

MATH635  
Riemannian Geometry

Pingbang Hu

January 7, 2023

## Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by [Lydia Bieri](#). Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [\[FC13\]](#) as our reference; while not required, but highly recommended have on.

# Contents

<b>1</b>	<b>Manifolds</b>	<b>2</b>
1.1	Differentiable Manifolds . . . . .	2

# Chapter 1

## Manifolds

### Lecture 1: A Foray to Smooth Manifolds

#### 1.1 Differentiable Manifolds

5 Jan. 14:30

##### 1.1.1 Topological Manifolds

Let's start with a common definition.

**Definition 1.1.1 (Topological manifold).** A *topological manifold*  $\mathcal{M}$  of dimension  $n$  is a (topological) Hausdorff space such that each point  $p \in \mathcal{M}$  has a neighborhood  $U$  homeomorphic via  $\varphi: U \rightarrow U'$  to an open subset  $U' \subseteq \mathbb{R}^n$ .

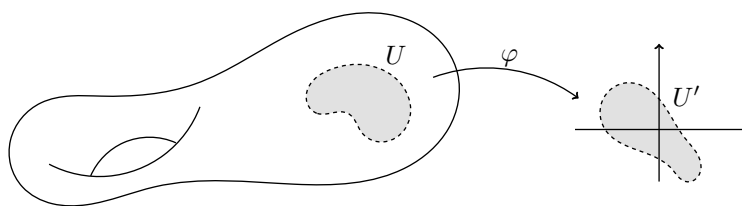
**Definition 1.1.2 (Local coordinate map).** For every  $p \in \mathcal{M}$ , the corresponding homeomorphism  $\varphi$  is called the *local coordinate map*.

**Definition 1.1.3 (Local coordinate).** The pull-back  $(x^1, \dots, x^n)$  of the *local coordinate map*  $\varphi$  from  $\mathbb{R}^n$  is called the *local coordinates* on  $U$ , given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

**Definition 1.1.4 (Coordinate chart).** The pair  $(U, \varphi)$  is called a *(coordinate) chart* on  $M$ .

In other words, a *topological manifold* can be thought of as a space such that it looks like  $\mathbb{R}^n$  locally.



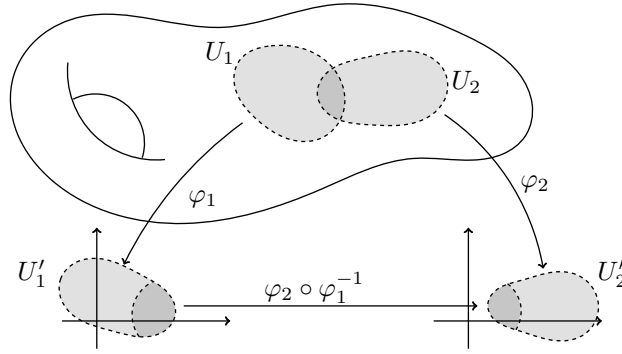
**Definition 1.1.5 (Atlas).** An *atlas*  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$  for a *manifold*  $\mathcal{M}$  is a collection of *charts* such that  $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$  are an open covering of  $\mathcal{M}$ , i.e.,  $\mathcal{M} = \bigcup_\alpha U_\alpha$ .

In other words, for all  $p \in \mathcal{M}$ , there exists a neighborhood  $U \subseteq \mathcal{M}$  and homeomorphism  $h: U \rightarrow U' \subseteq \mathbb{R}^n$  open.

**Definition 1.1.6 (Locally finite).** An *atlas* is said to be *locally finite* if each point  $p \in \mathcal{M}$  is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as  $\mathbb{R}^n$ , there's no clear way to make sense of differentiability of a **manifold**. But thankfully, we now have an explicit relation to the ambient space  $\mathbb{R}^n$  via  $\varphi_\alpha$ . To formalize, let  $\mathcal{A}$  be an **atlas** for a **manifold**  $\mathcal{M}$ , and assume that  $(U_1, \varphi_1), (U_2, \varphi_2)$  are 2 elements of  $\mathcal{A}$ . Then clearly, the map  $\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is a homeomorphism between 2 open sets of Euclidean spaces since both  $\varphi_1$  and  $\varphi_2$  are homeomorphism. Due to this map's importance, it has its own name.

**Definition 1.1.7 (Coordinate transition).** The map  $\varphi_2 \circ \varphi_1^{-1}$  is called the *coordinate transition* of  $\mathcal{A}$  for the pair of **charts**  $(U_1, \varphi_1), (U_2, \varphi_2)$ .



### 1.1.2 Differentiable Structures

Notice that the **coordinate transitions** are from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ; hence differentiability makes sense now, which induces the following.

**Definition 1.1.8 (Differentiable atlas).** The **atlas**  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$  is *differentiable* if all **transitions** are differentiable.

**Remark.** Here, the differentiability depends on the content. Sometimes, we may want it to be  $C^\infty$ , and sometimes may be  $C^k$  for some finite  $k$ . On the other hand, smooth always refers to  $C^\infty$ . We'll use them interchangeably if it's clear which case we're referring to.

**Definition 1.1.9 (Equivalence atlas).** Two **atlases**  $\mathcal{U}, \mathcal{V}$  of a **manifold** are equivalent if for every  $(U, \varphi) \in \mathcal{U}, (V, \psi) \in \mathcal{V}$ ,

$$\varphi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

**Notation (Smoothly compatible).** Two **charts**  $(U, \varphi)$  and  $(V, \psi)$  are *smoothly compatible* if either  $U \cap V = \emptyset$  or  $\psi \circ \varphi^{-1}$  is a diffeomorphism.

This suggests the following.

**Definition 1.1.10 (Smooth structure).** A *smooth structure* on  $\mathcal{M}$  is an equivalence class  $\mathcal{U}$  of **coordinate atlas** with the property that all **transition functions** are diffeomorphisms.

**Remark.** In the literature, we can also use the *maximal differentiable atlas* to be our differentiable structure. Either way is fine.

**Definition 1.1.11 (Smooth manifold).** A *smooth manifold* is a manifold  $\mathcal{M}$  with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function  $f: \mathcal{M} \rightarrow \mathbb{R}$  is differentiable (or  $C^\infty$ ) by considering differentiability of  $f \circ \varphi^{-1}$  around  $p$ .

**Notation.** The collection of smooth functions on smooth manifold  $\mathcal{M}$  is denoted by  $C^\infty(\mathcal{M}, \mathbb{R})$ , or  $C^k(\mathcal{M}, \mathbb{R})$ .

**Remark.** The class  $C^\infty(\mathcal{M}, \mathbb{R})$  consists of functions with property is well-defined.

**Proof.** Let  $\mathcal{A}$  be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if  $(U, \varphi) \in \mathcal{A}$ , then  $f \circ \varphi^{-1}$  is a smooth function on  $\mathbb{R}^n$ . This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.1.9 requirement that defines the equivalent manifolds.  $\circledast$

### 1.1.3 Orientation

Another essential property of a manifold is its orientability.

**Definition.** Consider an atlas  $\mathcal{A}$  for a differentiable manifold  $\mathcal{M}$ .

**Definition 1.1.12 (Oriented).**  $\mathcal{A}$  is *oriented* if all transitions have positive functional determinant.

**Definition 1.1.13 (Orientable).**  $\mathcal{M}$  is *orientable* if  $\mathcal{A}$  is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

**Definition 1.1.14 (Orientation).** Let  $\mathcal{M}$  be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of  $\mathcal{M}$ .

If  $\mathcal{M}$  possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.1.12.

**Remark.** Two differentiable structures obeying Definition 1.1.12 determine the same orientation if the union again satisfying Definition 1.1.12.

**Remark.** If  $\mathcal{M}$  is orientable and connected, then there exists exactly 2 distinct orientations on  $\mathcal{M}$ .

Now, we can see some examples of smooth manifolds.

**Example (Sphere).** The sphere  $S^n \subseteq \mathbb{R}^{n+1}$  given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Consider  $U_i^+ = \{x \in S^n \mid x_i > 0\}$ ,  $U_i^- = \{x \in S^n \mid x_i < 0\}$  for  $i = 1, \dots, n+1$ , and  $h_i^\pm: U_i^\pm \rightarrow \mathbb{R}^n$  such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Note that the minimum charts needed to cover  $S^n$  is 2.

**Example.** Let  $\mathcal{M} = U \subseteq \mathbb{R}^n$ , then  $\{(U, \varphi)\}$  is a smooth structure with  $\varphi = \text{id}$ .

**Example.** Open sets of  $C^\infty$ -manifolds are  $C^\infty$ -manifolds.

---

**Example** (General linear group).  $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ , open.

**Example** (Real projective space).  $\mathbb{R}P^n = S^n / \sim$  where  $x \sim -x$  with  $\pi: S^n \rightarrow \mathbb{R}P^n$ ,  $x \mapsto [x]$ .

**Proof.**  $\pi$  is a homeomorphism on each  $U_i^+$  for  $i = 1, \dots, n+1$ , with

$$\{(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}), i = 1, \dots, n+1\}$$

is a  $C^\infty$ -atlas for  $\mathbb{R}P^n$ .

⊛

**Note.** Observe that  $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ .

**Example** (Grassmannian manifold). Given  $m, n$ ,  $G(n, m)$  is the set of all  $n$ -dimensional subspaces of  $\mathbb{R}^{n+m}$ .

# Appendix



# Bibliography

- [FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: <https://books.google.com/books?id=ct91XCWkWEUC>.