# Assignment - Proof of Theorem

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### ① Prove the Fourier Series

* Given-

$$f(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \dots + a_n \cos(n\omega_0 t)$$
$$+ b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \dots + b_n \sin(n\omega_0 t) \quad \text{— (I)}$$

where $\omega_0 = \frac{2\pi}{T}$, n is a positive integer.

* Proof -

- To derive the coefficient $a_0$, we take integral of both sides. over one period.

$$\int_T f(t)dt = \int_T a_0 dt + \int_T a_1 \cos(\omega_1 t)dt + \int_T a_2 \cos(\omega_2 t)dt + \dots$$
$$\dots + \int_T a_n \cos(\omega_n t)dt$$
$$+ \int_T b_1 \sin(\omega_1 t)dt + \int_T b_2 \sin(\omega_2 t)dt + \dots +$$
$$\int_T b_n \sin(\omega_n t)dt + \dots$$

where $T = [t_0, t_0 + T]$

- After evaluation all the integrals on the right side with sine and cosine term drop out and finally we get,

$$\int_T f(t)dt = a_0 \int_T dt$$
$$\therefore \int_T f(t)\,dt = a_0 (T)$$
$$\therefore a_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} f(t).dt$$

• Now, to find $a_n$, we multiply both the sides of eq (I) by ~~cos (ω_n t)~~ $\cos(\omega_n t)$ —

$$f(t)\cos(\omega_n t) = a_0\cos(\omega_n t) + a_1\cos(\omega_1 t)\cos(\omega_n t) + a_2\cos(\omega_2 t).\cos(\omega_n t) + \dots + a_n\cos^2(\omega_n t) + \dots$$
$$+ b_1\sin(\omega_1 t).\cos(\omega_n t) + b_2\sin(\omega_2 t).\cos(\omega_n t) + \dots + b_n\sin(\omega_n t).\cos(\omega_n t) + \dots$$

• Now we take integrals of both sides of the equation over one period. -

$$\int_T f(t)\cos(\omega_n t)\,dt = \int_T a_0\cos(\omega_n t)\,dt + \int_T a_1\cos(\omega_1 t)\cos(\omega_n t)\,dt + \dots$$
$$\dots + \int_T a_n\cos^2(\omega_1 t) + \dots$$
$$+ \int_T b_1\sin(\omega_1 t).\cos(\omega_n t) + \int_T b_2\sin(\omega_2 t).\cos(\omega_n t) + \dots$$
$$\dots + \int_T b_n\sin(\omega_n t).\cos(\omega_n t) + \dots$$

• By evaluating, we get

$$\int_T f(t)\cos(\omega_n t)\,dt = \int_T a_n\cos^2(\omega_n t).dt$$
$$\int_T f(t)\cos(\omega_n t).dt = a_n\left(\frac{T}{2}\right)$$
$$\therefore\quad a_n = \frac{2}{T}\int_T f(t)\cos(\omega_n t)\,dt$$
$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(\omega_n t)\,dt}$$

• Now, to find $b_n$ we have to multiply the eq (I) by $\sin(\omega_n t)$ on both sides, we get -

$$f(t).\sin(\omega_n t) = a_0\cos(\omega_n t) + a_1\cos(\omega_1 t).\sin(\omega_n t) + \dots + a_n\cos(\omega_n t).\sin(\omega_n t) + \dots$$
$$+ b_1\sin(\omega_1 t)\sin(\omega_n t) + b_2\sin(\omega_2 t)\sin(\omega_n t) + \dots + b_n\sin^2(\omega_n t) + \dots$$

- Now we take the integrals on both sides, we get -

$$\int_T f(t)\sin(\omega_n t)\,dt = \int_T a_0\cos(\omega_n t)\,dt + \int_T a_1\cos(\omega_1 t).\sin(\omega_n t)\,dt +$$

$$\int_T a_2\cos(\omega_2 t).\sin(\omega_n t)\,dt + .. + \int_T a_n\cos(\omega_n t).\sin(\omega_n t)\,dt$$

$$+ \int_T b_1\sin(\omega_1 t).\sin(\omega_n t)\,dt + \int_T b_2\sin(\omega_2 t).\sin(\omega_n t)\,dt.$$

$$+ \int_T b_n\sin^2(\omega_n t)\,dt + ..$$

- By evaluating, we get -

$$\int_T f(t)\sin(\omega_n t)\,dt = \int_T b_n\sin^2(\omega_n t)\,dt$$

$$\int_T f(t)\sin(\omega_n t)\,dt = b_n\left(\frac{T}{2}\right)$$

$$\therefore \quad b_n = \frac{2}{T}\int_T f(t)\sin(\omega_n t)\,dt$$

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(\omega_n t)\,dt}$$

② Prove the convolution theorem -

Convolution theorem states that -

$$f(x) * g(x) \Longleftrightarrow F(u)\, G(u)$$

Proof-

* By shift theorem or shift-invariance we know that if fourier transform of $f(x) = F(u)$,

then $F[f(x-a)] = e^{-i2\pi u a} F(u)$ —— (I)

* By defination of convolution theorem -

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x)\, g(y-x)\, dx$$

The fourier transform of $f(x) * g(x)$ is

$$\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x)\, g(y-x)\, dx \right] e^{-i2\pi u y}\, dy$$

• By reversing the order of integration.

$$\int_{-\infty}^{\infty} f(x) \left[ \int_{-\infty}^{\infty} g(y-x)\, e^{-i2\pi u y}\, dy \right] dx$$

note: $f(x)$ can be treated as constant when integrating over $y$

• By shift theorem (I) -

$$\int_{-\infty}^{\infty} g(y-x)\, e^{-i2\pi u y}\, dy = F[g(y-x)] = e^{-i2\pi u x} G(u)$$

So $\displaystyle\int_{-\infty}^{\infty} f(x) \left[ \int_{-\infty}^{\infty} g(y-x)\, e^{-i2\pi u y}\, dy \right] dx = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi u x}\, G(u)\, dx$

$$= \left[ \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi u x}\, dx \right] G(u)$$

$$= F(u) . G(u)$$

Since the fourier transform $F[f*g] = F(u)\, G(u)$, Convolution theorem in time domain is equivalent to multiplication of fourier transform in frequency domain.

## ③ Properties of convolution -

### 1] Commutativity Property. or Reflexive property.

To prove : $x(t) * h(t) = h(t) * x(t)$

By defination -

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(T)\, h(t-T)\, dT$$

where $\lambda = t - T$ and by changing the variable it becomes -

$$x(t) * h(t) = -\int_{\infty}^{-\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

$$= \int_{\infty}^{-\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

$$\boxed{x(t) * h(t) = h(t) * x(t)}$$

Hence, Proved.

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### 2] Assosiative Property -

To prove -

$$[x(k) * h_1(k)] * h_2(k) = x(k) * [h_1(k) * h_2(k)]$$

Taking L.H.S -

$$[x(k) * h_1(k)] * h_2(k) = \left[\int_{-\infty}^{\infty} x(m)\, h_1(k-m)\right] * h_2(k)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(m)\, h_1(n-m)\, dm\right] * h_2(k-n)\, dn$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(m)\, h_1(n-m)\, h_2(k-n)\, dm.dn$$

Let $r = k-n \Rightarrow n = k-r$ where $n = \infty, r = -\infty$
$n = -\infty, r = \infty$

$$= \int_{-\infty}^{\infty} x(m)\,dm \int_{-\infty}^{\infty} h_1(n-m)\, h_2(k-n)\, dn$$

$$= \int_{-\infty}^{\infty} x(m)\,dm \int_{\infty}^{-\infty} h_1(k-r-m)\, h_2(r).\, dr$$

$$= \int_{-\infty}^{\infty} x(m)\,dm \int_{-\infty}^{\infty} h_2(r)\, h_1[(k-m)-r]\, dr$$

Let $z(k) = h_1(k) * h_2(k)$

$h_2(m) * h_1(k-m) = h_1(k-m) * h_2(m) = z[k-m]$ - By property (I)

$$\therefore = \int_{-\infty}^{\infty} x(m)\, z(k-m)$$

$$= x(k) * z(k)$$

$$\therefore \boxed{[x(k) * h_1(k)] * h_2(k) = x(k) * [h_1(k) * h_2(k)]}$$

Hence, Proved.

3] Distributive Property -

To Prove - $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

$$x(t) * [h_1(t) + h_2(t)] = \int_{-\infty}^{\infty} x(t)[h_1(t-T) + h_2(t-T)]\, dT$$

$$= \int_{-\infty}^{\infty} x(t)\, h_1(t-T)\, dT + \int_{-\infty}^{\infty} x(t)\, h_2(t-T)\, dT$$

$$= \cancel{x(t) * [h_1(t) + h_2(t)]}$$

$$\boxed{x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)}$$

Hence, Proved.