_V\}$, where V, W are vector spaces.

To make this concrete, let's compute the kernel of each of our examples.

Example. The kernel of any isomorphism $G \to H$ is trivial, since an isomorphism is injective. In particular, the kernel of the identity map $G \to G$ is $\{1_G\}$.

Example. The kernel of the trivial homomorphism $G \to H$ (by $g \mapsto 1_H$) is all of G.

Example. The kernel of the homomorphism $\mathbb{Z} \to \mathbb{Z}_{100}$ by $n \mapsto \overline{n}$ is precisely

$$100\mathbb{Z} = \{\ldots, -200, -100, 0, 100, 200, \ldots\}.$$

Example. The kernel of the map $\mathbb{Z} \to \mathbb{Z}$ by $x \mapsto 10x$ is trivial: $\{0\}$.

Example. There is a homomorphism from S_n to S_{n+1} by "embedding", but it also has trivial kernel because it is injective.

Example. A homomorphism $\phi\colon D_{12} \to D_6$ is given by $s_{12} \mapsto s_6$ and $r_{12} \mapsto r_6$. You can check that

$$\ker \phi = \{1, r_{12}^3\} \cong \mathbb{Z}_2.$$

Exercise. Fix any $g \in G$. Suppose we have a homomorphism $\mathbb{Z} \to G$ by $n \mapsto g^n$. What is the kernel?

Exercise. Show that for any homomorphism $\phi\colon G \to H$, the image $\phi(G)$ is a subgroup of H. Hence, we'll be especially interested in the case where ϕ is surjective.

Theorem B.3.2. Let H be a nonempty subset of G such that for all $h_1, h_2 \in H$ we have $h_1 h_2^{-1} \in H$. Then, H is a subgroup of G.

Proof. Since H is nonempty, let $h \in H$. It is obvious that $hh^{-1} = 1_G \in H$. Since $h \in H, 1_G \in H, 1_g h \in H$. $1_g h^{-1} = h^{-1} \in H$. Let $h_1, h_2, h_3 \in H$. Since these are elements of G, by associativity, $h_1(h_2 h_3) = (h_1 h_2) h_3$. Therefore, H is a group, and H is a subgroup of G. ■

B.4 Quotient Groups

B.4.1 Cosets

Example (Prototypical example). Modding out by n: $\mathbb{Z}/(n \cdot \mathbb{Z}) \cong \mathbb{Z}_n$.

Note. The next few sections are a bit dense.

Let G and Q be groups, and suppose there exists a *surjective* homomorphism

$$\phi\colon G \twoheadrightarrow Q.$$

In other words, if $\phi\colon G \to Q$ is injective then ϕ is a bijection, and hence an isomorphism.

Problem. Suppose we're not so lucky and $\ker \phi$ is bigger than just $\{1_G\}$, what is the correct interpretation of a more general homomorphism?

Let's look at the special case where $\phi\colon \mathbb{Z} \to \mathbb{Z}_{100}$ is "modding out by 100". We already saw that the kernel of this map is

$$\ker \phi = 100\mathbb{Z} = \{\ldots, -200, -100, 0, 100, 200, \ldots\}.$$

As previously seen. Recall that $\ker \phi$ is a subgroup of G.

What this means is that ϕ **is indifferent to the subgroup** $100\mathbb{Z}$ **of** $\mathbb{Z}$, i.e.,

$$\phi(15) = \phi(2000 + 15) = \phi(-300 + 15) = \phi(700 + 15) = \ldots.$$

So $\mathbb{Z}_{100}$ is what we get when we "mod out by 100". Cool.

In other words, let G be a group and $\phi\colon G \twoheadrightarrow Q$ be a surjective homomorphism with kernel $N \subseteq G$. We claim the following.

Intuition. Q should be thought of as the "quotient" of G by N.

To formalize this, we will define a so-called quotient group $G \,/\, N$ in terms of G and N only (without referencing Q) which will be naturally isomorphic to Q. For motivation, let's give a concrete description of Q using just ϕ and G. Continuing our previous example, let $N = 100\mathbb{Z}$ be our subgroup of G. Consider the sets

$$\begin{aligned} N &= \{\ldots, -200, -100, 0, 100, 200, \ldots\} \\ 1 + N &= \{\ldots, -199, -99, 1, 101, 201, \ldots\} \\ 2 + N &= \{\ldots, -198, -98, 2, 102, 202, \ldots\} \\ &\vdots \\ 99 + N &= \{\ldots, -101, -1, 99, 199, 299, \ldots\}. \end{aligned}$$

Note. The elements of each set all have the same image when we apply ϕ. Moreover, any two elements in different sets have different images.

Then the main idea is the following.

Intuition. We can think of Q as the group whose elements are the *sets* above.

Thus, given ϕ, we define an equivalence relation $\sim_N$ on G by saying $x \sim_N y$ for $\phi(x) = \phi(y)$. This $\sim_N$ divides G into several equivalence classes in G which are in obvious bijection with Q, as above. Now we claim that we can write these equivalence classes very explicitly.

Exercise. Show that $x \sim_N y$ if and only if $x = yn$ for some $n \in N$.[a] Thus, for any $g \in G$, the equivalence class of $\sim_N$ which contains g is given explicitly by

$$gN := \{gn \mid n \in N\}.$$

[a] In the mod 100 example, this means they "differ by some multiple of 100".

Here's the word that describes the types of sets we're running into now.

Definition B.4.1 (Left coset). Let H be any subgroup of G.[a] A set of the form gH is called a *left coset* of H.

[a] Not necessarily the kernel of some homomorphism.

Remark. Although the notation might not suggest it, keep in mind that g_1N is often equal to g_2N even if $g_1 \neq g_2$. In the "mod 100" example, $3 + N = 103 + N$. In other words, these cosets are *sets*.

This means that if I write "let gH be a coset" without telling you what g is, you can't figure out which g I chose from just the coset itself. If you don't believe me, here's an example of what I mean.

Example. Consider

$$x + 100\mathbb{Z} = \{\ldots, -97, 3, 103, 203, \ldots\} \Rightarrow x = ?$$

There's no reason to think I picked $x = 3$ (I actually picked $x = -13597$).

You might already notice it, but let's make the following explicit.

Lemma B.4.1. All cosets have the same cardinality.

Proof. Given cosets g_1H and g_2H, you can check that the map $x \mapsto g_2g_1^{-1}x$ is a bijection between them. ■

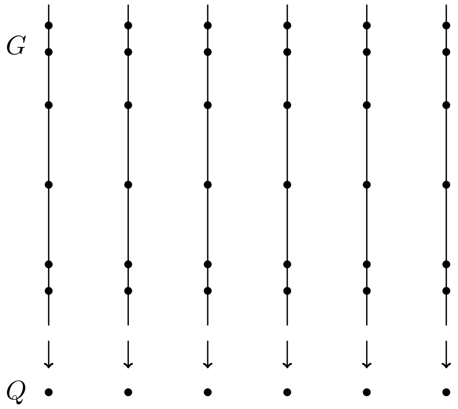
So, long story short,

Intuition. Elements of the group Q are naturally identified with left cosets of N.

In practice, people often still prefer to picture elements of Q as single points.

Example. It's easier to think of $\mathbb{Z}_2$ as $\{0,1\}$ rather than $\{\{\dots,-2,0,2,\dots\},\{\dots,-1,1,3,\dots\}\}$.

If you like this picture, then you might then draw G as a bunch of equally tall "fibers" (the cosets), which are then "collapsed" onto Q.



B.4.2 Quotient Groups

Now that we've done this, we can give an *intrinsic* definition for the quotient group we alluded to earlier.

Definition B.4.2 (Normal). A subgroup N of G is called *normal* if it is the kernel of some homomorphism.

Notation. We write $N \trianglelefteq G$ for N being normal to G.

We can now write down the first version of the definition of a quotient group.

Definition B.4.3 (Quotient group). Let $N \trianglelefteq G$. Then the *quotient group*, denoted G/N (and read "G mod N"), is the group defined as follows.

- The elements of G/N will be the left cosets of N.
- To define the product of two cosets C_1 and C_2 in G/N, recall that the cosets are in bijection with elements of Q. So let q_1 be the value associated to the coset C_1, and q_2 the one for C_2. Then we can take the product to be the cosets corresponding to $q_1 q_2$.

Quite importantly, to define the operation of a quotient group, **we can also do this in terms of representatives of the cosets**: Let $g_1 \in C_1$ and $g_2 \in C_2$, so $C_1 = g_1 N$ and $C_2 = g_2 N$. Then $C_1 \cdot C_2$ should be the coset which contains $g_1 g_2$.

Claim. This is the same as Definition B.4.3.

Proof. We simply observe that $\phi(g_1 g_2) = \phi(g_1)\phi(g_2) = q_1 q_2$, i.e., all we've done is define the product in terms of elements of G, rather than values in H. ⊛

Using the gN notation, and with the remark in mind, we can write this even more succinctly as

$$(g_1 N) \cdot (g_2 N) := (g_1 g_2) N.$$

We now write down the concise version of Definition B.4.3.

Definition B.4.4 (Quotient group). Let $N \trianglelefteq G$. Then the *quotient group* G/N is defined as $(\{gN\}_{g\in G}, \cdot)$ where

$$(g_1 N) \cdot (g_2 N) := (g_1 g_2) N.$$

Now, you see why the integers modulo n are often written as $\mathbb{Z}/n\mathbb{Z}$! By the way we've built it, the resulting group G/N is isomorphic to Q.

Intuition. In a sense we think of G / N as "G modulo the condition that $n = 1$ for all $n \in N$".

B.4.3 Lagrange's Theorem

As an aside, with the language of cosets we can now show Lagrange's theorem in the general case.

Theorem B.4.1 (Lagrange's theorem). Let G be a finite group, and let H be any subgroup. Then $|H|$ divides $|G|$.

Proof. We first note that $G / H = \{g_1 H, \ldots, g_n H\}$ (perhaps with repetitions) where $n = |G| < \infty$, and by Lemma B.4.1, for all $1 \leq i \leq j \leq n$, $|g_i H| = |g_j H|$. Moreover, $|g_i H| = |H|$.

Now, since G is finite, so it is the union of a finite number of disjoint cosets, i.e.,

$$G = \bigcup_{i=1}^{n} g_i H \Rightarrow |G| = \sum_{i=1}^{n} |g_i H|$$

since $g_i H$ are all disjoint. But with the fact that $|g_i H| = |H|$, $|G| = n|H|$. ■

Corollary B.4.1. $x^{|G|} = 1$.

Proof. By taking $H = \langle x \rangle \subseteq G$. ■

It should be mentioned at this point that in general, we have the following.

Remark. If G is a finite group and N is normal, then $|G / N| = |G|/|N|$.

B.4.4 Eliminating the Homomorphism

Example (Prototypical example). Again $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

As previously seen. Recall Definition B.4.4 of G / N:

- The elements of G / N are cosets gN, which you can think of as equivalence classes of a relation $\sim_N$ (where $g_1 \sim_N g_2$ if $g_1 = g_2 n$ for some $n \in N$).
- Given cosets $g_1 N$ and $g_2 N$ the group operation is $g_1 N \cdot g_2 N := (g_1 g_2) N$.

Problem. Where do we actually use the fact that N is normal? We don't talk about ϕ or Q anywhere in this definition.

Answer. The answer is in the remark. The group operation takes in two cosets, so it doesn't know what g_1 and g_2 are. But behind the scenes, **the normal condition guarantees that the group operation can pick any g_1 and g_2 it wants and still end up with the same coset.**

If we didn't have this property, then it would be hard to define the product of two cosets C_1 and C_2 because it might make a difference which $g_1 \in C_1$ and $g_2 \in C_2$ we picked (hence it's not *well-defined*). The fact that N came from a homomorphism meant we could pick any representatives g_1 and g_2 of the cosets we wanted, because they all had the same ϕ-value. ⊛

We want some conditions which force this to be true without referencing ϕ at all. Suppose $\phi \colon G \to K$ is a homomorphism of groups with $H = \ker \phi$. Aside from the fact H is a group, we can get an "obvious" property:

Exercise. Show that if $h \in H$, $g \in G$, then $ghg^{-1} \in H$. (Check $\phi(ghg^{-1}) = 1_K$.)

Example (Non-normal subgroup). Let $D_{12} = \langle r, s \mid r^6 = s^2 = 1, rs = sr^{-1}\rangle$. Consider the subgroup of order two $H = \{1, s\}$ and notice that

$$rsr^{-1} = r(sr^{-1}) = r(rs) = r^2 s \notin H.$$

Hence, H is not normal, and cannot be the kernel of any homomorphism.

Well, duh – so what? Amazingly it turns out that this is the *sufficient* condition we want. Specifically, it makes the nice "coset multiplication" we wanted work out. Thus we have a new criterion for "normal" subgroups which does not make any external references to ϕ.

Theorem B.4.2 (Algebraic condition for normal subgroups). Let H be a subgroup of G. Then $H \trianglelefteq G$ if and only if for every $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

Proof. We already showed the forward direction. For the other direction, we need to build a homomorphism with kernel H. So we simply *define* the group $G \,/\, H$ as the cosets. To put a group operation, we need to verify the following.

Claim. If $g_1' \sim_H g_1$ and $g_2' \sim_H g_2$ then $g_1'g_2' \sim_H g_1g_2$.

Proof. Let $g_1' = g_1h_1$ and $g_2' = g_2h_2$, so we want to show that $g_1h_1g_2h_2 \sim_H g_1g_2$. Since H has the property, $g_2^{-1}h_1g_2$ is some element of H, say h_3. Thus, $h_1g_2 = g_2h_3$, and the left-hand side becomes $g_1g_2(h_3h_2)$, which is fine since $h_3h_2 \in H$. ⊛

With that settled we can just *define* the product of two cosets (of normal subgroups) by

$$(g_1H) \cdot (g_2H) = (g_1g_2)H.$$

Thus, the claim above shows that this multiplication is well-defined, so $G \,/\, H$ is indeed a group! Moreover, there is an obvious "projection" homomorphism $G \to G \,/\, H$ (with kernel H), by $g \mapsto gH$. ■

Example (Modding out in the product group). Consider again the product group $G \times H$. Earlier we identified a subgroup

$$G' = \{(g, 1_H) \mid g \in G\} \cong G.$$

You can easily see that $G' \trianglelefteq G \times H$. Moreover, you can check that

$$(G \times H)/(G') \cong H.$$

Indeed, we have $(g, h) \sim_{G'} (1_G, h)$ for all $g \in G$ and $h \in H$.

It is not necessarily true that $(G \,/\, H) \times H \cong G$.

Example (Quotients and products don't necessarily cancel). Consider $G = \mathbb{Z}/4\mathbb{Z}$ and the normal subgroup $H = \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z}$. Then $G \,/\, H \cong \mathbb{Z}/2\mathbb{Z}$, but $\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.[a]

[a] The precise condition for this kind of "canceling" is called the Schur-Zassenhaus lemma.

Example (Another explicit computation). Let $\phi : D_8 \to \mathbb{Z}_4$ be defined by

$$r \mapsto \overline{2}, \quad s \mapsto \overline{2}.$$

The kernel of this map is $N = \{1, r^2, sr, sr^3\}$. We can do a quick computation of all the elements of D_8 to get

$$\phi(1) = \phi(r^2) = \phi(sr) = \phi(sr^3) = \overline{0} \text{ and } \phi(r) = \phi(r^3) = \phi(s) = \phi(sr^2) = \overline{2}.$$

The two relevant fibers are

$$\phi^{-1}(\overline{0}) = 1N = r^2N = srN = sr^3N = \{1, r^2, sr, sr^3\}$$

and

$$\phi^{-1}(\overline{2}) = rN = r^3N = sN = sr^2N = \{r, r^3, s, sr^2\}.$$

So we see that $|D_8 \,/\, N| = 2$ is a group of order two, or $\mathbb{Z}_2$. Indeed, the image of ϕ is

$$\{\overline{0}, \overline{2}\} \cong \mathbb{Z}_2.$$

Problem. Suppose G is Abelian. Why does it follow that any subgroup of G is normal?

Finally, here's some food for thought.

Problem. Suppose one has a group presentation for a group G that uses n generators. Can you write it as a quotient of the form $F_n \,/\, N$, where N is a normal subgroup of F_n?

B.4.5 (Digression) The First Isomorphism Theorem

One quick word about what other sources usually say. Most textbooks actually *define* normal using the $ghg^{-1} \in H$ property. Then they define $G \,/\, H$ for normal H in the way I did above, using the coset definition

$$(g_1H) \cdot (g_2H) = g_1g_2H.$$

Using purely algebraic manipulations (like I did) this is well-defined, and so now you have this group $G \,/\, H$ or something.

Remark. The underlying homomorphism isn't mentioned at all, or is just mentioned in passing.

I think this is *incredibly dumb*. The normal condition looks like it gets pulled out of thin air and no one has any clue what's going on, because no one has any clue what a normal subgroup actually should look like.

Other sources like to also write the so-called first isomorphism theorem.[3] It goes like this.

Theorem B.4.3 (First isomorphism theorem)**.** Let $\phi\colon G \to H$ be a homomorphism. Then $G/\ker\phi$ is isomorphic to $\phi(G)$.

Note. To me, this is just a clumsier way of stating the same idea.

About the only merit this claim has is that if ϕ is injective, then the image $\phi(G)$ is an *isomorphic copy* of G inside the group H (try to see this directly). This is a pattern we'll often see in other branches of mathematics:

Note. Whenever we have an *injective structure-preserving map*, often the image of this map will be some "copy" of G.[a] In that sense an injective homomorphism $\phi\colon G \hookrightarrow H$ is an *embedding* of G into H.

[a]Here "structure" refers to the group multiplication.

[3]There is a second and third isomorphism theorem. But four years after learning about them, I *still* don't know what they are. So I'm guessing they weren't very important.

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