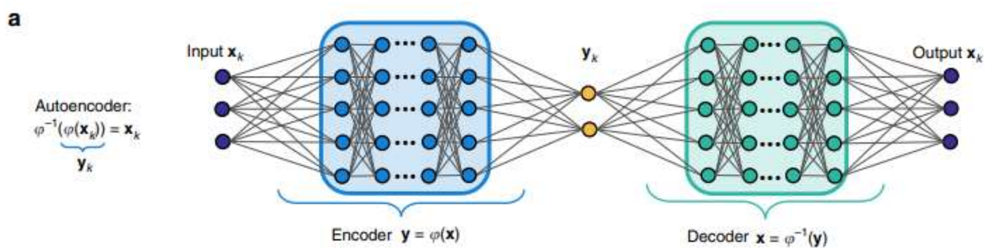
In [1]:
```python
# Importing all the necessary modules
import numpy as np
from keras.models import Model
from keras.layers import Input, Dense
from keras.optimizers import Adam
from sklearn.model_selection import train_test_split
```

I have taken the non linear function f(x) = x + sin(x) to find koopman operator with neural network. I have used an autoencoder architechture for estimating the linear weights and find out the koopman operator. The proposed model closely follows the following model picture where x(k) and x(k+1) are present state and the next state of the nonlinear function, given as input and output of the model respectively.



source : NATURE ARTICLE "Deep learning for universal linear embeddings of nonlinear dynamics" - Bethany Lusch 1,2, J. Nathan Kutz1 & Steven L. Brunton1,2

In [3]:
```python
# Define the nonlinear function
def nonlinear_function(x):
    return x+np.sin(x)

num_arrays = 5000 # Number of arrays
array_size = 100 # Size of each array

# Initialize the array
data_array = np.zeros((num_arrays, array_size))
# Generate the first array x1
x1 = np.linspace(-100, 100, num_arrays)

data_array[:,0] = x1

# Generate the remaining column arrays forrepresenting the next states
for i in range(1, array_size):
    data_array[:,i] = nonlinear_function(data_array[:,i-1])
print(data_array[0:5, 0:5])
```

```
[[-100.         -99.49363436  -98.63258453  -97.68576208  -97.39369273]
 [ -99.959992   -99.41954111  -98.52321022  -97.61716719  -97.39133722]
 [ -99.919984   -99.3463128   -98.4199447   -97.56235115  -97.39023361]
 [ -99.879976   -99.27400262  -98.32284565  -97.51915404  -97.38973628]
 [ -99.83996799  -99.20266226  -98.23192005  -97.4855788   -97.3895206 ]]
```

In [4]:
```python
# Initialize the output array where each column is next state of the corresponding column of the input array
data_array_op = np.zeros((num_arrays, array_size))

data_array_op[:,0:(array_size-1)] = data_array[:,1:array_size]
data_array_op[:,array_size-1] = nonlinear_function(data_array_op[:,array_size-2])
```

In [5]:
```python
print(data_array_op[0:5, 0:5])
```

```
[[-99.49363436 -98.63258453 -97.68576208 -97.39369273 -97.38937227]
 [-99.41954111 -98.52321022 -97.61716719 -97.39133722 -97.38937226]
 [-99.3463128  -98.4199447  -97.56235115 -97.39023361 -97.38937226]
 [-99.27400262 -98.32284565 -97.51915404 -97.38973628 -97.38937226]
 [-99.20266226 -98.23192005 -97.4855788  -97.3895206  -97.38937226]]
```

In [6]:
```
# Split the input data into training and testing sets
# Specify the split point
split_point = int(0.8 * num_arrays)

# Split the data into training and testing sets
train_data = data_array[:split_point, :]
test_data = data_array[split_point:, :]

print(train_data[0:5, 0:5])
print(test_data.shape)
```

```
[[-100.         -99.49363436  -98.63258453  -97.68576208  -97.39369273]
 [ -99.959992   -99.41954111  -98.52321022  -97.61716719  -97.39133722]
 [ -99.919984   -99.3463128   -98.4199447   -97.56235115  -97.39023361]
 [ -99.879976   -99.27400262  -98.32284565  -97.51915404  -97.38973628]
 [ -99.83996799  -99.20266226  -98.23192005  -97.4855788   -97.3895206 ]]
(1000, 100)
```

In [7]:
```
# Split the output data into training and testing sets
train_data_op = data_array_op[:split_point, :]
test_data_op = data_array_op[split_point:, :]

print(train_data_op[0:5, 0:5])
print(test_data_op.shape)
```

```
[[-99.49363436 -98.63258453 -97.68576208 -97.39369273 -97.38937227]
 [-99.41954111 -98.52321022 -97.61716719 -97.39133722 -97.38937226]
 [-99.3463128  -98.4199447  -97.56235115 -97.39023361 -97.38937226]
 [-99.27400262 -98.32284565 -97.51915404 -97.38973628 -97.38937226]
 [-99.20266226 -98.23192005 -97.4855788  -97.3895206  -97.38937226]]
(1000, 100)
```

In [8]:
```
print(train_data[0:5, 0:5])
print(train_data_op[0:5, 0:5])
```

```
[[-100.         -99.49363436  -98.63258453  -97.68576208  -97.39369273]
 [ -99.959992   -99.41954111  -98.52321022  -97.61716719  -97.39133722]
 [ -99.919984   -99.3463128   -98.4199447   -97.56235115  -97.39023361]
 [ -99.879976   -99.27400262  -98.32284565  -97.51915404  -97.38973628]
 [ -99.83996799  -99.20266226  -98.23192005  -97.4855788   -97.3895206 ]]
[[-99.49363436 -98.63258453 -97.68576208 -97.39369273 -97.38937227]
 [-99.41954111 -98.52321022 -97.61716719 -97.39133722 -97.38937226]
 [-99.3463128  -98.4199447  -97.56235115 -97.39023361 -97.38937226]
 [-99.27400262 -98.32284565 -97.51915404 -97.38973628 -97.38937226]
 [-99.20266226 -98.23192005 -97.4855788  -97.3895206  -97.38937226]]
```

In [9]:
```
shape=(array_size,)
shape
```

Out[9]: (100,)

Model Architechture

Input -> 3 dense layers -> 2 linear layers -> 3 dense layers -> output

In [10]:
```
## Making an auto encoder architechture for finding the koopman operator
# Define the encoder
input_layer = Input(shape=(array_size,))
encoded = Dense(256, activation='relu')(input_layer)
encoded = Dense(128, activation='relu')(encoded)
encoded = Dense(64, activation='relu')(encoded)
encoded_output = Dense(array_size, activation='relu')(encoded)

# Define linear layers in between
linear_layer = Dense(array_size, activation='linear')(encoded_output)
linear_layer = Dense(units=array_size, activation='linear')(linear_layer)

# Define the decoder with output matching the input shape
decoded = Dense(64, activation='relu')(linear_layer)
decoded = Dense(128, activation='relu')(decoded)
decoded = Dense(256, activation='relu')(decoded)
decoded_output = Dense(array_size, activation='tanh')(decoded)

# Combine the encoder and decoder into an autoencoder model
autoencoder = Model(inputs=input_layer, outputs=decoded_output)

# Extract the encoder model
encoder_model = Model(inputs=input_layer, outputs=encoded_output)
```

In [11]:
```
autoencoder.summary()
```

```
Model: "model"
_________________________________________________________________
 Layer (type)                Output Shape              Param #
=================================================================
 input_1 (InputLayer)        [(None, 100)]             0

 dense (Dense)               (None, 256)               25856

 dense_1 (Dense)             (None, 128)               32896

 dense_2 (Dense)             (None, 64)                8256

 dense_3 (Dense)             (None, 100)               6500

 dense_4 (Dense)             (None, 100)               10100

 dense_5 (Dense)             (None, 100)               10100

 dense_6 (Dense)             (None, 64)                6464

 dense_7 (Dense)             (None, 128)               8320

 dense_8 (Dense)             (None, 256)               33024

 dense_9 (Dense)             (None, 100)               25700

=================================================================
Total params: 167216 (653.19 KB)
Trainable params: 167216 (653.19 KB)
Non-trainable params: 0 (0.00 Byte)
_________________________________________________________________
```

In [12]:
```
encoder_model.summary()
```

```
Model: "model_1"
_________________________________________________________________
 Layer (type)                Output Shape              Param #
=================================================================
 input_1 (InputLayer)        [(None, 100)]             0

 dense (Dense)               (None, 256)               25856

 dense_1 (Dense)             (None, 128)               32896

 dense_2 (Dense)             (None, 64)                8256

 dense_3 (Dense)             (None, 100)               6500

=================================================================
Total params: 73508 (287.14 KB)
Trainable params: 73508 (287.14 KB)
Non-trainable params: 0 (0.00 Byte)
_________________________________________________________________
```

In [13]:
```
# Compile the autoencoder model
optimizer = 'Adam'
autoencoder.compile(optimizer=optimizer, loss='mean_squared_error') # using mse loss function
```

In [14]:
```
# Train the autoencoder
num_epochs = 25
batch_size = 32

autoencoder.fit(train_data, train_data_op, epochs=num_epochs, batch_size=batch_size, validation_data=(test_data, te
```

```
Epoch 1/25
125/125 [==============================] - 2s 4ms/step - loss: 2466.8433 - val_loss: 6368.3003
Epoch 2/25
125/125 [==============================] - 0s 3ms/step - loss: 2456.7915 - val_loss: 6368.3003
Epoch 3/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2844 - val_loss: 6368.3003
Epoch 4/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 5/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 6/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 7/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 8/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2114 - val_loss: 6368.3003
Epoch 9/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 10/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2114 - val_loss: 6368.3003
Epoch 11/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2112 - val_loss: 6368.3003
Epoch 12/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2112 - val_loss: 6368.3003
Epoch 13/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2114 - val_loss: 6368.3003
Epoch 14/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2112 - val_loss: 6368.3003
Epoch 15/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 16/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2119 - val_loss: 6368.3003
Epoch 17/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 18/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2112 - val_loss: 6368.3003
Epoch 19/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2112 - val_loss: 6368.3003
Epoch 20/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2109 - val_loss: 6368.3003
Epoch 21/25
125/125 [==============================] - 0s 3ms/step - loss: 2454.2117 - val_loss: 6368.3003
Epoch 22/25
125/125 [==============================] - 1s 4ms/step - loss: 2454.2117 - val_loss: 6368.3003
Epoch 23/25
125/125 [==============================] - 1s 4ms/step - loss: 2454.2114 - val_loss: 6368.3003
Epoch 24/25
125/125 [==============================] - 0s 4ms/step - loss: 2454.2107 - val_loss: 6368.3003
Epoch 25/25
125/125 [==============================] - 1s 4ms/step - loss: 2454.2117 - val_loss: 6368.3003
```

Out[14]:
```
<keras.src.callbacks.History at 0x7da6a8129270>
```

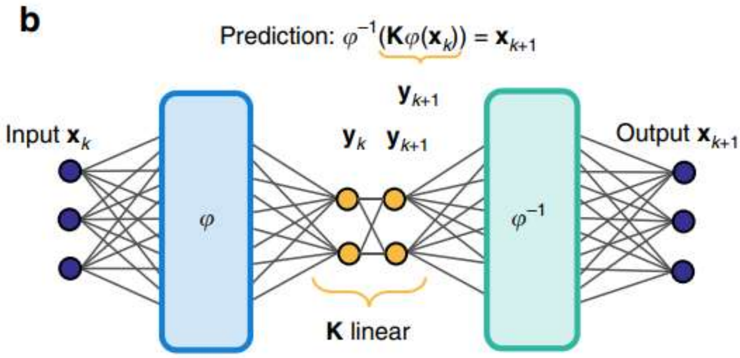
In [15]:
```
# Encode the data into lower dimension
encoded_data = encoder_model.predict(data_array)

# Print the shape of the encoded data as the input of the linear layer
print("Shape of encoded data:", encoded_data.shape)
```

```
157/157 [==============================] - 0s 1ms/step
Shape of encoded data: (5000, 100)
```

In [16]:
```
encoded_data
```

Out[16]:
```
array([[  0.       ,   0.       , 128.71175, ...,   0.       ,   0.       ,
          0.       ],
       [  0.       ,   0.       , 128.71695, ...,   0.       ,   0.       ,
          0.       ],
       [  0.       ,   0.       , 128.72198, ...,   0.       ,   0.       ,
          0.       ],
       ...,
       [  0.       , 298.57355,   0.       , ...,   0.       ,   0.       ,
          0.       ],
       [  0.       , 298.58386,   0.       , ...,   0.       ,   0.       ,
          0.       ],
       [  0.       , 298.5949 ,   0.       , ...,   0.       ,   0.       ,
          0.       ]], dtype=float32)
```

Extracting the linear model



extracting from input layer of y(k) to output layer of y(k+1)

In [17]:
```
# Extract the linear model from the model autoencoder to find out the weights
linear_model = Model(inputs=encoded_output, outputs=linear_layer)
linear_model.summary()
```

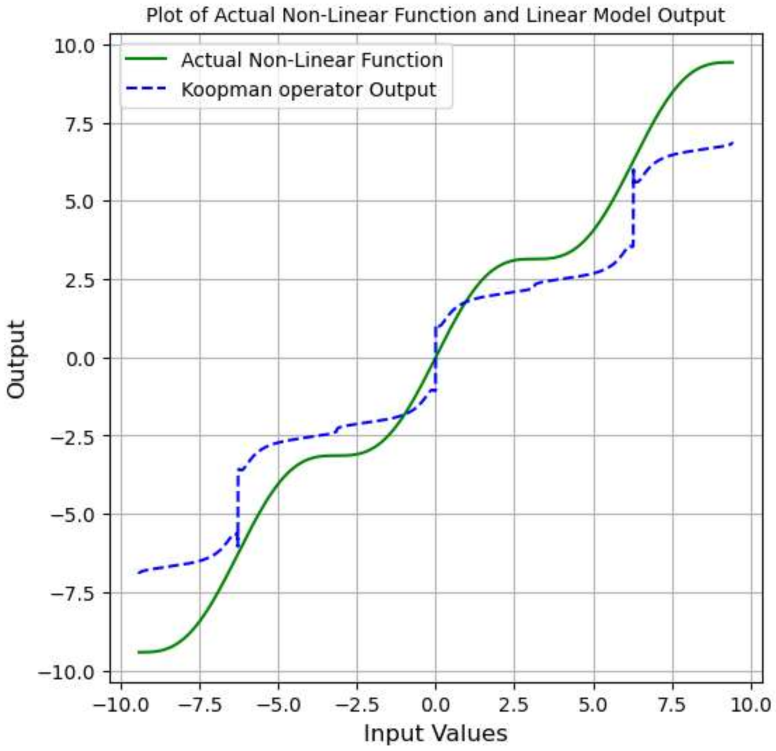
```
Model: "model_2"
_________________________________________________________________
 Layer (type)                Output Shape              Param #   
=================================================================
 input_2 (InputLayer)        [(None, 100)]             0         
                                                                 
 dense_4 (Dense)             (None, 100)               10100     
                                                                 
 dense_5 (Dense)             (None, 100)               10100     
                                                                 
=================================================================
Total params: 20200 (78.91 KB)
Trainable params: 20200 (78.91 KB)
Non-trainable params: 0 (0.00 Byte)
_________________________________________________________________
```

In [18]:
```
# Size of each array
array_size = array_size
# Initialize the array
lin_array = np.zeros((num_arrays, array_size))
# Generate the first array x1
x1 = np.linspace(-3*np.pi, 3*np.pi, num_arrays)

lin_array[:,0] = x1

# Generate the remaining arrays
for i in range(1, array_size):
    lin_array[:,i] = nonlinear_function(lin_array[:,i-1])
print(lin_array.shape)

# Initialize the array
lin_array_op = np.zeros((num_arrays, array_size))
lin_array_op[:,0:array_size-1] = lin_array[:,1:array_size]
lin_array_op[:,array_size-1] = nonlinear_function(lin_array_op[:,array_size-2])

# Encode the data into lower dimension
linear_data = linear_model.predict(lin_array)

# Print the shape of the encoded data
print("Shape of linear data:", linear_data.shape)
```

```
(5000, 100)
157/157 [==============================] - 0s 1ms/step
Shape of linear data: (5000, 100)
```

In [18]:

```
# Plot the actual non-linear function and the linear model's output
import matplotlib.pyplot as plt
import numpy as np

i=1
plt.figure(figsize=(6, 6))

plt.plot(lin_array[:, i], lin_array_op[:, i], label='Actual Non-Linear Function', color='green')
plt.plot(lin_array[:, i], linear_data[:, i], label='Koopman operator Output', linestyle='dashed', color='blue')

plt.title('Plot of Actual Non-Linear Function and Linear Model Output', fontsize=10)
plt.xlabel('Input Values', fontsize=12)
plt.ylabel('Output', fontsize=12)

plt.legend(loc='upper left')  # Adjust the legend location

plt.grid(True)  # Add grid lines

plt.show()
```



In [20]:

```
koopman_operator = linear_model.get_weights()
print(koopman_operator)
```

```
[array([[ 0.1311002 , -0.00418589,  0.1266438 , ...,  0.10871463,
          0.15435791,  0.06399512],
        [ 0.09753665,  0.10359903, -0.0288646 , ...,  0.11181825,
         -0.13761887,  0.00366623],
        [-0.07435913, -0.14408474,  0.07335582, ..., -0.08643966,
         -0.1617921 , -0.02294818],
        ...,
        [-0.03347534, -0.13689399, -0.06740212, ..., -0.12306757,
          0.11322278,  0.10246354],
        [ 0.11695228, -0.11887962, -0.09233312, ..., -0.07228058,
          0.12395415,  0.02010115],
        [ 0.03474497,  0.07707423,  0.0795785 , ..., -0.01999635,
         -0.09471869,  0.00284968]], dtype=float32), array([-1.1897674e-02, -9.6262246e-03, -2.8973811e-03,  9.6110
050e-03,
        -6.9848895e-03,  1.8641919e-02, -5.0925271e-05,  6.6038561e-03,
         5.2868817e-03, -4.0419488e-03,  5.3595980e-03, -1.9577409e-04,
         8.7194936e-04,  9.8337710e-04,  5.7031317e-03,  4.6505523e-03,
         3.1182368e-03, -7.1016937e-03, -8.1989467e-03,  2.3145517e-03,
         3.7311369e-03, -7.5154505e-03, -2.0236829e-03, -5.1815263e-03,
        -2.3176176e-03,  4.8162467e-03,  3.9004849e-03,  3.5193635e-03,
         9.8389611e-03, -9.9833412e-03, -9.9549117e-03,  6.6429428e-03,
         1.1133393e-02,  1.7104648e-03, -2.7114497e-03,  4.1131480e-03,
        -4.3522913e-04, -1.1425083e-02,  2.4416884e-03,  5.4279817e-03,
        -9.2966305e-03, -7.9120591e-04,  7.7449987e-03, -9.9323979e-03,
        -3.7408799e-03, -2.1021275e-03,  3.0477047e-03, -6.9491807e-03,
        -3.5881978e-03, -1.2408126e-02, -1.1144265e-03, -3.8660290e-03,
        -2.1488722e-03,  4.9907602e-03,  1.7479608e-03,  7.0261369e-03,
         5.3312338e-04, -7.5238920e-03,  6.3375714e-03, -2.9903417e-03,
         1.7374794e-03, -3.4676841e-03, -8.6804815e-03,  3.8424938e-03,
         2.2843468e-03,  1.1322858e-02, -1.1693384e-02,  3.4238941e-03,
         7.3275333e-03,  3.8637426e-03, -7.2382516e-03,  1.5390255e-03,
         4.9504014e-03, -8.0394773e-03,  4.3071462e-03, -4.2326627e-03,
         1.4398517e-03,  3.0769841e-04, -2.5659166e-03, -2.0755227e-03,
         3.2956349e-03, -8.8941125e-04, -1.3720836e-02, -9.1545321e-03,
        -2.1895582e-03, -3.5765332e-03, -3.6098796e-03, -5.2383221e-03,
         1.9221408e-03, -5.3502002e-04, -3.2410047e-03,  1.7319819e-03,
        -3.0768490e-03,  9.9812336e-03,  9.2114452e-03, -5.2624061e-03,
         6.7655351e-03, -6.1949380e-03, -1.4903434e-03, -6.0486556e-03],
      dtype=float32), array([[ 0.01224851,  0.01090407, -0.1515201 , ...,  0.05934076,
          0.1188776 ,  0.04618925],
        [-0.0608209 , -0.10668319, -0.00620097, ...,  0.08187099,
          0.03908557, -0.0850466 ],
        [ 0.16199456, -0.03823237, -0.08265487, ...,  0.16678156,
         -0.02752281, -0.05237859],
        ...,
        [-0.14767809,  0.01317098,  0.13925214, ...,  0.04138749,
          0.11358038,  0.0004531 ],
        [-0.06657568, -0.03318812,  0.06063652, ..., -0.02551206,
          0.15164824,  0.02924153],
        [ 0.1262106 ,  0.1536816 , -0.16327788, ...,  0.15373878,
         -0.0017263 , -0.14687234]], dtype=float32), array([-2.4877172e-03,  2.9793647e-03,  1.1472573e-02,  1.0561
763e-02,
        -6.6483091e-03, -1.6056994e-02,  5.4073399e-03,  4.3253419e-03,
         5.7702819e-03,  1.9109843e-03, -1.7034332e-03, -9.4154431e-03,
         1.4895925e-02,  6.4147701e-03, -3.4817019e-03, -8.4992062e-04,
        -6.0239243e-03,  6.2268311e-03, -2.3002899e-03,  8.8528721e-03,
         6.4099283e-04,  2.3840135e-03,  4.7360025e-03, -3.6791766e-03,
        -3.7170909e-03, -2.0027307e-03,  1.0600112e-02,  4.5840200e-03,
         1.7241030e-03,  4.3169921e-03, -7.9934476e-03,  4.4493177e-03,
        -4.5495033e-03, -3.0424071e-03, -1.5864109e-03,  3.4051680e-04,
         2.8804967e-03, -1.4627679e-03, -4.2894497e-03, -8.4803104e-03,
         3.6642100e-03, -3.3837291e-03, -4.0926267e-03,  5.1125777e-03,
        -6.0934536e-03,  5.4474534e-03,  8.9747855e-04, -1.1737868e-03,
         9.9079055e-04,  1.8266913e-03,  1.1415412e-02,  2.9731831e-03,
         4.0995385e-03, -8.5236002e-03,  7.4574468e-03,  2.6724394e-03,
         6.3035935e-03,  6.1217425e-03, -3.4509525e-03, -6.2926593e-03,
        -6.3697398e-03, -8.0583403e-03, -5.0759185e-03,  5.6246282e-03,
        -3.7832730e-03, -3.7922211e-05, -7.8141205e-03, -8.4226439e-03,
        -1.3091138e-02,  2.6703693e-03,  5.8164257e-03, -4.2863304e-04,
        -7.6781241e-03,  4.9776332e-03,  3.3931489e-04, -5.7363468e-03,
         9.4730724e-03, -4.1984636e-03, -3.6773563e-03,  9.6385954e-03,
         8.5481368e-03,  3.4619120e-03, -7.7489535e-03,  2.5052400e-03,
         4.6651918e-04, -4.8571457e-03, -4.0786401e-03, -1.4445902e-02,
        -2.3637556e-03, -6.5934109e-03, -1.8555560e-03,  4.7327532e-03,
         1.2245685e-02,  5.5526006e-03, -2.0513356e-04, -4.7205104e-03,
        -2.2517850e-03,  2.1931164e-03,  2.2269944e-03,  8.7238625e-03],
      dtype=float32)]
```

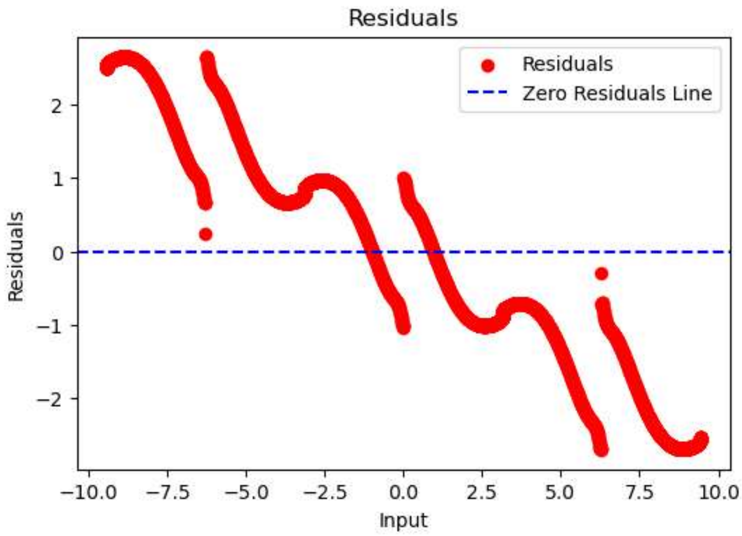
In [15]:
```
i=1
xt = lin_array[:,i]
yt = lin_array_op[:,i]
ypred = linear_data[:,i]
```

In [16]:
```
# Calculate Mean Squared Error (MSE)
from sklearn.metrics import mean_squared_error,r2_score
mse = mean_squared_error(yt,ypred)
print(f'Mean Squared Error: {mse}')

# Calculate R-squared
r2 = r2_score(yt,ypred)
print(f'R-squared: {r2}')
```

```
Mean Squared Error: 2.558672877611617
R-squared: 0.9247462013619882
```

In [17]:
```
# Plot Residuals
residuals = ypred - yt
plt.figure(figsize=(6, 4))
plt.scatter(xt, residuals, label='Residuals', color='red')
plt.axhline(y=0, color='blue', linestyle='--', label='Zero Residuals Line')
plt.title('Residuals')
plt.xlabel('Input')
plt.ylabel('Residuals')
plt.legend()
```

Out[17]: `<matplotlib.legend.Legend at 0x18c61cd8c10>`



In [24]:
```
# Get and save all model weights
weights_path = 'autoencodermodel_weights.h5'
autoencoder.save_weights(weights_path)

weights_path = 'linearmodel_weights.h5'
linear_model.save_weights(weights_path)
```

In [25]:
```
# Specify the file path where you want to save the array
file_path = 'lin_array.npy'
np.save(file_path, lin_array)
file_path = 'lin_array_op.npy'
np.save(file_path, lin_array_op)
file_path = 'linear_data.npy'
np.save(file_path, linear_data)
```