MATH635
Riemannian Geometry

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January 5, 2023

Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by Lydia Bieri. Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [FC13] as our reference; while not required, but highly recommended have on.

This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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Chapter 1

Manifolds

Lecture 1: Introduction

1.1 Introduction

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Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* $\mathcal{M}$ of dimension n is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic to $U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.2 (Coordinate chart). U' is called the *coordinate chart*.

Definition 1.1.3 (Local coordinate). The pull-back of the coordinate functions from $\mathbb{R}^n$ is called the *local coordinates*.

Definition 1.1.4 (Atlas). An *atlas* $\mathcal{A}$ is a collection such that $\mathcal{A} = \{U_\alpha, f_\alpha\}$ of charts for which the U_α are an open covering of $\mathcal{M}$, i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$, $U_\alpha \subseteq \mathcal{M}$ open.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h\colon U \to U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.5 (Locally finite). An atlas (coordinate atlas) is said to be *locally finite* if each point $p \in \mathcal{M}$ contained in only finite collection of its open sets.

Definition 1.1.6 (Smooth manifold). Let $\mathcal{A}$ be a coordinate atlas for a manifold $\mathcal{M}$. Assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of $\mathcal{A}$. The map $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces.

Definition 1.1.7 (Coodinate transition). The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of $\mathcal{A}$ for the pair of charts $(U_1, \varphi_1), (U_2, \varphi_2)$.

The atlas $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$ is called *differentiable* if all transitions are differentiable.
We can also talk about the equivalence between two atlases.

Definition 1.1.8 (Equivalence). Two atlases $\mathcal{U}, \mathcal{V}$ are equivalent if the following holds: Assume $(U_1, \varphi_1) \in \mathcal{U}$, $(V_1, \varphi_2) \in \mathcal{V}$, then

$$\varphi_1 \circ \varphi_2^{-1} \colon \varphi_2(U_1 \cap V_2) \to \varphi_1(U_1 \cap V_2)$$

and

$$\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap V_2) \to \varphi_2(U_1 \cap V_2)$$

are diffeomorphisms between subsets of Euclidean spaces.

Definition 1.1.9 (Smooth structure). A *smooth structure* on $\mathcal{M}$[a] is defined by an equivalence class $\mathcal{U}$ of coordinate atlas with property that all transition functions are diffeomorphisms. Then, the maximal differentiable atlas is our differentiable structure.

[a] Also called a *differentiable structure*.

A manifold $\mathcal{M}$ with a smooth structure is called a *smooth manifold*.[b]

[b] Also called a *differentiable manifold*.

In this way, we can do calculus on smooth manifolds! Furthermore, we can say that a function $f \colon \mathcal{M} \to \mathbb{R}$ is differentiable (or C^∞), and the collection of smooth functions of smooth manifold $\mathcal{M}$ is $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$ in general.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property: Let $\mathcal{A}$ be any given atlas from equivalence class that defines the smooth structure. If $(U_1, \varphi_1) \in \mathcal{A}$, then $f \circ \varphi_1^{-1}$ is a smooth function on $\mathbb{R}^n$. This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.1.8 requirement that defines the equivalence manifolds.

Definition 1.1.10 (Orientation). Consider an atlas for a differentiable manifold $\mathcal{M}$.

Definition 1.1.11 (Orientated). The atlas is called *orientated* if all transitions have positive functional determinant.

Definition 1.1.12 (Orientable). $\mathcal{M}$ is *orientable* if it possesses an orientated atlas.

Definition 1.1.13. Let $\mathcal{M}$ be an orientable manifold. Then a choice of a differentiable structure satisfying Definition 1.1.11 is called an *orientation* of $\mathcal{M}$, and then $\mathcal{M}$ is said to be *orientated*.

Remark. Two differentiable structures obeying Definition 1.1.11 determining the same orientation if the union again satisfying Definition 1.1.11.

Remark. If $\mathcal{M}$ is orientable and connected, then there exists exactly two distinct orientations on $\mathcal{M}$.

Example (Sphere). The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 g1\right\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \dots, n+1$, and $h_i^\pm \colon U_i^\pm \to \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, .., x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. $\mathcal{M} = \mathbb{R}^n$.

Example. $U \subseteq \mathbb{R}^n$ with $\varphi = \mathbb{1}$.

Example. Open sets of C^∞-manifolds are C^∞-manifolds.

Example. $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example. $\mathbb{R}P^n = S^n \,/\, \sim$ where $x \sim -x$ with $\pi\colon S^n \to \mathbb{R}P^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \ldots, n+1$, with

$$\left\{\left(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}\right), i = 1, \ldots, n+1\right\}$$

is a C^∞-atlas for $\mathbb{R}P^n$. ⊛

Note. $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \,/\, \sim$.

Example (Grassmannian manifolds)**.** Given m, n, $G(n, m)$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+m}$.

Appendix

Bibliography

[FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: https://books.google.com/books?id=ct91XCWkWEUC.