Mean-Reverting Portfolio Design with Budget Constraint

Ziping Zhao, *Student Member, IEEE*, and Daniel P. Palomar, *Fellow, IEEE*

Abstract—**This paper considers the mean-reverting portfolio design problem arising from statistical arbitrage in the financial markets. We first propose a general problem formulation aimed at finding a portfolio of underlying component assets by optimizing a mean-reversion criterion characterizing the mean-reversion strength, taking into consideration the variance of the portfolio and an investment budget constraint. Then several specific problems are considered based on the general formulation, and efficient algorithms are proposed. Numerical results on both synthetic and market data show that our proposed mean-reverting portfolio design methods can generate consistent profits and outperform the traditional design methods and the benchmark methods in the literature.**

Index Terms—**Portfolio optimization, mean-reversion, cointegration, pairs trading, statistical arbitrage, algorithmic trading, quantitative trading.**

I. INTRODUCTION

PAIRS trading [2]–[6] is a well-known trading strategy that was pioneered by scientists Gerry Bamberger and David Shaw, and the quantitative trading group led by Nunzio Tartaglia at Morgan Stanley in the mid 1980s. As indicated by the name, it is an investment strategy that focuses on a pair of assets at the same time. Investors or arbitrageurs embracing this strategy do not need to forecast the absolute price of every single asset in one pair, which by nature is hard to assess, but only the relative price of this pair. As a contrarian investment strategy, in order to arbitrage from the market, investors need to buy the under-priced asset and short-sell the over-priced one. Then profits are locked in after trading positions are unwound when the relative mispricing corrects itself in the future.

More generally, pairs trading with only two trading assets falls into the umbrella of statistical arbitrage [7], [8], where the underlying trading basket in general consists of three or more assets. Since profits from such arbitrage strategies do not depend on the movements and conditions of the general financial markets, statistical arbitrage is referred to as a kind of market neutral strategies [9], [10]. Nowadays, statistical arbitrage is widely used by institutional investors, hedge fund companies, and many individual investors in the financial markets.

In [11], [12], the authors first came up with the concept of cointegration to describe the linear stationary and hence mean-reverting relationship of underlying nonstationary time series which are named to be cointegrated. Later, the cointegrated vector autoregressive model [13]–[17] was proposed to describe the cointegration relations. Empirical and technical analyses [18]–[21] show that cointegration can be used to get statistical arbitrage opportunities and such relations really exist in financial markets. Taking the prices of common stocks for example, it is generally known that a stock price is observed and modeled as a nonstationary random walk process that can be hard to predict efficiently. However, companies in the same financial sector or industry usually share similar fundamental characteristics, then their stock prices may move in company with each other under the same trend, based on which cointegration relations can be established. Two examples are the stock prices of the two American famous consumer staple companies Coca-Cola and PepsiCo and those of the two energy companies Ensco and Noble Corporation. Some examples for other financial assets, to name a few, are the future contract prices of E-mini S&P 500 and E-mini Dow, the ETF prices of SPDR S&P 500 and SPDR DJIA, the US dollar foreign exchange rates for different countries, and the swap rates for US interest rates of different maturities.

Mean-reversion is a classic indicator of predictability in financial markets and used to obtain arbitrage opportunities. Assets in a cointegration relation can be used to form a portfolio or basket and traded based upon their stationary mean-reversion property. We call such a designed portfolio or basket of underlying assets a mean-reverting portfolio (MRP) or sometimes a long-short portfolio which is also called a "spread". An asset whose price shows naturally stationarity is a spread as well. The profits of statistical arbitrage come directly from trading on the mean-reversion of the spread around the long-run equilibrium. MRPs in practice are usually constructed using heuristic or statistical methods. Traditional statistical cointegration estimation methods are Engle-Granger ordinary least squares (OLS) method [12] and Johansen model-based method [14]. In practice, inherent correlations may exist among different MRPs. However, when having multiple MRPs, they are commonly traded separately with their possible connections neglected. So a natural and interesting question is whether we can design an optimized MRP based on the underlying spreads that could outperform every single one. In this paper, this issue is clearly addressed.

Designing one MRP by choosing proportions of various assets in general is a portfolio optimization or asset allocation problem [22]. Portfolio optimization today is considered to be an important part in portfolio management as well as in algorithmic trading. The seminal paper [23] by Markowitz in 1952 laid on the foundations of what is now popularly re-

The authors are with the Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology (HKUST), Clear Water Bay, Kowloon, Hong Kong (e-mail: ziping.zhao@connect.ust.hk; palomar@ust.hk).

ferred to as mean-variance portfolio optimization and modern portfolio theory. Given a collection of financial assets, the traditional mean-variance portfolio design problem is aimed at finding a tradeoff between the expected return and the risk measured by the variance. Different from the requirements for mean-variance portfolio design, in order to design a mean-reverting portfolio, there are two main factors to consider: i) the designed MRP should exhibit a strong mean-reversion indicating that it should have frequent mean-crossing points and hence bring in trading opportunities, and ii) the designed MRP should exhibit sufficient but controlled variance so that each trade can provide enough profit while controlling the probability that the believed mean-reversion equilibrium breaks down could be reduced.

In [24], the author first proposed to design an MRP by optimizing a criterion characterizing the mean-reversion strength which is a model-free method. Later, authors in [25] realized that solving the problem in [24] could result in a portfolio with very low variance, then the variance control was taken into consideration and also new criteria to characterize the mean-reversion property were proposed for the MRP design problem. In [24], [25], semidefinite programming (SDP) relaxation methods were used to solve the nonconvex problem formulations; however, these methods are very computationally costly in general. Besides that, the design methods in [24], [25] were all carried out by imposing an ℓ_2-norm constraint on the portfolio weights. This constraint brings mathematically convenience to the optimization problem, but its practical significance in financial applications is dubious since the ℓ_2-norm is not meaningful in a financial context. In this paper, we propose to use investment budget constraints in the design problems.

The contributions of this paper can be summarized as follows. First, a general problem formulation for MRP design problem is proposed based on which several specific problem formulations are elaborated by considering different mean-reversion criteria. Second, Two classes of commonly used investment budget constraints on portfolio weights are considered, namely, dollar neutral constraint and net budget constraint. Third, efficient algorithms are proposed for the proposed problem formulations, it is shown that some problems after reformulations can be tackled readily by solving the well-known generalized eigenvalue problem (GEVP) and the generalized trust region subproblem (GTRS). The other problems can be easily solved based on the majorization-minimization (MM) framework by solving a sequence of GEVPs and GTRSs, which are named iteratively reweighted generalized eigenvalue problem (IRGEVP) and iteratively reweighted generalized trust region subproblem (IRGTRS), respectively. An extension for IRGEVP with closed-form solution in every iteration named EIRGEVP (extended IRGEVP) is also proposed.

The remaining sections of this paper are organized as follows. In Section II, we introduce the design of mean-reverting portfolios. In Section III, we give out some mean-reversion criteria for an MRP, and a general formulation for the MRP design problem is proposed together with two commonly used investment budget constraint. Section IV develops the solving methods for GEVP and GTRS. The MM framework and MM-based solving algorithms are elaborated in Section V. The performance of the proposed algorithms are evaluated numerically in Section VI and, finally, the concluding remarks are drawn in Section VII.

Notation: Boldface upper case letters denote matrices, boldface lower case letters denote column vectors, and italics denote scalars. The notation $\mathbf{1}$ and $\mathbf{I}$ denote an all-one column vector and an identity matrix with proper size, respectively. $\mathbb{R}$ denotes the real field with $\mathbb{R}^+$ denoting positive real numbers and $\mathbb{R}^N$ denoting the N-dimensional real vector space. $\mathbb{N}$ denotes the natural field. $\mathbb{Z}$ denotes the integer circle with $\mathbb{Z}^+$ denoting positive integer numbers. $\mathbb{S}^K$ denotes the $K \times K$-dimensional symmetric matrices. The superscripts $(\cdot)^T$ and $(\cdot)^{-1}$ denote the matrix transpose and inverse operator, respectively. Due to the commutation of the inverse and the transpose for nonsingular matrices, the superscript $(\cdot)^{-T}$ denotes the matrix inverse and transpose operator. $x_{i,j}$ denotes the (ith, jth) element of matrix $\mathbf{X}$ and x_i denotes the ith element of vector $\mathbf{x}$. $\text{Tr}(\cdot)$ denotes the trace of a matrix. $\text{vec}(\cdot)$ denotes the vectorization of a matrix, i.e., $\text{vec}(\mathbf{X})$ is a column vector consisting of all the columns of $\mathbf{X}$ stacked. $\otimes$ denotes the Kronecker product of two matrices.

II. MEAN-REVERTING PORTFOLIO (MRP)

For a financial asset, e.g., a common stock, a future contract, an ETF, or a portfolio of them, its price at time index or holding period $t \in \mathbb{Z}^+$ is denoted by $p_t \in \mathbb{R}^+$, and the corresponding logarithmic price or log-price $y_t \in \mathbb{R}$ is computed as $y_t = \log(p_t)$, where $\log(\cdot)$ is the natural logarithm.

If we consider a collection of M assets in a basket, their log-prices can be accordingly denoted by $\mathbf{y}_t = [y_{1,t}, y_{2,t}, \ldots, y_{M,t}]^T \in \mathbb{R}^M$. Based on this basket, an MRP is accordingly defined by the portfolio weight or hedge ratio $\mathbf{w}_s = [w_{s,1}, w_{s,2}, \ldots, w_{s,M}]^T \in \mathbb{R}^M$ and its (log-price) spread s_t is defined as $s_t = \mathbf{w}_s^T \mathbf{y}_t = \sum_{m=1}^{M} w_{s,m} y_{m,t}$. Vector $\mathbf{w}_s$ indicates the market value proportion invested on the underlying asset[1]. For $m = 1, 2, \ldots, M$, $w_{s,m} > 0$, $w_{s,m} < 0$, and $w_{s,m} = 0$ mean a long position (i.e., the asset is bought), a short position (i.e., the asset is short-sold, or, more plainly, borrowed and sold), and no position, respectively.

In Figure 1, the spread of a designed MRP together with the log-prices of the two underlying assets is given. It is worth noting that an MRP can be interpreted as a synthesized stationary asset. The spread accordingly means its log-price which could be easier to predict and to make profits from in comparison with the underlying component assets in this MRP.

Suppose there exist N MRPs with their spreads denoted by $\mathbf{s}_t = [s_{1,t}, s_{2,t}, \ldots, s_{N,t}]^T \in \mathbb{R}^N$. Different spreads may possess different mean-reversion and variance properties in nature. Our objective is to design an MRP to combine such spreads into an improved overall spread with better properties. In particular, we denote the portfolio by $\mathbf{w} = [w_1, w_2, \ldots, w_N]^T \in \mathbb{R}^N$, where $\mathbf{w}$ denotes the market value

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[1]If the spread is designed based on asset price p_t instead of the log-price, $\mathbf{w}_s$ indicates the asset amount proportion measured in shares.
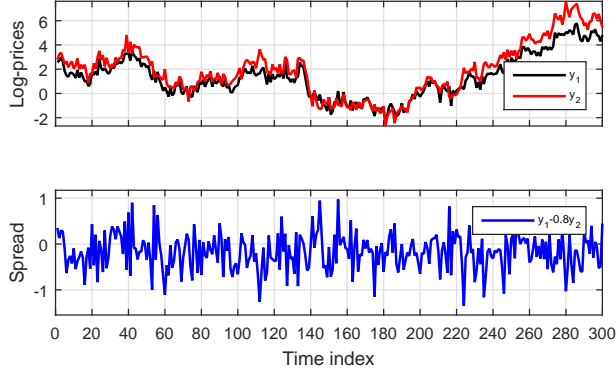
Fig. 1. An illustrative example of log-prices of two assets and a designed spread.

on the underlying spread. The resulting overall spread is then given by

$$z_t = \mathbf{w}^T \mathbf{s}_t = \sum_{n=1}^{N} w_n s_{n,t}. \tag{1}$$

III. MRP DESIGN PROBLEM FORMULATIONS

Traditional portfolio design problems are based on the Nobel prize-winning Markowitz portfolio theory [23], [26]–[28]. They aim at finding a desired trade-off between return and risk, the latter being measured traditionally by the variance or, in a more sophisticated way, by value-at-risk and conditional value-at-risk. The recently proposed risk-parity portfolios [29]–[31] can also be categorized into this design problem.

For the mean-reverting portfolio, we can formulate the design problem by optimizing some mean-reversion criterion quantifying the mean-reversion strength of the spread z_t, while controlling its variance and imposing an investment budget constraint.

A. Mean-Reversion Criteria

In this section, we introduce several mean-reversion criteria that can characterize the mean-reversion strength of the designed spread z_t. We start by defining the ith order (lag-i) autocovariance matrix for a stochastic process $\mathbf{s}_t$ as

$$
\begin{aligned}
\mathbf{M}_i &= \mathsf{Cov}\left(\mathbf{s}_t, \mathbf{s}_{t+i}\right) \\
&= \mathsf{E}\left[\left(\mathbf{s}_t - \mathsf{E}\left[\mathbf{s}_t\right]\right)\left(\mathbf{s}_{t+i} - \mathsf{E}\left[\mathbf{s}_{t+i}\right]\right)^T\right],
\end{aligned} \tag{2}
$$

where $i \in \mathbb{N}$. Specifically, when $i = 0$, $\mathbf{M}_0$ stands for the (positive definite) covariance matrix of $\mathbf{y}_t$.

Since for any random process $\mathbf{s}_t$, we can get its centered counterpart $\tilde{\mathbf{s}}_t$ as $\tilde{\mathbf{s}}_t = \mathbf{s}_t - \mathsf{E}\left[\mathbf{s}_t\right]$, in the following, we will use $\mathbf{s}_t$ to denote its centered form $\tilde{\mathbf{s}}_t$.

1) Predictability Statistics $\mathrm{pre}_z(\mathbf{w})$*:* Consider a centered univariate stationary autoregressive process written as follows:

$$z_t = \hat{z}_{t-1} + \epsilon_t, \tag{3}$$

where $\hat{z}_{t-1}$ is the prediction of z_t based on the information up to time $t - 1$, and ϵ_t denotes a white noise independent from $\hat{z}_{t-1}$. The predictability statistics [32] is defined as

$$\mathrm{pre}_z = \frac{\sigma_{\hat{z}}^2}{\sigma_z^2}, \tag{4}$$

where $\sigma_z^2 = \mathsf{E}\left[z_t^2\right]$ and $\sigma_{\hat{z}}^2 = \mathsf{E}\left[\hat{z}_{t-1}^2\right]$. If we define $\sigma_\epsilon^2 = \mathsf{E}\left[\epsilon_t^2\right]$, then from (4), we can have $\sigma_z^2 = \sigma_{\hat{z}}^2 + \sigma_\epsilon^2$ in the denominator. When pre_z is small, the variance of ϵ_t dominates that of $\hat{z}_{t-1}$, and z_t behaves like a white noise; when pre_z is large, the variance of $\hat{z}_{t-1}$ dominates that of ϵ_t, and z_t can be well predicted by $\hat{z}_{t-1}$. The predictability statistics is usually used to measure how close a random process is to a white noise.

Under this criterion, in order to design a spread z_t as close as possible to a white noise process, we need to minimize pre_z. For a spread $z_t = \mathbf{w}^T \mathbf{s}_t$, we assume the spread $\mathbf{s}_t$ follows a centered vector autoregressive model of order 1 (VAR(1)) as follows:

$$\mathbf{s}_t = \mathbf{A}\mathbf{s}_{t-1} + \mathbf{e}_t, \tag{5}$$

where $\mathbf{A}$ is the autoregressive coefficient and $\mathbf{e}_t$ denotes a white noise independent from $\mathbf{s}_{t-1}$. We can get $\mathbf{A}$ from the autocorrelation matrices as $\mathbf{A} = \mathbf{M}_1 \mathbf{M}_0^{-1}$. Multiplying (5) by $\mathbf{w}$ and further defining $\hat{z}_{t-1} = \mathbf{w}^T \mathbf{A}\mathbf{s}_{t-1}$ and $\epsilon_t = \mathbf{w}^T \mathbf{e}_t$, we can get $\sigma_z^2 = \mathbf{w}^T \mathbf{M}_0 \mathbf{w}$, and $\sigma_{\hat{z}}^2 = \mathbf{w}^T \mathbf{T}\mathbf{w}$, where $\mathbf{T} = \mathbf{A}\mathbf{M}_0\mathbf{A}^T = \mathbf{M}_1 \mathbf{M}_0^{-1} \mathbf{M}_1^T$. High order models VAR($p$), with $p > 1$, can be trivially reformulated into VAR(1) with proper reparametrization [33]. Then the estimator of predictability statistics for z_t is computed as

$$\mathrm{pre}_z(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{T}\mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}}. \tag{6}$$

2) Portmanteau Statistics $\mathrm{por}_z(p, \mathbf{w})$*:* The portmanteau statistics of order p [34] for a centered univariate stationary process z_t is defined as

$$\mathrm{por}_z(p) = T \sum_{i=1}^{p} \rho_i^2, \tag{7}$$

where ρ_i is the ith order autocorrelation (autocorrelation for lag i) of z_t defined as $\rho_i = \frac{\mathsf{E}[z_t z_{t+i}]}{\mathsf{E}[z_t^2]}$. The portmanteau statistics is used to test whether a random process is close to a white noise. From the above definition, we have $\mathrm{por}_z(p) \geq 0$ and the minimum of $\mathrm{por}_z(p)$ is attained by a white noise process, i.e., the portmanteau statistics for a white noise process is 0 for any p.

Under this criterion, in order to get a spread z_t close to a white noise process, we need to minimize $\mathrm{por}_z(p)$ for a prespecified order p. For an MRP $z_t = \mathbf{w}^T \mathbf{s}_t$, the ith order autocorrelation is given by

$$\rho_i = \frac{\mathsf{E}\left[z_t z_{t+i}\right]}{\mathsf{E}\left[z_t^2\right]} = \frac{\mathbf{w}^T \mathsf{E}\left[\mathbf{s}_t \mathbf{s}_{t+i}^T\right]\mathbf{w}}{\mathbf{w}^T \mathsf{E}\left[\mathbf{s}_t \mathbf{s}_t^T\right]\mathbf{w}} = \frac{\mathbf{w}^T \mathbf{M}_i \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}}. \tag{8}$$

Then we can get the expression for $\mathrm{por}_z(p, \mathbf{w})$ as

$$\mathrm{por}_z(p, \mathbf{w}) = T \sum_{i=1}^{p} \left(\frac{\mathbf{w}^T \mathbf{M}_i \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}}\right)^2. \tag{9}$$

3) Crossing Statistics $\mathrm{cro}_z(\mathbf{w})$ *and Penalized Crossing Statistics* $\mathrm{pcro}_z(p, \mathbf{w})$: Crossing statistics (zero-crossing rate) of a centered univariate stationary process z_t is defined as

$$\mathrm{cro}_z = \frac{1}{T-1} \mathsf{E} \left[\sum_{t=2}^{T} \mathbf{1}_{\{z_t z_{t-1} \leq 0\}} \right], \tag{10}$$

where $\mathbf{1}_E(z_t)$ is the indicator function defined as $\mathbf{1}_E(z_t) = \begin{cases} 1, & \text{if } z_t \in E \\ 0, & \text{if } z_t \notin E \end{cases}$, and the event here is $E = \{z_t z_{t-1} \leq 0\}$. Crossing statistics is used to test the probability that a stationary process crosses its mean per unit of time and it is easy to notice that $\mathrm{cro}_z \in [0, 1]$. According to [35], [36], for a centered stationary Gaussian process z_t, we have the following relationship:

$$\mathrm{cro}_z = \frac{1}{\pi} \arccos(\rho_1). \tag{11}$$

Remark 1. As a special case, if z_t is a centered stationary AR(1),

$$z_t = \phi z_{t-1} + \epsilon_t, \tag{12}$$

where $|\phi| < 1$ and ϵ_t is a Gaussian white noise, then $\phi = \rho_1$ and accordingly the crossing statistics is $\mathrm{cro}_z = \frac{1}{\pi} \arccos(\phi)$.

Using this criterion, in order to get a spread z_t having many zero-crossings, instead of directly maximizing cro_z, we can minimize ρ_1. For a spread $z_t = \mathbf{w}^T \mathbf{s}_t$, we can try to minimize the first order autocorrelation of z_t given in (8). In [25], besides minimizing the first order autocorrelation, it is also proposed to ensure the absolute autocorrelations of high orders $|\rho_i|$s $(i = 2, \ldots, p)$ are small at the same time which can result in good performance. In this paper, we also adopt this criterion and call it penalized crossing statistics of order p defined by

$$\mathrm{pcro}_z(p, \mathbf{w}) = \frac{\mathbf{w}^T \mathbf{M}_1 \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}} + \eta \sum_{i=2}^{p} \left(\frac{\mathbf{w}^T \mathbf{M}_i \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}} \right)^2, \tag{13}$$

where η is a positive prespecified penalization factor.

B. General MRP Design Problem Formulation

The MRP design problem is formulated as the optimization of a mean-reversion criterion denoted in general as $F_z(\mathbf{w})$, which can be taken to be any of the criteria mentioned before. This unified criterion can be written into a compact form as

$$F_z(\mathbf{w}) = \xi \frac{\mathbf{w}^T \mathbf{H} \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}} + \zeta \left(\frac{\mathbf{w}^T \mathbf{M}_1 \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}} \right)^2 + \eta \sum_{i=2}^{p} \left(\frac{\mathbf{w}^T \mathbf{M}_i \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}} \right)^2, \tag{14}$$

which particularizes to i) $\mathrm{pre}_z(\mathbf{w})$, when $\xi = 1$, $\mathbf{H} = \mathbf{T}$, and $\zeta = \eta = 0$; ii) $\mathrm{por}_z(p, \mathbf{w})$, when $\xi = 0$, and $\zeta = \eta = 1$; iii) $\mathrm{cro}_z(\mathbf{w})$, when $\xi = 1$, $\mathbf{H} = \mathbf{M}_1$, and $\zeta = \eta = 0$; and iv) $\mathrm{pcro}_z(p, \mathbf{w})$, when $\xi = 1$, $\mathbf{H} = \mathbf{M}_1$, $\zeta = 0$, and $\eta > 0$. The matrices $\mathbf{M}_i$s in (14) are assumed symmetric without loss of generality since they can be symmetrized.

The variance of the spread should also be controlled to a certain level which can be represented as $\mathrm{Var}\left[\mathbf{w}^T \mathbf{s}_t\right] = \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu$. Due to this variance constraint, the denominators of $F_z(\mathbf{w})$ can be removed. Denoting the portfolio

investment budget constraint by $\mathcal{W}$, the general MRP design problem can be formulated as follows:

$$\begin{aligned} \underset{\mathbf{w}}{\text{mininize}} \quad & \xi \mathbf{w}^T \mathbf{H} \mathbf{w} + \zeta \left(\mathbf{w}^T \mathbf{M}_1 \mathbf{w} \right)^2 + \eta \sum_{i=2}^{p} \left(\mathbf{w}^T \mathbf{M}_i \mathbf{w} \right)^2 \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu \\ & \mathbf{w} \in \mathcal{W}, \end{aligned} \tag{15}$$

where the objective function is denoted by $f_z(\mathbf{w})$ in the following. The problem in (15) is a nonconvex problem due to the nonconvexity of the objective function and the constraint set.

C. Investment Budget Constraint $\mathcal{W}$

In portfolio optimization, constraints are usually imposed to represent the specific investment guidelines. In this paper, we use $\mathcal{W}$ to denote it and we focus on two types of budget constraints: dollar neutral constraint and net budget constraint.

Dollar neutral constraint, denoted by $\mathcal{W}_0$, means the net investment or net portfolio position is zero; in other words, all the long positions are financed by the short positions, commonly termed self-financing.[2] It is represented mathematically by

$$\mathcal{W}_0 = \left\{ \mathbf{1}^T \mathbf{w} = 0 \right\}. \tag{16}$$

Net budget constraint, denoted by $\mathcal{W}_1$, means the net investment or net portfolio position is nonzero and equal to the current budget which is normalized to one.[3] It is represented mathematically by

$$\mathcal{W}_1 = \left\{ \mathbf{1}^T \mathbf{w} = 1 \right\}. \tag{17}$$

It is worth noting that, for two trading spreads defined by $\mathbf{w}^T \mathbf{y}_t$ and $-\mathbf{w}^T \mathbf{y}_t$, they are naturally the same, because in statistical arbitrage the actual investment not only depends on $\mathbf{w}$, which defines a spread, but also on whether a long or short position is taken on this spread in the trading.

IV. PROBLEM SOLVING ALGORITHMS VIA GEVP AND GTRS ALGORITHMS

In this section, solving methods for the MRP design problem formulations using $\mathrm{pre}_z(\mathbf{w})$ and $\mathrm{cro}_z(\mathbf{w})$ (i.e., (15) with $\zeta = \eta = 0$) are introduced.

A. GEVP - Solving Algorithm for MRP Design Using $\mathrm{pre}_z(\mathbf{w})$ and $\mathrm{cro}_z(\mathbf{w})$ with $\mathbf{w} \in \mathcal{W}_0$

For notational simplicity, we denote the matrices $\mathbf{T}$ in $\mathrm{pre}_z(\mathbf{w})$ and $\mathbf{M}_1$ in $\mathrm{cro}_z(\mathbf{w})$ by matrix $\mathbf{H}$ in general and recast the problem as follows:

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^T \mathbf{H} \mathbf{w} \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu \\ & \mathbf{1}^T \mathbf{w} = 0, \end{aligned} \tag{18}$$

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[2]Dollar neutral constraint generally cannot be satisfied by the traditional design methods, like methods in [12] and [14], and the methods in [25].

[3]The net portfolio position can be positive or negative under net budget constraint. Since the problem formulation in (15) is invariant to the sign of $\mathbf{w}$, only the case that budget is normalized to positive 1 is considered.

where ν is a positive constant. The above problem is equivalent to the following nonconvex quadratically constrained quadratic programming (QCQP) [37] formulation:

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^T \mathbf{H} \mathbf{w} \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu \\ & \mathbf{w}^T \mathbf{E} \mathbf{w} = 0, \end{aligned} \quad (19)$$

where $\mathbf{E} = \mathbf{1}\mathbf{1}^T$. By using the matrix lifting technique, i.e., defining $\mathbf{W} = \mathbf{w}\mathbf{w}^T$, the above problem can be solved by the following convex SDP relaxation problem:

$$\begin{aligned} \underset{\mathbf{W}}{\text{minimize}} \quad & \mathrm{Tr}\,(\mathbf{H}\mathbf{W}) \\ \text{subject to} \quad & \mathrm{Tr}\,(\mathbf{M}_0 \mathbf{W}) = \nu \\ & \mathrm{Tr}\,(\mathbf{E}\mathbf{W}) = 0 \\ & \mathbf{W} \succeq \mathbf{0}. \end{aligned} \quad (20)$$

The following theorem gives a useful relationship between the number of variables and the number of equality constraints.

Theorem 2 ([38, Theorem 3.2]). *Given a separable SDP as follows:*

$$\begin{aligned} \underset{\mathbf{X}_1,\ldots,\mathbf{X}_L}{\text{minimize}} \quad & \sum_{l=1}^{L} \mathrm{Tr}\,(\mathbf{A}_l \mathbf{X}_l) \\ \text{subject to} \quad & \sum_{l=1}^{L} \mathrm{Tr}\,(\mathbf{B}_{ml}\mathbf{X}_l) = b_m,\ m = 1,\ldots,M \\ & \mathbf{X}_l \succeq \mathbf{0},\ l = 1,\ldots,L. \end{aligned} \quad (21)$$

Suppose that the separable SDP are strictly feasible. Then, the problem has always an optimal solution $(\mathbf{X}_1^\star,\ldots,\mathbf{X}_L^\star)$ such that

$$\sum_{l=1}^{L} \left[rank\,(\mathbf{X}_l^\star) \right]^2 \leq M.$$

Observe that if there is only one variable $\mathbf{X}$, that is to say, $L = 1$, we can get $\mathrm{rank}\,(\mathbf{X}^\star) \leq \sqrt{M}$. Further, if the number of constraints $M \leq 3$, a rank-1 solution can always be attainable.

Lemma 3. *The nonconvex problem in* (18) *or* (19) *has no duality gap.*

Proof: This lemma directly follows from Theorem 2 and the equivalence of problems (18) and (19). ∎

In other words, by solving the convex SDP in (20), there always exists a rank-1 solution for $\mathbf{W}$ which is the solution for the original problem (18), however, in practice, to find such a solution, rank reduction methods [39] should be applied which could be computationally expensive.

As an alternative to the SDP procedure mentioned above, we find the problem in (18) can be efficiently solved as a generalized eigenvalue problem (GEVP) [40] by reformulation. Considering $\mathbf{w} = \mathbf{F}\mathbf{x}$, where $\mathbf{F}$ is the kernel that spans the null space of $\mathbf{1}^T$, i.e., $\mathbf{1}^T \mathbf{F} = \mathbf{0}$, and also required to be semi-unitary, i.e., $\mathbf{F}^T \mathbf{F} = \mathbf{I}$, we can define $\mathbf{N} = \mathbf{F}^T \mathbf{H} \mathbf{F}$ and $\mathbf{N}_0 = \mathbf{F}^T \mathbf{M}_0 \mathbf{F}$ which is positive definite, then the problem (18) is equivalent to the following problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^T \mathbf{N} \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}^T \mathbf{N}_0 \mathbf{x} = \nu, \end{aligned} \quad (22)$$

which is still a nonconvex QCQP. However, this problem becomes the classical GEVP problem and can be easily dealt with using tailored algorithms. Here, we will apply the steepest descent algorithm [41] to solve it. The procedure to solve problem (18) is summarized in Algorithm 1.

Algorithm 1 GEVP - Algorithm for MRP design problems using $\mathrm{pre}_z\,(\mathbf{w})$ and $\mathrm{cro}_z\,(\mathbf{w})$ with $\mathbf{w} \in \mathcal{W}_0$.

Require: $\mathbf{N}$, $\mathbf{N}_0$, and $\nu > 0$.
1: Set $k = 0$, and choose $\mathbf{x}^{(k)} \in \left\{ \mathbf{x} : \mathbf{x}^T \mathbf{N}_0 \mathbf{x} = \nu \right\}$;
2: **repeat**
3:     Compute $R\left(\mathbf{x}^{(k)}\right) = \mathbf{x}^{(k)T} \mathbf{N} \mathbf{x}^{(k)} / \mathbf{x}^{(k)T} \mathbf{N}_0 \mathbf{x}^{(k)}$;
4:     Compute $\mathbf{d}^{(k)} = \mathbf{N}\mathbf{x}^{(k)} - R\left(\mathbf{x}^{(k)}\right) \mathbf{N}_0 \mathbf{x}^{(k)}$;
5:     $\mathbf{x} = \mathbf{x}^{(k)} + \tau \mathbf{d}^{(k)}$ with τ chosen to minimize $R\left(\mathbf{x}^{(k)} + \tau \mathbf{d}^{(k)}\right)$;
6:     $\mathbf{x}^{(k+1)} = \sqrt{\nu}\mathbf{x}/\sqrt{\mathbf{x}^T \mathbf{N}_0 \mathbf{x}}$;
7:     $k = k + 1$;
8: **until** convergence

B. GTRS - Solving Algorithm for MRP Design Using $\mathrm{pre}_z\,(\mathbf{w})$ and $\mathrm{cro}_z\,(\mathbf{w})$ with $\mathbf{w} \in \mathcal{W}_1$

As before, for generality, we denote matrices $\mathbf{T}$ in $\mathrm{pre}_z\,(\mathbf{w})$ and $\mathbf{M}_1$ in $\mathrm{cro}_z\,(\mathbf{w})$ as $\mathbf{H}$. Then the problems can be rewritten as

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^T \mathbf{H} \mathbf{w} \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu \\ & \mathbf{1}^T \mathbf{w} = 1, \end{aligned} \quad (23)$$

where ν is a positive constant. As before, rewriting the constraint $\mathbf{1}^T \mathbf{w} = 1$ as $\mathbf{w}^T \mathbf{E} \mathbf{w} = 1$ (since the problem is invariant with respect to a sign change in $\mathbf{w}$) and using the matrix lifting technique, the problem in (23) can be solved by the following convex SDP problem:

$$\begin{aligned} \underset{\mathbf{W}}{\text{minimize}} \quad & \mathrm{Tr}\,(\mathbf{H}\mathbf{W}) \\ \text{subject to} \quad & \mathrm{Tr}\,(\mathbf{M}_0 \mathbf{W}) = \nu \\ & \mathrm{Tr}\,(\mathbf{E}\mathbf{W}) = 1 \\ & \mathbf{W} \succeq \mathbf{0}. \end{aligned} \quad (24)$$

Like before, the nonconvex problem in (23) has no duality gap. Besides the above SDP method, here we introduce an efficient solving approach by reformulating (23) into a generalized trust region subproblem (GTRS) [42]. Considering $\mathbf{w} = \mathbf{w}_0 + \mathbf{F}\mathbf{x}$ where $\mathbf{w}_0$ is any vector satisfying $\mathbf{1}^T \mathbf{w}_0 = 1$ and $\mathbf{F}$ is the kernel of $\mathbf{1}^T$ satisfying $\mathbf{1}^T \mathbf{F} = \mathbf{0}$ and a semi-unitary matrix satisfying $\mathbf{F}^T \mathbf{F} = \mathbf{I}$. Let us define $\mathbf{N} = \mathbf{F}^T \mathbf{H} \mathbf{F}$, $\mathbf{p} = \mathbf{F}^T \mathbf{H} \mathbf{w}_0$, $b = \mathbf{w}_0^T \mathbf{H} \mathbf{w}_0$, $\mathbf{N}_0 = \mathbf{F}^T \mathbf{M}_0 \mathbf{F}$ which is positive definite, $\mathbf{p}_0 = \mathbf{F}^T \mathbf{M}_0 \mathbf{w}_0$, and $b_0 = \mathbf{w}_0^T \mathbf{M}_0 \mathbf{w}_0$, then the problem in (23) is equivalent to the following problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^T \mathbf{N} \mathbf{x} + 2\mathbf{p}^T \mathbf{x} + b \\ \text{subject to} \quad & \mathbf{x}^T \mathbf{N}_0 \mathbf{x} + 2\mathbf{p}_0^T \mathbf{x} + b_0 = \nu, \end{aligned} \quad (25)$$

which is a nonconvex QCQP and QCQPs of this type are specially named GTRSs. Such problems are usually nonconvex but possess necessary and sufficient optimality conditions based on which efficient solving methods can be derived. We first introduce the following useful theorem.

Theorem 4 ([42, Theorem 3.2]). *Consider the following QCQP:*

$$\begin{aligned}\underset{\mathbf{x}}{\text{minimize}} \quad & q\left(\mathbf{x}\right) \triangleq \mathbf{x}^T \mathbf{A} \mathbf{x} + 2\mathbf{a}^T \mathbf{x} + a \\ \text{subject to} \quad & c\left(\mathbf{x}\right) \triangleq \mathbf{x}^T \mathbf{B} \mathbf{x} + 2\mathbf{b}^T \mathbf{x} + b = 0.\end{aligned} \quad (26)$$

Assume that the constraint set $c\left(\mathbf{x}\right)$ is nonempty and that $\nabla^2 c\left(\mathbf{x}\right) = 2\mathbf{B} \neq \mathbf{0}$. A vector $\mathbf{x}^\star$ is a global minimizer of the problem (26) together with a multiplier $\xi^\star$ if and only if the following conditions are satisfied:

$$\begin{cases} \nabla q\left(\mathbf{x}^\star\right) + \xi^\star \nabla c\left(\mathbf{x}^\star\right) = \mathbf{0} \\ c\left(\mathbf{x}^\star\right) = 0 \\ \nabla^2 q\left(\mathbf{x}^\star\right) + \xi^\star \nabla^2 c\left(\mathbf{x}^\star\right) \succeq \mathbf{0}, \end{cases}$$

and the interval set defined by

$$\mathcal{I} = \{\xi \mid \mathbf{A} + \xi \mathbf{B} \succ \mathbf{0}\}$$

is not empty.

According to Theorem 4, the optimality conditions for the primal and dual variables $(\mathbf{x}^\star, \xi^\star)$ of problem (25) are given as follows:

$$\begin{cases} \left(\mathbf{N} + \xi^\star \mathbf{N}_0\right) \mathbf{x}^\star + \mathbf{p} + \xi^\star \mathbf{p}_0 = 0 \\ \mathbf{x}^{\star T} \mathbf{N}_0 \mathbf{x}^\star + 2\mathbf{p}_0^T \mathbf{x}^\star + b_0 - \nu = 0 \\ \mathbf{N} + \xi^\star \mathbf{N}_0 \succeq \mathbf{0}. \end{cases} \quad (27)$$

We assume $\mathbf{N} + \xi \mathbf{N}_0 \succ \mathbf{0}^4$, then we can see that the optimal solution is given by

$$\mathbf{x}\left(\xi\right) = -\left(\mathbf{N} + \xi \mathbf{N}_0\right)^{-1} \left(\mathbf{p} + \xi \mathbf{p}_0\right), \quad (28)$$

and ξ is the unique solution of the following equation with definition on the interval $\mathcal{I}$:

$$\phi\left(\xi\right) = 0, \ \xi \in \mathcal{I}, \quad (29)$$

where the function $\phi\left(\xi\right)$ is defined by

$$\phi\left(\xi\right) = \mathbf{x}\left(\xi\right)^T \mathbf{N}_0 \mathbf{x}\left(\xi\right) + 2\mathbf{p}_0^T \mathbf{x}\left(\xi\right) + b_0 - \nu, \quad (30)$$

and the interval $\mathcal{I}$ consists of all ξ for which $\mathbf{N} + \xi \mathbf{N}_0 \succ \mathbf{0}$, which implies that

$$\mathcal{I} = \left(-\lambda_{\min}\left(\mathbf{N}, \mathbf{N}_0\right), \infty\right), \quad (31)$$

where $\lambda_{\min}\left(\mathbf{N}, \mathbf{N}_0\right)$ is the minimum generalized eigenvalue of matrix pair $(\mathbf{N}, \mathbf{N}_0)$.

Theorem 5 ([42, Theorem 5.2]). *Assume $\mathcal{I}$ is not empty, then the function $\phi\left(\xi\right)$ is strictly decreasing on $\mathcal{I}$ unless $\mathbf{x}\left(\xi\right)$ is constant on $\mathcal{I}$.*

In practice, the case $\mathbf{x}\left(\xi\right)$ is constant on $\mathcal{I}$ cannot happen. So from Theorem (5), we know when $\phi\left(\xi\right)$ is strictly decreasing on $\mathcal{I}$, then a simple line search algorithm like bisection algorithm can be used to find the optimal ξ over $\mathcal{I}$. The algorithm for problem (23) is summarized in Algorithm 2.

---

[4]The limiting case $\mathbf{N} + \xi \mathbf{N}_0$ being singular (i.e., $\xi = -\lambda_{\min}\left(\mathbf{N}, \mathbf{N}_0\right)$) can be treated separately. The assumption here is reasonable since the case when $\xi = -\lambda_{\min}\left(\mathbf{N}, \mathbf{N}_0\right)$ is very rare to occur theoretically and practically.

Algorithm 2 GTRS - Algorithm for MRP design problems using $\text{pre}_z\left(\mathbf{w}\right)$ and $\text{cro}_z\left(\mathbf{w}\right)$ with $\mathbf{w} \in \mathcal{W}_1$.

Require: $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{p}$, $\mathbf{p}_0$, b_0, $\lambda_{min}\left(\mathbf{N}, \mathbf{N}_0\right)$, and $\nu > 0$.
1: Set $k = 0$, and choose $\xi^{(k)} \in \left(-\left(\mathbf{N}, \mathbf{N}_0\right), \infty\right)$;
2: **repeat**
3:     Compute $\phi\left(\xi^{(k)}\right)$ according to (30);
4:     Update $\xi^{(k+1)}$ according to the value of $\phi\left(\xi^{(k)}\right)$ by a line search algorithm;
5:     $k = k + 1$;
6: **until** convergence
7: Compute $\mathbf{x}$ according to (28).

V. Problem Solving Algorithms via Majorization-Minimization Method

In this section, we first discuss the majorization-minimization or minorization-maximization (MM) method briefly, and then solving algorithms for the MRP design problem formulations using $\text{por}_z\left(p, \mathbf{w}\right)$ (i.e., (15) with $\xi = 0$ and $\zeta = \eta = 1$) and $\text{pcro}_z\left(p, \mathbf{w}\right)$ (i.e., (15) with $\xi = 1$, $\mathbf{H} = \mathbf{M}_1$, $\zeta = 0$ and $\eta > 0$) are derived based on the MM framework and the GEVP and GTRS algorithms mentioned in the previous section.

A. The MM Method

The MM method [43]–[45] refers to the majorization-minimization or minorization-maximization which is a generalization of the well-known expectation-maximization (EM) algorithm. The idea behind MM is that instead of dealing with the original optimization problem which could be difficult to tackle directly, it solves a series of simple surrogate subproblems.

Suppose the optimization problem is

$$\begin{aligned}\underset{\mathbf{x}}{\text{minimize}} \quad & f\left(\mathbf{x}\right) \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X},\end{aligned} \quad (32)$$

where the constraint set $\mathcal{X} \subseteq \mathbb{R}^N$. In general, there is no assumption about the convexity and differentiability on $f\left(\mathbf{x}\right)$. The MM method aims to solve this problem by optimizing a sequence of surrogate functions that majorize the objective function $f\left(\mathbf{x}\right)$ over the set $\mathcal{X}$. More specifically, starting from an initial feasible point $\mathbf{x}^{(0)}$, the algorithm produces a sequence $\{\mathbf{x}^{(k)}\}$ according to the following update rule:

$$\mathbf{x}^{(k+1)} \in \arg\min_{\mathbf{x} \in \mathcal{X}} u\left(\mathbf{x}, \mathbf{x}^{(k)}\right), \quad (33)$$

where $\mathbf{x}^{(k)}$ is the point generated by the update rule at the kth iteration and the surrogate function $u\left(\mathbf{x}, \mathbf{x}^{(k)}\right)$ is the corresponding majorizing function of $f\left(\mathbf{x}\right)$ at point $\mathbf{x}^{(k)}$. A surrogate function is called a majorizing function of $f\left(\mathbf{x}\right)$ at point $\mathbf{x}^{(k)}$ if it satisfies the following properties:

$$\begin{aligned} u\left(\mathbf{x}, \mathbf{x}^{(k)}\right) &\geq f\left(\mathbf{x}\right), \qquad \forall \mathbf{x} \in \mathcal{X}, \\ u\left(\mathbf{x}^{(k)}, \mathbf{x}^{(k)}\right) &= f\left(\mathbf{x}^{(k)}\right). \end{aligned} \quad (34)$$

That is to say, the surrogate function $u\left(\mathbf{x}, \mathbf{x}^{(k)}\right)$ should be an upper bound of the original function $f\left(\mathbf{x}\right)$ over $\mathcal{X}$ and coincide with $f\left(\mathbf{x}\right)$ at point $\mathbf{x}^{(k)}$. Although the definition of $u\left(\mathbf{x}, \mathbf{x}^{(k)}\right)$

gives us a great deal of flexibility for choosing it, in practice, the surrogate function $u\left(\mathbf{x}, \mathbf{x}^{(k)}\right)$ must be properly chosen so as to make the iterative update in (33) easy to compute while maintaining a fast convergence over the iterations.

The MM method iteratively runs until some convergence criterion is met. Under this MM method, the objective function value is decreased monotonically in every iteration, i.e.,

$$f\left(\mathbf{x}^{(k+1)}\right) \leq u\left(\mathbf{x}^{(k+1)}, \mathbf{x}^{(k)}\right) \leq u\left(\mathbf{x}^{(k)}, \mathbf{x}^{(k)}\right) = f\left(\mathbf{x}^{(k)}\right). \tag{35}$$

The first inequality and the third equality follow from the first and second properties of the majorizing function in (34) respectively and the second inequality follows from (33).

B. IRGEVP and IRGTRS - Solving Algorithms for MRP Design Using $\mathrm{por}_z\left(p, \mathbf{w}\right)$ and $\mathrm{pcro}_z\left(p, \mathbf{w}\right)$

We rewrite the problems using $\mathrm{por}_z\left(p, \mathbf{w}\right)$ and $\mathrm{pcro}_z\left(p, \mathbf{w}\right)$ in the general formulation as follows:

$$
\begin{aligned}
\underset{\mathbf{w}}{\text{minimize}} \quad & \xi \mathbf{w}^T \mathbf{M}_1 \mathbf{w} + \zeta \left(\mathbf{w}^T \mathbf{M}_1 \mathbf{w}\right)^2 \\
& + \eta \sum_{i=2}^p \left(\mathbf{w}^T \mathbf{M}_i \mathbf{w}\right)^2 \\
\text{subject to} \quad & \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu \\
& \mathbf{w} \in \mathcal{W},
\end{aligned}
\tag{36}
$$

where the specific portfolio weight constraints are implicitly replaced by $\mathcal{W}$.

To solve the problems in (36) via majorization-minimization, the key step is to find a majorizing function of the objective function such that the majorized subproblem is easy to solve. Observe that the objective function is quartic in $\mathbf{w}$. The following mathematical manipulations are necessary. We first compute the Cholesky decomposition of $\mathbf{M}_0$ which is $\mathbf{M}_0 = \mathbf{L}\mathbf{L}^T$, where $\mathbf{L}$ is a lower triangular with positive diagonal elements. Let us define $\bar{\mathbf{w}} = \mathbf{L}^T \mathbf{w}$, $\bar{\mathbf{M}}_i = \mathbf{L}^{-1}\mathbf{M}_i \mathbf{L}^{-T}$, and $\bar{\mathbf{W}} = \bar{\mathbf{w}}\bar{\mathbf{w}}^T$. The portfolio weight set $\mathcal{W}$ is mapped to $\bar{\mathcal{W}}$ under the linear transformation $\mathbf{L}$. Then problem (36) can be written as

$$
\begin{aligned}
\underset{\bar{\mathbf{w}}, \bar{\mathbf{W}}}{\text{minimize}} \quad & \xi \operatorname{Tr}\left(\bar{\mathbf{M}}_1 \bar{\mathbf{W}}\right) + \zeta \left(\operatorname{Tr}\left(\bar{\mathbf{M}}_1 \bar{\mathbf{W}}\right)\right)^2 \\
& + \eta \sum_{i=1}^p \left(\operatorname{Tr}\left(\bar{\mathbf{M}}_i \bar{\mathbf{W}}\right)\right)^2 \\
\text{subject to} \quad & \bar{\mathbf{W}} = \bar{\mathbf{w}}\bar{\mathbf{w}}^T \\
& \bar{\mathbf{w}}^T \bar{\mathbf{w}} = \nu \\
& \bar{\mathbf{w}} \in \bar{\mathcal{W}}.
\end{aligned}
\tag{37}
$$

Since $\operatorname{Tr}\left(\bar{\mathbf{M}}_i \bar{\mathbf{W}}\right) = \operatorname{vec}\left(\bar{\mathbf{M}}_i\right)^T \operatorname{vec}\left(\bar{\mathbf{W}}\right)$ (recall the $\mathbf{M}_i$s are assumed symmetric), problem (36) can be reformulated as

$$
\begin{aligned}
\underset{\bar{\mathbf{w}}, \bar{\mathbf{W}}}{\text{minimize}} \quad & \xi \operatorname{vec}\left(\bar{\mathbf{M}}_1\right)^T \operatorname{vec}\left(\bar{\mathbf{W}}\right) + \operatorname{vec}\left(\bar{\mathbf{W}}\right)^T \bar{\mathbf{M}} \operatorname{vec}\left(\bar{\mathbf{W}}\right) \\
\text{subject to} \quad & \bar{\mathbf{W}} = \bar{\mathbf{w}}\bar{\mathbf{w}}^T \\
& \bar{\mathbf{w}}^T \bar{\mathbf{w}} = \nu \\
& \bar{\mathbf{w}} \in \bar{\mathcal{W}},
\end{aligned}
\tag{38}
$$

where in the objective function

$$\bar{\mathbf{M}} = \zeta \operatorname{vec}\left(\bar{\mathbf{M}}_1\right) \operatorname{vec}\left(\bar{\mathbf{M}}_1\right)^T + \eta \sum_{i=2}^p \operatorname{vec}\left(\bar{\mathbf{M}}_i\right) \operatorname{vec}\left(\bar{\mathbf{M}}_i\right)^T. \tag{39}$$

Specifically, we can have the expressions for portmanteau statistics $\mathrm{por}_z\left(p, \mathbf{w}\right)$ (i.e., $\zeta = 1$ and $\eta = 1$) and penalized crossing statistics $\mathrm{pcro}_z\left(p, \mathbf{w}\right)$ (i.e., $\zeta = 0$ and $\eta > 0$) as follows:

$$
\begin{cases}
\begin{aligned}
\bar{\mathbf{M}}_{\mathrm{por}_z} &= \sum_{i=1}^p \operatorname{vec}\left(\bar{\mathbf{M}}_i\right) \operatorname{vec}\left(\bar{\mathbf{M}}_i\right)^T \\
&= \left(\mathbf{L} \otimes \mathbf{L}\right)^{-1} \sum_{i=1}^p \operatorname{vec}\left(\mathbf{M}_i\right) \cdot \\
& \quad \operatorname{vec}\left(\mathbf{M}_i\right)^T \left(\mathbf{L} \otimes \mathbf{L}\right)^{-T} \\
\bar{\mathbf{M}}_{\mathrm{pcro}_z} &= \eta \sum_{i=2}^p \operatorname{vec}\left(\bar{\mathbf{M}}_i\right) \operatorname{vec}\left(\bar{\mathbf{M}}_i\right)^T \\
&= \eta \left(\mathbf{L} \otimes \mathbf{L}\right)^{-1} \sum_{i=2}^p \operatorname{vec}\left(\mathbf{M}_i\right) \cdot \\
& \quad \operatorname{vec}\left(\mathbf{M}_i\right)^T \left(\mathbf{L} \otimes \mathbf{L}\right)^{-T}.
\end{aligned}
\end{cases}
\tag{40}
$$

Now, the objective function in (38) is a quadratic function of $\bar{\mathbf{W}}$, however, this problem is still hard to solve due to the rank-1 constraint $\bar{\mathbf{W}} = \bar{\mathbf{w}}\bar{\mathbf{w}}^T$. We then consider the application of the MM trick on this problem (38) based on the following simple result.

Lemma 6 ([41, Lemma 1]). *Let $\mathbf{A} \in \mathbb{S}^K$ and $\mathbf{B} \in \mathbb{S}^K$ such that $\mathbf{B} \succeq \mathbf{A}$. Then for any point $\mathbf{x}_0 \in \mathbb{R}^K$, the quadratic function $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is majorized by $\mathbf{x}^T \mathbf{B} \mathbf{x} + 2\mathbf{x}_0^T \left(\mathbf{A} - \mathbf{B}\right) \mathbf{x} + \mathbf{x}_0^T \left(\mathbf{B} - \mathbf{A}\right) \mathbf{x}_0$ at $\mathbf{x}_0$.*

According to Lemma 6, given $\bar{\mathbf{W}}^{(k)}$ at the kth iteration, we know the second part in the objective function of problem (38) is majorized by the following majorizing function at $\bar{\mathbf{W}}^{(k)}$:

$$
\begin{aligned}
& u_1\left(\bar{\mathbf{W}}, \bar{\mathbf{W}}^{(k)}\right) \\
= \quad & \psi\left(\bar{\mathbf{M}}\right) \operatorname{vec}\left(\bar{\mathbf{W}}\right)^T \operatorname{vec}\left(\bar{\mathbf{W}}\right) \\
& + 2\operatorname{vec}\left(\bar{\mathbf{W}}^{(k)}\right)^T \left(\bar{\mathbf{M}} - \psi\left(\bar{\mathbf{M}}\right) \mathbf{I}\right) \operatorname{vec}\left(\bar{\mathbf{W}}\right) \\
& + \operatorname{vec}\left(\bar{\mathbf{W}}^{(k)}\right)^T \left(\psi\left(\bar{\mathbf{M}}\right) \mathbf{I} - \bar{\mathbf{M}}\right) \operatorname{vec}\left(\bar{\mathbf{W}}^{(k)}\right),
\end{aligned}
\tag{41}
$$

where $\psi\left(\bar{\mathbf{M}}\right)$ is a scalar number depending on $\bar{\mathbf{M}}$ and satisfying $\psi\left(\bar{\mathbf{M}}\right) \mathbf{I} \succeq \bar{\mathbf{M}}$. Since the first term $\operatorname{vec}\left(\bar{\mathbf{W}}\right)^T \operatorname{vec}\left(\bar{\mathbf{W}}\right) = \left(\bar{\mathbf{w}}^T \bar{\mathbf{w}}\right)^2 = \nu^2$ and the last term only depends on $\bar{\mathbf{W}}^{(k)}$, they are just two constants.

On the choice of $\psi\left(\bar{\mathbf{M}}\right)$, according to Lemma 6, it is obvious to see that $\psi\left(\bar{\mathbf{M}}\right)$ can be easily chosen to be $\lambda_{\max}\left(\bar{\mathbf{M}}\right) = \left\|\bar{\mathbf{M}}\right\|_2$. In the implementation of the algorithm, although $\left\|\bar{\mathbf{M}}\right\|_2$ only needs to be computed once for the whole algorithm, it is still not computationally easy to get. In view of this, we introduce the following lemma to obtain more possibilities for $\psi\left(\bar{\mathbf{M}}\right)$ which could be relatively easy to compute.

Lemma 7 ([40]). *For any matrix $\mathbf{B} \in \mathbb{R}^{P \times Q}$, the following inequalities about $\left\|\mathbf{B}\right\|_2$ hold:*

$$
\left\|\mathbf{B}\right\|_2
\leq
\begin{cases}
\left\|\mathbf{B}\right\|_F = \sqrt{\sum_{i=1}^P \sum_{j=1}^Q \left|b_{ij}\right|^2} \\
\sqrt{P} \left\|\mathbf{B}\right\|_\infty = \sqrt{P} \max_{i=1,\ldots,P} \sum_{j=1}^Q \left|b_{ij}\right| \\
\sqrt{Q} \left\|\mathbf{B}\right\|_1 = \sqrt{Q} \max_{j=1,\ldots,Q} \sum_{i=1}^P \left|b_{ij}\right| \\
\sqrt{PQ} \left\|\mathbf{B}\right\|_{max} = \sqrt{PQ} \max_{i=1,\ldots,P} \max_{j=1,\ldots,Q} \left|b_{ij}\right| \\
\sqrt{\left\|\mathbf{B}\right\|_\infty \left\|\mathbf{B}\right\|_1} \\
\quad = \sqrt{\left(\max_{i=1,\ldots,P} \sum_{j=1}^Q \left|b_{ij}\right|\right) \left(\max_{j=1,\ldots,Q} \sum_{i=1}^P \left|b_{ij}\right|\right)}.
\end{cases}
$$

According to the above relations, $\psi\left(\bar{\mathbf{M}}\right)$ can be chosen to be any number is larger than $\left\|\bar{\mathbf{M}}\right\|_2$ but much easier to compute.

After ignoring the constants in (41), the majorized problem of problem (38) is given by

$$
\begin{aligned}
\underset{\bar{\mathbf{w}}, \bar{\mathbf{W}}}{\operatorname{minimize}} \quad & \xi \operatorname{vec}\left(\bar{\mathbf{M}}_1\right)^T \operatorname{vec}\left(\bar{\mathbf{W}}\right) \\
& +2 \operatorname{vec}\left(\bar{\mathbf{W}}^{(k)}\right)^T\left(\bar{\mathbf{M}}-\psi\left(\bar{\mathbf{M}}\right) \mathbf{I}\right) \operatorname{vec}\left(\bar{\mathbf{W}}\right) \\
\text{subject to} \quad & \bar{\mathbf{W}}=\bar{\mathbf{w}} \bar{\mathbf{w}}^T \\
& \bar{\mathbf{w}}^T \bar{\mathbf{w}}=\nu \\
& \bar{\mathbf{w}} \in \bar{\mathcal{W}},
\end{aligned}
\tag{42}
$$

which can be further written as

$$
\begin{aligned}
\underset{\bar{\mathbf{w}}, \bar{\mathbf{W}}}{\operatorname{minimize}} \quad & \xi \operatorname{Tr}\left(\bar{\mathbf{M}}_1 \bar{\mathbf{W}}\right)+2 \zeta \operatorname{Tr}\left(\bar{\mathbf{M}}_1 \bar{\mathbf{W}}^{(k)}\right) \operatorname{Tr}\left(\bar{\mathbf{M}}_1 \bar{\mathbf{W}}\right) \\
& +2 \eta \sum_{i=2}^p \operatorname{Tr}\left(\bar{\mathbf{M}}_i \bar{\mathbf{W}}^{(k)}\right) \operatorname{Tr}\left(\bar{\mathbf{M}}_i \bar{\mathbf{W}}\right) \\
& -2 \psi\left(\bar{\mathbf{M}}\right) \operatorname{Tr}\left(\bar{\mathbf{W}}^{(k)} \bar{\mathbf{W}}\right) \\
\text{subject to} \quad & \bar{\mathbf{W}}=\bar{\mathbf{w}} \bar{\mathbf{w}}^T \\
& \bar{\mathbf{w}}^T \bar{\mathbf{w}}=\nu \\
& \bar{\mathbf{w}} \in \bar{\mathcal{W}}.
\end{aligned}
\tag{43}
$$

By changing $\bar{\mathbf{W}}$ back to $\bar{\mathbf{w}}$, problem (43) becomes

$$
\begin{aligned}
\underset{\bar{\mathbf{w}}}{\operatorname{minimize}} \quad & \bar{\mathbf{w}}^T \bar{\mathbf{H}}^{(k)} \bar{\mathbf{w}} \\
\text{subject to} \quad & \bar{\mathbf{w}}^T \bar{\mathbf{w}}=\nu \\
& \bar{\mathbf{w}} \in \bar{\mathcal{W}},
\end{aligned}
\tag{44}
$$

where in the objective function, $\bar{\mathbf{H}}^{(k)}$ is defined in this way $\bar{\mathbf{H}}^{(k)}=\xi \bar{\mathbf{M}}_1+2 \zeta\left(\bar{\mathbf{w}}^{(k)T} \bar{\mathbf{M}}_1 \bar{\mathbf{w}}^{(k)}\right) \bar{\mathbf{M}}_1+2 \eta \sum_{i=2}^p\left(\bar{\mathbf{w}}^{(k)T} \bar{\mathbf{M}}_i \bar{\mathbf{w}}^{(k)}\right) \bar{\mathbf{M}}_i-2 \psi\left(\bar{\mathbf{M}}\right) \bar{\mathbf{w}}^{(k)} \bar{\mathbf{w}}^{(k)T}$. Finally, we can undo the change of variable $\bar{\mathbf{w}}=\mathbf{L}^T \mathbf{w}$, obtaining

$$
\begin{aligned}
\underset{\mathbf{w}}{\operatorname{minimize}} \quad & \mathbf{w}^T \mathbf{H}^{(k)} \mathbf{w} \\
\text{subject to} \quad & \mathbf{w}^T \mathbf{M}_0 \mathbf{w}=\nu \\
& \mathbf{w} \in \mathcal{W},
\end{aligned}
\tag{45}
$$

where in the objective function

$$
\begin{aligned}
\mathbf{H}^{(k)}= & \xi \mathbf{M}_1+2 \zeta\left(\mathbf{w}^{(k)T} \mathbf{M}_1 \mathbf{w}^{(k)}\right) \mathbf{M}_1 \\
& +2 \eta \sum_{i=2}^p\left(\mathbf{w}^{(k)T} \mathbf{M}_i \mathbf{w}^{(k)}\right) \mathbf{M}_i \\
& -2 \psi\left(\bar{\mathbf{M}}\right) \mathbf{M}_0 \mathbf{w}^{(k)} \mathbf{w}^{(k)T} \mathbf{M}_0 .
\end{aligned}
\tag{46}
$$

More specifically, for portmanteau statistics $\operatorname{por}_z(p, \mathbf{w})$ (i.e., $\xi=0$, $\zeta=1$ and $\eta=1$) and penalized crossing statistics $\operatorname{pcro}_z(p, \mathbf{w})$ (i.e., $\xi=1$, $\zeta=0$ and $\eta>0$), we have the following expressions:

$$
\left\{\begin{aligned}
\mathbf{H}_{\mathrm{por}_z}^{(k)}= & 2 \sum_{i=1}^p\left(\mathbf{w}^{(k)T} \mathbf{M}_i \mathbf{w}^{(k)}\right) \mathbf{M}_i \\
& -2 \psi\left(\bar{\mathbf{M}}\right) \mathbf{M}_0 \mathbf{w}^{(k)} \mathbf{w}^{(k)T} \mathbf{M}_0, \\
\mathbf{H}_{\mathrm{pcro}_z}^{(k)}= & \mathbf{M}_1+2 \eta \sum_{i=2}^p\left(\mathbf{w}^{(k)T} \mathbf{M}_i \mathbf{w}^{(k)}\right) \mathbf{M}_i \\
& -2 \psi\left(\bar{\mathbf{M}}\right) \mathbf{M}_0 \mathbf{w}^{(k)} \mathbf{w}^{(k)T} \mathbf{M}_0 .
\end{aligned}\right.
\tag{47}
$$

Finally, in the majorization problems (44) and (45), the objective functions become quadratic in the variable rather than quartic in the variable as in the original problem (36). Depending on the specific form of $\mathcal{W}$, problem (45) is either the GEVP or GTRS problems discussed in the previous sections. So, in order to handle the original problem (36) directly which could be difficult, we just need to iteratively solve a sequence of GEVPs or GTRSs. We call these MM-based algorithms iteratively reweighted GEVP (IRGEVP) and iteratively reweighted GTRS (IRGTRS) respectively which are summarized in Algorithm 3.

Algorithm 3 IRGEVP and IRGTRS - Algorithms for MRP design problems using $\operatorname{por}_z(p, \mathbf{w})$ and $\operatorname{pcro}_z(p, \mathbf{w})$.

Require: p, $\mathbf{M}_i$ with $i=1, \ldots, p$, and $\nu>0$.
1: Set $k=0$, and choose initial value $\mathbf{w}^{(k)} \in \mathcal{W}$;
2: Compute $\bar{\mathbf{M}}$ according to (39) and $\psi\left(\bar{\mathbf{M}}\right)$;
3: **repeat**
4:    Compute $\mathbf{H}^{(k)}$ according to (46);
5:    Update $\mathbf{w}^{(k+1)}$ by solving the GEVP in (22) or the GTRS in (25);
6:    $k=k+1$;
7: **until** convergence

C. EIRGEVP - An Extended Algorithm for IRGEVP

In the MM-based algorithms mentioned above, it would be much desirable if we could get a closed-form solution for the subproblems in every iteration. In fact, for IRGEVPs, applying the MM trick once again, a closed-form solution is attainable at every iteration. To illustrate this, we rewrite the subproblem (44) of IRGEVP again as follows:

$$
\begin{aligned}
\underset{\mathbf{w}}{\operatorname{minimize}} \quad & \mathbf{w}^T \mathbf{H}^{(k)} \mathbf{w} \\
\text{subject to} \quad & \mathbf{w}^T \mathbf{M}_0 \mathbf{w}=\nu \\
& \mathbf{1}^T \mathbf{w}=0 .
\end{aligned}
\tag{48}
$$

Considering the trick used to eliminate the linear constraint to get problem (22), we can get the following equivalent formulation:

$$
\begin{aligned}
\underset{\mathbf{x}}{\operatorname{minimize}} \quad & \mathbf{x}^T \mathbf{N}^{(k)} \mathbf{x} \\
\text{subject to} \quad & \mathbf{x}^T \mathbf{N}_0 \mathbf{x}=\nu,
\end{aligned}
\tag{49}
$$

where $\mathbf{F}$ and $\mathbf{N}_0$ are defined as before; $\mathbf{N}^{(k)}=\mathbf{F}^T \mathbf{H}^{(k)} \mathbf{F}$. Considering the Cholesky decomposition $\mathbf{N}_0=\mathbf{R} \mathbf{R}^T$ with $\mathbf{R}$ to be a lower triangular with positive diagonal elements, we can have the variable transformation $\bar{\mathbf{x}}=\mathbf{R}^T \mathbf{x}$. Then the problem (48) becomes

$$
\begin{aligned}
\underset{\bar{\mathbf{x}}}{\operatorname{minimize}} \quad & \bar{\mathbf{x}}^T \bar{\mathbf{N}}^{(k)} \bar{\mathbf{x}} \\
\text{subject to} \quad & \bar{\mathbf{x}}^T \bar{\mathbf{x}}=\nu,
\end{aligned}
\tag{50}
$$

where $\bar{\mathbf{N}}^{(k)}=\mathbf{R}^{-1} \mathbf{N}^{(k)} \mathbf{R}^{-T}$.

Applying Lemma 6 again, the objective function of problem (50) is majorized by the following majorizing function at $\bar{\mathbf{x}}^{(k)}$:

$$
\begin{aligned}
& u_2\left(\bar{\mathbf{x}}, \bar{\mathbf{x}}^{(k)}\right) \\
= & \psi\left(\bar{\mathbf{N}}^{(k)}\right) \bar{\mathbf{x}}^T \bar{\mathbf{x}} \\
& +2\left[\left(\bar{\mathbf{N}}^{(k)}-\psi\left(\bar{\mathbf{N}}^{(k)}\right) \mathbf{I}\right) \bar{\mathbf{x}}^{(k)}\right]^T \bar{\mathbf{x}} \\
& +\bar{\mathbf{x}}^{(k)T}\left[\psi\left(\bar{\mathbf{N}}^{(k)}\right) \mathbf{I}-\bar{\mathbf{N}}^{(k)}\right] \bar{\mathbf{x}}^{(k)},
\end{aligned}
\tag{51}
$$

where $\psi\left(\bar{\mathbf{N}}^{(k)}\right)$ can be chosen using the results from Lemma 7. The first and last parts are just two constants. Note that although in the derivation we have applied the MM scheme

twice, it can be viewed as a direct majorization for the objective of the original problem at $\mathbf{w}^{(k)}$. The following lemma summarizes the overall majorizing function.

Lemma 8. *For problem* (36) *with* $\mathbf{w} \in \mathcal{W}_0$, *the majorization in* (41) *together with* (51) *can be shown to be a majorization for the objective function of the original problem at* $\mathbf{w}^{(k)}$ *over the constraint set by the following function:*

$$
\begin{aligned}
& u_2\left(\mathbf{w}, \mathbf{w}^{(k)}\right) \\
= \ & 2\left[\left(\mathbf{H}^{(k)} - \psi\left(\mathbf{R}^{-1}\mathbf{F}^T\mathbf{H}^{(k)}\mathbf{F}\mathbf{R}^{-T}\right)\mathbf{M}_0\right)\mathbf{w}^{(k)}\right]^T \mathbf{w} \\
& + 2\psi\left(\mathbf{R}^{-1}\mathbf{F}^T\mathbf{H}^{(k)}\mathbf{F}\mathbf{R}^{-T}\right)\nu - \mathbf{w}^{(k)T}\mathbf{H}^{(k)}\mathbf{w}^{(k)}.
\end{aligned}
\tag{52}
$$

where the last two terms are constants.

Proof: See Appendix A. ∎

Then, the majorized problem of (50) becomes

$$
\begin{aligned}
\underset{\bar{\mathbf{x}}}{\text{minimize}} \quad & \mathbf{e}^{(k)T}\bar{\mathbf{x}} \\
\text{subject to} \quad & \bar{\mathbf{x}}^T\bar{\mathbf{x}} = \nu,
\end{aligned}
\tag{53}
$$

where $\mathbf{e}^{(k)} = 2\left(\bar{\mathbf{N}}^{(k)} - \psi\left(\bar{\mathbf{N}}^{(k)}\right)\mathbf{I}\right)\bar{\mathbf{x}}^{(k)}$ for the majorization in (51). By Cauchy-Schwartz inequality, we have $\mathbf{e}^T\bar{\mathbf{x}} \geq -\|\mathbf{e}\|_2\|\bar{\mathbf{x}}\|_2 = -\nu\|\mathbf{e}\|_2$, and the equality holds only when $\bar{\mathbf{x}}$ and $\mathbf{e}$ are aligned in the opposite direction. Considering the constraint, we can get the optimal solution of (53) as $\bar{\mathbf{x}} = -\sqrt{\nu}\frac{\mathbf{e}}{\|\mathbf{e}\|_2}$. We call this algorithm extended IRGEVP (EIRGEVP) which is summarized in Algorithm 4.

Algorithm 4 EIRGEVP - An extended algorithm for IRGEVP.

Require: p, $\mathbf{M}_i$ with $i = 1, \ldots, p$, and $\nu > 0$.
1: Set $k = 0$, and choose initial value $\mathbf{w}^{(k)} \in \mathcal{W}$;
2: Compute $\bar{\mathbf{M}}$ and $\psi\left(\bar{\mathbf{M}}\right)$;
3: **repeat**
4:     Compute $\bar{\mathbf{N}}^{(k)}$ and $\psi\left(\bar{\mathbf{N}}^{(k)}\right)$;
5:     Update $\mathbf{w}^{(k+1)}$ with a closed-form solution;
6:     $k = k + 1$;
7: **until** convergence

VI. Numerical Experiments

A statistical arbitrage strategy involves several steps of which the MRP design is a central one. Here, we divide the whole strategy into four sequential steps, namely: assets pool construction, MRP design, unit-root test, and mean-reversion trading. In the first step, we select a collection of possibly cointegrated asset candidates to construct an asset pool, on which we will not elaborate in this paper. In the second step, based on the candidate assets from the asset pool, MRPs are designed using either traditional design methods like Engle-Granger OLS method [12] and Johansen method [13] or the proposed methods in this paper. In the third step, unit-root test procedures like Augmented Dickey-Fuller test [46] and Phillips-Perron test [47] are applied to test the stationarity or mean-reversion property of the designed MRPs. In the fourth step, MRPs passing the unit-root tests will be traded based on a designed mean-reversion trading strategy.

In this section, we first illustrate a mean-reversion trading strategy and based on that the performance of our proposed

MRP design methods in Sections IV and V using both synthetic data and real market data are shown accordingly.

A. Mean-Reversion Trading Design

In this paper, we use a simple trading strategy where the trading signals, i.e., to buy, to sell, or simply to hold, are designed based on simple event triggers. Mean-reversion trading is carried out on the designed spread z_t which is tested to be unit-root stationary. A trading position (either a long position denoted by 1 or a short position denoted by -1) denotes a state for investment and it is opened when the spread z_t is away from its long-run equilibrium μ_z by a predefined trading threshold Δ and closed (denoted by 0) when z_t crosses its equilibrium μ_z. (A common variation is to close the position after the spread crossed the equilibrium by more than another threshold Δ'.) The time period from position opening to position closing is defined as a trading period.

In order to get a standard trading rule, we introduce a standardization technique by defining $z - \text{score}$ which is a normalized spread as follows:

$$
\tilde{z}_t = \frac{z_t - \mu_z}{\sigma_z},
\tag{54}
$$

where μ_z and σ_z are the mean and the standard deviation of the spread z_t and computed over an in-sample look-back period in practice. For $\tilde{z}_t$, it follows that $\mathsf{E}\left[\tilde{z}_t\right] = 0$ and $\mathsf{Std}\left[\tilde{z}_t\right] = 1$. Then, we can define $\Delta = d \times \sigma_z$, for some value of d (e.g., $d = 1$).

In a trading stage, based on the trading position and observed (normalized) spread value at holding period t, we can get the trading actions at the next consecutive holding period $t + 1$. The mean-reversion trading strategy is summarized in Table I and a simple trading example based on this strategy is illustrated in Figure 2.
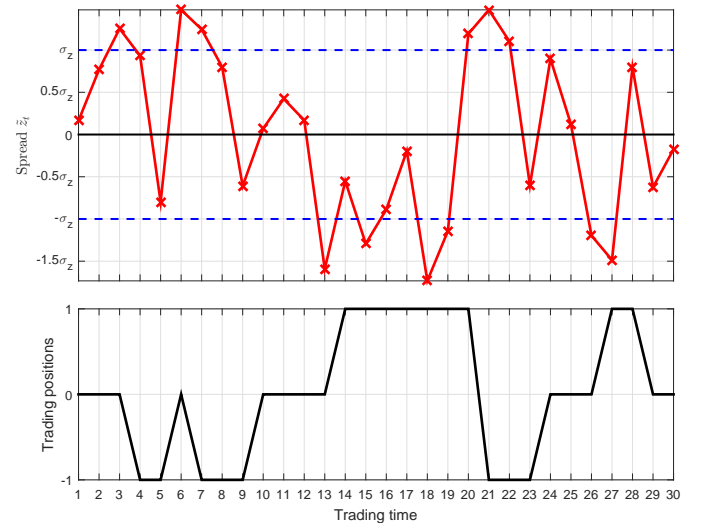

Fig. 2. A simple example for mean-reversion trading strategy design (trading threshold $\Delta = \sigma_z$).

TABLE I
TRADING POSITIONS, NORMALIZED SPREAD, AND TRADING ACTIONS OF A MEAN-REVERSION TRADING STRATEGY

Trading Position at t	Normalized Spread $\tilde{z}_t$	Action(s) Taken within Holding Period $t+1$	Trading Position at $t+1$
1	$+d \leq \tilde{z}_t$	Close the long pos. & Open a short pos.	-1
	$0 \leq \tilde{z}_t < +d$	Close the long pos.	0
	$\tilde{z}_t < 0$	No action	1
0	$+d \leq \tilde{z}_t$	Open a short pos.	-1
	$-d < \tilde{z}_t < +d$	No action	0
	$\tilde{z}_t \leq -d$	Open a long pos.	1
-1	$0 < \tilde{z}_t$	No action	-1
	$-d < \tilde{z}_t \leq 0$	Close the short pos.	0
	$\tilde{z}_t \leq -d$	Close the short pos. & Open a long pos.	1

B. Performance Metrics

After an MRP is constructed, we need to define the relation between the designed MRP with the underlying financial assets. Recall that the spread for the designed MRP is $z_t = \mathbf{w}^T \mathbf{s}_t$, where $\mathbf{s}_t = [s_{1,t}, s_{2,t}, \ldots, s_{N,t}]^T$ with $s_{n,t} = \mathbf{w}_{s_n}^T \mathbf{y}_t$ for $n = 1, 2, \ldots, N$. By defining $\mathbf{s}_t = \mathbf{W}_s^T \mathbf{y}_t$ with $\mathbf{W}_s = [\mathbf{w}_{s_1}, \mathbf{w}_{s_2}, \ldots, \mathbf{w}_{s_N}]$, we get the spread $z_t = \mathbf{w}_p^T \mathbf{y}_t$, where $\mathbf{w}_p = \mathbf{W}_s \mathbf{w}$ denotes the portfolio weight directly defined on the underlying assets.

Based on the mean-reversion trading strategy introduced before and the MRP defined by $\mathbf{w}_p$ here, we employ the following performance metrics in the numerical experiments.

1) Portfolio Return Measures: In the following, we first give the return definition for one single asset, and after that, several different return measures for an MRP are talked about.

For one single asset, the return or cumulative return at time t for τ holding periods is defined as $r_t(\tau) = \frac{p_t - p_{t-\tau}}{p_{t-\tau}}$, where τ in the parentheses denotes the period length and is usually omitted when the length is one. Here, the return $r_t(\tau)$ as a rate of return is used to measure the aggregate amount of profits or losses (in percentage) of an investment strategy on one asset over a time period τ.

a) Profit and Loss (P&L): The profit and loss (P&L) measures the amount of profits or losses (in units of dollars) of an investment on the portfolio for some holding periods.

Within one trading period, if a long position is opened on an MRP at time t_o and closed at time t_c, then the multi-period P&L of this MRP at time t ($t_o \leq t \leq t_c$) accumulated from t_o is computed as $\text{P\&L}_t(\tau) = \mathbf{w}_p^T \mathbf{r}_t(\tau) = \mathbf{w}_p^T \mathbf{r}_t(t - t_o)$, where $\tau = t - t_o$ denotes the length of the holding periods, and $\mathbf{r}_t(\tau) = [r_{1,t}(\tau), r_{2,t}(\tau), \ldots, r_{M,t}(\tau)]^T$ is the return vector. More generally, the cumulative P&L of this MRP at time t for τ ($0 \leq \tau \leq t - t_o$) holding periods is defined as

$$\text{P\&L}_t(\tau) = \mathbf{w}_p^T \mathbf{r}_t(t - t_o) - \mathbf{w}_p^T \mathbf{r}_{t-\tau}(t - \tau - t_o), \quad (55)$$

where we define $\mathbf{r}_t(0) = \mathbf{0}$. Then we have the single-period P&L (e.g., daily P&L, monthly P&L) denoted by $\text{P\&L}_t$ at time t (i.e., $\tau = 1$) is computed as

$$\text{P\&L}_t = \mathbf{w}_p^T \mathbf{r}_t(t - t_o) - \mathbf{w}_p^T \mathbf{r}_{t-1}(t - 1 - t_o). \quad (56)$$

Likewise, within one trading period, if a short position is opened on this MRP, then multi-period P&L is $\text{P\&L}_t(\tau) = \mathbf{w}_p^T \mathbf{r}_{t-\tau}(t - \tau - t_o) - \mathbf{w}_p^T \mathbf{r}_t(t - t_o)$ and the single-period P&L is $\text{P\&L}_t = \mathbf{w}_p^T \mathbf{r}_{t-1}(t - 1 - t_o) - \mathbf{w}_p^T \mathbf{r}_t(t - t_o)$. About

the portfolio P&L calculation within the trading periods, we have the following lemma.

Lemma 9 (P&L Calculation for Mean-Reversion Trading). *Within one trading period, if the price change of every asset in an MRP is small enough, then the P&L in (55) can be approximately calculated by the change of the log-price spread z_t. Specifically,*

1) for a long position opened on the MRP, $\text{P\&L}_t(\tau) \approx z_t - z_{t-\tau}$; and

2) for a short position opened on the MRP, $\text{P\&L}_t(\tau) \approx z_{t-\tau} - z_t$.

Proof: See Appendix B. ∎

This lemma reveals the philosophy of the MRP design and also the mean-reversion trading by showing the connection between the log-price spread value and the portfolio return.

Since there is no trading conduct between two trading periods, the P&L measures (both the multi-period P&L and single-period P&L) are simply defined to be 0.

b) Cumulative P&L: In order to measure the cumulative return performance for an MRP, we define the cumulative P&L in one trading from time t_1 to t_2 as

$$\text{Cum. P\&L}(t_1, t_2) = \sum_{t=t_1}^{t_2} \text{P\&L}_t. \quad (57)$$

c) Return on Investment (ROI): Since different MRPs may have different leverage properties due to $\mathbf{w}_p$, we introduce another portfolio return measure (rate of return) called return on investment (ROI).

Within one trading period, the ROI at time t ($t_o \leq t \leq t_c$) is defined to be the single-period P&L at time t normalized by the gross investment deployed which is $\|\mathbf{w}_p\|_1$ (that is the gross investment exposure to the market including the long position investment and the short position investment) written as

$$\text{ROI}_t = \text{P\&L}_t / \|\mathbf{w}_p\|_1. \quad (58)$$

Like the P&L measures, between two trading periods, ROI_t is defined to be 0.

2) Sharpe Ratio (SR): The Sharpe ratio (SR) [48] is a measure for calculating risk-adjusted return. It describes how much excess return one can receive for the extra volatility (square root of variance).

Here, the Sharpe ratio of ROI (or, equivalently, Sharpe ratio of P&L) for a trading stage from time t_1 to t_2 is defined as follows:
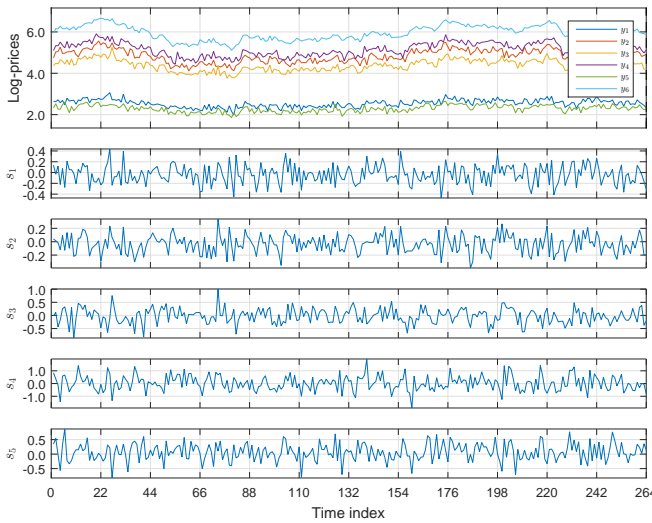
Fig. 3. Log-prices and five estimated spreads. (The sample length for in-sample training is chosen to be $5 \times 12 \times 22$, and the sample length for out-of-sample trading is 12×22.)

$$\mathrm{SR}_{\mathrm{ROI}}(t_1, t_2) = \frac{\mu_{\mathrm{ROI}}}{\sigma_{\mathrm{ROI}}}, \qquad (59)$$

where $\mu_{\mathrm{ROI}} = \frac{1}{t_2 - t_1} \sum_{t=t_1}^{t_2} \mathrm{ROI}_t$ and $\sigma_{\mathrm{ROI}} = \left[\frac{1}{t_2 - t_1} \sum_{t=t_1}^{t_2} (\mathrm{ROI}_t - \mu_{\mathrm{ROI}})^2 \right]^{1/2}$. In the computation of the Sharpe ratio, we set the risk-free return to 0, in which case it reduces to the information ratio.

C. Synthetic Data Experiments

For synthetic data experiments, we generate the sample path of log-prices for M financial assets using a multivariate cointegrated systems [33], [49], where there are r long-run cointegration relations and $M - r$ common trends. We divide the sample path into two stages: in-sample training stage and out-of-sample backtesting or trading stage. All the parameters like spread equilibrium μ_z, trading threshold Δ, and portfolio weight $\mathbf{w}$ are decided in the training stage. The out-of-sample performance of our design methods are tested in the trading stage.

In the synthetic experiments, we set $M = 6$ and $r = 5$ and only show the performance of the MRP design methods under net budget constraint $\mathcal{W}_1$. We estimate $N = 5$ spreads using the generated sample path. Based on these 5 spreads, an MRP is designed as $z_t = \mathbf{w}^T \mathbf{s}_t$. The simulated log-prices and the spreads for the trading stage are shown in Figure 3.

The performance of the MRP designed using our proposed methods are compared with those of one underlying spread and the method in [25] based on $\mathrm{pcro}_z(5, \mathbf{w})$ and $\mathrm{pre}_z(\mathbf{w})$, which are shown in Figure 4 and Figure 5. From our simulations, we can conclude that our designed MRPs do generate consistent positive profits. And simulation results also show that our designed portfolios can outperform the underlying spreads and the MRPs designed using methods in [25] with higher Sharpe ratios of ROIs and higher cumulative P&Ls.
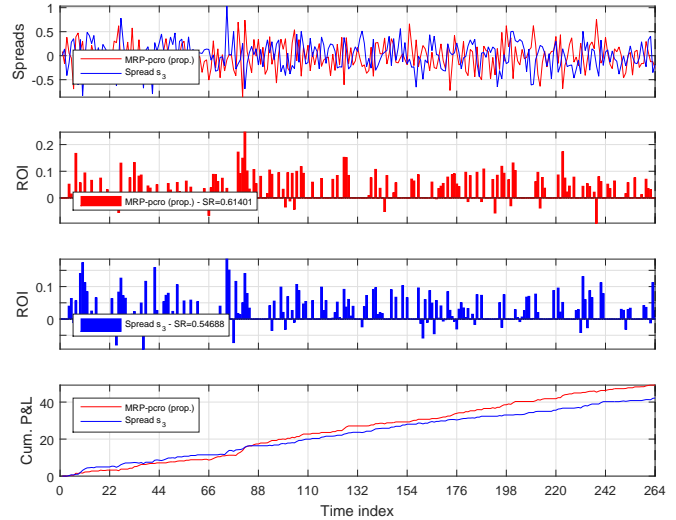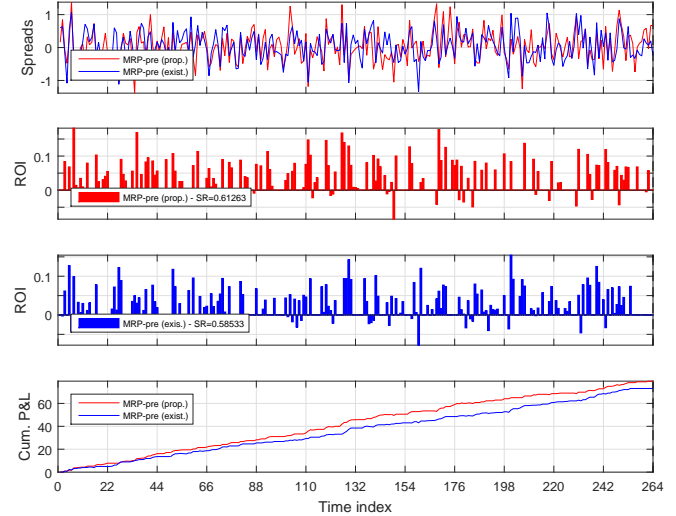

Fig. 4. Comparisons of ROIs, Sharpe ratios of ROIs, and cumulative P&Ls between the MRP designed using our proposed method denoted as MRP-pre (prop.) with one underlying spread denoted as Spread s_3.

Fig. 5. Comparisons of ROIs, Sharpe ratios of ROIs, and cumulative P&Ls between the MRP designed using our proposed method denoted as MRP-pcro (prop.) and one existing benchmark method in [25] denoted as MRP-pcro (exist.).

D. Market Data Experiments

We also test our methods using real market data from the Standard & Poor's 500 (S&P 500) Index, which is usually considered as one of the best representatives for the U.S. stock markets. The data are retrieved from Yahoo! Finance[5] and adjusted daily closing stock prices are employed. We first choose stock candidates which are possibly cointegrated to form stock asset pools. One stock pool is $\{\mathrm{APA}, \mathrm{AXP}, \mathrm{CAT}, \mathrm{COF}, \mathrm{FCX}, \mathrm{IBM}, \mathrm{MMM}\}$, where the stocks are denoted by their ticker symbols. Three spreads are constructed from this pool. Then MRP design methods are employed and unit-root tests are used to test their tradability. The log-prices of the stocks and the log-prices for the three spreads
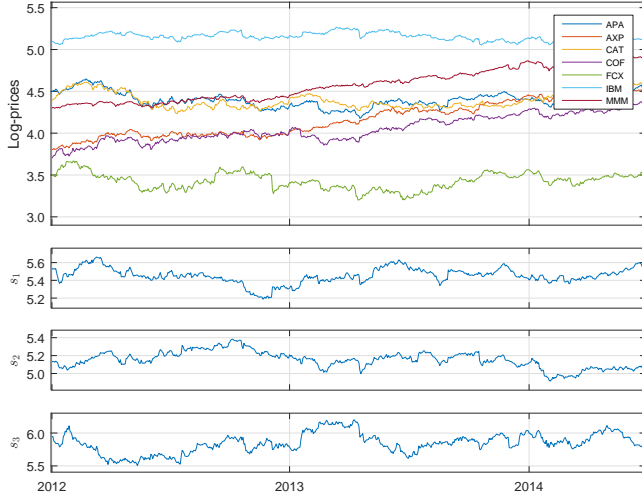
[5] http://finance.yahoo.com

Fig. 6. Log-prices for $\{\text{APA}, \text{AXP}, \text{CAT}, \text{COF}, \text{FCX}, \text{IBM}, \text{MMM}\}$ and three spreads s_1, s_2, and s_3.

are shown in Figure 6.

Based on the mean-reversion trading framework mentioned before, one trading experiment is carried out from February 1st, 2012 to June 30th, 2014. In Figure 7, we compare the performance of our designed MRP with the underlying spread s_1. The log-prices for the designed spreads, and the out-of-sample performance like ROIs, Sharpe ratios of ROIs, and cumulative P&Ls are reported. It is shown that using our method, the designed MRP can achieve a higher Sharpe ratio and a better final cumulative return.
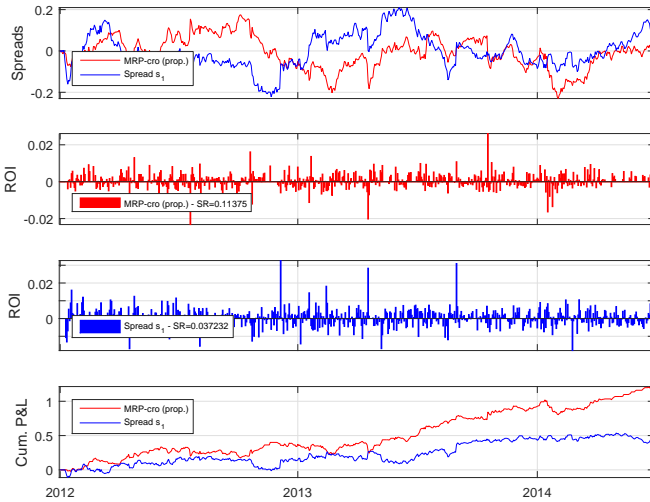

Fig. 7. Comparisons of ROIs, Sharpe ratios of ROIs, and cumulative P&Ls between the MRP designed using our proposed method denoted as MRP-cro (prop.) with one underlying spread denoted as Spread s_1.

We also compare our proposed design method with the method in [25] using $\text{por}_z\left(5, \mathbf{w}\right)$ in Figure 8. We can see that our method can outperform the benchmark method in terms of Sharpe ratio and the return performance.

VII. CONCLUSIONS

The mean-reverting portfolio design problem arising from statistical arbitrage has been considered in this paper. We
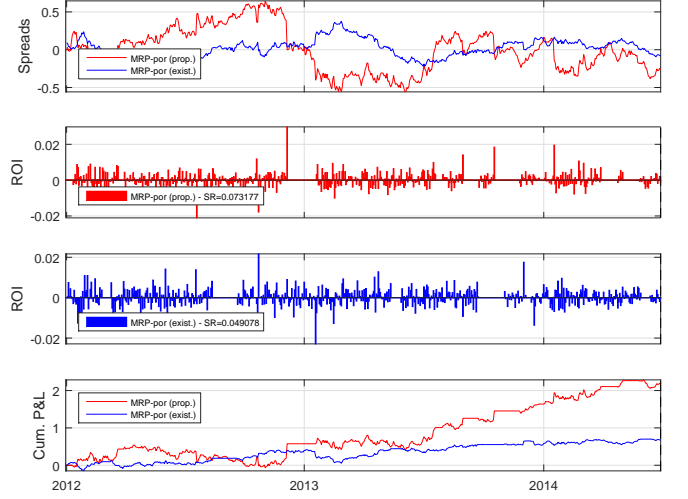

Fig. 8. Comparisons of ROIs, Sharpe ratios of ROIs, and cumulative P&Ls between the MRP designed using our proposed method denoted as MRP-por (prop.) and one existing benchmark method in [25] denoted as MRP-por (exist.).

have formulated the MRP design problem in a general form by optimizing a mean-reversion criterion characterizing the mean-reversion strength of the portfolio and, at the same time, taking into consideration the variance of the portfolio and an investment budget constraint. Several specific optimization problems have been proposed based on the general design idea. Efficient algorithms have been derived to solve the design problems. Numerical results show that our proposed methods are able to generate consistent positive profits and significantly outperform the the design methods in literature.

APPENDIX A
PROOF FOR LEMMA 8

For problem (36) with $\mathbf{w} \in \mathcal{W}_0$, the majorizing function in the first majorization step (41) is denoted by $u_1\left(\mathbf{w}, \mathbf{w}^{(k)}\right)$, and the majorizing function in the second step (51) is denoted by $u_2\left(\mathbf{w}, \mathbf{w}^{(k)}\right)$. From the majorization properties in (34), then we can have have this relationship: $f_z\left(\mathbf{w}\right) \leq u_1\left(\mathbf{w}, \mathbf{w}^{(k)}\right) \leq u_2\left(\mathbf{w}, \mathbf{w}^{(k)}\right)$.

Then we can get the overall majorization in $\mathbf{w}$ for the objective function of the original problem at $\mathbf{w}^{(k)}$ over the constraint set by the following function:

$$
\begin{aligned}
& u_2\left(\mathbf{w}, \mathbf{w}^{(k)}\right) \\
\overset{(\text{in } \bar{\mathbf{x}})}{=} \ & 2\left[\left(\bar{\mathbf{N}}^{(k)} - \psi\left(\bar{\mathbf{N}}^{(k)}\right)\mathbf{I}\right)\bar{\mathbf{x}}^{(k)}\right]^T \bar{\mathbf{x}} \\
& +2\psi\left(\bar{\mathbf{N}}^{(k)}\right)\nu - \bar{\mathbf{x}}^{(k)T}\bar{\mathbf{N}}^{(k)}\bar{\mathbf{x}}^{(k)} \\
\overset{(\text{in } \mathbf{x})}{=} \ & 2\left[\left(\mathbf{N}^{(k)} - \psi\left(\mathbf{R}^{-1}\mathbf{N}^{(k)}\mathbf{R}^{-T}\right)\mathbf{N}_0\right)\mathbf{x}^{(k)}\right]^T \mathbf{x} \\
& +2\psi\left(\mathbf{R}^{-1}\mathbf{N}^{(k)}\mathbf{R}^{-T}\right)\nu - \mathbf{x}^{(k)T}\mathbf{N}^{(k)}\mathbf{x}^{(k)} \\
\overset{(\text{in } \mathbf{w})}{=} \ & 2\left[\left(\mathbf{H}^{(k)} - \psi\left(\mathbf{R}^{-1}\mathbf{F}^T\mathbf{H}^{(k)}\mathbf{F}\mathbf{R}^{-T}\right)\mathbf{M}_0\right)\mathbf{w}^{(k)}\right]^T \mathbf{w} \\
& +2\psi\left(\mathbf{R}^{-1}\mathbf{F}^T\mathbf{H}^{(k)}\mathbf{F}\mathbf{R}^{-T}\right)\nu - \mathbf{w}^{(k)T}\mathbf{H}^{(k)}\mathbf{w}^{(k)},
\end{aligned}
$$

where the last two terms in every step of the derivations are constants since they are independent of the optimization variables.

APPENDIX B
PROOF FOR LEMMA 9

Since the spread is defined as $z_t = \mathbf{w}_p^T \mathbf{y}_t$, then the multi-period P&L at time t for τ holding periods is given by

$$
\begin{aligned}
\text{P\&L}_t(\tau) =\;& \mathbf{w}_p^T \mathbf{r}_t(t-t_o) - \mathbf{w}_p^T \mathbf{r}_{t-\tau}(t-\tau-t_o) \\
=\;& \sum_{m=1}^M w_{p,m} r_{m,t}(t-t_o) \\
& - \sum_{m=1}^M w_{p,m} r_{m,t-\tau}(t-\tau-t_o) \\
=\;& \sum_{m=1}^M w_{p,m} \left(\frac{p_{m,t}}{p_{m,t_o}} - 1 \right) \\
& - \sum_{m=1}^M w_{p,m} \left(\frac{p_{m,t-\tau}}{p_{m,t_o}} - 1 \right) \\
\approx\;& \sum_{m=1}^M w_{p,m} \left[\log(p_{m,t}) - \log(p_{m,t_o}) \right] \\
& - \sum_{m=1}^M w_{p,m} \left[\log(p_{m,t-\tau}) - \log(p_{m,t_o}) \right] \\
=\;& \sum_{m=1}^M w_{p,m} \log(p_{m,t}) - w_{p,m} \log(p_{m,t-\tau}) \\
=\;& \sum_{m=1}^M w_{p,m} y_{m,t} - \sum_{m=1}^N w_{p,m} y_{m,t-\tau} \\
=\;& \mathbf{w}_p^T \mathbf{y}_t - \mathbf{w}_p^T \mathbf{y}_{t-\tau} \\
=\;& z_t - z_{t-\tau}.
\end{aligned}
$$

The approximation in the fourth step follows from $\log(1+x) \approx x$ when $x \to 0$, where $\log(\cdot)$ denotes the natural logarithm. Similarly, for a short position on the MRP, the calculation of $\text{P\&L}_t(\tau)$ is given by $z_{t-\tau} - z_t$.

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