VECTOR

Question: $\boldsymbol{P}$ and $\boldsymbol{Q}$ are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$.Show that, $ar(\triangle APB) = ar(\triangle BQC)$.

Figure:

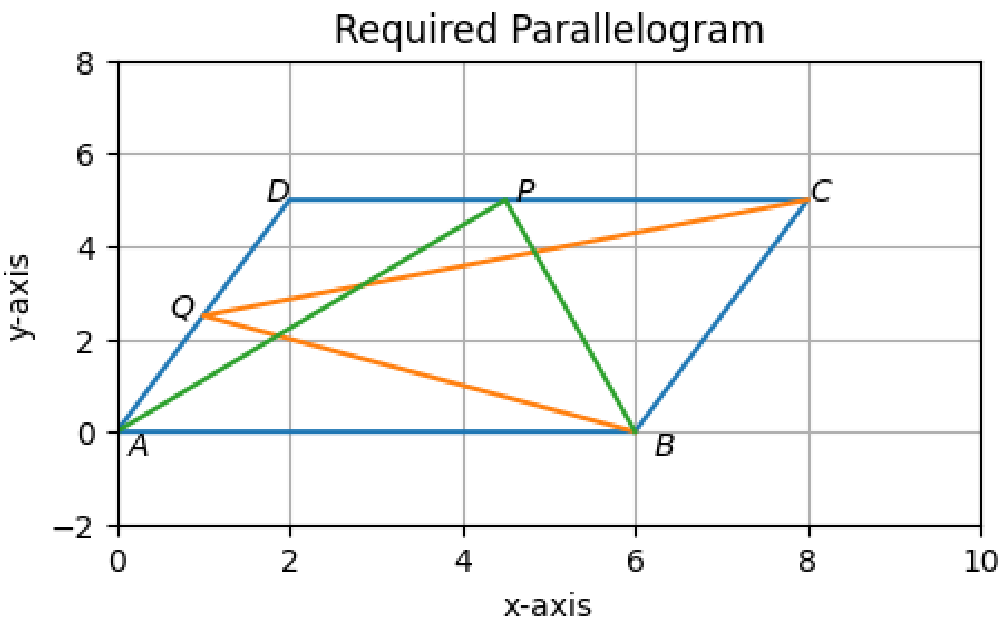

Figure 1

Solution:

Using formulas:

1. Area of a parallelogram with sides $\boldsymbol{a}$ and $\boldsymbol{b}$ is $= \frac{1}{2}\left|\boldsymbol{a} \times \boldsymbol{b}\right|$ square units $= \frac{1}{2}\left|a.b\sin\theta\right|$

2. Area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is $= \frac{1}{2} \times \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$ square units . (only positive values will be taken as negative value of area is not possible)

3. If a triangle and a parallelogram are on the same base and between the same parallels, then area of the triangle is equal to half the area of the parallelogram.

From figure 1 , considering $\triangle BQC$ and parallelogram $ABCD$, having $\underline{\text{same base } BC}$ and, $\underline{AD \text{ is parallel to } BC}$.

$$\implies \triangle BQC = \frac{1}{2} \times ar ABCD. \tag{1}$$

Considering $\triangle APB$ and parallelogram $ABCD$, having $\underline{\text{same base } AB}$ and,$\underline{DC \text{ is parallel to } AB}$.

$$\implies \triangle APB = \frac{1}{2} \times ar ABCD. \tag{2}$$

By comparing equation (1) and equation (2) we obtain that,

$$ar(\triangle APB) = ar(\triangle BQC). \tag{3}$$

Hence proved.

Proof by the help of diagram :(figure 1,table 1,table 2)
Let the points $\boldsymbol{Q}$ and $\boldsymbol{P}$ divide AD and CD by $k_1 : 1$ and $k_2 : 1$ ratio respectively.

Symbols	**Description**	**Value**
$AB = DC$	Sides of parallelogram	$a = 6$(unit)
$AD = BC$	Sides of parallelogram	$b = \sqrt{29}$
θ	$\angle BAD$	$\sin^{-1}(\frac{5}{\sqrt{29}})$
$\boldsymbol{A}$	vertex (at origin)	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\boldsymbol{Q}$	Divides AD	$k_1 : 1$ ratio
$\boldsymbol{P}$	Divides CD	$k_2 : 1$ ratio

Table 1: Table of input parameters

Symbols	Description	Value
$\boldsymbol{B}$	vertex	$a\begin{pmatrix}1\\0\end{pmatrix}$
$\boldsymbol{D}$	vertex	$b\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$
$\boldsymbol{C}$	vertex	$\boldsymbol{B}+\boldsymbol{D}$
$\boldsymbol{Q}$	Vertex($\triangle BQC$)	$\begin{pmatrix}\frac{2k_1}{k_1+1}\\ \frac{5k_1}{k_1+1}\end{pmatrix}$
$\boldsymbol{P}$	Vertex($\triangle APB$)	$\begin{pmatrix}\frac{8k_2+2}{k_2+1}\\ 5\end{pmatrix}$

Table 2: Table of output parameters

For the $\triangle BQC$, the vertices of the triangle are

$$\boldsymbol{B}=\begin{pmatrix}6\\0\end{pmatrix},\boldsymbol{Q}=\begin{pmatrix}\frac{2k_1}{k_1+1}\\ \frac{5k_1}{k_1+1}\end{pmatrix},\boldsymbol{C}=\begin{pmatrix}8\\5\end{pmatrix}$$

$$\implies \triangle BQC=\begin{vmatrix}1 & 1 & 1\\ 6 & \frac{2k_1}{k_1+1} & 8\\ 0 & \frac{5k_1}{k_1+1} & 5\end{vmatrix}$$

$\frac{1}{2}\times\left|1(\frac{2k_1}{k_1+1}.5-\frac{5k_1}{k_1+1}.8)-6(5.1-\frac{5k_1}{k_1+1}.1)+0\right|$
$=\frac{1}{2}\times\left|\frac{-40k_1+10k_1-30}{k_1+1}\right|$
$=\frac{1}{2}\times 30$
$=15$ square units.

For the $\triangle APB$, the vertices of the triangle are

$$\boldsymbol{A}=\begin{pmatrix}0\\0\end{pmatrix},\boldsymbol{P}=\begin{pmatrix}\frac{8k_2+2}{k_2+1}\\ 5\end{pmatrix},\boldsymbol{B}=\begin{pmatrix}6\\0\end{pmatrix}$$

$$\implies \triangle APB=\begin{vmatrix}1 & 1 & 1\\ 0 & \frac{8k_2+2}{k_2+1} & 6\\ 0 & 5 & 0\end{vmatrix}$$

$=\frac{1}{2}\times\left|1(\frac{8k_2+2}{k_2+1}.0-6.5)+0+0\right|$
$=\frac{1}{2}\times\left|-30\right|$
$=\frac{1}{2}\times 30$
$=15$ square units.

For parallelogram $ABCD$,the vertices are

$$\boldsymbol{A}=\begin{pmatrix}0\\0\end{pmatrix},\boldsymbol{B}=\begin{pmatrix}6\\0\end{pmatrix},\boldsymbol{C}=\begin{pmatrix}8\\5\end{pmatrix},\boldsymbol{D}=\begin{pmatrix}2\\5\end{pmatrix}$$

$\implies arABCD = a.b.\sin\theta$
$=6.\sqrt{29}.\sin(\sin^{-1}\frac{5}{\sqrt{29}})$
$=6.\sqrt{29}.\frac{5}{\sqrt{29}}$
=30 square units.

Therefore,$arABCD = \frac{1}{2} \times ar\triangle APB = \frac{1}{2} \times ar\triangle BQC$(proved).