VECTOR

Question: $\vec{P}$ and $\vec{Q}$ are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$.Show that, $ar(\triangle APB) = ar(\triangle BQC)$.

Figure:

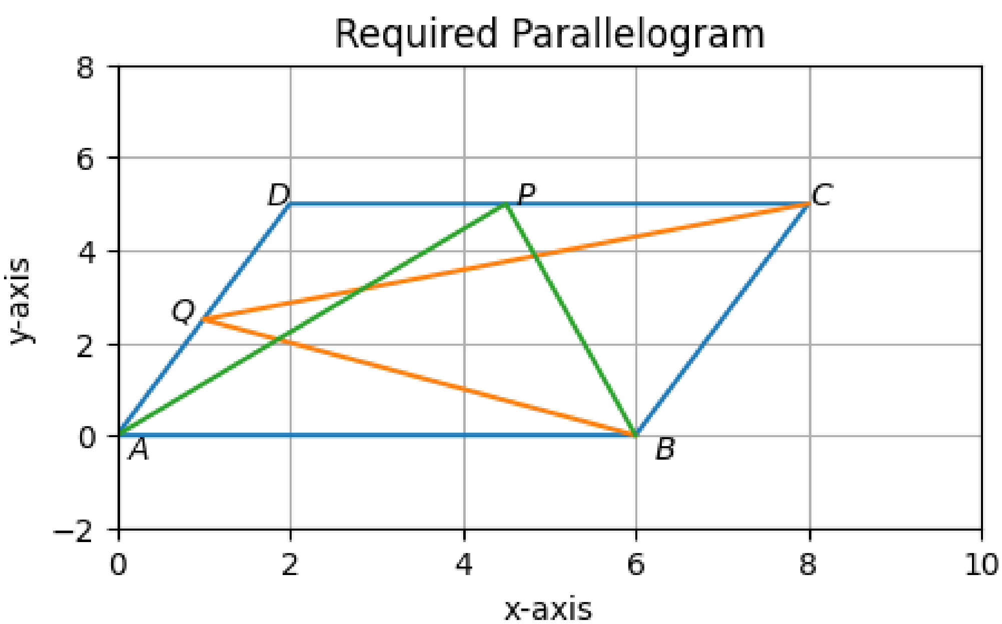

Figure 1

Solution:

Using formulas:

1. Area of a parallelogram with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ is $= \frac{1}{2} \times \left| [(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)] \right|$ square units.

2. Area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is $= \frac{1}{2} \times \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$ square units.

3. If a triangle and a parallelogram are on the same base and between the same parallels, then area of the triangle is equal to half the area of the parallelogram.

From figure 1 , considering $\triangle BQC$ and parallelogram $ABCD$, having $\underline{\text{same base } BC}$ and, $\underline{AD \text{ is parallel to } BC}$.

$$\implies \triangle BQC = \frac{1}{2} \times ar ABCD. \tag{1}$$

Considering $\triangle APB$ and parallelogram $ABCD$, having $\underline{\text{same base } AB}$ and,$\underline{DC \text{ is parallel to } AB}$.

$$\implies \triangle APB = \frac{1}{2} \times ar ABCD. \tag{2}$$

By comparing equation (1) and equation (2) we obtain that,

$$ar(\triangle APB) = ar(\triangle BQC). \tag{3}$$

Hence proved.

Proof by the help of diagram :(figure 1,table 1)

Symbols	**Description**	**Value**
$AB = DC$	Sides of parallelogram	$a = 6$(unit)
$AD = BC$	Sides of parallelogram	$d = \sqrt{29}$
θ	$\angle BAD$	$\sin^{-1}(\frac{5}{\sqrt{29}})$
$\vec{A}$	vertex (at origin)	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\vec{P}$	Divides CD	$m_1 : n_1$ ratio
$\vec{Q}$	Divides AD	$m_2 : n_2$ ratio

Table 1: Table of input parameters

Let the points $\vec{P}$ and $\vec{Q}$ divide CD and AD by $m_1 : n_1$ and $m_2 : n_2$ ratio respectively.

So,the coordinates of $\vec{P}$ will be $(\frac{8m_1+2n_1}{m_1+n_1}, \frac{5m_1+5n_1}{m_1+n_1}) = (\frac{8m_1+2n_1}{m_1+n_1}, 5)$ and, the coordinates of $\vec{Q}$ will be $(\frac{2m_2}{m_2+n_2}, \frac{5m_2}{m_2+n_2})$.
For the $\triangle APB$, the vertices of the triangle are

$$\vec{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec{P} = \begin{pmatrix} \frac{8m_1+2n_1}{m_1+n_1} \\ 5 \end{pmatrix}, \vec{B} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$\implies \triangle APB = \frac{1}{2} \times \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$
$= \frac{1}{2} \times \left| 0(5-0) + \frac{8m_1+2n_1}{m_1+n_1}(0-0) + 6(0-5) \right|$
$= \frac{1}{2} \times \left| 0 + 0 - 30 \right|$
$= \frac{1}{2} \times 30$
$= 15$ square units.

For the $\triangle BQC$, the vertices of the triangle are

$$\vec{B} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \vec{Q} = \begin{pmatrix} \frac{2m_2}{m_2+n_2} \\ \frac{5m_2}{m_2+n_2} \end{pmatrix}, \vec{C} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

$\implies \triangle BQC = \frac{1}{2} \times \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$
$= \frac{1}{2} \times \left| 6(\frac{5m_2}{m_2+n_2} - 5) + \frac{2m_2}{m_2+n_2}(5-0) + 8(0 - \frac{5m_2}{m_2+n_2}) \right|$
$= \frac{1}{2} \times \left| \frac{30m_2+10m_2-40m_2}{m_2+n_2} - 30 \right|$
$= \frac{1}{2} \times 30$
$= 15$ square units.

For parallelogram $ABCD$,the vertices are

$$\vec{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec{B} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \vec{C} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \vec{D} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$\implies arABCD = \frac{1}{2} \times \left| [(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)] \right|$
$= \frac{1}{2} \times \left| [(0.0 - 6.0) + (6.5 - 8.0) + (8.5 - 2.5) + (2.0 - 0.5)] \right| = \frac{1}{2} \times \left| 30 + 30 \right|$
$= \frac{1}{2} \times 60$
$= 30$ square units.

Or,By another method the area of the parallelogram $ABCD$ will be $= a \times b \times \sin\theta$
$=6 \times \sqrt{29} \times \sin\left(\sin^{-1}\frac{5}{\sqrt{29}}\right)$
$=6 \times \sqrt{29} \times \frac{5}{\sqrt{29}}$
$=30$ square units.

Therefore,$arABCD = \frac{1}{2} \times ar\triangle APB = \frac{1}{2} \times ar\triangle BQC$(proved).