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Assignment – 6

Parameter Estimation and Hypothesis Testing

Question 1: Let $(X_1, X_2, \ldots, X_n)$ be a random sample of size 'n' taken from a Normal population with parameter mean $= \theta_1$ and variance $= \theta_2$. Find the maximum likelihood estimates of these two parameters.

Answer: $\theta_1 = \mu$, $\theta_2 = \sigma^2$

$$PDF = f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Now, $$f(x, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\frac{(x-\theta_1)^2}{\theta_2}}$$

$\theta_1 \in (-\infty, \infty)$
$\theta_2 \in (0, \infty)$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\frac{(x_i-\theta_1)^2}{\theta_2}} = (\theta_2)^{-n/2} (2\pi)^{-n/2} \cdot e^{-\frac{1}{2\theta_2}\sum(x_i-\theta_1)^2}$$

taking natural log on both sides,

$$\ln L(\theta_1, \theta_2) = -\frac{n}{2} \ln(2\pi\theta_2) - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

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Differentiating w.r.t. θ_1

$$\frac{dl}{d\theta_1} = \frac{1}{\theta_1} \sum_{i=1}^{n} (x_i - \theta_1) = 0$$

$$\theta_1 = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{—} \quad (1)$$

Differentiating w.r.t. θ_2

$$\frac{dl}{d\theta_2} = -\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = 0$$

Solving for θ_2:

$$\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta_1)^2 \quad \text{—} \quad (2)$$

∴ Maximum likelihood estimators are:

$$\boxed{\theta_1 = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad \theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta_1)^2}$$

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Question 2: Let $X_1, X_2, \ldots, X_n$ be a random sample from $B(m, \theta)$ distribution, where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compute value of θ using the M.L.E.

$$B(m, \theta) = {}^mC_x \, \theta^x (1-\theta)^{m-x}$$

$$f(x_i) = {}^mC_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i}$$

$$L(\theta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} {}^mC_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i}$$

Taking log on both sides,

$$l(\theta \mid x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \left[\log({}^mC_{x_i}) + x_i \log(\theta) + (m-x_i)\log(1-\theta) \right]$$

derivative w.r.t. θ,

$$\frac{dl}{d\theta} = \sum_{i=1}^{n} \left[\frac{x_i}{\theta} - \left(\frac{m-x_i}{1-\theta} \right) \right] = 0$$

Solving for θ,

$$\sum_{i=1}^{n} \left[\frac{x_i}{\theta} - \frac{m}{1-\theta} \right] = 0$$

$$\left(\frac{1}{\theta}\right) \sum_{i=1}^{n} x_i - \left(\frac{nm}{1-\theta} \right) = 0$$

$$\Rightarrow \theta nm = (1-\theta) \sum_{i=1}^{n} x_i$$

$$\boxed{\therefore \theta = \frac{\sum_{i=1}^{n} x_i}{nm}} \qquad \boxed{\therefore MLE = \frac{\sum_{i=1}^{n} x_i}{nm}}$$

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