10.6. Show that the canonical correlations are invariant under nonsingular linear transformations of the $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$ variables of the form $\underset{(p\times p)}{\mathbf{C}}\;\underset{(p\times 1)}{\mathbf{X}^{(1)}}$ and $\underset{(q\times q)}{\mathbf{D}}\;\underset{(q\times 1)}{\mathbf{X}^{(2)}}$.

Hint: Consider $\mathrm{Cov}\left(\begin{bmatrix} \mathbf{CX}^{(1)} \\ \hline \mathbf{DX}^{(2)} \end{bmatrix}\right) = \left[\begin{array}{c|c} \mathbf{C\Sigma}_{11}\mathbf{C}' & \mathbf{C\Sigma}_{12}\mathbf{D}' \\ \hline \mathbf{D\Sigma}_{21}\mathbf{C}' & \mathbf{D\Sigma}_{22}\mathbf{D}' \end{array}\right]$. Consider any linear combination $\mathbf{a}_1'(\mathbf{CX}^{(1)}) = \mathbf{a}'\mathbf{X}^{(1)}$ with $\mathbf{a}' = \mathbf{a}_1'\mathbf{C}$. Similarly, consider $\mathbf{b}_1'(\mathbf{DX}^{(2)}) = \mathbf{b}'\mathbf{X}^{(2)}$ with $\mathbf{b}' = \mathbf{b}_1'\mathbf{D}$. The choices $\mathbf{a}_1' = \mathbf{e}'\mathbf{\Sigma}_{11}^{-1/2}\mathbf{C}^{-1}$ and $\mathbf{b}_1' = \mathbf{f}'\mathbf{\Sigma}_{22}^{-1/2}\mathbf{D}^{-1}$ give the maximum correlation.

Hints:

10.6 $\mathrm{Cov}\begin{bmatrix} CX^{(1)} \\ \hline DX^{(2)} \end{bmatrix} = \left[\begin{array}{c|c} C\Sigma_{11}C' & C\Sigma_{12}D' \\ \hline D\Sigma_{21}C' & D\Sigma_{22}D' \end{array}\right]$

$a_1'(CX^1) = a'X^1 \quad a' = a_1'C \quad b_1'(DX^2) = b'X^2 \quad b' = b_1'D$

$a_1' = e'\Sigma_{11}^{-1/2}C^{-1} \quad b_1' = f'\Sigma_{22}^{-1/2}D^{-1}$ max correlation

Show that the canonical correlation are invariant under non singular linear transformations of the X^1, X^2 variables of the form $CX^{(1)}$ and $DX^{(2)}$.

$a'X^1 = a_1'CX^1 = \underbrace{e'\Sigma_{11}^{-1/2}C^{-1}}_{a_1'}CX^1 = e'\Sigma_{11}^{-1/2}X^1$

So $a'X^1 = e'\Sigma_{11}^{-1/2}X^1$, and with out loss of generality

$b'X^2 = f'\Sigma_{22}^{-1/2}X^2$

$$\mathrm{Corr}(CX^1, DX^2) = \frac{a'\Sigma_{12}b}{\sqrt{a'\Sigma_{11}a}\cdot\sqrt{b'\Sigma_{22}b}} = \frac{e\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1/2}f'}{\sqrt{ee'}\cdot\sqrt{ff'}}$$

Given the non singular linear transformations with variables CX^1 and DX^2, since the C and D cancel out the canonical correlation $\mathrm{Corr}(CX^1, DX^2)$ is same as $\mathrm{Corr}(X^1, X^2)$

10.9. H. Hotelling [5] reports that $n = 140$ seventh-grade children received four tests on $X_1^{(1)}$ = reading speed, $X_2^{(1)}$ = reading power, $X_1^{(2)}$ = arithmetic speed, and $X_2^{(2)}$ = arithmetic power. The correlations for performance are

$$\mathbf{R} = \left[\begin{array}{c|c} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \hline \mathbf{R}_{21} & \mathbf{R}_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1.0 & .6328 & .2412 & .0586 \\ .6328 & 1.0 & -.0553 & .0655 \\ \hline .2412 & -.0553 & 1.0 & .4248 \\ .0586 & .0655 & .4248 & 1.0 \end{array}\right]$$

(a) Find all the sample canonical correlations and the sample canonical variates.

```
# Define the correlation matrices
Rxx <- matrix(data = c(1.000, 0.6328, 0.6328, 1.000), nrow = 2, ncol = 2)
Rxy <- matrix(data = c(0.2412, -0.0553,0.0586, 0.0655), nrow = 2, ncol = 2)
Ryy <- matrix(data = c(1.0, 0.4248, 0.4248,1.0), nrow = 2, ncol = 2)
Ryx <- t(Rxy)

# Calculate the eigenvalues and eigenvectors of each correlation matrix
exx <- eigen(Rxx)
exy <- eigen(Rxy)
eyy <- eigen(Ryy)
eyx <- eigen(Ryx)

# Calculate the square root of the inverse of each eigenvalue matrix
sqrEXX <- (exx$vectors) %*% diag(exx$values^(-0.5)) %*% solve(exx$vectors)
sqrEXY <- (exy$vectors) %*% diag(exy$values^(-0.5)) %*% solve(exy$vectors)
sqrEYY <- (eyy$vectors) %*% diag(eyy$values^(-0.5)) %*% solve(eyy$vectors)
sqrEYX <- (eyx$vectors) %*% diag(eyx$values^(-0.5)) %*% solve(eyx$vectors)
# Calculate the canonical correlation matrices
m1 <- sqrEXX %*% Rxy %*% solve(Ryy) %*% t(Rxy) %*% sqrEXX
m2 <- sqrEYY %*% t(Rxy) %*% solve(Rxx) %*% Rxy %*% sqrEYY

# Calculate the eigenvectors of the canonical correlation matrices
#canonical coefficients:
#p1= 0.39451 p2=0.06885
m1e <- eigen(m1)
sqrt(m1e$values)
m2e <- eigen(m2)
# Calculate the canonical variates
U <- sqrEXX %*% m1e$vectors
V <- sqrEYY %*% m2e$vectors

#the Canonical Variates:
#U1= 1.26reading_speed - 1.03reading_power
#U2 = 0.29reading_speed +0.79reading_power
#V1= 1.105arithmetic_speed - 0.45arithmetic_power
#V2= -0.2arithmetic_speed +1.0028arithmetic_power
```