we can define a *local* wave zone, as the region where the distance to the source is sufficiently large so that the gravitational field already has the $1/r$ behavior characteristic of waves, but still sufficiently small, so that the expansion of the Universe is negligible. During the propagation of the GW in the local wave zone, the scale factor $a(t)$ does not change appreciably, so in the local wave zone physical distances can be written as $r_{\rm phys} = a(t_{\rm emis})r$, where r is the comoving distance and $t_{\rm emis}$ is (any) time of emission,[17] so $r_{\rm phys}$ differs from r by just a constant normalization factor. Using eq. (4.29) we see that the GW produced by a binary inspiral, at a distance $r_{\rm phys} = a(t_{\rm emis})r$ in the local wave zone, can be written as

[17] We do not need to track the change of the scale factor during the observed part of the emission process, so we do not need to be more precise about $t_{\rm emis}$.

$$h_+(t_s) = h_c(t_s^{\rm ret})\,\frac{1+\cos^2\iota}{2}\,\cos\left[2\pi\int^{t_s^{\rm ret}} dt'_s f_{\rm gw}^{(s)}(t'_s)\right], \quad (4.170)$$

$$h_\times(t_s) = h_c(t_s^{\rm ret})\,\cos\iota\,\sin\left[2\pi\int^{t_s^{\rm ret}} dt'_s f_{\rm gw}^{(s)}(t'_s)\right], \quad (4.171)$$

where

$$h_c(t_s^{\rm ret}) = \frac{4}{a(t_{\rm emis})r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}^{(s)}(t_s^{\rm ret})}{c}\right)^{2/3}. \quad (4.172)$$

Here time is the time t_s measured by the clock of the source (and $t_s^{\rm ret}$ is the corresponding value of retarded time) and the GW frequency $f_{\rm gw}$ is the one associated to this definition of time, that we denote by $f_{\rm gw}^{(s)}$. They are related to the quantities measured by the observer which is at a cosmological distance, very far from the local wave zone, by eqs. (4.150) and (4.151). The dependence of $f_{\rm gw}^{(s)}$ on t_s is given by eq. (4.19), that we rewrite as

$$f_{\rm gw}^{(s)}(\tau_s) = \frac{1}{\pi}\left(\frac{5}{256}\,\frac{1}{\tau_s}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}, \quad (4.173)$$

where τ_s is the time to coalescence measured by the source's clock.

To compute how this waveform propagates across cosmological distances to reach the observer, we should use eq. (1.179), that describes the propagation of GWs in a curved space-time, specializing it to the FRW metric. Actually, it is instructive to start from a simpler problem, namely the propagation of a *scalar* perturbation ϕ in the FRW metric. In this case, the propagation equation is simply $\Box\phi = 0$ where, on scalar functions, the curved-space d'Alembertian $\Box = D_\mu D^\mu$ can be written as

$$\Box = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu)\,. \quad (4.174)$$

To solve this wave equation in the FRW metric, it is convenient to introduce the conformal time η , from

$$d\eta = \frac{dt}{a(t)}, \quad (4.175)$$

i.e.

$$\eta = \int^t \frac{dt'}{a(t')}, \tag{4.176}$$

so the FRW metric reads (limiting ourselves for simplicity to $k = 0$)

$$ds^2 = a^2(\eta)\left[-c^2 d\eta^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]. \tag{4.177}$$

We want to know how a solution which decreases as $1/r$ evolves in this space-time. We therefore search for spherically symmetric solutions of the form $\phi(r,\eta) = (1/r)f(r,\eta)$. The equation $\Box\phi = 0$ in this metric then becomes

$$\begin{aligned} 0 &= \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi \\ &= -\frac{1}{c^2}\partial_\eta\left[a^2(\eta)r^2\partial_\eta\phi\right] + \partial_r\left[a^2(\eta)r^2\partial_r\phi\right] \\ &= \partial_r^2 f - f'' - 2\frac{a'}{a}f', \end{aligned} \tag{4.178}$$

where the prime denotes derivation with respect to $c\eta$, $f' = (1/c)\partial f/\partial\eta$. It is convenient to search for the solution in the form

$$f(r,\eta) = \frac{1}{a(\eta)}\, g(r,\eta). \tag{4.179}$$

Then $g(r,\eta)$ satisfies the equation

$$\partial_r^2 g - g'' + \frac{a''}{a}\, g = 0. \tag{4.180}$$

Now observe that $a''/a \sim \eta^{-2}$, for dimensional reasons.[18] Then we see that eq. (4.180) has the approximate solutions

$$g(r,\eta) \simeq e^{\pm i\omega(\eta - r/c)}, \tag{4.181}$$

as long as $\omega^2 \gg 1/\eta^2$, since in this case in eq. (4.180) the term $(a''/a)g \sim g/\eta^2$ is negligible with respect to $-g'' = \omega^2 g$, and we are left with a simple wave equation $\partial_r^2 g - g'' \simeq 0$. More generally, any function of the form $g(\eta - r/c)$ is a solution, as long as in Fourier space it contains only frequencies such that $\eta^2\omega^2 \gg 1$. In conclusion, we have the approximate solution

$$\phi(r,\eta) \simeq \frac{1}{ra(\eta)}\, g(\eta - r/c). \tag{4.182}$$

We can normalize conformal time so that, at the present epoch, $\eta = t$. Then, the wave observed today at a detector reads[19]

$$\phi(r,t) \simeq \frac{1}{ra(t_0)}\, g(t - r/c). \tag{4.183}$$

Thus, the propagation of a scalar wave through a FRW background is very simple. Compared to the solution in the absence of cosmological expansion, we just need to replace the factor $1/r$ with $1/[ra(t)]$.

Now we can turn to the propagation equation of a tensor perturbation $h_{\mu\nu}$, eq. (1.179). It is in principle straightforward to write down this

[18] More precisely, we are interested in the evolution in a matter-dominated Universe (since we need that stars already formed!). In this case the FRW scale factor evolves as $a(\eta) \sim \eta^2$ and $a''/a = 2/\eta^2$. Instead, in a radiation-dominated Universe, $a(\eta) \sim \eta$ and $a''/a = 0$.

[19] Obviously, since $a(\eta)$ evolves appreciably only on a cosmological time-scale, once we fix $\eta = t$ at one moment of time in the present epoch, we have $\eta = t$, with exceedingly good accuracy, over the whole time-scale relevant for GW observation at a detector, so we can write $g(\eta - r/c) = g(t - r/c)$ for all these values of time. For the same reason, $a(\eta)$ can be written simply as a constant factor $a(t_0)$.

or as

$$ds^2 = R^2(\eta)[-d\eta^2 + d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

where $\Sigma^2(\chi) = \sin^2\chi$ or χ^2 or $\sinh^2\chi$ $(k = 1, 0, -1)$.

Problem 19.6. Show that the spacelike 3-surfaces of a closed, isotropic, homogeneous universe possess a translation symmetry which leaves no points fixed. (Notice that this is not true in 2 dimensions; a 2 sphere cannot be combed smooth!)

Problem 19.7. A bullet is shot out into an expanding Robertson-Walker universe with a velocity V_1 (relative to cosmological observers). Later, when the universe has expanded by a scale factor $(1+z)^{-1}$, it has a different velocity V_2 with respect to cosmological observers. Find V_2 in terms of z and V_1. Show that in the limit $V_1 \to c$, the formula for the redshift of photons is obtained.

Problem 19.8. Show by an explicit coordinate transformation that the Robertson-Walker metric is conformally flat. Write $R_{\mu\nu\alpha\beta}$ in terms of $g_{\mu\nu}$ and ρ, p and the 4-velocity u^μ of the matter.

Problem 19.9. In a Robertson-Walker metric show that angular diameter distance (d_A), luminosity distance (d_L) and proper motion distance (d_M) are related by

$$(1+z)^2 d_A = (1+z) d_M = d_L \ .$$

Problem 19.10. Suppose astronomers are able to find a family of objects whose absolute luminosities L are known. Suppose their apparent luminosities ℓ (or equivalently their luminosity distance d_L) and their redshift z are measured. Using the Robertson-Walker line element, find an expression for ℓ (or d_L) as a function of L, z, H_0 and q_0 for small z.

Problem 19.11. Let $n(t_0)$ be the number density at the present epoch of a (mythical) family of identical light or radio sources distributed uniformly throughout the universe.

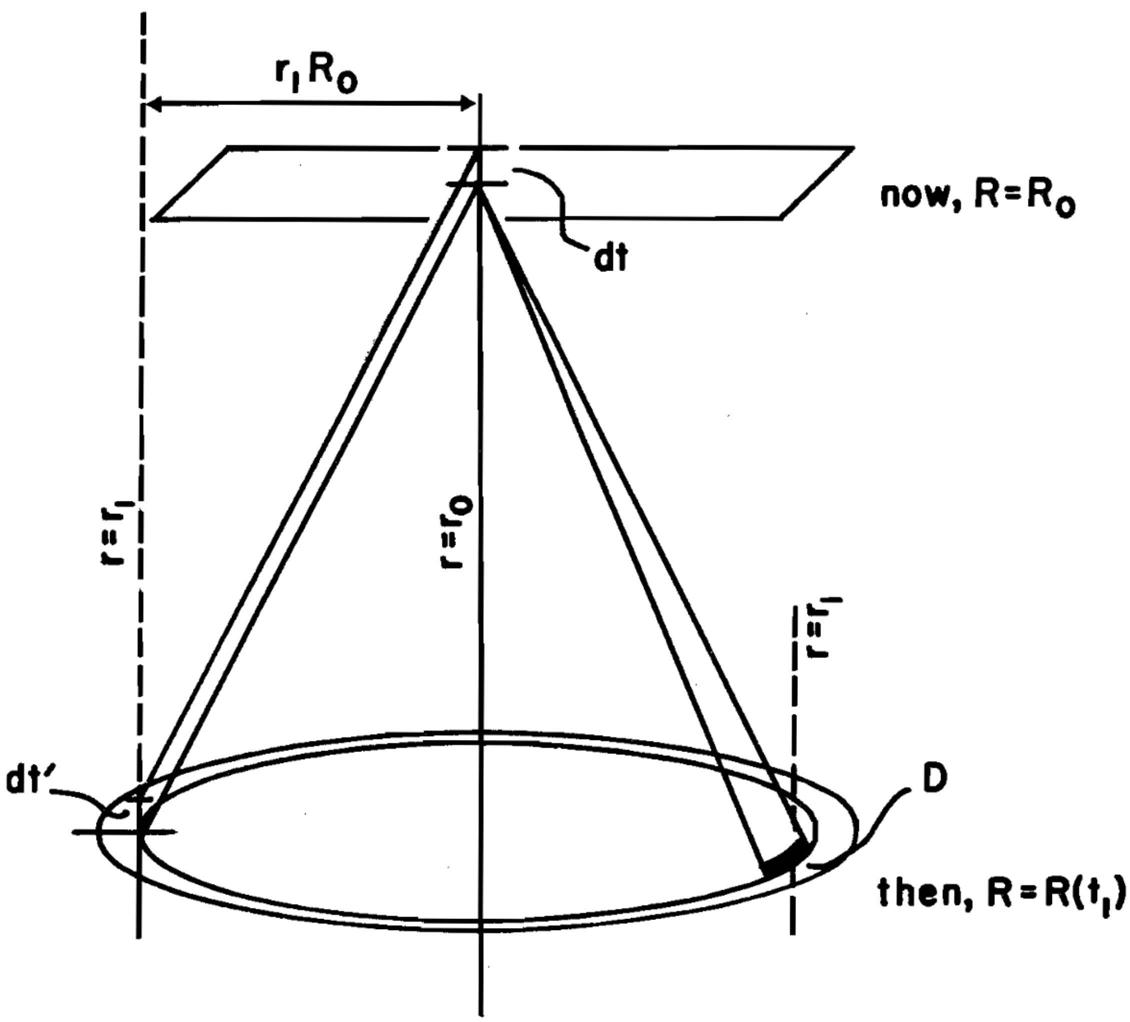
$$R^{\alpha}{}_{\beta} = 8\pi\left(T^{\alpha}{}_{\beta} - \frac{1}{2}\, g^{\alpha}{}_{\beta}\, T\right)$$

and the stress energy tensor

$$T^{\alpha}{}_{\beta} = (\rho + p)\, u^{\alpha} u_{\beta} + p g^{\alpha}{}_{\beta}$$

give $R^{\alpha\beta}{}_{\gamma\delta}$ in terms of $g_{\mu\nu}, u^{\mu}, \rho, p$.

Solution 19.9. If D is the physical size of an object, and δ is its angle subtended, $d_A \equiv D/\delta$. From the figure, and the Robertson-Walker metric, we see that $D = R(t)\, r_1 \delta$, so $d_A = r_1 R(t_1)$. If the object is moving transversely at a proper velocity V and with an apparent angular motion



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$d\delta/dt$, then $d_M \equiv V/(d\delta/dt)$. Let t' denote time measured at the emission of photons. Since $V = d(R(t_1)r_1\delta)/dt'$, and since $dt'/dt = R(t_1)/R_0$ because of the cosmological redshift, and noting that $R(t_1)$ can be considered constant since its change induces no transverse motion, we get $d_M = R_0 r_1$. If the object has an intrinsic luminosity L, and we receive a flux ℓ, then $d_L \equiv (L/4\pi\ell)^{\frac{1}{2}}$. In a time dt' it emits an energy $L\,dt'$. This energy is redshifted to the present by a factor $R(t_1)/R_0$, and is now distributed over a sphere of proper area $4\pi(r_1R_0)^2$ (see figure). Thus, $\ell = (L\,dt'\,R/R_0)(4\pi r_1 R_0)^{-2}/dt$ and $d_L = R_0^2 r_1/R(t_1)$. Using $R_0/R(t_1) = 1+z$, we have now obtained $(1+z)^2 d_A = (1+z)\,d_M = d_L$.

Solution 19.10. From the solution to Problem 19.9 we have $d_L = R_0^2 r_1/R(t_1)$ where r_1 is the radial coordinate of the object and $R(t_1)$ is the scale factor at the time t_1 at which the light was emitted. Since $H_0 \equiv \dot{R}/R$ and $q_0 \equiv -\ddot{R}R/\dot{R}^2$, the first terms in the power series expansion of R are

$$R(t) = R_0\left(1 + H_0(t-t_0) - \frac{1}{2}\,q_0H_0^2(t-t_0)^2 + \cdots\right). \qquad (1)$$

We can eliminate the factor $R(t_1)$ from d_L using $R_0/R(t_1) = 1+z$, but we still need to find expressions for R_0 and for r_1. Setting $ds^2 = 0$ to get the path of light rays in the Robertson-Walker metric gives

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{(1-kr^2)^{\frac{1}{2}}}\;. \qquad (2)$$

Integrating Equation (2) to the lowest 2 orders in (t_0-t_1) and r_1 by means of Equation (1) gives

$$r_1 = \frac{1}{R_0}\left[(t_0-t_1) + \frac{1}{2}\,H_0(t_0-t_1)^2 + \cdots\right] \qquad (3)$$

and inverting Equation (1) to get $t-t_0$ in terms of $1+z = R_0/R(t_1)$ gives

$$R_0 r_1 = \frac{1}{H_0}\left[z - \frac{1}{2}\,(1+q_0)\,z^2 + \cdots\right]$$

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so that

$$d_L = (1+z)R_0 r_1 = \frac{1}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + \cdots\right]$$

or equivalently

$$\ell = \frac{L}{4\pi d_L^2} = \frac{L H_0^2}{4\pi z^2}\,[1 + (q_0 - 1)z + \cdots]\ .$$

Solution 19.11.

(a) Let $n(t_1)$ be the number of sources per unit volume at time t_1. The volume element is

$$|{}^{(3)}g|^{\frac{1}{2}} dr_1\, d\theta_1\, d\phi_1 = R^3(t_1)(1-kr_1^2)^{-\frac{1}{2}} r_1^2\, dr_1 \sin\theta_1\, d\theta_1\, d\phi_1\ .$$

Thus the number of sources between r_1 and $r_1 + dr_1$ at time t_1 is

$$dN = 4\pi R^3(t_1)(1-kr_1^2)^{-\frac{1}{2}} r_1^2\, n(t_1)\, dr_1\ .$$

The quantities r_1 and t_1 are related by the equation for null rays propagating in the Robertson-Walker metric (Equation (2) in the solution of Problem 19.10) i.e. $r_1 = r(t_1)$ where

$$dr_1 = (1-kr_1^2)^{\frac{1}{2}}\, dt_1/R(t_1)\ .$$

We have, therefore

$$dN = 4\pi R^2(t_1) r^2(t_1) n(t_1)\, |dt_1|$$

and

$$N(z) = \int_{t_z}^{t_0} 4\pi R^2(t_1) r^2(t_1) n(t_1)\, dt_1$$

where t_z, the cosmological time corresponding to redshift z, is defined implicitly by

$$\frac{R(t_z)}{R(t_0)} = \frac{1}{1+z}\ . \qquad (1)$$

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