VECTOR

Question: $\boldsymbol{P}$ and $\boldsymbol{Q}$ are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$.Show that, $ar\left(\triangle APB\right) = ar\left(\triangle BQC\right)$.

Figure:

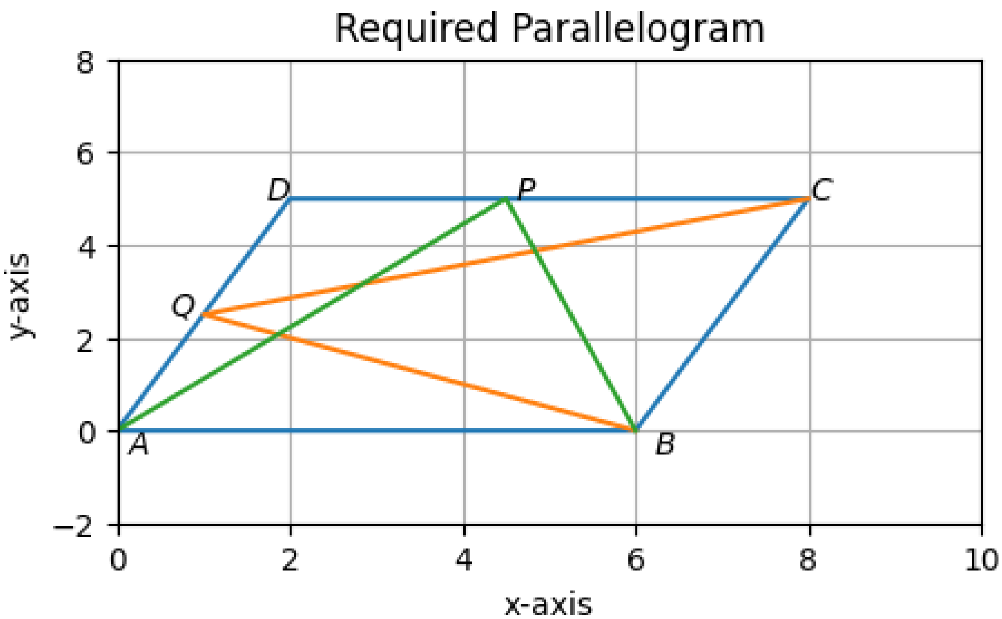

Figure 1

Solution:

From figure 1 , considering $\triangle BQC$ and parallelogram $ABCD$, having <u>same base BC</u> and, <u>AD is parallel to BC</u>.

$$\implies \triangle BQC = \frac{1}{2} \times arABCD. \tag{1}$$

Considering $\triangle APB$ and parallelogram $ABCD$, having <u>same base AB</u> and,<u>DC is parallel to AB</u>.

$$\implies \triangle APB = \frac{1}{2} \times arABCD. \tag{2}$$

By comparing equation (1) and equation (2) we obtain that,

$$ar\left(\triangle APB\right) = ar\left(\triangle BQC\right). \tag{3}$$

Hence proved.

Proof by the help of diagram :$(figure\ 1, table\ 1, table\ 2)$
Let the points $\boldsymbol{Q}$ and $\boldsymbol{P}$ divide AD and CD by $k_1 : 1$ and $k_2 : 1$ ratio respectively.

Input Parameters	**Description**	**Value**
$\boldsymbol{A}$	Vertex(at origin)	$\mathbf{0}$
a	Side of the parallelogram	$AB\,(= DC = 6unit)$
b	Side of the parallelogram	$AD\left(= BC = \sqrt{29}unit\right)$
θ	Angle of parallelogram	$\angle BAD\left(= \sin^{-1}\left(\frac{5}{\sqrt{29}}\right)\right)$
$k_1 : 1$	Ratio by which $\boldsymbol{Q}$ divides AD	$AQ : QD$
$k_2 : 1$	Ratio by which $\boldsymbol{P}$ divides DC	$DP : PC$

Table 1: Table of input parameters

Output Parameters	**Description**	**Value**
$\boldsymbol{B}$	Vertex of parallelogram	$a\begin{pmatrix}1\\0\end{pmatrix}$
$\boldsymbol{D}$	Vertex of parallelogram	$b\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$
$\boldsymbol{C}$	Vertex of parallelogram	$\boldsymbol{B} + \boldsymbol{D}$
$\boldsymbol{Q}$	Vertex of $\triangle BQC$	$\frac{k_1.\boldsymbol{D}+\boldsymbol{A}}{k_1+1}$
$\boldsymbol{P}$	Vertex of $\triangle APB$	$\frac{k_2.\boldsymbol{C}+\boldsymbol{D}}{k_2+1}$

Table 2: Table of output parameters

For the $\triangle BQC$, the vertices of the triangle are $\boldsymbol{B} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \boldsymbol{Q} = \begin{pmatrix} \frac{2k_1}{k_1+1} \\ \frac{5k_1}{k_1+1} \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

$$\implies \triangle BQC = \begin{vmatrix} 1 & 1 & 1 \\ 6 & \frac{2k_1}{k_1+1} & 8 \\ 0 & \frac{5k_1}{k_1+1} & 5 \end{vmatrix}$$
$= \frac{1}{2} \times \left|1\left(\frac{2k_1}{k_1+1}.5 - \frac{5k_1}{k_1+1}.8\right) - 6\left(5.1 - \frac{5k_1}{k_1+1}.1\right) + 0\right|$
$= \frac{1}{2} \times \left|\frac{-40k_1+10k_1-30}{k_1+1}\right|$
$= \frac{1}{2} \times 30$
$= 15$square units.

For the $\triangle APB$, the vertices of the triangle are
$\boldsymbol{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{P} = \begin{pmatrix} \frac{8k_2+2}{k_2+1} \\ 5 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$$\implies \triangle APB = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \frac{8k_2+2}{k_2+1} & 6 \\ 0 & 5 & 0 \end{vmatrix}$$
$= \frac{1}{2} \times \left|1\left(\frac{8k_2+2}{k_2+1}.0 - 6.5\right) + 0 + 0\right|$
$= \frac{1}{2} \times \left| - 30\right|$
$= \frac{1}{2} \times 30$
$= 15$ square units.

For parallelogram $ABCD$,the vertices are
$\boldsymbol{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \boldsymbol{D} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$\implies arABCD = ab\sin\theta$
$=6\sqrt{29}\sin\left(\sin^{-1}\frac{5}{\sqrt{29}}\right)$
$=6\sqrt{29}\frac{5}{\sqrt{29}}$
$=30$ square units.

Therefore,$arABCD = 2 \times ar\triangle APB = 2 \times ar\triangle BQC(proved)$.