

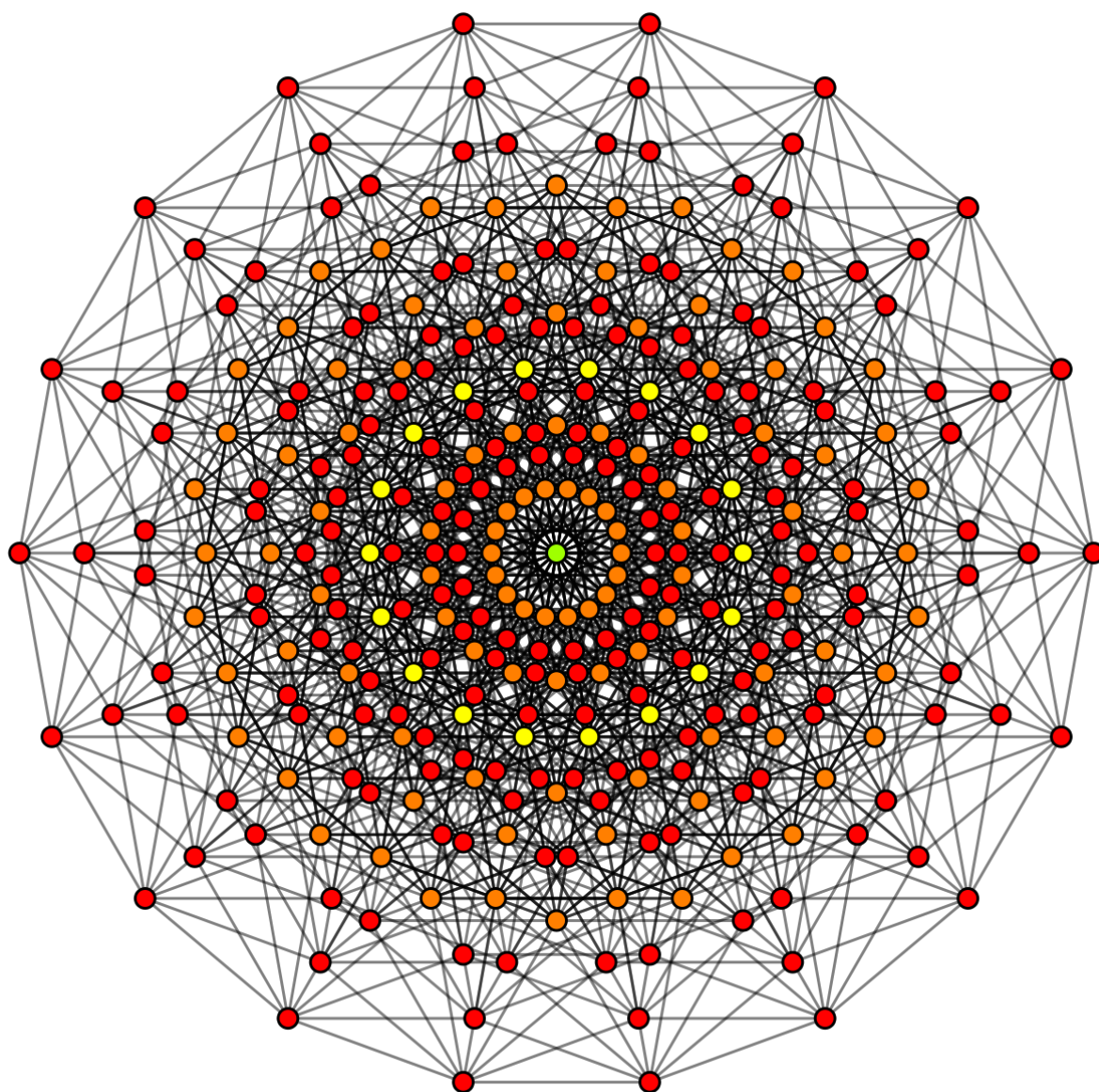
High-Dimensional Probability Solution Manual

Pingbang Hu

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Abstract

This is the solution I write for the reading group on [Roman Vershynin's](#) *High Dimensional Probability* [Ver24], where I serve as the lead. It may contain factual and/or typographic errors, and some exercises are omitted.



The reading group is held from Spring 2024, and the date on the cover page is the last updated time.

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Appetizer: using probability to cover a geometric set

Week 1: Appetizer and Basic Inequalities

19 Jan. 2024

Problem (Exercise 0.0.3). Check the following variance identities that we used in the proof of Theorem 0.0.2.

(a) Let Z_1, \dots, Z_k be independent mean zero random vectors in \mathbb{R}^n . Show that

$$\mathbb{E} \left[\left\| \sum_{j=1}^k Z_j \right\|_2^2 \right] = \sum_{j=1}^k \mathbb{E} [\|Z_j\|_2^2].$$

(b) Let Z be a random vector in \mathbb{R}^n . Show that

$$\mathbb{E} [\|Z - \mathbb{E}[Z]\|_2^2] = \mathbb{E} [\|Z\|_2^2] - \|\mathbb{E}[Z]\|_2^2.$$

Answer. (a) If Z_1, \dots, Z_k are independent mean zero random vectors in \mathbb{R}^n , then

$$\mathbb{E} \left[\left\| \sum_{j=1}^k Z_j \right\|_2^2 \right] = \mathbb{E} \left[\sum_{i=1}^n \left(\sum_{j=1}^k (Z_j)_i \right)^2 \right] = \sum_{i=1}^n \mathbb{E} \left[\left(\sum_{j=1}^k (Z_j)_i \right)^2 \right].$$

From the assumption, $\mathbb{E} [(Z_j)_i (Z_{j'})_i] = \mathbb{E} [(Z_j)_i] \mathbb{E} [(Z_{j'})_i] = 0$, hence

$$\sum_{i=1}^n \mathbb{E} \left[\left(\sum_{j=1}^k (Z_j)_i \right)^2 \right] = \sum_{i=1}^n \mathbb{E} \left[\sum_{j=1}^k (Z_j)_i^2 \right] = \sum_{j=1}^k \mathbb{E} \left[\sum_{i=1}^n (Z_j)_i^2 \right] = \sum_{j=1}^k \mathbb{E} [\|Z_j\|_2^2],$$

proving the result.

(b) If Z is a random vector in \mathbb{R}^n , then

$$\begin{aligned} \mathbb{E} [\|Z - \mathbb{E}[Z]\|_2^2] &= \mathbb{E} \left[\sum_{i=1}^n (Z_i - \mathbb{E}[Z_i])^2 \right] \\ &= \sum_{i=1}^n \mathbb{E} [Z_i^2 - 2Z_i \mathbb{E}[Z_i] + (\mathbb{E}[Z_i])^2] \\ &= \sum_{i=1}^n \mathbb{E} [Z_i^2] - 2 \sum_{i=1}^n \mathbb{E}[Z_i] \mathbb{E}[Z_i] + \sum_{i=1}^n \mathbb{E}[Z_i]^2 \\ &= \mathbb{E} [\|Z\|_2^2] - \|\mathbb{E}[Z]\|_2^2. \end{aligned}$$

⊛

Problem (Exercise 0.0.5). Prove the inequalities

$$\left(\frac{n}{m}\right)^m \leq \binom{n}{m} \leq \sum_{k=0}^m \binom{n}{k} \leq \left(\frac{en}{m}\right)^m$$

for all integers $m \in [1, n]$.

Answer. Fix some $m \in [1, n]$. We first show $(n/m)^m \leq \binom{n}{m}$. This is because

$$\frac{(n/m)^m}{\binom{n}{m}} = \prod_{j=0}^{m-1} \left(\frac{n}{m} \frac{m-j}{n-j} \right) \leq 1$$

as $\frac{n-j}{m-j} \geq \frac{n}{m}$ for all j . The second inequality $\binom{n}{m} \leq \sum_{k=0}^m \binom{n}{k}$ is trivial since $\binom{n}{k} \geq 1$ for all k . The last inequality is due to

$$\frac{\sum_{k=0}^m \binom{n}{k}}{\left(\frac{n}{m}\right)^m} \leq \sum_{k=0}^n \binom{n}{k} \left(\frac{m}{n}\right)^k = \left(1 + \frac{m}{n}\right)^n \leq e^m.$$

⊛

Problem (Exercise 0.0.6). Check that in Corollary 0.0.4,

$$(C + C\epsilon^2 N)^{\lceil 1/\epsilon^2 \rceil}$$

suffice. Here C is a suitable absolute constant.

Answer. Omit.

⊛

Chapter 1

Preliminaries on random variables

1.1 Basic quantities associated with random variables

No Exercise!

1.2 Some classical inequalities

Problem (Exercise 1.2.2). Prove the following extension of Lemma 1.2.1, which is valid for any random variable X (not necessarily non-negative):

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t) dt - \int_{-\infty}^0 \mathbb{P}(X < t) dt.$$

Answer. Separating X into the plus and minus parts would do the job. Specifically, let $X = X_+ - X_-$ where $X_+ = \max(X, 0)$ and $X_- = \max(-X, 0)$, both are non-negative. Then, we see that by applying Lemma 1.2.1,

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X_+] - \mathbb{E}[X_-] \\ &= \int_0^\infty \Pr(t < X_+) dt - \int_0^\infty \Pr(t < X_-) dt \\ &= \int_0^\infty \Pr(X > t) dt - \int_0^\infty \Pr(X < -t) dt \\ &= \int_0^\infty \Pr(X > t) dt - \int_{-\infty}^0 \Pr(X < t) dt. \end{aligned}$$

⊛

Problem (Exercise 1.2.3). Let X be a random variable and $p \in (0, \infty)$. Show that

$$\mathbb{E}[|X|^p] = \int_0^\infty pt^{p-1} \mathbb{P}(|X| > t) dt$$

whenever the right-hand side is finite.

Answer. Since $|X|^p$ is non-negative, from Lemma 1.2.1, we have

$$\mathbb{E}[|X|^p] = \int_0^\infty \Pr(t < |X|^p) dt = \int_0^\infty pt^{p-1} \Pr(|X| > t) dt$$

where we let $t \leftarrow t^p$, hence $dt \leftarrow pt^{p-1}dt$.

⊛

Week 2: Basic Inequalities and Limit Theorems

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Problem (Exercise 1.2.6). Deduce Chebyshev's inequality by squaring both sides of the bound $|X - \mu| \geq t$ and applying Markov's inequality.

Answer. From Markov's inequality, for any $t > 0$,

$$\Pr(|X - \mu| \geq t) = \Pr(|X - \mu|^2 \geq t^2) \leq \frac{\mathbb{E}[|X - \mu|^2]}{t^2} = \frac{\sigma^2}{t^2}.$$

*

1.3 Limit theorems

Problem (Exercise 1.3.3). Let X_1, X_2, \dots be a sequence of i.i.d. random variables with mean μ and finite variance. Show that

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N X_i - \mu \right| \right] = O\left(\frac{1}{\sqrt{N}}\right) \text{ as } N \rightarrow \infty.$$

Answer. We see that

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N X_i - \mu \right| \right] \leq \sqrt{\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N X_i - \mu \right|^2 \right]} = \sqrt{\text{Var} \left[\frac{1}{N} \sum_{i=1}^N X_i \right]} = \frac{\sigma}{\sqrt{N}}.$$

As $\sigma < \infty$ is a constant, the rate is exactly $O(1/\sqrt{N})$.

*

Chapter 2

Concentration of sums of independent random variables

Week 3: More Powerful Concentration Inequalities

2.1 Why concentration inequalities?

2 Feb. 2024

Problem (Exercise 2.1.4). Let $g \sim \mathcal{N}(0, 1)$. Show that for all $t \geq 1$, we have

$$\mathbb{E}[g^2 \mathbb{1}_{g>t}] = t \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} + \mathbb{P}(g > t) \leq \left(t - \frac{1}{t}\right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Answer. Denote the standard normal density as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Since we have $\Phi'(x) = -x\Phi(x)$, by integration by part,

$$\begin{aligned} \mathbb{E}[g^2 \mathbb{1}_{g>t}] &= \int_0^\infty x^2 \mathbb{1}_{x>t} \Phi(x) \, dx \\ &= - \int_t^\infty x \Phi'(x) \, dx \\ &= -x\Phi(x)|_t^\infty + \int_t^\infty \Phi(x) \, dx \\ &= t \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} + \mathbb{P}(g > t), \end{aligned}$$

which gives the first equality. Furthermore, as $t \geq 1$, we trivially have

$$\int_t^\infty \Phi(x) \, dx \leq \int_t^\infty \frac{x}{t} \Phi(x) \, dx = \frac{1}{t} \int_t^\infty -\Phi'(x) \, dx = \frac{\Phi(t)}{t},$$

implying that

$$\mathbb{E}[g^2 \mathbb{1}_{g>t}] = t \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} + \int_t^\infty \Phi(x) \, dx \leq \left(t + \frac{1}{t}\right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

which gives the second inequality. ⊗

2.2 Hoeffding's inequality

Problem (Exercise 2.2.3). Show that

$$\cosh(x) \leq \exp(x^2/2) \text{ for all } x \in \mathbb{R}.$$

Answer. Omit. ⊛

The next exercise is to prove Theorem 2.2.5 ([Hoeffding's inequality for general bounded random variables](#)), which we restate it for convenience.

Theorem 2.2.1 (Hoeffding's inequality for general bounded random variables). Let X_1, \dots, X_N be independent random variables. Assume that $X_i \in [m_i, M_i]$ for every i . Then, for any $t > 0$, we have

$$\mathbb{P}\left(\sum_{i=1}^N (X_i - \mathbb{E}[X_i]) \geq t\right) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^N (M_i - m_i)^2}\right).$$

Problem (Exercise 2.2.7). Prove the [Hoeffding's inequality for general bounded random variables](#), possibly with some absolute constant instead of 2 in the tail.

Answer. Since raising both sides to p -th power doesn't work since we're now working with sum of random variables, so we instead consider the MGF trick (also known as Cramer-Chernoff method):

Lemma 2.2.1 (Cramer-Chernoff method). Given a random variable X ,

$$\mathbb{P}(X - \mu \geq t) = \mathbb{P}(e^{\lambda(X-\mu)} \geq e^{\lambda t}) \leq \inf_{\lambda > 0} \frac{\mathbb{E}[e^{\lambda(X-\mu)}]}{e^{\lambda t}}.$$

Proof. This directly follows from the Markov's inequality. ■

Hence, we see that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^N (X_i - \mathbb{E}[X_i]) \geq t\right) &\leq \inf_{\lambda > 0} e^{-\lambda t} \mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^N (X_i - \mathbb{E}[X_i])\right)\right] \\ &= \inf_{\lambda > 0} e^{-\lambda t} \prod_{i=1}^N \mathbb{E}[e^{\lambda(X_i - \mathbb{E}[X_i])}]. \end{aligned}$$

So now everything left is to bound $\mathbb{E}[e^{\lambda(X_i - \mathbb{E}[X_i])}]$. Before we proceed, we need one lemma.

Lemma 2.2.2. For any bounded random variable $Z \in [a, b]$,

$$\text{Var}[Z] \leq \frac{(b-a)^2}{4}.$$

Proof. Since

$$\text{Var}[Z] = \text{Var}\left[Z - \frac{a+b}{2}\right] \leq \mathbb{E}\left[\left(Z - \frac{a+b}{2}\right)^2\right] \leq \frac{(b-a)^2}{4}.$$
■

Claim. Given $X \in [a, b]$ such that $\mathbb{E}[X] = 0$, for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}[e^{\lambda X}] \leq \exp\left(\lambda^2 \frac{(b-a)^2}{8}\right).$$

Proof. We first define $\psi(\lambda) = \ln \mathbb{E}[e^{\lambda X}]$, and compute

$$\psi'(\lambda) = \frac{\mathbb{E}[X e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}, \quad \psi''(\lambda) = \frac{\mathbb{E}[X^2 e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} - \left(\frac{\mathbb{E}[X e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} \right)^2.$$

Now, observe that ψ'' is the variance under the law of X re-weighted by $\frac{e^{\lambda X}}{\mathbb{E}[e^{\lambda X}]}$, i.e., by a change of measure, consider a new distribution \mathbb{P}_λ (w.r.t. the original distribution \mathbb{P} of X) as

$$d\mathbb{P}_\lambda(x) := \frac{e^{\lambda x}}{\mathbb{E}_\mathbb{P}[e^{\lambda X}]} d\mathbb{P}(x),$$

then

$$\psi'(\lambda) = \frac{\mathbb{E}_\mathbb{P}[X e^{\lambda X}]}{\mathbb{E}_\mathbb{P}[e^{\lambda X}]} = \int \frac{x e^{\lambda x}}{\mathbb{E}_\mathbb{P}[e^{\lambda X}]} d\mathbb{P}(x) = \mathbb{E}_{\mathbb{P}_\lambda}[X]$$

and

$$\psi''(\lambda) = \frac{\mathbb{E}_\mathbb{P}[X^2 e^{\lambda X}]}{\mathbb{E}_\mathbb{P}[e^{\lambda X}]} - \left(\frac{\mathbb{E}_\mathbb{P}[X e^{\lambda X}]}{\mathbb{E}_\mathbb{P}[e^{\lambda X}]} \right)^2 = \mathbb{E}_{\mathbb{P}_\lambda}[X^2] - \mathbb{E}_{\mathbb{P}_\lambda}[X]^2 = \text{Var}_{\mathbb{P}_\lambda}[X].$$

From [Lemma 2.2.2](#), since X under the new distribution \mathbb{P}_λ is still bounded between a and b ,

$$\psi''(\lambda) = \text{Var}_{\mathbb{P}_\lambda}[X] \leq \frac{(b-a)^2}{4}.$$

Then by Taylor's theorem, there exists some $\tilde{\lambda} \in [0, \lambda]$ such that

$$\psi(\lambda) = \psi(0) + \psi'(0)\lambda + \frac{1}{2}\psi''(\tilde{\lambda})\lambda^2 = \frac{1}{2}\psi''(\tilde{\lambda})\lambda^2$$

since $\psi(0) = \psi'(0) = 0$. By bounding $\psi''(\tilde{\lambda})\lambda^2/2$, we finally have

$$\ln \mathbb{E}[e^{\lambda X}] = \psi(\lambda) \leq \frac{1}{2} \cdot \frac{(b-a)^2}{4} \lambda^2 = \lambda^2 \frac{(b-a)^2}{8},$$

raising both sides by e shows the desired result. \circledast

Say given $X_i \in [m_i, M_i]$ for every i , then $X_i - \mathbb{E}[X_i] \in [m_i - \mathbb{E}[X_i], M_i - \mathbb{E}[X_i]]$ with mean 0 for every i . Then given any of the two bounds, for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[e^{\lambda(X_i - \mathbb{E}[X_i])}\right] \leq \exp\left(\lambda^2 \frac{(M_i - m_i)^2}{8}\right).$$

Then we simply recall that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^N (X_i - \mathbb{E}[X_i]) \geq t\right) &= \inf_{\lambda > 0} e^{-\lambda t} \prod_{i=1}^N \exp(\lambda(X_i - \mathbb{E}[X_i])) \\ &\leq \inf_{\lambda > 0} \exp\left(-\lambda t + \sum_{i=1}^N \lambda^2 \frac{(M_i - m_i)^2}{8}\right) \\ &= \exp\left(-\frac{4t^2}{\sum_{i=1}^N (M_i - m_i)^2} + \frac{2t^2}{\sum_{i=1}^N (M_i - m_i)^2}\right) \\ &= \exp\left(-\frac{2t^2}{\sum_{i=1}^N (M_i - m_i)^2}\right) \end{aligned}$$

since infimum is achieved at $\lambda = 4t/(\sum_{i=1}^N (M_i - m_i)^2)$. \circledast

Problem (Exercise 2.2.8). Imagine we have an algorithm for solving some decision problem (e.g., is a given number p a prime?). Suppose the algorithm makes a decision at random and returns the correct answer with probability $\frac{1}{2} + \delta$ with some $\delta > 0$, which is just a bit better than a random guess. To improve the performance, we run the algorithm N times and take the majority vote. Show that, for any $\epsilon \in (0, 1)$, the answer is correct with probability at least $1 - \epsilon$, as long as

$$N \geq \frac{1}{2\delta^2} \ln\left(\frac{1}{\epsilon}\right).$$

Answer. Consider $X_1, \dots, X_N \stackrel{\text{i.i.d.}}{\sim} \text{Ber}(\frac{1}{2} + \delta)$, which is a series of indicators indicating whether the random decision is correct or not. Note that $\mathbb{E}[X_i] = \frac{1}{2} + \delta$.

We see that by taking majority vote over N times, the algorithm makes a mistake if $\sum_{i=1}^N X_i \leq N/2$ (let's not consider tie). This happens with probability

$$\mathbb{P}\left(\sum_{i=1}^N X_i \leq \frac{N}{2}\right) = \mathbb{P}\left(\sum_{i=1}^N (X_i - \mathbb{E}[X_i]) \leq -N\delta\right) \leq \exp\left(-\frac{2(N\delta)^2}{N}\right) = e^{-2N\delta^2}$$

from Hoeffding's inequality.^a Requiring $e^{-2N\delta^2} \leq \epsilon$ is equivalent to requiring $N \geq \frac{1}{2\delta^2} \ln(1/\epsilon)$. \circledast

^aNote that the sign is flipped. However, Hoeffding's inequality still holds (why?).

Problem (Exercise 2.2.9). Suppose we want to estimate the mean μ of a random variable X from a sample X_1, \dots, X_N drawn independently from the distribution of X . We want an ϵ -accurate estimate, i.e., one that falls in the interval $(\mu - \epsilon, \mu + \epsilon)$.

- Show that a sample of size $N = O(\sigma^2/\epsilon^2)$ is sufficient to compute an ϵ -accurate estimate with probability at least $3/4$, where $s^2 = \text{Var}[X]$.
- Show that a sample of size $N = O(\log(\delta^{-1})\sigma^2/\epsilon^2)$ is sufficient to compute an ϵ -accurate estimate with probability at least $1 - \delta$.

Answer. (a) Consider using the sample mean $\hat{\mu} = \frac{1}{N} \sum_{i=1}^N X_i$ as an estimator of μ . From the Chebyshev's inequality,

$$\mathbb{P}(|\hat{\mu} - \mu| > \epsilon) \leq \frac{\sigma^2/N}{\epsilon^2}.$$

By requiring $\sigma^2/(N\epsilon^2) \leq 1/4$, i.e., $N \geq 4\sigma^2/\epsilon^2 = O(\sigma^2/\epsilon^2)$, suffices.

- Consider gathering k estimator from the above procedure, i.e., we now have $\hat{\mu}_1, \dots, \hat{\mu}_k$ such that each are an ϵ -accurate mean estimator with probability at least $3/4$. This requires $k \cdot 4\sigma^2/\epsilon^2 = O(k\sigma^2/\epsilon^2)$ samples. We claim that the median $\hat{\mu} := \text{median}(\hat{\mu}_1, \dots, \hat{\mu}_k)$ is an ϵ -accurate mean estimator with probability at least $1 - \delta$ for some k (depends on δ). Consider a series of indicators $X_i = \mathbb{1}_{|\hat{\mu}_i - \mu| > \epsilon}$, indicating if $\hat{\mu}_i$ is not ϵ -accurate. Then $X_i \sim \text{Ber}(1/4)$. Then, our median estimator $\hat{\mu}$ fails with probability

$$\mathbb{P}(|\hat{\mu} - \mu| > \epsilon) = \mathbb{P}\left(\sum_{i=1}^k X_i > \frac{k}{2}\right) = \mathbb{P}\left(\sum_{i=1}^k (X_i - \mathbb{E}[X_i]) > \frac{k}{4}\right)$$

as $\mathbb{E}[X_i] = 1/4$. From Hoeffding's inequality, the above probability is bounded above by $\exp(-2(k/4)^2/k)$, setting it to be less than δ we have

$$\exp\left(-\frac{2(k/4)^2}{k}\right) \leq \delta \Leftrightarrow \ln\left(\frac{1}{\delta}\right) \geq \frac{k}{8} \Leftrightarrow k = O(\ln(\delta^{-1})),$$

i.e., the total number of samples required is $O(k\sigma^2/\epsilon^2) = O(\ln(\delta^{-1})\sigma^2/\epsilon^2)$.

\circledast

Problem (Exercise 2.2.10). Let X_1, \dots, X_N be *non-negative* independent random variables with continuous distributions. Assume that the densities of X_i are uniformly bounded by 1.

(a) Show that the MGF of X_i satisfies

$$\mathbb{E}[\exp(-tX_i)] \leq \frac{1}{t} \text{ for all } t > 0.$$

(b) Deduce that, for any $\epsilon > 0$, we have

$$\mathbb{P}\left(\sum_{i=1}^N X_i \leq \epsilon N\right) \leq (e\epsilon)^N.$$

Answer. (a) Since X_i 's are non-negative and the densities $f_{X_i} \leq 1$ uniformly, for every $t > 0$,

$$\mathbb{E}[\exp(-tX_i)] = \int_0^\infty e^{-tx} f_{X_i}(x) dx \leq \int_0^\infty e^{-tx} dx = -\frac{1}{t} e^{-tx} \Big|_0^\infty = \frac{1}{t}.$$

(b) From [Chernoff's inequality](#), for any $\epsilon > 0$,

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^N X_i \leq \epsilon N\right) &= \mathbb{P}\left(\sum_{i=1}^N -\frac{X_i}{\epsilon} \geq -N\right) \\ &\leq \inf_{\lambda > 0} e^{\lambda N} \mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^N -\frac{X_i}{\epsilon}\right)\right] \\ &= \inf_{\lambda > 0} e^{\lambda N} \prod_{i=1}^N \mathbb{E}\left[\exp\left(-\lambda \frac{X_i}{\epsilon}\right)\right] \\ &\leq \inf_{\lambda > 0} e^{\lambda N} \prod_{i=1}^N \frac{\epsilon}{\lambda} && \text{Part (a) with } t = \lambda/\epsilon \\ &= \inf_{\lambda > 0} \left(e^\lambda \frac{\epsilon}{\lambda}\right)^N \\ &= (e\epsilon)^N \end{aligned}$$

since the infimum is achieved when $\lambda = 1$.

⊛

2.3 Chernoff's inequality

Problem (Exercise 2.3.2). Modify the proof of Theorem 2.3.1 to obtain the following bound on the lower tail. For any $t < \mu$, we have

$$\mathbb{P}(S_N \leq t) \leq e^{-\mu} \left(\frac{e\mu}{t}\right)^t.$$

Answer. A direct modification is that considering for any $\lambda > 0$,

$$\mathbb{P}(S_N \leq t) = \mathbb{P}(-S_N \geq -t) = \mathbb{P}(e^{-\lambda S_N} \geq e^{-\lambda t}) \leq e^{\lambda t} \prod_{i=1}^N \mathbb{E}[\exp(-\lambda X_i)].$$

A direct computation gives

$$\mathbb{E}[\exp(-\lambda X_i)] = e^{-\lambda} p_i + (1 - p_i) = 1 + (e^{-\lambda} - 1)p_i \leq \exp((e^{-\lambda} - 1)p_i),$$

hence

$$\mathbb{P}(S_N \leq t) \leq e^{\lambda t} \prod_{i=1}^N \exp((e^{-\lambda} - 1)p_i) = e^{\lambda t} \exp((e^{-\lambda} - 1)\mu) = \exp(\lambda t + (e^{-\lambda} - 1)\mu).$$

Minimizing the right-hand side, we see that

$$t + (-\mu e^{-\lambda}) = 0 \Leftrightarrow t = \mu e^{-\lambda} \Leftrightarrow \lambda = \ln \frac{\mu}{t}$$

achieves the infimum. And since $t < \mu$, $\lambda > 0$ as required, which gives

$$\mathbb{P}(S_N \leq t) \leq \exp\left(t \ln \frac{\mu}{t} + \left(\frac{t}{\mu} - 1\right)\mu\right) = \exp\left(t \ln \frac{\mu}{t} + t - \mu\right) = e^{-\mu} \left(\frac{e\mu}{t}\right)^t.$$

⊛

Problem (Exercise 2.3.3). Let $X \sim \text{Pois}(\lambda)$. Show that for any $t > \lambda$, we have

$$\mathbb{P}(X \geq t) \leq e^{-\lambda} \left(\frac{e\lambda}{t}\right)^t.$$

Answer. From [Chernoff's inequality](#), for any $\theta > 0$, we have

$$\mathbb{P}(X \geq t) \leq e^{-\theta t} \mathbb{E}[\exp(\theta X)].$$

Then the Poisson moment can be calculated as

$$\mathbb{E}[\exp(\theta X)] = \sum_{k=0}^{\infty} e^{\theta k} \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{\theta} \lambda)^k}{k!} = e^{-\lambda} \exp(e^{\theta} \lambda) = \exp((e^{\theta} - 1)\lambda),$$

hence

$$\mathbb{P}(X \geq t) \leq e^{-\theta t} \exp((e^{\theta} - 1)\lambda) = \left(\frac{\lambda}{t}\right)^t \exp(t - \lambda) = e^{-\lambda} \left(\frac{e\lambda}{t}\right)^t$$

where we take the minimizing $\theta = \ln(t/\lambda) > 0$ as $t > \lambda$.

⊛

Alternatively, we can also solve [Exercise 2.3.3](#) directly as follows.

Answer. Consider a series of independent Bernoulli random variables $X_{N,i}$ for a fixed N such that the Poisson limit theorem applies to approximate $X \sim \text{Pois}(\lambda)$, i.e., as $N \rightarrow \infty$, $\max_{i \leq N} p_{N,i} \rightarrow 0$ and $\lambda_N := \mathbb{E}[S_N] \rightarrow \lambda < \infty$, $S_N \rightarrow \text{Pois}(\lambda)$. From Chernoff's inequality, for any $t > \lambda_N$,

$$\mathbb{P}(S_N > t) \leq e^{-\lambda_N} \left(\frac{e\lambda_N}{t}\right)^t.$$

We then see that

$$\mathbb{P}(X > t) = \lim_{N \rightarrow \infty} \mathbb{P}(S_N > t) \leq \lim_{N \rightarrow \infty} e^{-\lambda_N} \left(\frac{e\lambda_N}{t}\right)^t = e^{-\lambda} \left(\frac{e\lambda}{t}\right)^t$$

since $\lambda_N \rightarrow \lambda$ as $N \rightarrow \infty$.

⊛

Week 4: Chernoff's Inequality and Degree Concentration

Problem (Exercise 2.3.5). Show that, in the setting of Theorem 2.3.1, for $\delta \in (0, 1]$ we have

$$\mathbb{P}(|S_N - \mu| \geq \delta \mu) \leq 2e^{-c\mu\delta^2}$$

where $c > 0$ is an absolute constant.

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Answer. From Chernoff's inequality (right-tail), for $t = (1 + \delta)\mu$, we have

$$\begin{aligned}\ln \mathbb{P}(S_N \geq (1 + \delta)\mu) &\leq -\mu + (1 + \delta)\mu(1 + \ln \mu - \ln(1 + \delta) - \ln \mu) \\ &= \delta\mu - (1 + \delta)\mu(\ln(1 + \delta)) \\ &= \mu(\delta - (1 + \delta)\ln(1 + \delta)).\end{aligned}$$

A classic bound for $\ln(1 + \delta)$ is the following.

Claim. For all $x > 0$,

$$\frac{2x}{2+x} \leq \ln(1+x).$$

Proof. As $(1 + x/2)^2 = 1 + x + x^2/4 \geq 1 + x$,

$$[\log(1+x)]' = \frac{1}{1+x} \geq \frac{1}{(1+x/2)^2} = \left(\frac{x}{1+x/2}\right)'.$$

Note that $\log(1+x) = x/(1+x/2) = 0$ at $x = 0$, so for all $x > 0$

$$\log(1+x) \geq \frac{x}{1+x/2}.$$

⊗

Hence, as our $\delta \in (0, 1]$, we have

$$\ln \mathbb{P}(S_N \geq (1 + \delta)\mu) \leq \mu(\delta - (1 + \delta)\ln(1 + \delta)) \leq \mu\delta - \mu(1 + \delta)\frac{2\delta}{2 + \delta} = -\frac{\mu\delta^2}{2 + \delta} \leq -\frac{\mu\delta^2}{3}.$$

Similarly, from Chernoff's inequality (left-tail), for $t = (1 - \delta)\mu$, we have

$$\begin{aligned}\ln \mathbb{P}(S_N \leq (1 - \delta)\mu) &\leq -\mu + (1 - \delta)\mu(1 + \ln \mu - \ln(1 - \delta) - \ln \mu) \\ &= -\delta\mu - (1 - \delta)\mu \ln(1 - \delta) \\ &= \mu(-\delta - (1 - \delta)\ln(1 - \delta)).\end{aligned}$$

Another classic bound for $\ln(1 - \delta)$ is the following.

Claim. For all $x \in [-1, 1)$,

$$-x - \frac{x^2}{2} \leq \ln(1 - x).$$

Proof. This one is even easier: since $\ln(1 - x) = -x - x^2/2 - x^3/3 - \dots$

⊗

Hence, if $\delta \in (0, 1]$,^a we have

$$\ln \mathbb{P}(S_N \leq (1 - \delta)\mu) \leq \mu(-\delta - (1 - \delta)\ln(1 - \delta)) \leq -\mu\delta - \mu(1 - \delta)\left(-\delta - \frac{\delta^2}{2}\right) \leq -\frac{\mu\delta^2}{2}.$$

Combining two tails, we then see that

$$\begin{aligned}\mathbb{P}(|S_N - \mu| > \delta\mu) &\leq \mathbb{P}(S_N \geq (1 + \delta)\mu) + \mathbb{P}(S_N \leq (1 - \delta)\mu) \\ &\leq \exp\left(-\frac{\mu\delta^2}{3}\right) + \exp\left(-\frac{\mu\delta^2}{2}\right) \\ &\leq 2\exp\left(-\frac{\mu\delta^2}{3}\right),\end{aligned}$$

which almost complete the proof for $c = 1/3$.

⊗

^aWhen $\delta = 1$, $\ln \mathbb{P}(S_N \leq (1 - \delta)\mu) \leq -\frac{\mu\delta^2}{2}$ holds trivially since $\mathbb{P}(S_N = 0) \leq \exp(-\mu/2)$.

Problem (Exercise 2.3.6). Let $X \sim \text{Pois}(\lambda)$. Show that for $t \in (0, \lambda]$, we have

$$\mathbb{P}(|X - \lambda| \geq t) \leq 2 \exp\left(-\frac{ct^2}{\lambda}\right).$$

Answer. Fix some $t =: \delta\lambda \in (0, \lambda]$ for some $\delta \in (0, 1]$ first. Consider a series of independent Bernoulli random variables $X_{N,i}$ for a fixed N such that the Poisson limit theorem applies to approximate $X \sim \text{Pois}(\lambda)$, i.e., as $N \rightarrow \infty$, $\max_{i \leq N} p_{N,i} \rightarrow 0$ and $\lambda_N := \mathbb{E}[S_N] \rightarrow \lambda < \infty$, $S_N \rightarrow \text{Pois}(\lambda)$. From multiplicative form of Chernoff's inequality, for $t_N := \delta\lambda_N$,

$$\mathbb{P}(|S_N - \lambda_N| \geq t_N = \delta\lambda_N) \leq 2 \exp\left(-\frac{ct_N^2}{\lambda_N}\right).$$

It then follows that from the Poisson limit theorem,

$$\mathbb{P}(|X - \lambda| \geq t) = \lim_{N \rightarrow \infty} \mathbb{P}(|S_N - \lambda_N| \geq t_N) = \lim_{N \rightarrow \infty} 2 \exp\left(-\frac{ct_N^2}{\lambda_N}\right) = 2 \exp\left(-\frac{ct^2}{\lambda}\right)$$

since $t_N = \delta\lambda_N \rightarrow \delta\lambda = t$. *

Problem (Exercise 2.3.8). Let $X \sim \text{Pois}(\lambda)$. Show that, as $\lambda \rightarrow \infty$, we have

$$\frac{X - \lambda}{\sqrt{\lambda}} \xrightarrow{D} \mathcal{N}(0, 1).$$

Answer. Since $X := \sum_{i=1}^{\lambda} X_i \sim \text{Pois}(\lambda)$ if $X_i \stackrel{\text{i.i.d.}}{\sim} \text{Pois}(1)$ for all i , from Lindeberg-Lévy central limit theorem, we have

$$\frac{X - \mathbb{E}[X]}{\sqrt{\text{Var}[X]}} = \frac{X - \lambda}{\sqrt{\lambda}} \xrightarrow{d} \mathcal{N}(0, 1)$$

as $\mathbb{E}[X_i] = \text{Var}[X_i] = 1$. *

2.4 Application: degrees of random graphs

Problem. Consider a random graph $G \sim G(n, p)$ with expected degrees $d = O(\log n)$. Show that with high probability (say, 0.9), all vertices of G have degrees $O(\log n)$.

Answer. Since $d = O(\log n)$, there exists an absolute constant $M > 0$ such that $d = (n-1)p \leq M \log n$ for all large enough n . Now, consider some $C > 0$ such that $eM/C =: \alpha < 1$. From Chernoff's inequality,

$$\mathbb{P}(d_i \geq C \log n) \leq e^{-d} \left(\frac{ed}{C \log n}\right)^{C \log n} \leq e^{-d} \left(\frac{eM}{C}\right)^{C \log n} \leq \alpha^{C \log n}.$$

Hence, from union bound, we have

$$\mathbb{P}(\forall i: d_i \leq C \log n) \geq 1 - n\alpha^{C \log n},$$

which can be arbitrarily close to 1 as C is sufficiently large. *

Problem (Exercise 2.4.3). Consider a random graph $G \sim G(n, p)$ with expected degrees $d = O(1)$. Show that with high probability (say, 0.9), all vertices of G have degrees

$$O\left(\frac{\log n}{\log \log n}\right).$$

Answer. Since now $d = (n-1)p \leq M$ for some absolute constant $M > 0$ for all large n , from Chernoff's inequality,

$$\mathbb{P}\left(d_i \geq C \frac{\log n}{\log \log n}\right) \leq e^{-d} \left(\frac{ed}{C \frac{\log n}{\log \log n}}\right)^{C \frac{\log n}{\log \log n}} \leq e^{-d} \left(\frac{eM \log \log n}{C \log n}\right)^{C \frac{\log n}{\log \log n}}$$

for some $C > 0$. This implies that

$$\mathbb{P}\left(\forall i: d_i \leq C \frac{\log n}{\log \log n}\right) \geq 1 - ne^{-d} \left(\frac{eM \log \log n}{C \log n}\right)^{C \frac{\log n}{\log \log n}}.$$

Now, considering $C = M$, we have

$$ne^{-d} \left(\frac{eM \log \log n}{C \log n}\right)^{C \frac{\log n}{\log \log n}} \leq ne^{-d} \left(\frac{e \log \log n}{\log n}\right)^{M \frac{\log n}{\log \log n}}.$$

Taking logarithm, we observe that

$$\begin{aligned} & \log n - d + M \frac{\log n}{\log \log n} (1 + \log \log \log n - \log \log n) \\ &= (1 - M) \log n - d + M \frac{\log n}{\log \log n} (1 + \log \log \log n) \\ &= \left[1 - M \left(1 + \frac{1}{\log \log n} + \frac{\log \log \log n}{\log \log n}\right)\right] \log n - d \rightarrow -\infty \end{aligned}$$

as $n \rightarrow \infty$, i.e.,

$$ne^{-d} \left(\frac{eM \log \log n}{C \log n}\right)^{C \frac{\log n}{\log \log n}} \rightarrow 0,$$

which is what we want to prove. *

Problem (Exercise 2.4.5). Consider a random graph $G \sim G(n, p)$ with expected degrees $d = O(1)$. Show that with high probability, (say, 0.9), G has a vertex with degree

$$\Omega\left(\frac{\log n}{\log \log n}\right).$$

Answer. Firstly, note that the question is ill-defined in the sense that if $d = (n-1)p = O(1)$, it can be $d = 0$ (with $p = 0$), which is impossible to prove the claim. Hence, consider the non-degenerate case, i.e., $d = \Theta(1)$.

We want to prove that there exists some absolute constant $C > 0$ such that with high probability G has a vertex with degree at least $C \log n / \log \log n$. First, consider separate the graph randomly into two parts A, B , each of size $n/2$. It's then easy to see by dropping every inner edge in A and B , the graph becomes bipartite such that now A and B forms independent sets. Consider working on this new graph (with degree denoted as d'), we have

$$\begin{aligned} \mathbb{P}(d'_i = k) &= \binom{n/2}{k} \left(\frac{d}{n-1}\right)^k \left(1 - \frac{d}{n-1}\right)^{n/2-k} \geq \left(\frac{n}{2k}\right)^k \cdot \frac{d^k}{n^k} \cdot e^{-d} \\ &= d^k n^{-k} \left(\frac{n}{2k}\right)^k e^{-d} = \left(\frac{d}{2k}\right)^k e^{-d}. \end{aligned}$$

Let $k = C \log n / \log \log n$ such that $d/2k > 1/\log n$ for large enough n ,^a we have

$$\begin{aligned} \mathbb{P}\left(d'_i = \frac{C \log n}{\log \log n}\right) &\geq e^{-d} \left(\frac{d}{2k}\right)^k \geq e^{-d} (\log n)^{-k} = \exp(-d - k \log \log n) \\ &= \exp(-d - C \log n) = e^{-d} n^{-C}. \end{aligned}$$

Let this probability be q , and focus on A . We can then define $X_i = \mathbb{1}_{d'_i=k}$ for $i \in A$, and note that X_i are all independent as A being an independent set. Then, the number of vertices in A , denoted as X , with degree exactly k follows $\text{Bin}(n/2, q)$ with $X = \sum_{i \in A} X_i$ and mean $nq/2$, variance $nq(1-q)/2$. From Chebyshev's inequality,

$$\mathbb{P}(X = 0) \leq \mathbb{P}(|X - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} = \frac{nq(1-q)/2}{(nq/2)^2} = 2 \frac{1-q}{nq} \leq \frac{2}{nq} \leq \frac{2}{ne^{-d} n^{-C}} = \frac{2e^d}{n^{1-C}}.$$

Now, by setting $C < 1$, say $1/2$, then

$$\mathbb{P}(X = 0) \leq 2e^d n^{-1/2} \rightarrow 0$$

as $n \rightarrow \infty$, which means $\mathbb{P}(X \geq 1) \rightarrow 1$, i.e., with probability 1, there are at least one point with degree $\log n / 2 \log \log n$. Now, by considering the deleting edges in the beginning, we conclude that there will be a vertex with degree

$$\Omega\left(\frac{\log n}{\log \log n}\right)$$

with overwhelming probability. *

^aSince this is equivalent as $k < d \log n / 2$. As k has a $\log \log n \rightarrow \infty$ factor in the denominator, the claim holds.

Week 5: Sub-Gaussian Random Variables

2.5 Sub-gaussian distributions

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Problem (Exercise 2.5.1). Show that for each $p \geq 1$, the random variable $X \sim \mathcal{N}(0, 1)$ satisfies

$$\|X\|_{L^p} = (\mathbb{E}[|X|^p])^{1/p} = \sqrt{2} \left(\frac{\Gamma((1+p)/2)}{\Gamma(1/2)} \right)^{1/p}.$$

Deduce that

$$\|X\|_{L^p} = O(\sqrt{p}) \text{ as } p \rightarrow \infty.$$

Answer. We see that for $p \geq 1$, we have

$$(\mathbb{E}[|X|^p])^{1/p} = \left(\int_{-\infty}^{\infty} |x|^p \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)^{1/p} = \left(2 \int_0^{\infty} |x|^p \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)^{1/p}$$

from the symmetry around 0. Next, consider a change of variable $x^2 =: u$, we have

$$= \left(2 \frac{1}{\sqrt{2\pi}} \int_0^{\infty} u^{p/2} e^{-u/2} \frac{1}{2\sqrt{u}} du \right)^{1/p} = \left(\frac{1}{\sqrt{2\pi}} \int_0^{\infty} u^{(p-1)/2} e^{-u/2} du \right)^{1/p}$$

with another change of variable $u/2 =: t$,

$$\begin{aligned} &= \left(\frac{1}{\sqrt{2\pi}} \int_0^{\infty} (2t)^{(p-1)/2} e^{-t} 2 dt \right)^{1/p} = \left(\frac{1}{\sqrt{2\pi}} \cdot 2^{(p-1)/2} \cdot 2 \int_0^{\infty} t^{(p-1)/2} e^{-t} dt \right)^{1/p} \\ &= \left(\frac{1}{\sqrt{2\pi}} 2^{(p+1)/2} \Gamma\left(\frac{p+1}{2}\right) \right)^{1/p} = \left(\frac{1}{\sqrt{2}} \sqrt{2^{p+1}} \frac{\Gamma((p+1)/2)}{\Gamma(1/2)} \right)^{1/p} \end{aligned}$$

as $\Gamma(1/2) = \sqrt{\pi}$, we finally have

$$= \sqrt{2} \left(\frac{\Gamma((p+1)/2)}{\Gamma(1/2)} \right)^{1/p},$$

where we recall that

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

To show that $\|X\|_{L^p} = O(\sqrt{p})$ as $p \rightarrow \infty$, we first note the following.

Lemma 2.5.1. We have that for $p \geq 1$,

$$\Gamma\left(\frac{1+p}{2}\right) = \begin{cases} 2^{-p/2} \sqrt{\pi} (p-1)!!, & \text{if } p \text{ is even;} \\ 2^{-(p-1)/2} (p-1)!!, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Consider the Legendre duplication formula, i.e.,

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z} \sqrt{\pi} \Gamma(2z).$$

We see that for p being even, $(1+p)/2 = p/2 + 1/2$, by letting $z := p/2 \in \mathbb{N}$,

$$\begin{aligned} \Gamma((1+p)/2) &= \frac{2^{1-p} \sqrt{\pi} \Gamma(p)}{\Gamma(p/2)} = 2^{1-p} \sqrt{\pi} \frac{(p-1)!}{(p/2-1)!} \\ &= 2^{1-p} \sqrt{\pi} \frac{(p-1)!}{(1/2)^{p/2-1} (p-2)!!} = 2^{-p/2} \sqrt{\pi} (p-1)!!. \end{aligned}$$

For odd p , recall the identity $\Gamma(z+1) = z\Gamma(z)$. We then have

$$\begin{aligned} \Gamma((1+p)/2) &= \frac{p-1}{2} \cdot \Gamma((p-1)/2) \\ &= \frac{(p-1)(p-3)}{2^2} \cdot \Gamma((p-3)/2) \\ &\vdots \\ &= \frac{(p-1)(p-3)\dots(p-(p-2))}{2^{(p-1)/2}} \cdot \Gamma(1) & 2 = (p - (p-2)) \\ &= 2^{-(p-1)/2} (p-1)(p-3)\dots(2) \\ &= 2^{-(p-1)/2} (p-1)!!. \end{aligned}$$

■

We then see that as $p \rightarrow \infty$,

$$\|X\|_{L^p} = \sqrt{2} \left(\frac{\Gamma((1+p)/2)}{\Gamma(1/2)} \right)^{1/p} \lesssim ((p-1)!!)^{1/p} = O(\sqrt{p}^{1/p}) = O(\sqrt{p}).$$

⊛

Problem (Exercise 2.5.4). Show that the condition $\mathbb{E}[X] = 0$ is necessary for property v to hold.

Answer. Since if $\mathbb{E}[\exp(\lambda X)] \leq \exp(K_5^2 \lambda^2)$ for all $\lambda \in \mathbb{R}$, we see that from Jensen's inequality,

$$\exp(\mathbb{E}[\lambda X]) \leq \mathbb{E}[\exp(\lambda X)] \leq \exp(K_5^2 \lambda^2),$$

i.e.,

$$\lambda \mathbb{E}[X] \leq K_5^2 \lambda^2.$$

Since this holds for every $\lambda \in \mathbb{R}$, if $\lambda > 0$, $\mathbb{E}[X] \leq K_5^2 \lambda$; on the other hand, if $\lambda < 0$, $\mathbb{E}[X] \geq K_5^2 \lambda$. In either case, as $\lambda \rightarrow 0$ (from both sides, respectively), $0 \leq \mathbb{E}[X] \leq 0$, hence $\mathbb{E}[X] = 0$. ⊛

Problem (Exercise 2.5.5). (a) Show that if $X \sim \mathcal{N}(0, 1)$, the function $\lambda \mapsto \mathbb{E}[\exp(\lambda^2 X^2)]$ is only finite in some bounded neighborhood of zero.

- (b) Suppose that some random variable X satisfies $\mathbb{E}[\exp(\lambda^2 X^2)] \leq \exp(K\lambda^2)$ for all $\lambda \in \mathbb{R}$ and some constant K . Show that X is a bounded random variable, i.e., $\|X\|_\infty < \infty$.

Answer. (a) If $X \sim \mathcal{N}(0, 1)$, we see that

$$\mathbb{E}[\exp(\lambda^2 X^2)] = \int_{-\infty}^{\infty} \exp(\lambda^2 x^2) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp((\lambda^2 - 1/2)x^2) dx.$$

It's obvious that if $\lambda^2 - 1/2 \geq 0$, the above integral doesn't converge simply because $e^{\epsilon x^2}$ for any $\epsilon \geq 0$ is unbounded. On the other hand, if $\lambda^2 - 1/2 < 0$, then this is just a (scaled) Gaussian integral, which converges. Hence, this function is only finite in $\lambda \in (-1/\sqrt{2}, 1/\sqrt{2})$.

- (b) Simply because that for any t , we have that for any λ ,

$$\mathbb{P}(|X| > t) \leq \frac{\mathbb{E}[\exp(\lambda^2 X^2)]}{\exp(\lambda^2 t^2)} \leq \frac{\exp(K\lambda^2)}{\exp(\lambda^2 t^2)} = \exp(\lambda^2(K - t^2)).$$

Now, let's pick $t > \sqrt{K}$ (as K being a constant, t can be any constant greater than $t > \sqrt{K}$), so $\lambda^2(K - t^2) < 0$. By letting $\lambda \rightarrow \infty$, we see that $\mathbb{P}(|X| > t) = 0$, i.e., $\mathbb{P}(|X| \leq t) = 1$. Since we're in one-dimensional, $|X| = \|X\|_\infty$, hence we're done.

⊛

Problem (Exercise 2.5.7). Check that $\|\cdot\|_{\psi_2}$ is indeed a norm on the space of sub-gaussian random variables.

Answer. It's clear that $\|X\|_{\psi_2} = 0$ if and only if $X = 0$. Also, for any $\lambda > 0$, $\|\lambda X\|_{\psi_2} = \lambda\|X\|_{\psi_2}$ is obvious. Hence, we only need to verify triangle inequality, i.e., for any sub-gaussian random variables X and Y ,

$$\|X + Y\|_{\psi_2} \leq \|X\|_{\psi_2} + \|Y\|_{\psi_2}.$$

Firstly, we observe that since $\exp(x)$ and x^2 are both convex (hence their composition),

$$\begin{aligned} \exp\left(\left(\frac{X + Y}{\|X\|_{\psi_2} + \|Y\|_{\psi_2}}\right)^2\right) &\leq \frac{\|X\|_{\psi_2}}{\|X\|_{\psi_2} + \|Y\|_{\psi_2}} \exp((X/\|X\|_{\psi_2})^2) \\ &\quad + \frac{\|Y\|_{\psi_2}}{\|X\|_{\psi_2} + \|Y\|_{\psi_2}} \exp((Y/\|Y\|_{\psi_2})^2). \end{aligned}$$

Then, by taking expectation on both sides,

$$\mathbb{E}\left[\exp\left(\left(\frac{X + Y}{\|X\|_{\psi_2} + \|Y\|_{\psi_2}}\right)^2\right)\right] \leq 2 \frac{\|X\|_{\psi_2}}{\|X\|_{\psi_2} + \|Y\|_{\psi_2}} + 2 \frac{\|Y\|_{\psi_2}}{\|X\|_{\psi_2} + \|Y\|_{\psi_2}} = 2.$$

Now, we see that from the definition of $\|X + Y\|_{\psi_2}$ and $t := \|X\|_{\psi_2} + \|Y\|_{\psi_2}$, the above implies

$$\|X + Y\|_{\psi_2} \leq \|X\|_{\psi_2} + \|Y\|_{\psi_2},$$

hence the triangle inequality is verified.

⊛

Problem (Exercise 2.5.10). Let X_1, X_2, \dots , be a sequence of sub-gaussian random variables, which are not necessarily independent. Show that

$$\mathbb{E}\left[\max_i \frac{|X_i|}{\sqrt{1 + \log i}}\right] \leq CK,$$

where $K = \max_i \|X_i\|_{\psi_2}$. Deduce that for every $N \geq 2$ we have

$$\mathbb{E} \left[\max_{i \leq N} |X_i| \right] \leq CK \sqrt{\log N}.$$

Answer. Let $Y_i := |X_i|/K\sqrt{1+\log i}$ (which is always positive) for all $i \geq 1$. Then for all $t \geq 0$,

$$\begin{aligned} \mathbb{P}(Y_i \geq t) &= \mathbb{P} \left(\frac{|X_i|}{K\sqrt{1+\log i}} \geq t \right) \\ &= \mathbb{P} \left(|X_i| \geq tK\sqrt{1+\log i} \right) \\ &\leq 2 \exp \left(-\frac{ct^2 K^2 (1+\log i)}{\|X_i\|_{\psi_2}^2} \right) \leq 2 \exp(-ct^2(1+\log i)) = 2(ei)^{-ct^2} \end{aligned}$$

as $K := \max_i \|X_i\|_{\psi_2}$. Then, our goal now is to show that $\mathbb{E}[\max_i Y_i] \leq C$ for some absolute constant C . Consider $t_0 := \sqrt{1/c}$, then we have

$$\begin{aligned} \mathbb{E} \left[\max_i Y_i \right] &= \int_0^\infty \mathbb{P} \left(\max_i Y_i \geq t \right) dt \\ &\leq \int_0^{t_0} \mathbb{P} \left(\max_i Y_i \geq t \right) dt + \int_{t_0}^\infty \sum_{i=1}^\infty \mathbb{P}(Y_i \geq t) dt && \text{union bound} \\ &\leq t_0 + \int_{t_0}^\infty \sum_{i=1}^\infty 2(ei)^{-ct^2} dt \\ &\leq \sqrt{1/c} + 2 \int_{t_0}^\infty e^{-ct^2} \sum_{i=1}^\infty i^{-2} dt \\ &\leq \sqrt{1/c} + 2 \cdot \frac{\pi^2}{6} \int_0^\infty e^{-ct^2} dt = \sqrt{1/c} + \frac{\pi^2}{3} \cdot \frac{\sqrt{\pi}}{2\sqrt{c}} = \frac{1 + \frac{\pi^{5/2}}{6}}{\sqrt{c}} =: C. \end{aligned}$$

Finally, for every $N \geq 2$,

$$\mathbb{E} \left[\max_{i \leq N} \frac{|X_i|}{\sqrt{1+\log i}} \right] \leq \mathbb{E} \left[\max_{i \leq N} \frac{|X_i|}{\sqrt{1+\log i}} \right] \leq \mathbb{E} \left[\max_i \frac{|X_i|}{\sqrt{1+\log i}} \right] \leq CK,$$

i.e., $\mathbb{E}[\max_{i \leq N} |X_i|] \leq CK\sqrt{1+\log N} \leq CK\sqrt{2\log N}$ for all $N \geq 2$. By letting $C' := \sqrt{2}C$,

$$\mathbb{E} \left[\max_{i \leq N} |X_i| \right] \leq C' K \sqrt{\log N},$$

which is exactly what we want. *

Problem (Exercise 2.5.11). Show that the bound in [Exercise 2.5.10](#) is sharp. Let X_1, X_2, \dots, X_N be independent $\mathcal{N}(0, 1)$ random variables. Prove that

$$\mathbb{E} \left[\max_{i \leq N} X_i \right] \geq c\sqrt{\log N}.$$

Answer. Again, let's first write

$$\mathbb{E} \left[\max_{i \leq N} X_i \right] = \int_0^\infty \mathbb{P} \left(\max_{i \leq N} X_i \geq t \right) dt,$$

and observe that for any $t \geq 0$,

$$\begin{aligned}
 \mathbb{P}(X_i \geq t) &= \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{(x+t)^2}{2}\right) dx && x \leftarrow x + t \\
 &\geq \frac{1}{\sqrt{2\pi}} \int_0^1 \exp\left(-\frac{(x+t)^2}{2}\right) dx \\
 &\geq Ce^{-t^2}
 \end{aligned}$$

for some constant $C > 0$. Since X_i 's are i.i.d.,

$$\mathbb{P}\left(\max_{i \leq N} X_i \geq t\right) = 1 - (\mathbb{P}(X_1 < t))^N = 1 - (1 - \mathbb{P}(X_1 \geq t))^N,$$

so

$$\begin{aligned}
 \mathbb{E}\left[\max_{i \leq N} X_i\right] &= \int_0^\infty 1 - (1 - \mathbb{P}(X_1 \geq t))^N dt \\
 &\geq \int_0^\infty 1 - (1 - Ce^{-t^2})^N dt \\
 &= \sqrt{\log N} \int_0^\infty 1 - \left(1 - \frac{C}{Nu^2}\right)^N du. && t =: \sqrt{\log N} u
 \end{aligned}$$

Finally, as the final integral can be further bounded below by some absolute constant c depending only on C , hence we obtain the desired result. \circledast

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