The University of Sydney

MOOC Introduction to Calculus

Notes for 'Lines and circles in the plane'

Important Ideas and Useful Facts:

(i) Lines in the plane: A *line* in the xy-plane is the set of points (x, y) satisfying an equation either of the form $y = mx + k$ (the nonvertical case), where m is the *slope* and k is the *y-intercept*, or $x = \ell$ (the vertical case), where ℓ is the *x-intercept*. All lines may be put in the form

$$ax + by \;=\; c$$

for some constants a, b and c.

This equation is only interesting by assuming that at least one of a or b is nonzero. If $a = 0$ then this assumption forces b to be nonzero, in which case the equation describes a horizontal line ($y = \frac{c}{b}$). If $b = 0$ then this assumption forces a to be nonzero, in which case the equation describes a vertical line ($x = \frac{c}{a}$).

(ii) Slope of a line determined by two distinct points: A line is completely determined by two distinct points. Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be distinct points in the plane. The slope of the line joining P to Q is

$$\frac{\text{vertical rise}}{\text{horizontal run}} \;=\; \frac{y_2 - y_1}{x_2 - x_1} \;=\; \frac{y_1 - y_2}{x_1 - x_2}\,,$$

if $x_1 \neq x_2$, and infinite if $x_1 = x_2$ (when the line is vertical). An equation of the line through P and Q is

$$y - y_1 \;=\; \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

provided that $x_1 \neq x_2$.

If the slope is positive then the line slopes upwards as one moves from left to right. If the slope is negative then the line slopes downwards as one moves from left to right. If the slope is zero then the line is horizontal.

If the slope is m, then one moves m units vertically (upwards if m is positive and downwards if m is negative) for each unit that one moves horizontally towards the right.

(iii) Parallel and perpendicular lines: Two nonvertical lines in the xy-plane with slopes m_1 and m_2 respectively are *parallel* if $m_1 = m_2$, and *perpendicular* if $m_1 m_2 = -1$, in which case m_1 and m_2 become negative reciprocals of each other.

(iv) Equation of a circle: A *circle* in the xy-plane, centred at $P(x_0, y_0)$ with radius r is the set of points (x, y) satisfying the equation

$$(x - x_0)^2 + (y - y_0)^2 \;=\; r^2\,.$$

Any tangent to the circle is perpendicular to the radius joining the centre of the circle to the intersection point.

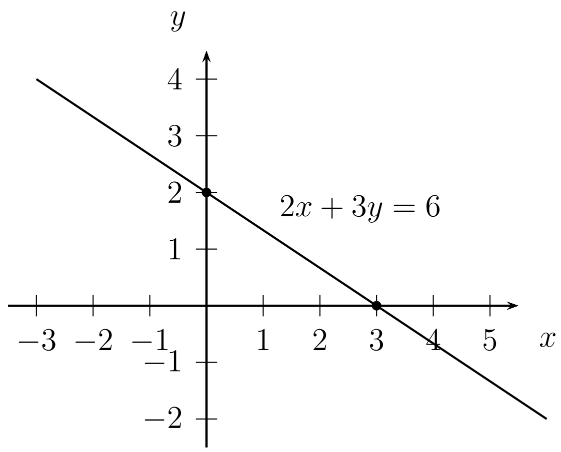
Examples:

1. Consider the line described by the equation

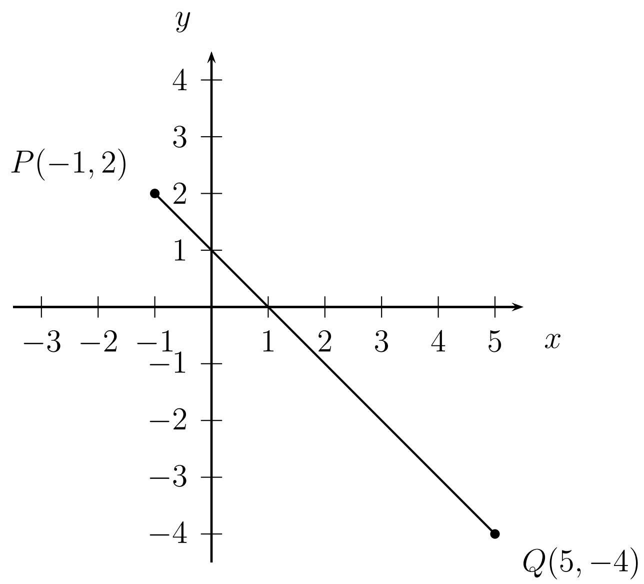
$$2x + 3y \; = \; 6 \; .$$

The linear relationship can be visualised in the xy-plane by finding two points that satisfy this equation, and then drawing a line between them.

The intercepts with the axes are easy to find: to find the y-intercept, we put $x = 0$, so that $3y = 6$, yielding $y = 2$. To find the x-intercept, we put $y = 0$, so that $2x = 6$, yielding $x = 3$. We can then draw a line that passes through these intercepts.



2. Consider the line passing through the points $P(-1, 2)$ and $Q(5, -4)$. The segment of this line joining P and Q is shown below.



The slope is

$$m \; = \; \frac{-4 - 2}{5 - (-1)} \; = \; \frac{-6}{6} \; = \; -1 \; .$$

This matches the fact that the line has x and y-intercepts both equal to 1, so that in moving from the y-intercept to the x-intercept, one moves down one unit vertically as one moves one unit horizontally to the right. The equation of the line passing through P and Q must have equation $y = -x + k$, where k is the y-intercept. Then $k = 1$, so the equation of the line is $y = -x + 1$.

3. Consider the line passing through the points $P(2,3)$ and $Q(7,13)$. The slope is

$$m \;=\; \frac{13-3}{7-2} \;=\; \frac{10}{5} \;=\; 2\,,$$

so the equation has the form $y = 2x + k$ for some k. To find k we can substitute coordinates from either point. If we use the coordinates for P we get

$$3 \;=\; 2(2) + k \;=\; 4 + k\,,$$

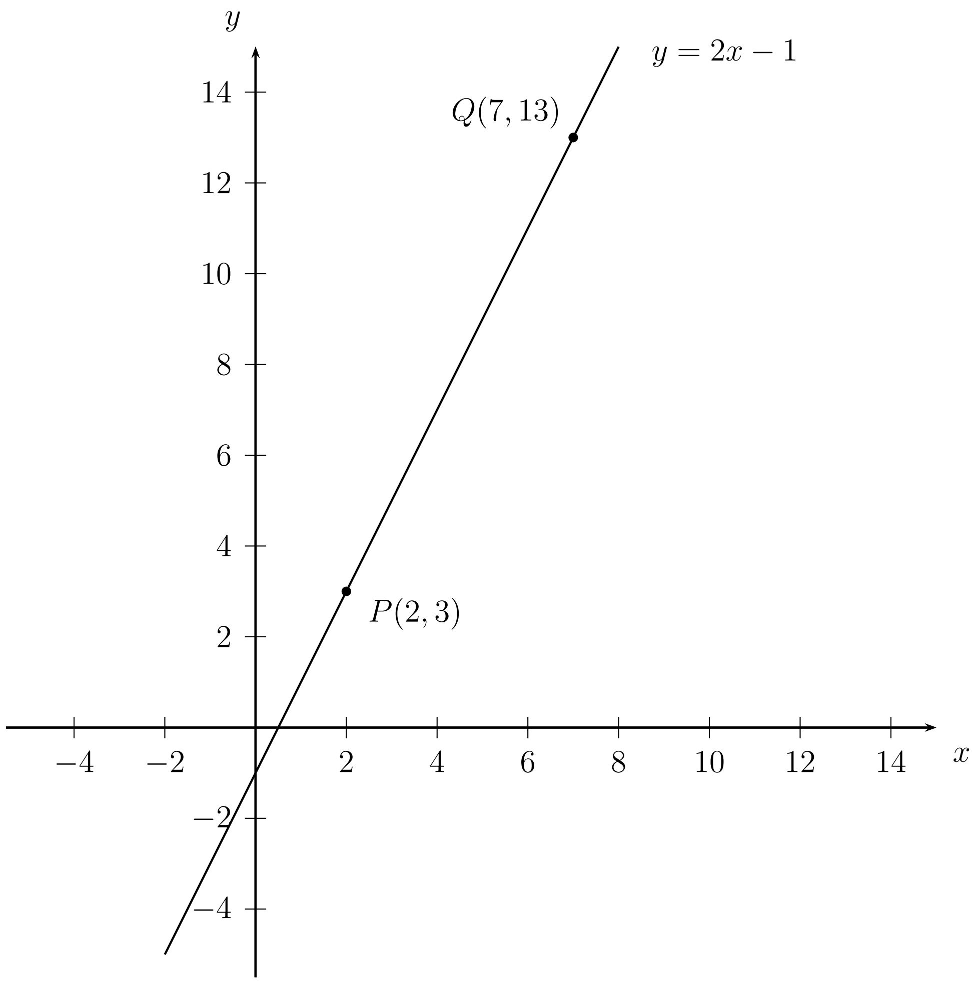
so that $k = -1$. The equation of the line therefore is

$$y \;=\; 2x - 1\,.$$

We can check that the coordinates of Q satisfy this equation:

$$2(7) - 1 \;=\; 14 - 1 \;=\; 13\,,$$

so indeed $Q(7,13)$ lies on the line with this equation, and all is well.



4. Consider the line passing through the points $P(2,2)$ and $Q(5,6)$. The slope is

$$m \;=\; \frac{6-2}{5-2} \;=\; \frac{4}{3}\,,$$

so the equation has the form $y = \frac{4}{3}x + k$ for some k. To find k we can substitute coordinates from either point. If we use the coordinates for P we get

$$2 \;=\; \frac{4}{3}(2) + k \;=\; \frac{8}{3} + k\,,$$

so that $k = -\frac{2}{3}$. The equation of the line therefore is

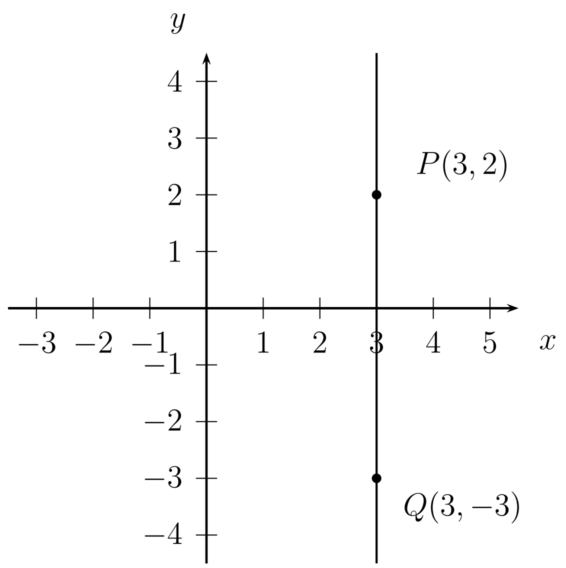
$$y \;=\; \frac{4}{3}x - \frac{2}{3} \;=\; \frac{4x-2}{3}\,.$$

We can check that the coordinates of Q satisfy this equation:

$$\frac{4(5)-2}{3} \;=\; \frac{18}{3} \;=\; 6\,,$$

so indeed $Q(5,6)$ lies on the line with this equation, and all is well.

5. Consider the line passing through the points $P(3,2)$ and $Q(3,-3)$. Because the x-coordinates are the same, the line is vertical (with infinite slope), and has equation $x = 3$.



6. Consider the circle $\mathcal{C}$ centred at $P(3,4)$ with radius 5 units. We can read off the equation from the formula given above:

$$(x-3)^2 + (y-4)^2 \;=\; 5^2 \;=\; 25\,.$$

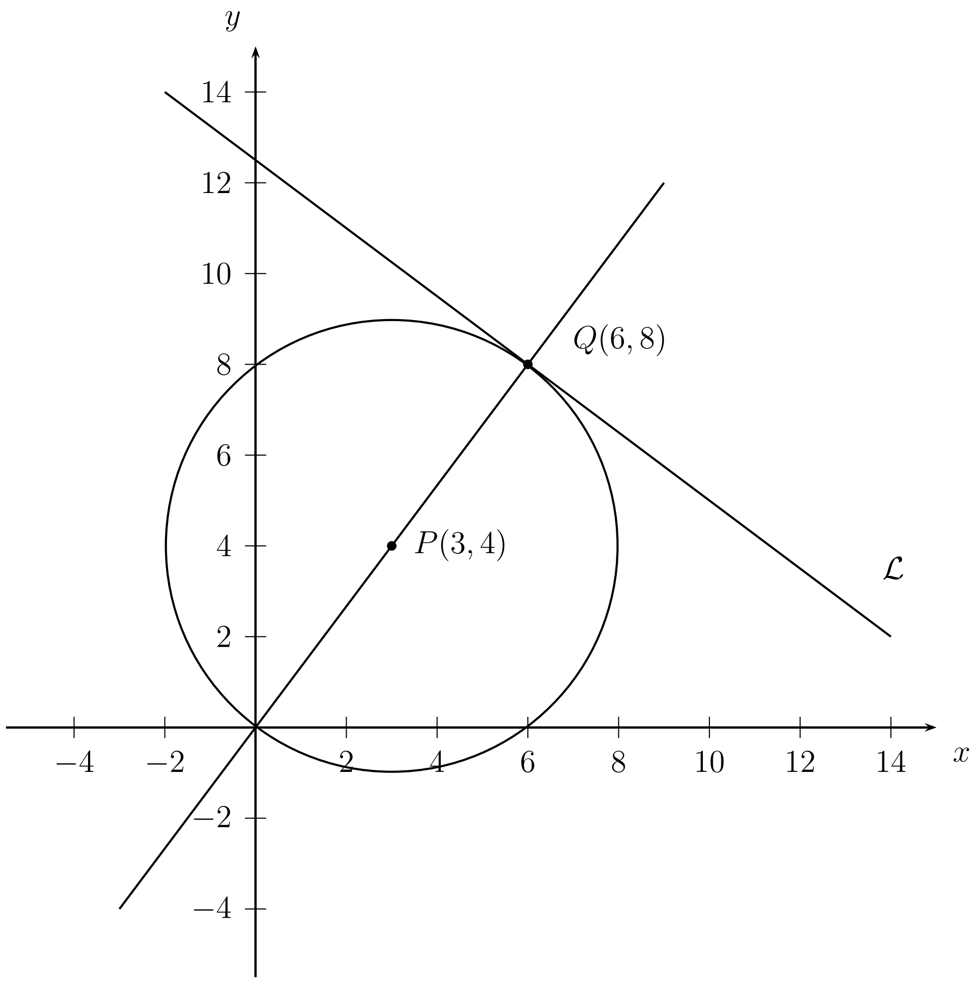
To see where the circle crosses the y-axis, we put $x = 0$, giving

$$(-3)^2 + (y-4)^2 \;=\; 25\,,$$

so that $(y-4)^2 = 25 - 9 = 16$, so that $y - 4 = \pm\sqrt{16} = \pm 4$, so that $y = 0$ or $y = 8$. To see where the circle crosses the x-axis, we put $y = 0$, giving

$$(x-3)^2 + (-4)^2 \;=\; 25\,,$$

so that $(x-3)^2 = 25 - 16 = 9$, so that $x - 3 = \pm\sqrt{9} = \pm 3$, so that $x = 0$ or $x = 6$. In particular, the circle passes through the origin (where the axes intersect).



Observe that the point $Q(6,8)$ lies on the circle, as

$$(6-3)^2 + (8-4)^2 \;=\; 9 + 16 \;=\; 25 \;.$$

In fact the origin $O(0,0)$, $P(3,4)$ and $Q(6,8)$ all lie on the same line, with slope $\frac{4}{3}$ and equation $y = \frac{4x}{3}$.

Consider the tangent line $\mathcal{L}$ to the circle at Q. Then $\mathcal{L}$ is perpendicular to the line passing through P and Q, so must have slope $-\frac{3}{4}$ (because $\frac{4}{3} \times \left(-\frac{3}{4}\right) = -1$). Hence $\mathcal{L}$ has equation

$$y \;=\; \frac{-3x}{4} + k$$

for some constant k. But Q lies on $\mathcal{L}$, so that $8 = \frac{-3\times 6}{4} + k = \frac{-9}{2} + k$, yielding $k = \frac{25}{2}$. Hence the tangent line $\mathcal{L}$ has equation

$$y \;=\; \frac{-3x}{4} + \frac{25}{2} \;=\; \frac{50-3x}{4} \;.$$

7. In this next example, we find the points, if any, where the line $\mathcal{L}$ with equation

$$4x + 3y \;=\; 24$$

intersects the circle $\mathcal{C}$ with equation

$$(x-3)^2 + (y-4)^2 \;=\; 25 \;.$$

In fact, $\mathcal{C}$ is the circle of the previous example, with centre $(3,4)$ and radius 5. For a point (x,y) to lie on $\mathcal{L}$ and $\mathcal{C}$ simultaneously, both equations must hold. Though two variables x and y are involved, we can use the equation of $\mathcal{L}$ to express y in terms of x, namely

$$y = \frac{24-4x}{3} ,$$

and then substitute this expression for y in the equation for $\mathcal{C}$, to get a single equation involving x:

$$(x-3)^2 + \left(\frac{24-4x}{3} - 4\right)^2 = 25 .$$

This becomes

$$(x-3)^2 + \left(\frac{24-4x-12}{3}\right)^2 = (x-3)^2 + \left(\frac{12-4x}{3}\right)^2 = 25 ,$$

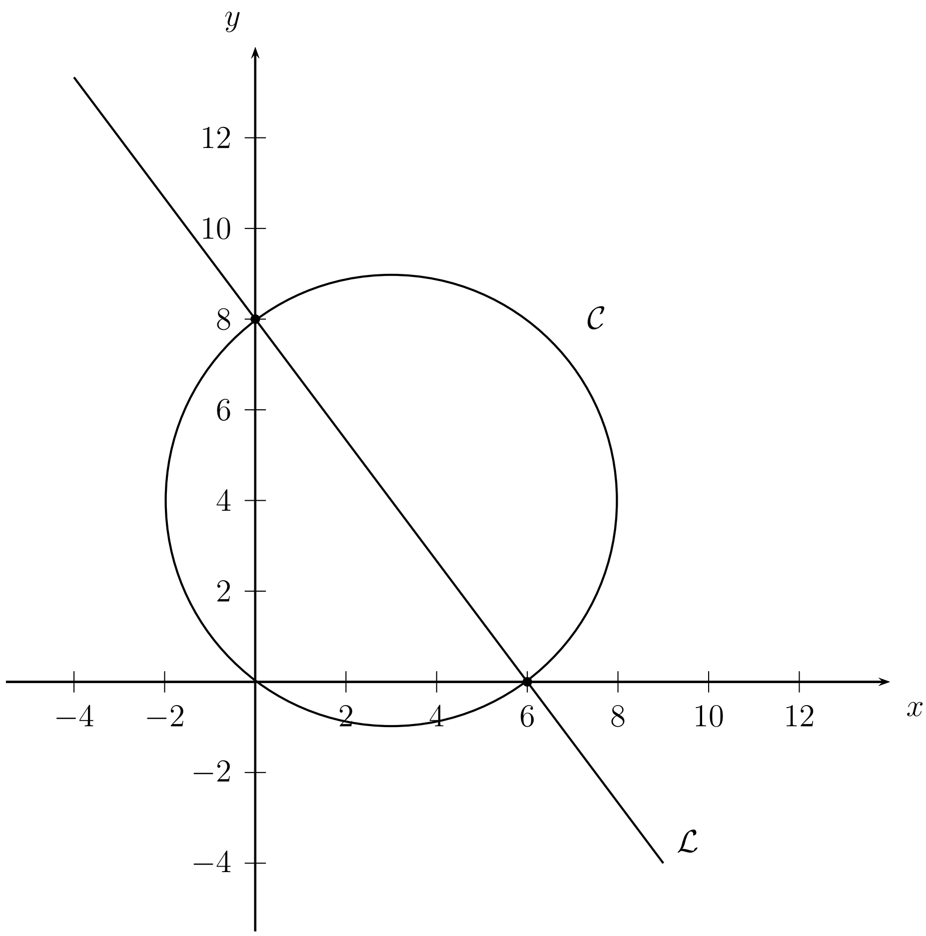
that is,

$$(x-3)^2 + \frac{16(3-x)^2}{9} = (x-3)^2 + \frac{16(x-3)^2}{9} = \frac{25(x-3)^2}{9} = 25 .$$

Hence $(x-3)^2 = 9$, and it follows quickly that $x = 0$ or $x = 6$. Thus, from the equation for $\mathcal{L}$, we get that $y = 8$ when $x = 0$, and that $y = 0$ when $x = 6$. The intersection points are therefore

$$(0,8) \quad \text{and} \quad (6,0) .$$

We can visualise the line, circle and intersection points in the xy-plane:



8. **(The following remarks are for interest only, and go beyond the scope of this course.)** The general equation of a line in the plane has the form

$$ax + by = c$$

where a, b and c are constants, where we may assume at least one of a or b is nonzero. This equation generalises to higher dimensions. For example, if we introduce a third variable z, then the pattern suggests considering an equation of the form

$$ax + by + cz = d ,$$

where now a, b, c and d are constants such that not all of a, b and c are zero. Previously, we have discussed coordinates in space using x, y and z-axes. The set of triples (x, y, z) that satisfy this equation in fact form a *plane* in space (not a line). There are three variables, and we may choose values for any two of them freely, and then the third value is constrained by the equation. For example, if we choose x and y then, rearranging the equation, we get (assuming c is nonzero)

$$z = \frac{d - ax - by}{c} .$$

If instead, for example, we choose x and z freely, then y is forced (assuming b is nonzero):

$$y = \frac{d - ax - cz}{b} .$$

Because we can choose values for two of the variables independently and freely (though the third is constrained by the equation), we say that there are *two degrees of freedom*, or that the configuration in space defined by the equation is *two-dimensional*. Because we think of a line in space as "one-dimensional", at least in some informal sense, this equation cannot possibly describe a line in space. Instead, it describes something that we think of as "two-dimensional". The notion of *dimension* is formally described and discussed in detail in more advanced courses in mathematics.

There is no reason to stop at three variables. If $x_1, x_2, x_3, \ldots, x_n$ are variables, where n is a large positive integer, then we can consider the following *linear equation*:

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b ,$$

where $a_1, a_2, a_3, \ldots, a_n, b$ are constants, and not all of $a_1, \ldots, a_n$ are zero. This in fact describes a configuration in a higher dimensional space known as a *hyperplane*. Because there are n variables and one equation that places a constraint, there are $n - 1$ *degrees of freedom* or $n - 1$ *dimensions*. If one chooses values for all but one of the variables freely, then the final variable (assuming its coefficient is nonzero) is then completely determined by the equation. The study of hyperplanes and their intersections is really the study of *systems of simultaneous linear equations*, which is a topic in *linear algebra*, which you will meet if you study further mathematics at university.