VECTOR

Question: $\boldsymbol{P}$ and $\boldsymbol{Q}$ are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$. Show that, $ar(\triangle APB) = ar(\triangle BQC)$.
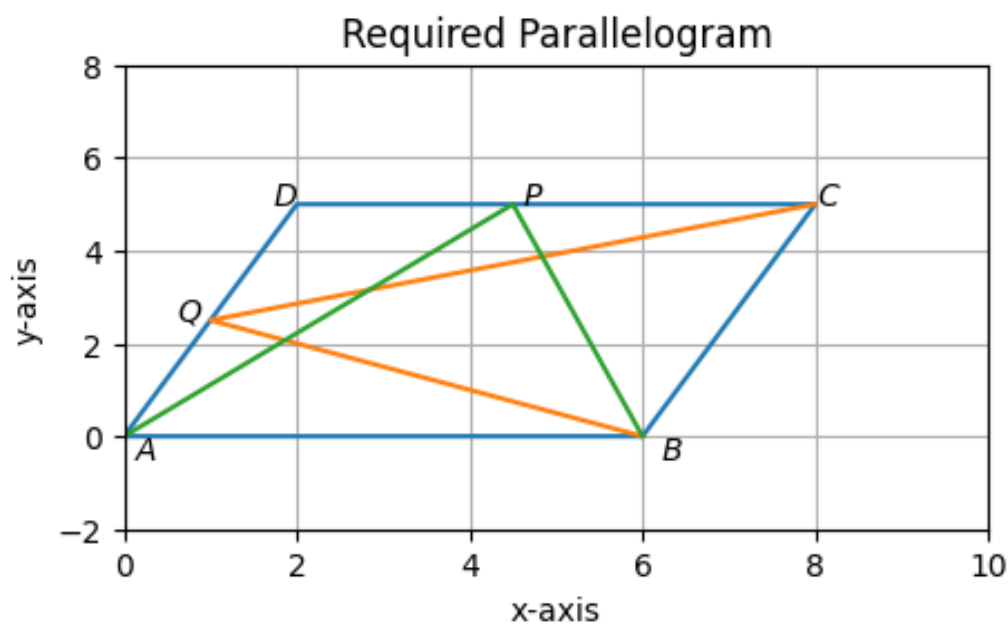
Figure:

Figure 1

Solution:

Using formulas:

1. Area of a parallelogram with sides $\boldsymbol{a}$ and $\boldsymbol{b}$ is $= \frac{1}{2}\left|\boldsymbol{a} \times \boldsymbol{b}\right|$ square units
   $= \frac{1}{2}\left|a.b\sin\theta\right|$

2. Area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is $= \frac{1}{2} \times \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$ square units . (only positive values will be taken as negative value of area is not possible)

3. If a triangle and a parallelogram are on the same base and between the same parallels, then area of the triangle is equal to half the area of the parallelogram.

From figure 1 , considering $\triangle BQC$ and parallelogram $ABCD$, having <u>same base BC</u> and, <u>AD is parallel to BC</u>.

$$\implies \triangle BQC = \frac{1}{2} \times ar\,ABCD. \tag{1}$$

Considering $\triangle APB$ and parallelogram $ABCD$, having <u>same base AB</u> and, <u>DC is parallel to AB</u>.

$$\implies \triangle APB = \frac{1}{2} \times ar\,ABCD. \tag{2}$$

By comparing equation (1) and equation (2) we obtain that,

$$ar(\triangle APB) = ar(\triangle BQC). \tag{3}$$

Hence proved.

Proof by the help of diagram :(figure 1,table 1,table 2)
Let the points $\boldsymbol{Q}$ and $\boldsymbol{P}$ divide AD and CD by $k_1 : 1$ and $k_2 : 1$ ratio respectively.

Symbols	Description	Value
$AB = DC$	Sides of parallelogram	$a = 6$(unit)
$AD = BC$	Sides of parallelogram	$b = \sqrt{29}$
θ	$\angle BAD$	$\sin^{-1}(\frac{5}{\sqrt{29}})$
$\boldsymbol{A}$	vertex (at origin)	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\boldsymbol{Q}$	Divides AD	$k_1 : 1$ ratio
$\boldsymbol{P}$	Divides CD	$k_2 : 1$ ratio

Table 1: Table of input parameters

Symbols	Description	Value
B	vertex	$a \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
D	vertex	$b \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
C	vertex	$B + D$
Q	Vertex($\triangle BQC$)	$\begin{pmatrix} \frac{2k_1}{k_1+1} \\ \frac{5k_1}{k_1+1} \end{pmatrix}$
P	Vertex($\triangle APB$)	$\begin{pmatrix} \frac{8k_2+2}{k_2+1} \\ 5 \end{pmatrix}$

Table 2: Table of output parameters

For the $\triangle BQC$, the vertices of the triangle are

$$B = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, Q = \begin{pmatrix} \frac{2k_1}{k_1+1} \\ \frac{5k_1}{k_1+1} \end{pmatrix}, C = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

$$\implies \triangle BQC = \begin{vmatrix} 1 & 1 & 1 \\ 6 & \frac{2k_1}{k_1+1} & 8 \\ 0 & \frac{5k_1}{k_1+1} & 5 \end{vmatrix}$$

$\frac{1}{2} \times \left| 1(\frac{2k_1}{k_1+1}.5 - \frac{5k_1}{k_1+1}.8) - 6(5.1 - \frac{5k_1}{k_1+1}.1) + 0 \right|$

$= \frac{1}{2} \times \left| \frac{-40k_1+10k_1-30}{k_1+1} \right|$

$= \frac{1}{2} \times 30$

$= 15$ square units.

For the $\triangle APB$, the vertices of the triangle are

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, P = \begin{pmatrix} \frac{8k_2+2}{k_2+1} \\ 5 \end{pmatrix}, B = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$\implies \triangle APB = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \frac{8k_2+2}{k_2+1} & 6 \\ 0 & 5 & 0 \end{vmatrix}$$

$= \frac{1}{2} \times \left| 1(\frac{8k_2+2}{k_2+1}.0 - 6.5) + 0 + 0 \right|$

$= \frac{1}{2} \times | -30 |$

$= \frac{1}{2} \times 30$

$= 15$ square units.

For parallelogram $ABCD$,the vertices are

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 8 \\ 5 \end{pmatrix}, D = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\implies ar\,ABCD = a.b.\sin\theta$$
$$=6.\sqrt{29}.\sin(\sin^{-1}\tfrac{5}{\sqrt{29}})$$
$$=6.\sqrt{29}.\tfrac{5}{\sqrt{29}}$$
$=30$ square units.

Therefore,$ar\,ABCD = 2 \times ar\triangle APB = 2 \times ar\triangle BQC$(proved).