MODEL REDUCTION AND RECONSTRUCTION USING SYSTEM EQUIVALENT REDUCTION AND EXPANSION PROCESS

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Abstract

A large structural model has to be reduced to a manageable scale to facilitate computational effort and correlation studies. System Equivalent Reduction and Expansion Process (SEREP) is used to reduce a large linear system-equations of motions. Numerical example with finite element modelling of a plane frame structure shows the effectiveness of the process.

1. Introduction

One of the most important parameters in Finite Element Modelling are the degrees of freedom. The degrees of freedom signify the number of possible motions at a particular node. Higher the number of degrees of freedom, more accurate will be our calculations. However finite element models only give a theoretical result. They have to be verified with the physical model. In order to do so, sensors have to be installed at each degree of freedom to find out their displacements and correlate with the theoretical result. The number of degrees of freedom has to be high to guarantee accuracy in the theoretical result. However, motion parameters at each and every one these degrees of freedom is not possible to be measured because then the cost of the sensors will be very high. Also, in theoretical model rotational degrees of freedom can be found. However practically there is no method to measure the amount of rotation at a particular node. So, we are compelled to take measurements at only a limited number of degrees of freedom. Reduction of problem size or the number of co-ordinates in a system of equations is very important for reduction of computational effort and the ease or capability of measurement of co-ordinates for future processing. System Equivalent Reduction Expansion Process (SEREP) is one such technique among few others to reduce large analytical equations of motion preserving the information about selected important modes of a system. The limited number of degrees of freedom system will be termed as the reduced system henceforth and the system with all the degrees of freedom as the full system. This paper aims to find out how accurately the full system model can be predicted by analysing the reduced system.

2. Modal Analysis

Modal analysis is the study of the dynamic properties of structures under vibrational excitation.

Modal analysis, or the mode-superposition method, is a linear dynamic-response procedure which evaluates and superimposes free-vibration mode shapes to characterize displacement patterns. Mode shapes describe the configurations into which a structure will naturally displace. Typically, lateral displacement patterns are of primary concern. Mode shapes of low-order mathematical expression tend to provide the greatest contribution to structural response. As orders increase, mode shapes contribute less, and are predicted less reliably. It is reasonable to truncate analysis when the number of mode shapes is sufficient.

A structure with N degrees of freedom will have N corresponding mode shapes. Each mode shape is an independent and normalized displacement pattern which may be amplified and superimposed to create a resultant displacement pattern.

The equation of motion for any mechanical system can be written as

$$[M]_{n\times n}\{\ddot{X}\}_{n\times 1} + [C]_{n\times n}\{\dot{X}\}_{n\times 1} + [K]_{n\times n}\{X\}_{n\times 1} = \{f\}_{n\times 1}$$

In our case [C]=0. For modal analysis {f}=0

$$\Rightarrow [M]_{n\times n}\{\ddot{X}\}_{n\times 1} + [K]_{n\times n}\{X\}_{n\times 1} = \{0\}$$

Let,

$$\{X\} = X_0 e^{wt}$$

$$\Rightarrow \{\dot{X}\} = X_0 w e^{wt} = w\{X\}$$

$$\Rightarrow \{\ddot{X}\} = X_0 w^2 e^{wt} = w^2\{X\}$$

Substituting this in the equation of motion we get,

$$[M].w^2\{X\} + [K]\{X\} = \{0\}$$

$$\Rightarrow w^2\{X\} = -[M]^{-1}[K]\{X\}$$

$$\Rightarrow [A]\{X\} = \lambda\{X\}$$

This is a standard eigen value problem. Here λ is the eigen value and {X} is the corresponding eigen vector.

The eigen values found by solving this problem gives the natural frequencies of vibrations while the eigen vectors corresponding to these eigen values gives the corresponding mode shapes.

In all practical problems, high frequency mode shapes are not useful to us because such high frequency will seldom be achieved. So, our main area of interest is the low frequency domain, particularly the first 10 natural frequencies.

3. Model Reduction

Dynamic reduction means reducing a given finite element model to one with fewer degrees of freedom while preserving the dynamic characteristics of the system.

It becomes necessary to reduce a finite element model especially when correlation studies are done. There are several model reduction techniques. Some of them are

- System Equivalent Reduction and Expansion Process (SEREP)
- Modified System Equivalent Reduction and Expansion Process (Modified SEREP)
- Guyan Condensation
- Dynamic Condensation
- Improved Reduced System

For all model reduction/expansion techniques there are is a set of deleted DOFs (ddof) and a set of retained DOFs or master DOFs (adof) which can be written in general terms as

$$\{X_n\} = \begin{Bmatrix} X_a \\ X_d \end{Bmatrix} = [T]\{X_a\}$$

Here n denotes the total number of DOFs

a denotes the retained DOFs

d denotes the deleted DOFs

The transformation matrix [T] will take on various forms depending on the transformation technique used.

Guyan Condensation (Guyan,1965): It is one of the oldest and most popular methods of model reduction. It is also called static condensation. Here the inertia and damping terms associated with the discarded degrees of freedom are neglected.

Dynamic Condensation (Paz Mario,1984): It is an alternative of static reduction where the frequency at which the reduction is exact may be chosen arbitrarily.

Improved Reduction System Method (O'Callahan.1989): In this approach an extra term is added to the static reduction transformation to make allowance for inertia forces. The inertia terms allow the modal vectors of interest in the full model to approximated more accurately but relies on statically reduced models.

4. System Equivalent Reduction and Expansion Process (SEREP)

Most reduction techniques will affect the dynamic character contained in the original full analytical model. Generally, the estimated frequencies in the reduced model are higher than those of the original model. However, System Equivalent Reduction and Expansion Process (Kammer 1987,O'Callahan 1989) is such a technique in which reduced model containing arbitrarily selected DOF's preserves the dynamic characteristic of the full system model for the selected mode of interest.

SEREP was originally formulated as a global mapping technique that was used to develop rotational DOF for modal test data. SEREP is also used for checking correlations and orthogonality between analytical and experimental modal vectors, linear and nonlinear forced response studies and analytical model improvement.

SEREP provides features that other reduction techniques do not such as

- Modes can be selected arbitrarily
- Active DOFs can be selected arbitrarily
- The frequency and mode shapes of the reduced system are exactly equal to the frequencies and mode shapes (for the selected modes) of the full system model
- The reduction/expansion process is irreversible; expanding the reduced system's mode shapes back to the full system's space develops mode shapes that are exactly the same as the original mode shapes of the full system model.

General Applications for SEREP:

- Expansion of experimental modal data to include rotational DOF as well as unmeasured translator DOF information needed for system modelling and structural dynamic modification studies
- Correlation of analytical and experimental modal data and pseudo orthogonality checks can be performed at either the reduced state corresponding to the tested DOF or at the full state corresponding to the full system model.
- Reduction of large analytical models to a much smaller model used for the study of both linear and nonlinear response
- Improvement of analytical models using the measured modal data can be made at either the test set of DOF using the reduced mass and stiffness matrices or at the full set of DOF using the expanded modal vectors.

The equation of motion for an undamped system is

$$M_n\ddot{X}_n + K_nX_n = 0 \quad (1)$$

The eigen solution of the n dimensional system based on the selected m modal vectors is given by

$$X_n = U_nP \quad (2)$$

Where U_n is the $(n \times n)$ modal matrix whose columns are made up of the m nodal vectors. P is the $(m \times 1)$ displacement vector in the modal coordinate system. The columns of U_n are linearly independent and therefore the rank of U_n is m.

To formulate the reduced system, the full system model is partitioned so as to separate those DOFs in the full system which will be tracked in the reduced system from those DOFs which will not be tracked. Equation (2) is rewritten as

$$\{X_n\} = \begin{Bmatrix} X_a \\ X_d \end{Bmatrix} = \begin{bmatrix} U_a \\ U_d \end{bmatrix} P \quad (3)$$

Where the subscript a denotes the active or tracked DOF of the full system model and subscript denotes the deleted or untracked DOF. Considering only the active DOFs in equation (3) yields

$$X_a = U_aP \quad (4)$$

Which is a description of the system response at the a active DOFs in terms of the m modal variables. The U_a matrix of dimension $(a \times m)$ is generally not square. Solving for the modal vector P requires that a generalised inverse of the matrix U_a be formed.

Two cases arise. One where the number of active DOF is greater than or equal to the number of selected modes and another where the number of active DOF is less than the number of selected modes.

For $a \geq m$

$$U_a^g = (U_a^TU_a)^{-1}U_a^T \quad (5)$$

Where U_a^g is the generalised inverse of U_a. So the modal displacement vector can be solved as

$$P = U_a^gX_a \quad (6)$$

For $a < m$,

$$U_a^g = U_a^T(U_aU_a^T)^{-1} \quad (7)$$

Substituting in equation (2), gives an expression of full systems displacement vector in terms of the reduced systems displacement vector. Substituting equation (6) in equation (2),

$$X_n = U_nU_a^gX_a \quad (8)$$

The global mapping transformation matrix relating the reduced system to the full system is then defined as,

$$T_u = U_n U_a^g = \begin{bmatrix} U_a U_a^g \\ U_d U_a^g \end{bmatrix} \quad (9)$$

The subscript u denotes that this transformation matrix is based on the analytical modal vector set U_n. Substituting equation (9) in equation (8) yields,

$$X_n = T_u X_a \quad (10)$$

The T_u matrix is used to form the reduced mass and stiffness matrices as

$$M_a = T_u^T M_n T_u \quad (11)$$

And

$$K_a = T_u^T K_n T_u \quad (12)$$

Where M_a and K_a are the equivalently reduced mass and stiffness matrices respectively. The equations of motion for the a dimensional system can be written as,

$$M_a \ddot{X}_a + K_a X_a = 0 \quad (13)$$

The eigen values of this a DOF system are equal to the eigen values of the n DOF system which correspond to the modes employed in the formulation of T_u. Further,

$$X_a = U_a' P \quad (14)$$

While

$$U_a = U_a' \quad (15)$$

Where $U_a{}'$ is the $(a \times m)$ modal matrix formed from the eigen solution of a DOF system. Finally the modal vector matrix of the n DOF system can be obtained by

$$U_n = T_u U_a{}' \quad (16)$$

A simple planar frame model was used for the analysis. Material is steel with Youngs Modulus 200GPa, density 7700 kg/m^3. Cross section of the frame is rectangular of dimensions 25mmX25mm. The finite element model comprised of 14 nodes and 14 planar beam elements with 3 DOFs per node. Length of each element is 500mm. This 42 DOF model is considered as the reference to which all the reduced models are compared. Figure 1. shows the reference model

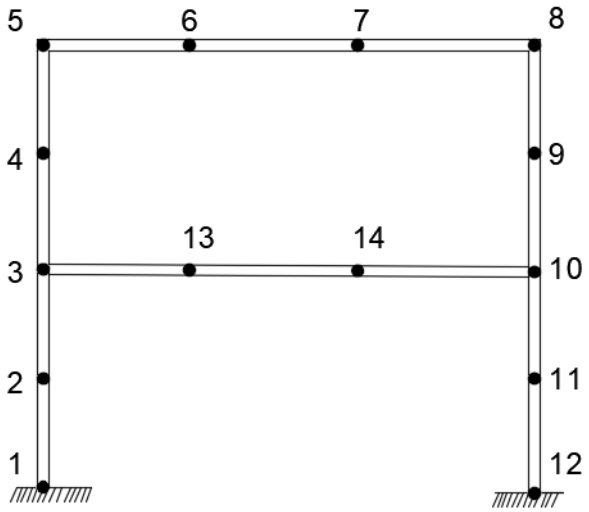
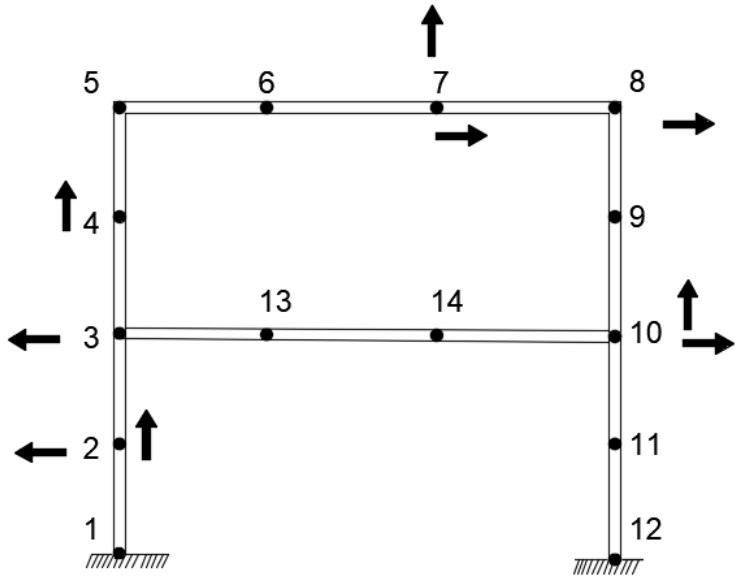


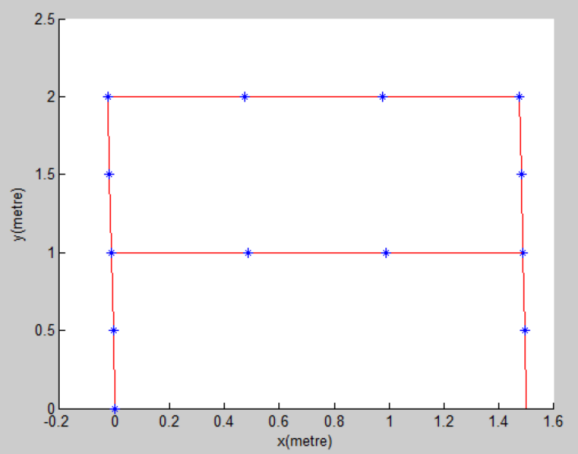
Figure 1: Reference Model

The reduced model shown in Figure 2. has the following active DOFs.

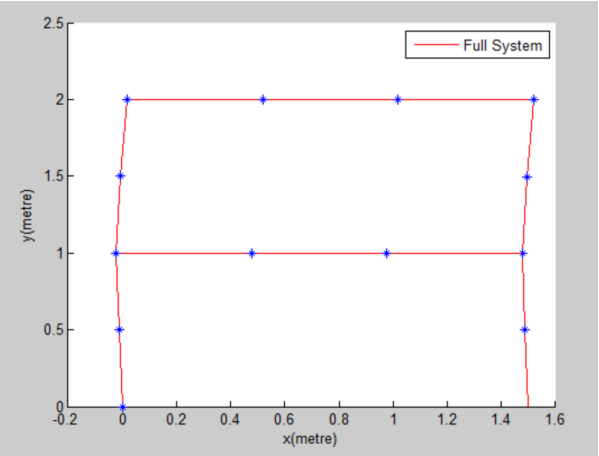


5. Result and Discussion

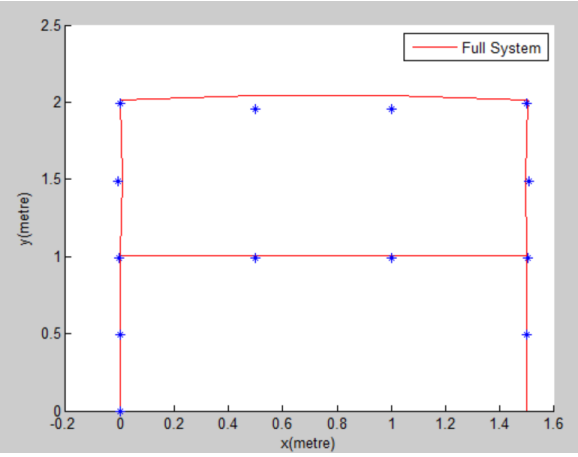
The mode shapes corresponding to the first 10 natural frequencies are shown below. The solid line represents the mode shape obtained from full system analysis. The starred points represent the mode shapes obtained from the SEREP reduced system.



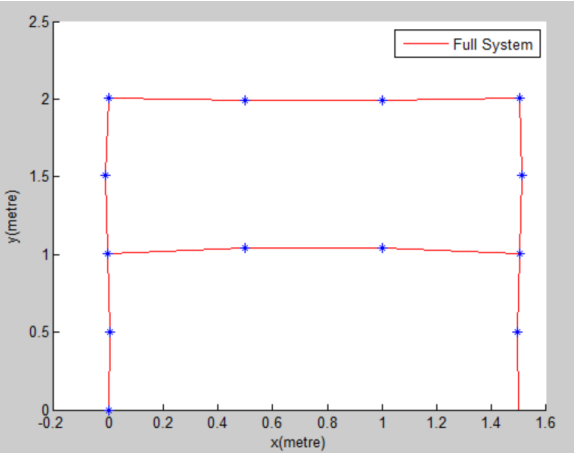
Mode 1



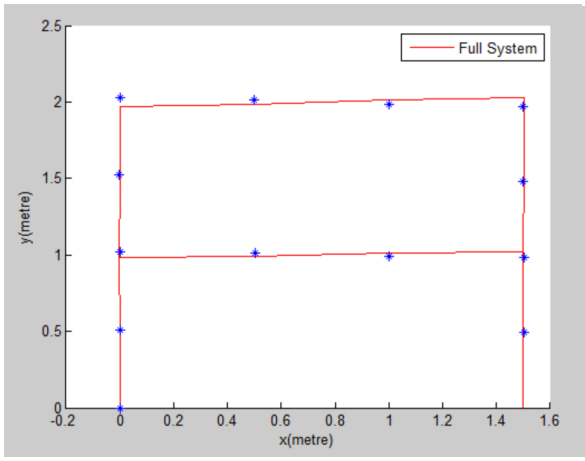
Mode 2



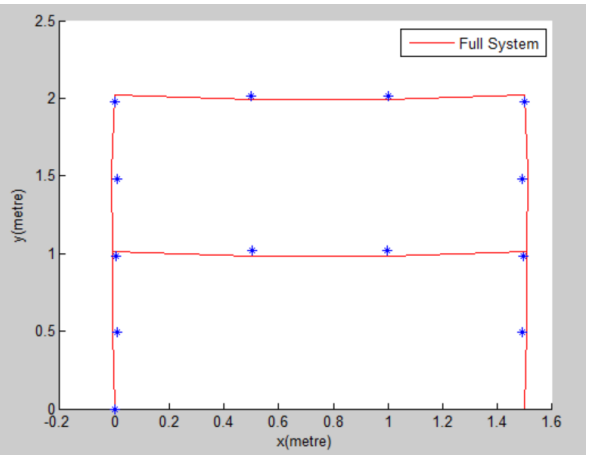
Mode 3



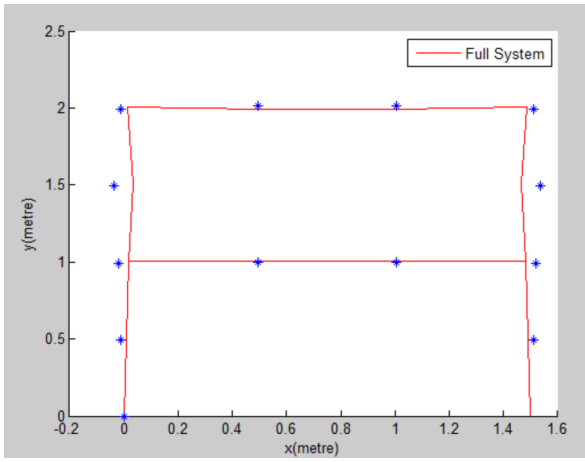
Mode 4



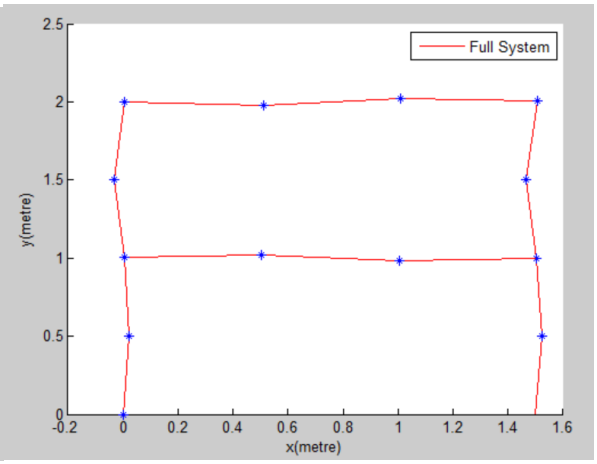
Mode 5



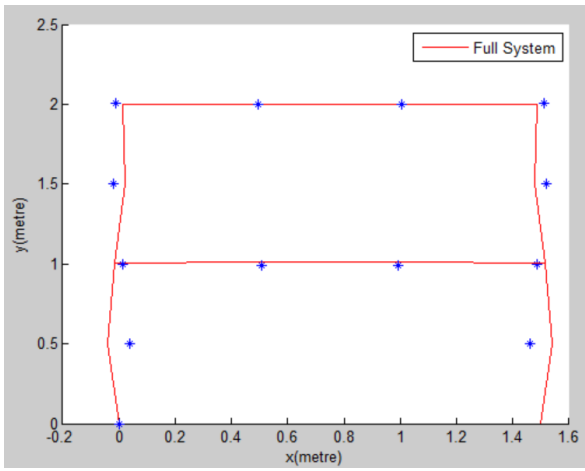
Mode 6



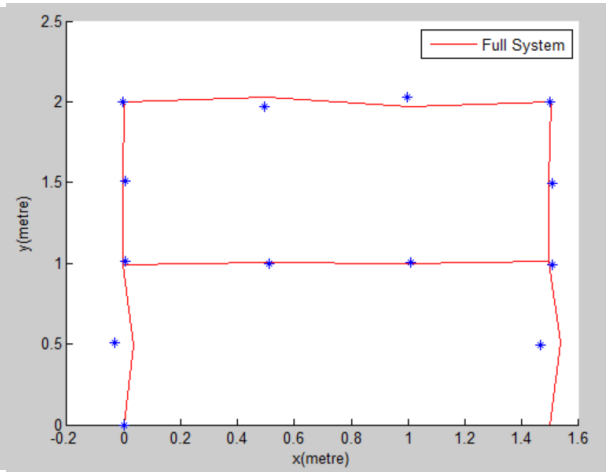
Mode 7



Mode 8



Mode 9



Mode 10

The comparison of full system natural frequency and the reduced system natural frequency for the 1st 10 modes and a given set of active DOFs is shown in Table 1.

The active DOFs in this reduced model are 4,5,7,11,17,19,20,22,28,29.

Table 1: Comparison of Natural Frequency of Full System and Reduced System

Mode	Full System Model	Reduced Model
1	448	448
2	1572	1572
3	2090	2090
4	2471	2471
5	3304	3304
6	4380	4380
7	5177	5177
8	5779	5779
9	6509	6509
10	8078	8078

Comparison of different models for the same active DOF but arbitrary modes is shown in table 2.

Model 1: Modes 1,2,3,4,5,6,7,8,9,10

Model 2: Modes 2,3,4,7,8,11,12,13,14,15

Model 3: Modes 3,4,5,7,8,9,13,14,17,18

Table 2: Comparison of different models for the same DOF but different modes

Mode	Reference Model	Model 1	Model 2	Model 3
1	448	448		
2	1572	1572	1572	
3	2090	2090	2090	2090
4	2471	2471	2471	2471

Mode	Reference Model	Model 1	Model 2	Model 3
5	3304	3304		3304
6	4380	4380		
7	5177	5177	5177	5177
8	5779	5779	5779	5779
9	6509	6509		6509
10	8078	8078		
11	8392		8392	
12	9708		9708	
13	9883		9883	9883
14	10341		10341	10341
15	12143		12143	
17	12690			12690
18	16417			16417

The active DOF in each of the above models are 4,5,7,11,17,19,20,22,28,29.

Accuracy of prediction as shown in Table 3. of full system from the reduced system is measured in terms of the angle and length parameters e_1 and e_2 respectively.

$$e_1 = \frac{\left(2 \times \left|\left|u_i\right|\right| \times \left|\left|u_i'\right|\right|\right)}{\left|\left|u_i\right|\right|^2 + \left|\left|u_i'\right|\right|^2}$$

$$e_2 = \frac{\left|\left|u_i\right|\right| - \left|\left|u_i'\right|\right|}{\left|\left|u_i\right|\right|}$$

Equality of 2 vectors is predicted when e_1 is close to unity and e_2 approaches 0.

Table 3: Accuracy of prediction of full system from reduced system

Models Being Compared	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9	Mode 10
Full System and SEREP reduced System	e_1									
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	e_2 $(\times 10^{-12})$									
	-0.1709	-0.0046	-0.1374	0.0090	-0.0045	-0.0252	-0.3966	0.0869	0.0184	-0.0150
Full System and SEREP reversed System	e_1									
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$e_2(\times 10^{-12})$									
	0.1475	0.0327	0.0591	0.0430	0.0118	-0.0172	0.0904	0.1880	0.0298	0.0071

From the comparison of the full system with the SEREP reduced system, it is evident that the reduced system correctly predicts both the natural frequencies and the mode shapes of the full system for the selected modes.

From the comparison of the full system and SEREP reversed system it is evident that the SEREP is a reversible process.

The System Equivalent Reduction and Expansion Process is an exact method for mapping large analytical models down to much smaller equivalent models. Unlike other reduction processes which develop reduced system whose eigen solutions approximate the eigen solutions of the full system, SEREP develops reduced models whose frequencies and mode shapes are exactly the same as the full system model for the selected modes of interest.

6. Conclusion

The following conclusions can be drawn from this study

- The natural frequency and mode shape obtained using SEREP matches very closely with that of the original system model which proves usefulness of the reduction process
- As predicted, results obtained from SEREP are independent of the choice of active degrees of freedom
- Both the reduced system and reconstructed system using SEREP can predict the original mode shape closely as indicated by the angle and length estimators e_1 and e_2

REFERENCES

- **Guyan J.** Reduction of stiffness and mass matrices. AIAA, Vol. 3, 1965
- **O'Callahan, J., Avitabile, P., Riemer**, R., 1989. System equivalent reduction expansion process (SEREP). In: Proceedings of the Seventh International Modal Analysis Conference, Las Vegas
- **Paz Mario** Dynamic condensation. AIAA, Vol. 22, Issue 5, 1984
- **Wang Fei, Luo Guihuo**, 2015.A comparison study on application of model reduction methods in rotor dynamics

APPENDIX

```
%%%%%%%%%%%%%%%%%%%%%Pre-processing%%%%%%%%%%%%%%%%%%%%%
clear all; clc;close all;
nelem=input('Enter the number of elements');
fileID=fopen('lnode.txt');
tline=fgets(fileID);
k=1;
element_nodes=zeros(k,3);
while ischar(tline)
   p=str2num(tline);
   element_nodes(k,1)=p(1);
   element_nodes(k,2)=p(2);
   element_nodes(k,3)=p(3);
   tline=fgets(fileID);
   k=k+1;
end
fclose(fileID);
fileID=fopen('nodex.txt');
tline=fgets(fileID);
k=1;
while ischar(tline)
   p=str2num(tline);
   coordinates(k,1)=p(1);
   coordinates(k,2)=p(2);
   coordinates(k,3)=p(3);
   tline=fgets(fileID);
   k=k+1;
end
fclose(fileID);
element_dofs=zeros(nelem,7);
l=size(element_dofs);
for i=1:l(1)
   element_dofs(i,1)=element_nodes(i,1);
   element_dofs(i,2)=3*(element_nodes(i,2)-1)+1;
   element_dofs(i,3)=3*(element_nodes(i,2)-1)+2;
   element_dofs(i,4)=3*(element_nodes(i,2)-1)+3;
```

```
    element_dofs(i,5)=3*(element_nodes(i,3)-1)+1;
    element_dofs(i,6)=3*(element_nodes(i,3)-1)+2;
    element_dofs(i,7)=3*(element_nodes(i,3)-1)+3;
end
p=1;
m=element_dofs(1,2);
for i=1:l(1)
    for j=2:l(2)
        if(element_dofs(i,j)>m)
            p=p+1;
            m=element_dofs(i,j);
        else
            continue;
        end
    end
end
nnode=max(max(element_nodes));
%%%%%%%%%%%%%%Stiffness and Mass Matrix Assembly%%%%%%%%%%%%%%%
xk=zeros(p,p);
mk=zeros(p,p);
for ielem=1:nelem
    n1=element_nodes(ielem,2);
    n2=element_nodes(ielem,3);
    x1=sqrt((coordinates(n1,2)-coordinates(n2,2))^2+(coordinates(n1,3)-coordinates(n2,3))^2);
    c=(coordinates(n2,2)-coordinates(n1,2))/x1;
    s=(coordinates(n2,3)-coordinates(n1,3))/x1;
    width=0.25;
    depth=0.25;
    A=width*depth;
    I=1/12*width*depth^3;
    E=200*10^9;
    density=7700;
    T1=[c,s,0;-s,c,0;0,0,1];
    z=zeros(3,3);
    T=[T1,z;z,T1];
    st=[A*E/x1,0,0,-A*E/x1,0,0;0,12*E*I/x1^3,6*E*I/x1^2,0,-
12*E*I/x1^3,6*E*I/x1^2;0,6*E*I/x1^2,4*E*I/x1,0,-6*E*I/x1^2,2*E*I/x1;-
```

```
A*E/x1,0,0,A*E/x1,0,0;0,-12*E*I/x1^3,-6*E*I/x1^2,0,12*E*I/x1^3,-
6*E*I/x1^2;0,6*E*I/x1^2,2*E*I/x1,0,-6*E*I/x1^2,4*E*I/x1];
   stiff=T'*st*T;
   mass=[140,0,0,70,0,0;0,156,22*x1,0,54,-13*x1;0,22*x1,4*x1^2,0,13*x1,-
3*x1^2;70,0,0,140,0,0;0,54,13*x1,0,156,-22*x1;0,-13*x1,-3*x1^2,0,-
22*x1,4*x1^2]*x1*density*A/420;
   for i=1:6
      for j=1:6
         idof=element_dofs(ielem,i+1);
         jdof=element_dofs(ielem,j+1);
         xk(idof,jdof)=xk(idof,jdof)+stiff(i,j);
         mk(idof,jdof)=mk(idof,jdof)+mass(i,j);
      end
   end
end
%%%%%%%%%%Finding eigen vector and eigen frequency of complete system%%%%%%%%
fn=[1,12];
fd=[1,2,3,34,35,36];
xk(:,fd)=[];
xk(fd,:)=[];
mk(:,fd)=[];
mk(fd,:)=[];
[V2,D2]=eig(-xk,mk);
[D2,I1]=sort(diag(D2));
V2=V2(:,I1);
D2=sqrt(D2);
zz=size(V2);
V2_size=zz(1);
%%%%%%%%%%%%%%%%%%%%%%SEREP for a=m%%%%%%%%%%%%%%%%%%%%%%
adof1=[4,5,7,11,17,19,20,22,28,29];
adof=adof1-3;
asize=size(adof);
modeset=[27:36];
m=size(modeset);
msize=m(2);
U_n=V2(:,(modeset));
U_ng=U_n;
```

```
aa=max(abs(U_ng));
U_a=U_n((adof),:);
for i=1:msize
   U_n(:,i)=U_n(:,i)/max(abs(U_n(:,i)));
end
U_ag=inv(U_a'*U_a)*U_a';
T_u=U_n*U_ag;
M_a=T_u'*mk*T_u;
K_a=T_u'*xk*T_u;
[V_r,D_r]=eig(-K_a,M_a);
U_exp=T_u*V_r;
D_r=diag(D_r);
[D_r,I]=sort(D_r);
V_r=V_r(:,I);
U_exp=U_exp(:,I);
for h=1:msize
    U_exp(:,h)=U_exp(:,h)/max(abs(U_exp(:,h)));
     V_r(:,h)=V_r(:,h)/max(abs(V_r(:,h)));
   U_a(:,h)=U_a(:,h)/max(abs(U_a(:,h)));
end
D_r=sqrt(D_r);
%%%%%%%%%%Comparison of original system with SEREP reduced system%%%%%%%%%
for i=1:msize
   e1_s(i)=2*norm(U_a(:,i))*norm(V_r(:,i))/(norm(V_r(:,i))^2+norm(V_r(:,i))^2)
e2_s(i)=(norm(U_a(:,i))-norm(V_r(:,i)))/norm(U_a(:,i))
end
%%%%%%%%%%Comparison of original system with SEREP reversed system%%%%%%%%%
for i=1:msize
   e1_sr(i)=2*norm(U_n(:,i))*norm(U_exp(:,i))/(norm(U_n(:,i))^2+norm(U_exp(:,i))^2)
e2_sr(i)=(norm(U_n(:,i))-norm(U_exp(:,i)))/norm(U_n(:,i))
end
%%%%%%%%%%Plotting Mode Shape for Full System and Reduced System%%%%%%%%%%
for i=1:10
   U_exp(:,i)=U_exp(:,i)*aa(i);
end
zz=size(fn);
fn_size=zz(2);
```

```
x_degree=[1:3:36];
y_degree=[2:3:36];
x_cord=U_ng(x_degree,:);
y_cord=U_ng(y_degree,:);
x_cord_exp=U_exp(x_degree,:);
y_cord_exp=U_exp(y_degree,:);
k=1;
m=0;
for q=1:10
   for w=1:nnode
      for e=1:fn_size
         if(w==fn(e))
            m=1;
         end
      end
      if(m==1)
            new_cord_x(w,q)=coordinates(w,2);
            new_cord_y(w,q)=coordinates(w,3);
             new_cord_x_exp(w,q)=coordinates(w,2);
            new_cord_y_exp(w,q)=coordinates(w,3);
         else
           new_cord_x(w,q)=coordinates(w,2)+x_cord(k,q);
           new_cord_y(w,q)=coordinates(w,3)+y_cord(k,q);
           new_cord_x_exp(w,q)=coordinates(w,2)+x_cord_exp(k,q);
           new_cord_y_exp(w,q)=coordinates(w,3)+y_cord_exp(k,q);
           k=k+1;
      end
         m=0;
   end
   k=1;
end
zz=size(element_nodes);
lnode_size=zz(1);
for q=1:asize(2)
   figure
 hold on;
 xlabel('x(metre)');
```

```
 ylabel('y(metre)');
    for w=1:lnode_size
        e=element_nodes(w,2);
        f=element_nodes(w,3);
        plot([new_cord_x(e,q) new_cord_x(f,q)],[new_cord_y(e,q) new_cord_y(f,q)],'r');
        plot(new_cord_x_exp(e,q),new_cord_y_exp(e,q),'*');
    end
end
hold off
%%%%%%%%%%%%%%%%%% %%END%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Files lnode and nodex are given below:

lnode (element no, node 1, node 2)	nodex (node no, x coordinate, y coordinate)
1 1 2	1 0 0
2 2 3	2 0 0.5
3 3 4	3 0 1
4 4 5	4 0 1.5
5 5 6	5 0 2
6 6 7	6 0.5 2
7 7 8	7 1 2
8 8 9	8 1.5 2
9 9 10	9 1.5 1.5
10 10 11	10 1.5 1
11 11 12	11 1.5 0.5
12 3 13	12 1.5 0
13 13 14	13 0.5 1
14 14 10	14 1 1