

Pragmatic Information Rates, Generalizations of the Kelly Criterion, and Financial Market Efficiency

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Abstract

This paper is part of an ongoing investigation of “pragmatic information,” defined in Weinberger (2002) as “the amount of information actually used in making a decision.” Because a study of information *rates* led to the Noiseless and Noisy Coding Theorems, two of the most important results of Shannon’s theory, we begin the paper by defining a pragmatic information rate, showing that all of the relevant limits make sense, and interpreting them as the improvement in compression obtained from using the correct distribution of transmitted symbols.

The first of two applications of the theory extends the information theoretic analysis of the Kelly Criterion, and its generalization, the “horse race”, to a series of races where the stochastic process of winning horses and strategies depend on some stationary process, including, but not limited to the history of previous races. If the bettor is receiving messages (side information) about the probability distribution of winners, the increase in the doubling rate of the bettor’s winnings is the pragmatic information of the messages. The results presented here are well known in the literature, but they are included as an important illustration of the relevance of pragmatic information to finance.

A second application is to the question of market efficiency. An efficient market is, by definition, a market in which the pragmatic information of the “tradable past” with

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respect to current prices is zero. Under this definition, we show that markets whose returns are characterized by a GARCH(1,1) process cannot be efficient.

Finally, a pragmatic informational analogue to Shannon's Noisy Coding Theorem suggests that a cause of market inefficiency is that the underlying fundamentals are changing so fast that the price discovery mechanism simply cannot keep up. This may happen most readily in the run-up to a financial bubble, where investors' willful ignorance degrades the information processing capabilities of the market.

Key Words and Phrases

Shannon-McMillan-Breiman Theorem; mutual information; pragmatic information; expected log return; log-optimal portfolio; ergodic stock market; asymptotic optimality principle; efficient market hypothesis.

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Introduction

Weinberger (2002) published a definition of “pragmatic information” as “the amount of information in a message that is actually used to make a decision”, or as a measure of the amount of *meaningful* information in a message. This definition distinguishes pragmatic information from the usual (Shannon) measure of information, which refers merely to the reduction in uncertainty resulting from the receipt of a message, and not to the *meaning* that the uncertainty reduction has to the receiver. The most obvious, if not the only objective meaning that a message can have to a receiver is in observed changes in subsequent behavior; indeed, Weaver made precisely this point in his extended introduction to the Shannon paper that began information theory (Shannon and Weaver, 1962). Although Shannon’s paper began with a demonstration that his celebrated entropy measure was, given a few reasonable conditions, unique, Shannon wrote that the ultimate importance of this quantity was not that it obeyed the uniqueness theorem, but that it represented the minimum compressed length of a message, a result now familiar as the Noiseless Coding Theorem. Therefore it seems reasonable to ask whether there is an equivalent “Noiseless Coding Theorem for Pragmatic Information” and what that might mean.

Answering the above question requires some background. First, the Noiseless Coding Theorem was only the first of a series of successively more general results; in that Shannon’s result only applied to

- symbol sequences generated from a finite alphabet, as opposed to the infinite set of real numbers,

- measurements of the *expected* length of the encoding, thus leaving open the possibility that any individual sequence could have a markedly different compressed length.
- sequences generated by a so-called discrete Markov source, *i.e.* one in which the (stationary) conditional probability of observing each symbol, given the entire past history of the sequence, is identical to the conditional probability of observing that symbol, given only the most recent $N < \infty$ symbols.

Subsequent attempts to generalize the Noiseless Coding Theorem beyond the above restrictions culminated in the Shannon-McMillan-Breiman Theorem (for finite alphabet sequences) and its generalization to potentially real valued sequences by Barron (1985). These theorems guarantee that, in the limit of long sequences, individual sequences must *all*² have the compressed length per input symbol predicted by the Noiseless Coding Theorem. Furthermore, the class of sequences to which the sequence applies is considerably broader than those produced by discrete Markov sources.

An obvious application of the present work is in the prediction, or, per the efficient market hypothesis (Brigham and Ehrhardt, 2005), difficulty of prediction of financial time series. Indeed,

- the weak form of market efficiency is simply the statement that the pragmatic information of *past returns* in predicting future returns is zero;
- the semi-strong form of market efficiency is simply the statement that the pragmatic information of *all public information* for predicting future returns is zero;
- the strong form of market efficiency is simply the statement that the pragmatic information of *all information, including insider information*, for predicting future returns is zero.

In addition to the Noiseless Coding Theorem, the other landmark result in Shannon's paper was his so-called "Noisy Coding Theorem", a fundamental limitation on the rate at which information can be transmitted through a noisy communications channel. This theorem states that every such channel has a well defined "channel capacity", a rate below which the receiver can detect and correct any transmission errors, so that transmission can be made effectively noiseless at any rate below the channel capacity. The theorem also states that the receiver cannot, in general, do the same for transmissions above the channel capacity. Again, is there a pragmatic informational analogue to such a theorem, and, if so, what might it mean?

This paper is therefore organized as follows: The section after this introduction provides some relevant background, including the formal statements of the above mentioned information theoretic results and the ideas surrounding them. As part of that review, we consider a class of processes that are provably not discrete Markov sources and we argue that they are, in fact, better models for financial price series than random walks. Thus motivated, a third section discusses the theoretical underpinnings of measuring pragmatic information rates, followed by a section that relates pragmatic information to the

² Actually "almost all," in the sense that the set of all such sequences has probability 1.

generalized Kelly Criterion and to efficient market theory. The final two sections of the paper are considerably less concerned with specific results; the first of these sections briefly considers the implications of Shannon's Noisy Coding Theorem for both the theory of pragmatic information in general and its specific application to finance, the second presents a summary and some conclusions to be drawn from this work.

We adopt the information theoretic convention that all logarithms are base 2 logarithms, unless otherwise noted. Thus, the information theoretic expressions in this paper have units of "bits" (as opposed to the units of "nats" if natural logs were used).

Background

An Introduction to Pragmatic Information

It is tempting to define the pragmatic information of a message, m , as the relative entropy, $\mathcal{D}(P \parallel Q)$, of the probabilities of the decision maker's actions before and after receiving m . This quantity is defined over a finite symbol string A , consisting of concatenations of the symbols $A_1, A_2, A_3, \dots, A_{|\mathcal{A}|}$, drawn from the finite alphabet \mathcal{A} , as³

$$\mathcal{D}(P \parallel Q) = \sum_A P(A) \log [P(A)/Q(A)], \quad (1)$$

where $P(A)$ is the probability of A subsequent to the receipt of m and $Q(A)$ is the probability of Q beforehand. The use of the character A to represent this symbol string serves as a reminder that these symbols actually represent *actions*, albeit actions broadly construed.

A theorem similar to the Noiseless Coding Theorem, albeit less well known, states that a message erroneously compressed using the wrong probabilities, $Q(A)$, when the actual symbol probabilities, $P(A)$, will require an additional compressed length equal in expected value to the relative entropy between the wrong distribution and the right one. Surely, some kind of learning is clearly taking place if the act of compressing the sequence can be improved, and the additional amount of compression measures, in some sense, the amount that has been learned. The formal statement of the theorem would therefore seem to provide a useful quantification of this learning, as in the

"Wrong Code" Theorem (Cover and Thomas, 1991). If an encoding that would be optimal assuming the wrong probability measure $Q(A)$ is used in place of the optimal encoding for the finite symbol string A using the correct probability, $P(A)$, then the expected length of the resulting encoding, $E_P[\ell(\alpha)] = \sum_A P(A) \ell(A)$, satisfies the bounds

³ Here, and in the sequel, all logarithms are to the base $|\mathcal{A}|$, where $|\mathcal{A}|$ is the size of \mathcal{A} , unless otherwise indicated. We also adopt the convention that α denotes a randomly selected symbol, as opposed to A , which would indicate that the symbol is, for example, an index in a sum, as above.

$$\mathcal{H}(P) + \mathcal{D}(P \| Q) \leq E_P[\ell(\alpha)] \leq \mathcal{H}(P) + \mathcal{D}(P \| Q) + 1,$$

where $\mathcal{H}(P)$ is the entropy of α , $\mathcal{D}(P \| Q)$ is defined as in (1), and $\ell(\alpha)$ is the length, in symbols, of the encoding of α using Q .

However, as Weinberger (2002) makes clear, pragmatic information is more sensibly defined over the *ensemble*, \mathcal{M} , of possible messages. There, it is assumed that the actual message sent, which we will denote as μ , is a random variable with known distribution $Pr\{\mu = m\} = \varphi(m)$. In particular, it is shown there that the definition of pragmatic information as the mutual information between \mathcal{M} and the ensemble of outcomes, \mathcal{A} , i.e.

$$\mathcal{I}(\mathcal{A}; \mathcal{M}) = \sum_m \sum_A Pr(A, m) \log \left[\frac{Pr(A, m)}{Pr(m)Pr(A)} \right] = \sum_m Pr(m) \sum_A Pr(A|m) \log \left[\frac{Pr(A|m)}{Pr(A)} \right] \quad (2)$$

has the all-important property that the joint pragmatic information from independent message ensembles is the sum of the pragmatic information values from each ensemble. It is also the relative information of the probability distribution of outputs subsequent to the receipt of each message with respect to the prior distribution, averaged over all possible messages.

Another problem with using (1) as a definition of pragmatic information emerges in the limit of long sequences. Recall that the Noiseless Coding Theorem states that

$$\mathcal{H}(P) \leq E[\ell_P(\alpha)] \leq \mathcal{H}(P) + 1,$$

where $\ell_P(\alpha)$ is the length, in symbols, of the optimal (shortest) encoding of α . Dividing through by N , the length of α , would give the exact result that the information rate, \mathcal{I} , is exactly equal to the expected encoded length per symbol in the limit $N \rightarrow \infty$, if the limit exists. In the case of Shannon entropy, it does exist for all stationary sequences (Cover and Thomas, 1991). In contrast, Shields (1992) shows that the corresponding “relative information rate”, $\lim_{N \rightarrow \infty} N^1 \mathcal{D}(P \| Q)$, need not exist.

Also, Shannon’s result applies only to the *expected* length of the encoding, thus providing no assurance that any individual sequence will not have a markedly different compressed length. The most general statement of the Noiseless Coding Theorem that could be hoped for is that effectively *every* sufficiently long symbol sequence would have the right encoded length, provided that it is generated from an ergodic source⁴. That this is actually true is the content of the

⁴ An ergodic source is, for our purposes, a stationary stochastic process in which no set, \mathcal{S} , of symbol sequences is mapped via time translation onto itself, except for sets of zero probability and sets of probability one. Note that stationarity requires that the probability of \mathcal{S} also remains invariant under the time translation. Per Birkhoff’s ergodic theorem (Billingsly, 1978), these conditions guarantee that averages taken over a sufficiently long individual sequence will correspond to averages taken over the

Theorem (Shannon-McMillan-Breiman).

$$\lim_{N \rightarrow \infty} -\frac{1}{N} \log [P\{\alpha_0, \alpha_1, \dots, \alpha_{N-1}\}] = \lim_{n \rightarrow \infty} -\frac{1}{N} E_P [\log P(\alpha)]$$

with probability 1 (Billingsley, 1977) for any ergodic source, α , generated from a finite alphabet.

Note that $\log[P\{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{N-1}\}]/N$ is a random variable because $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{N-1}$ is a particular realization of a stochastic process. Yet the Shannon-McMillan-Breiman (SMB) Theorem states that, in the limit as $N \rightarrow \infty$, the length of the optimally encoded version of the substring is the (non-random) log of the probability of the observed substring, almost surely under P . This is precisely the result needed to establish the strong form of the Noiseless Coding Theorem. Indeed, per Cover and Thomas, (1991), the length, $\ell(\alpha_0 \dots \alpha_{N-1})$, of the optimal encoding of α_0 thru α_{N-1} , satisfies

$$-\log P\{\alpha_0 \dots \alpha_{N-1}\} \leq \ell(\alpha_0 \dots \alpha_{N-1}) \leq -\log P\{\alpha_0 \dots \alpha_{N-1}\} + 1,$$

so

$$\lim_{N \rightarrow \infty} -\frac{1}{N} \log [P\{\alpha_0, \alpha_1, \dots, \alpha_{N-1}\}] = \lim_{N \rightarrow \infty} \frac{1}{N} \ell(\alpha_0, \alpha_1, \dots, \alpha_{N-1}),$$

as required. This result argues powerfully in favor of using Shannon's definition as *the* per-symbol measure of information in a long message, especially since, per Barron (1985), the theorem remains true if α is a sequence of real random variables. Clearly, a similar argument could be made for any definition of pragmatic information for which a similar result is true, and, as will be shown shortly, this is the case for the per-symbol version of (2).

From Random Walks to Random Grammars

The simplest kind of random symbol sequence is a sequence of independently chosen symbols, or, more generally, random real variables. The only "information" in the sequence is *which* values were chosen for which position in the sequence, and the only parameters needed to characterize the process are the probabilities of observing these values. Evidently, the pragmatic information that an observer would use in predicting subsequent values in the sequence, given its history, is zero, corresponding to the fact that a price series with independent changes is, by definition, weakly efficient. Arguments over whether markets really *are* efficient have been raging for decades in such works as *A Random Walk Down Wall Street* (Malkiel, 1991) and *A Non-Random Walk Down Wall*

ensemble of all such sequences. Otherwise, it is possible that even stationary processes could get trapped in some subset of the ensemble.

Street (Lo and MacKinlay, 1999). Previous attempts to resolve such arguments generally assume certain kinds of intertemporal dependence, such as serial correlation. Nevertheless, true market efficiency rules out *any* kind of intertemporal dependence, including that implied by the “technical analysis” of complex price patterns (Murphy, 1999).

A step up in complexity is the class of symbol sequences generated by a Markov chain, in which successive symbols are chosen as a function of a finite number, n , of previous symbols. Again, the class of such random processes can be generalized somewhat by allowing each symbol to be a real valued random variable. Textbook treatments of Markov processes almost always consider so-called first order Markov processes, in which the probability that a value appears in the sequence can only depend on the value immediately preceding it. While first order Markov processes can be generalized to so-called n^{th} order Markov processes, in which the probability that a value appears in the sequence depends on the n values immediately preceding it, n^{th} order Markov processes are isomorphic to vector valued first order processes. Therefore, k step transition probabilities for n^{th} order Markov processes also decay exponentially with k .

Trading systems are often built on rules such as “if a price exceeds a given threshold, then sell,” rules which are effectively the transition rules of a finite state automaton. Also, most popular models of price volatility are Markov processes. Typical of these is the familiar univariate GARCH(1,1) process, effectively a random walk with variable step size, given at step n by

$$\begin{aligned} R_n &= R_0 + \sigma_{n-1} B_n \\ P_n &= (1 + R_{n-1}) P_{n-1} \\ \sigma_n^2 &= \alpha \sigma_{n-1}^2 + \beta \sigma_0^2 + \gamma R_n^2, \end{aligned}$$

where R_0 , α , β , and γ are constants and P_n , R_n , and σ_n are, respectively, the price, per-period return, and per period volatility. B_n is a normal random variable with mean zero and unit variance in the time units of the problem. Again, all of these processes are characterized by exponential decay of transients.

However, financial markets consist of literally millions of coupled trading systems, both explicitly implemented in computer software and implicit in human traders’ minds. It therefore seems likely that a more complex stochastic process would emerge. There is, in fact, a large literature suggesting fractal and/or multi-fractal scaling in financial markets, including an analysis of over a million increments of the Standard and Poor’s 500 Index from the years 1984-1989 (Mantegna & Stanley, 1995). This literature includes a series of models in Calvet and Fisher (2008), in which transients decay as an inverse power of the elapsed time, rather than exponentially.

It is precisely because financial markets could exhibit these more complex kinds of dependencies that a study of the pragmatic information of historical price series for predicting future prices is relevant.

Limit Theorems for Pragmatic Information

In this section, we establish first, a version of the “wrong code theorem” for pragmatic information, proving along the way that a pragmatic information rate for stationary sequences and stationary side messages must exist. For ergodic sequences, we need only quote the known result that, just as with the Shannon-McMillan-Breiman Theorem, the pragmatic information rate is, in the limit of long sequences and side messages, the observable increase in the entropy rate of the sequence.

First, the formal statement of the

“Wrong Code” Theorem for pragmatic information. Suppose an encoding of a finite symbol string α that would be optimal assuming the wrong probability measure $Q(\alpha)$ is used in place of the optimal encoding. Suppose further that, given the (random) side message μ , the actual probability of observing α could be determined to be $P(\alpha = A | \mu = m)$. Then $E_\mu[\ell(\alpha)]$, the expected length of the resulting encoding, averaged over all possible side messages μ , satisfies the bounds

$$\mathcal{H}(\alpha | \mu) + \mathcal{I}(\alpha; \mu) \leq E_\mu[\ell(\alpha)] \leq \mathcal{H}(\alpha | \mu) + \mathcal{I}(\alpha; \mu) + 1,$$

where $\mathcal{H}(\alpha | \mu)$ is the conditional entropy of α , given side message μ , $\mathcal{I}(\alpha; \mu)$ is defined as in (2), and $\ell(\alpha)$ is the length, in symbols, of the encoding of α using Q .

Proof: Let μ be one of the possible side messages, and write P_μ and E_μ in place of the more cumbersome $P(\alpha | \mu)$ and $E_{P(\alpha | \mu)}$, respectively. As guaranteed by the Wrong Code Theorem of Cover and Thomas (1991),

$$\mathcal{H}(P_\mu) + \mathcal{D}(P_\mu \| Q) \leq E_\mu[\ell(\alpha)] \leq \mathcal{H}(P_\mu) + \mathcal{D}(P_\mu \| Q) + 1.$$

Recalling that μ is, in fact, one possible value of the random variable μ and taking expectations over this random variable, the resulting inequalities are

$$\sum_\mu \varphi_\mu \mathcal{H}(P_\mu) + \sum_\mu \varphi_\mu \mathcal{D}(P_\mu \| Q) \leq \sum_\mu \varphi_\mu E_\mu[\ell(\alpha)] \leq \sum_\mu \varphi_\mu \mathcal{H}(P_\mu) + \sum_\mu \varphi_\mu \mathcal{D}(P_\mu \| Q) + 1.$$

The first two sums in the leftmost and rightmost inequalities are, respectively, $\mathcal{H}(\alpha | \mu)$ and $\mathcal{I}(\alpha; \mu)$, and the expectation in the middle sum is indeed $E_\mu[\ell(\alpha)]$. ■

The next step is to consider what happens to pragmatic information in the limit of long sequences. However, we must also consider what happens with the side messages, as different applications seem to demand different versions of the limit. For example, a receiver/decision maker may have already received an entire side message μ before any

output information is generated, perhaps because the set of possible side messages is finite. We are then interested in $\lim_{N \rightarrow \infty} \frac{1}{N} \mathcal{I}(\alpha_1 \dots \alpha_N; \mu)$. Another possibility is that the decision maker is adding characters to the α sequence information at the same time it is receiving information via characters in μ . In this case, we are interested in $\lim_{N \rightarrow \infty} \frac{1}{N} \mathcal{I}(\alpha_1 \dots \alpha_N; m_1 \dots m_N)$. In contrast to relative entropy, these limits do exist, per the

Existence Theorem for Pragmatic Information Rates for Stationary Processes. If n is the length of α , the limits

$\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{I}(\alpha_1 \dots \alpha_n; \mu)$, $\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{I}(\alpha; \mu_1 \dots \mu_n)$, $\lim_{n \rightarrow \infty} \mathcal{I}(\alpha_n | \alpha_1 \dots \alpha_{n-1}; \mu)$, $\lim_{n \rightarrow \infty} \mathcal{I}(\alpha_1 \dots \alpha_n; \mu_n | \mu_1 \dots \mu_{n-1})$, and $\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{I}(\alpha_1 \dots \alpha_n; \mu_1 \dots \mu_n)$ all exist, thus justifying the use of the notation $i(\alpha; \mu)$ to represent the common limit.

Proof: The proof of the existence and equality of the first and third limits parallels the proof of the existence and equality of the two versions of the entropy rate, $\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{H}(\alpha_1 \dots \alpha_n)$ and $\lim_{n \rightarrow \infty} \mathcal{H}(\alpha_n | \alpha_1 \dots \alpha_{n-1})$. Additional details can be found in Cover and Thomas (1991).

As with the entropy rate, the terms in the sequence comprising $\mathcal{I}(\alpha_n | \alpha_1 \dots \alpha_{n-1}; \mu)$, for a fixed random variable μ , are monotonically decreasing towards zero with increasing n , since we must have, for stationary α ,

$$0 \leq \mathcal{I}(\alpha_n | \alpha_1 \dots \alpha_{n-1}; \mu) \leq \mathcal{I}(\alpha_n | \alpha_2 \dots \alpha_{n-1}; \mu) = \mathcal{I}(\alpha_{n-1} | \alpha_1 \dots \alpha_{n-2}; \mu).$$

A basic theorem in real analysis guarantees that such sequences converge to a definite limit. Another such theorem also guarantees that Cesaro sums of convergent sequences converge to the limit of the summands. Since the chain rule for mutual information,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{I}(\alpha_1 \dots \alpha_n; \mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^N \mathcal{I}(\alpha_i | \alpha_1 \dots \alpha_{i-1}; \mu),$$

is such a sum, the sequence of such left sides must also converge to the same quantity, which we denote as $i(\alpha; \mu)$.

The convergence of the second and fourth limits follows from the complete symmetry between side messages and what we have called “actions”. The convergence of the fifth follows from the fact that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{I}(\alpha_1 \dots \alpha_n; \mu_1 \dots \mu_n) =$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[\mathcal{H}(\alpha_1 \dots \alpha_n) + \mathcal{H}(\mu_1 \dots \mu_n) - \mathcal{H}(\alpha_1 \dots \alpha_n; \mu_1 \dots \mu_n) \right]$$

and that the individual limits of the three terms on the right exist and sum to the quantity defined as $i(\alpha; \mu)$ above. ■

We can then use the above results to conclude that

$$i(\alpha; \mu) = \lim_{n \rightarrow \infty} \frac{1}{n} E_P [\ell_Q(\alpha) - \ell_P(\alpha; \mu)],$$

where $\ell_P(\alpha; \mu)$ is the length, in symbols, of the optimal (shortest) encoding of α because it uses the information in μ to determine the actual probability distribution of α . In contrast, $\ell_Q(\alpha)$ is the length, in symbols, of an encoding of α using $Q(\alpha)$, the probability distribution that would be used for encoding without benefit of μ . This result is sufficient to conclude that, for all stationary α and μ , the pragmatic information rate is precisely the expected improvement in the length per encoded symbol as the result of receiving μ .

The equivalent to the Shannon-McMillan-Breiman Theorem for mutual information does not seem to be named after anyone. We therefore state it as the

Ergodic Theorem for Mutual Information Rates (Barron, 1985). Let P_α be the probability measure of the ergodic random sequence α , let P_μ be the probability measure of the ergodic random sequence μ , and let $P_{\alpha, \mu}$ be the joint probability measure of the ergodic random sequences α and μ . Then, with $P_{\alpha, \mu}$ probability one,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \left[\frac{P_{\mu, \alpha}(\alpha, \mu)}{P_\mu(\mu)P_\alpha(\alpha)} \right] = i(\alpha, \mu)$$

In particular, with probability one in the measure $P_{\alpha, \mu}$,

$$i(\alpha; \mu) = \lim_{n \rightarrow \infty} \frac{1}{n} [\ell_Q(\alpha) - \ell_P(\alpha; \mu)].$$

Note that α and μ can be either finite alphabet or real valued stationary processes.

A point that was insufficiently emphasized in previous work about pragmatic information is the tie-in with the theory of noisy communications channels. Central to the latter is a careful formalization of an information source and a communications channel. The

source is characterized by the probability triple, (Y, \mathcal{Y}, ψ) , consisting of a probability space Y , the collection, \mathcal{Y} , of measurable subsets of Y , and the relevant measure, ψ . Gray (2009) defines the formal channel as the triple $[Y, v_y, Z]$, where Y and Z are probability spaces and, for each $y \in Y$, v_y is a probability measure on Z , the collection of measurable subsets of Z . The subscript attached to v_y captures the intuition that there will be different probabilities of receiving $z \in Z$, depending on which y was sent. All of this formalism makes possible the definition of what Billingsley (1978) calls the “input-output measure,” Ψ as

$$\Psi(B, C) = \int_B v_y(C) \mu(dy)$$

for each B in \mathcal{Y} and C in \mathcal{Z} . If the receiver could know the channel input, y_0, y_1, \dots, y_{n-1} , the amount of information transmitted through the channel would be $\mathcal{H}(y_0, y_1, \dots, y_{n-1})$, computed via the measure μ . However, the receiver only knows the channel output, z_0, z_1, \dots, z_{n-1} , so the actual amount of information transmitted is

$$\mathcal{I}(y_0, y_1, \dots, y_{n-1}; z_0, z_1, \dots, z_{n-1}) = \mathcal{H}(y_0, y_1, \dots, y_{n-1}) - \mathcal{H}(y_0, y_1, \dots, y_{n-1} | z_0, z_1, \dots, z_{n-1}),$$

i.e. the reduction in the uncertainty of the original message, less the remaining uncertainty, given the output of the channel, computed using Ψ . In other words, the amount of information transmitted is precisely the pragmatic information of the input and output of the channel.

An Application to Portfolio Management

Following Cover and Thomas (1991), we apply the above results to the ongoing performance of a portfolio of M assets that we are allowed to trade at the beginning of each period. Let the fraction of the portfolio allocated to the m^{th} asset at the n^{th} period be $0 \leq b_n^m \leq 1$, with $\sum_m b_n^m = 1$ and suppose that no short sales are allowed. Denote the so-called “wealth relatives” of the assets for the n^{th} period as the (random) vector X_n , where the wealth relative of each asset at each period is the ratio of its price at the beginning of the period to its price at the end. It follows that each component of X_n and b_n are non-negative real numbers and that the value of the portfolio at the beginning of the N^{th} period (for $N > 0$) is

$$S_N = \prod_{n=0}^N b_n^T X_n.$$

Cover and Thomas define the doubling rate

$$W_N = E\left[\frac{\log_2 S_N}{N}\right],$$

and consider the limit of W_N as $N \rightarrow \infty$, after a suitable warning that the limit may or may not exist. One sufficient condition for the limit to exist is if X_n is ergodic and exactly one of the components of X_n has a non-zero value for each n (the so-called “horse race”, a generalization of the more familiar Kelly criterion). Then⁵, the fractional change in the portfolio after the n^{th} race is $\sum_i b_i X_i(n)$, where the random variable $X_i(n)$ is defined as

$$X_i(n) = \begin{cases} R_i & \text{if horse } i \text{ wins the } n^{th} \text{ race} \\ 0 & \text{otherwise} \end{cases}$$

with probability $p_i(n)$. In general, $p_i(n)$ and b_i can depend on previous races, as well as other history-specific conditions, though we assume that R_i cannot. In any case, for a single race,

$$E\left[\log_2 \sum_{i=1}^M b_i X_i(n)\right] = \sum_{i=1}^M p_i(n) \log_2(b_i R_i)$$

By setting $q_i = 1/(R_i T)$, with $\sum_k (1/R_k) = T$, we ensure that $0 \leq q_i \leq 1$, for all i , and that $\sum_k q_k = 1$. We can therefore interpret the q 's as “track probabilities,” the odds that the bookies are prepared to offer. The appropriate substitutions give

$$\begin{aligned} \sum_{i=1}^M p_i \log_2(b_i R_i) &= \sum_{i=1}^M p_i \log_2\left(\frac{b_i}{p_i} \frac{p_i}{T q_i}\right) \\ &= \mathcal{D}(\mathbf{p} \| \mathbf{q}) - \mathcal{D}(\mathbf{p} \| \mathbf{b}) - \log_2 T, \end{aligned}$$

where we have introduced the notation $\mathbf{p}(n)$ for the vector of the $p_i(n)$'s, \mathbf{q} for the vector of the q_i 's, and \mathbf{b} for the vector of the b_i 's. The case where $T = 1$ corresponds to the situation where the race is fair with respect to the track probabilities (and is the case considered by Cover and Thomas). Indeed, for $T = 1$ and the bet allocations $b_i = q_i$, $S(n+1) = S(n)$ for all n and for all values of the X 's. When $T > 1$, the race is rigged against the bettor, who can only make money (via a positive doubling rate) if his betting allocation is a better estimate of $\mathbf{p}(n)$ than the track probabilities \mathbf{q} . This is the situation of a “price taker” who must buy at an “offered price” that is higher than the market maker's selling or “bid price” and who must sell at the market maker's “bid price” (Figure 1 graphically illustrates the relevant conventions.). On the other hand, when $T < 1$, the race is rigged in favor of the bettor, who can make money with estimates of $\mathbf{p}(n)$ no better than, and perhaps even somewhat worse than \mathbf{q} . This is the situation of the market maker, who buys at the bid and sells at the offer. However, regardless of the value of T , the best policy for the bettor is always to choose $\mathbf{b} = \mathbf{p}(n)$ because

⁵ For $M = 2$, this model corresponds to the situation where a trader has decided to commit a portion $b < 1$ of his wealth in the hope of realizing a gain of bR , with $R > 1$, the usual Kelly criterion situation. It is the experience of the author that professional traders actually think this way. Sometimes, a trader might decide to close out a trade at one or more intermediate points, *i.e.* before either b is lost or bR is won, a situation that can be incorporated into the present model by assuming that $M > 2$.

$$\mathcal{D}(\mathbf{p}(n) \parallel \mathbf{b}) \geq 0,$$

with equality if and only if $\mathbf{b} = \mathbf{p}(n)$. The optimal doubling rate after N races is therefore

$$\begin{aligned} W_N^* &= E\left[\frac{1}{N} \log_2 S_N\right] \\ &= \frac{1}{N} \sum_n \mathcal{D}(\mathbf{p}(n) \parallel \mathbf{q}) - \log_2 T \\ &= \frac{1}{N} \sum_n \sum_i p_i(n) \log_2 \frac{p_i(n)}{q_i} - \log_2 T \end{aligned}$$

We note that $(\log_2 S_N)/N \rightarrow W_N^*$ almost surely as $N \rightarrow \infty$ if X_n , the sequence of race outcomes, is ergodic. Note also that, in the special case where $q_i = 1/M$ for all i (“even odds” for each horse), the above expression for the optimal doubling rate approaches $\log_2 M - h$, where h is the per-symbol entropy of the sequence of race outcomes, shown to have a definite value by the Shannon-McMillan-Breiman Theorem. We recognize the quantity $\log_2 M - h$ as the channel capacity associated with the probability vectors $\mathbf{p}(n)$.

Pragmatic information enters the picture by supposing that the factors that influence $\mathbf{p}(n)$ at the n^{th} race are made available to the bettor via a “side message”, $\mu(n)$, which could include such things as the previous history of the process. We reflect the knowledge of these factors by re-writing $\mathbf{p}(n)$ as $\mathbf{p}^{(\mu)}$, the probability vector given the side message μ (with its argument n suppressed) with components $(p_{1|\mu}, p_{2|\mu}, \dots, p_{M|\mu})$. At the n^{th} race, we now have

$$\begin{aligned} E\left[\log_2 \sum_{i=1}^M b_i X_i(n)\right] &= \sum_{i=1}^M p_{i|\mu} \log_2 (b_i R_i) \\ &= \mathcal{D}(\mathbf{p}^{(\mu)} \parallel \mathbf{q}) - \mathcal{D}(\mathbf{p}^{(\mu)} \parallel \mathbf{b}) - \log_2 T \end{aligned}$$

It follows that at the n^{th} race, the bettor should choose $\mathbf{b} = \mathbf{p}^{(\mu)}$ for all n . If φ_μ is the probability of side message μ , the optimal expected doubling rate, now averaged over all such side messages, is given by

$$\begin{aligned}
W_N^* &= E\left[\frac{1}{N} \log_2 S_N\right] \\
&= E_{\mu}\left[\frac{1}{N} \sum_n \mathcal{D}\left(\mathbf{p}^{(\mu)} \| \mathbf{q}\right) - \log_2 T\right] \\
&= E_{\mu}\left[\frac{1}{N} \sum_n \sum_i p_{i|\mu} \log_2 \frac{p_{i|\mu}}{q_i}\right] - \log_2 T \\
&= \frac{1}{N} \sum_n \sum_i p_{i,\mu} \log_2 \frac{p_{i,\mu} p_i}{\varphi_{\mu} q_i p_i} - \log_2 T \\
&= \frac{1}{N} \sum_n \sum_i p_{i,\mu} \log_2 \frac{p_{i,\mu}}{\varphi_{\mu} p_i} + \frac{1}{N} \sum_n \sum_i p_i \log_2 \frac{p_i}{q_i} - \log_2 T
\end{aligned}$$

Comparing the above expression for W_N^* with the equivalent expression without side information, we see that the additional term in the former is precisely the pragmatic information of the side messages. Furthermore, if the side messages are an ergodic message stream, then, once more, the convergence, as $N \rightarrow \infty$, of $(\log_2 S_N)/N$ to its expected value is almost sure.

In the more general case where more than one of the X_i 's can be positive, and \mathbf{X} is merely stationary, Cover and Thomas's only guidance as to finding the optimal allocation is that, if $\mathbf{c}_n(\mathbf{X})$ is optimal and $\mathbf{a}_n(\mathbf{X})$ is another allocation, then $E[\mathbf{a}_n(\mathbf{X})^T X_n / \mathbf{c}_n(\mathbf{X})^T X_n] \leq 1$. Nevertheless, if $\mathbf{b}_n(\mathbf{X})$ is optimal, given side information $\boldsymbol{\mu}_n$, and \mathbf{c}_n is optimal without it, the mutual information between $\boldsymbol{\mu}$ and \mathbf{X} still bounds the difference in the optimal doubling rates, $W_b - W_c = \Delta W$, provided the expectation is taken over both $\boldsymbol{\mu}$ and \mathbf{X} . In demonstrating this result, let $f(\mathbf{v}_n | \mathbf{X}_n, \boldsymbol{\mu}_n)$ be the conditional density of X_n , given \mathbf{X}_n and $\boldsymbol{\mu}_n$, $g(\mathbf{v}_n | \mathbf{X}_n)$ for the conditional density of X_n , given only \mathbf{X}_n , and let $\varphi(\boldsymbol{\mu}_n)$ be the unconditional density of $\boldsymbol{\mu}_n$. It is also convenient to suppress the \mathbf{X} dependence of \mathbf{b} and \mathbf{c} . We have, if we use subscripts for variables over which expectations are taken,

$$\begin{aligned}
\Delta W &= \lim_{N \rightarrow \infty} \frac{1}{N} E_{\mathbf{X}_N \boldsymbol{\mu}_N} \left\{ \sum_{n=1}^N E_{X_n} [\log_2 (\mathbf{b}^T X_n) \| \mathbf{X}_{n-1}] - \sum_{n=1}^N E_{X_n} [\log_2 (\mathbf{c}^T X_n) \| \mathbf{X}_{n-1}] \right\} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} E_{\mathbf{X}_N} \left\{ \sum_{n=1}^N E_{X_n \boldsymbol{\mu}_N} [\log_2 (\mathbf{b}^T X_n) \| \mathbf{X}_{n-1}] - \sum_{n=1}^N E_{X_n \boldsymbol{\mu}_N} [\log_2 (\mathbf{c}^T X_n) \| \mathbf{X}_{n-1}] \right\} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} E_{\mathbf{X}_N} \left\{ \sum_{n=1}^N E_{X_n \boldsymbol{\mu}_N} \left[\log_2 \left(\frac{\mathbf{b}^T X_n}{\mathbf{c}^T X_n} \right) \| \mathbf{X}_{n-1} \right] \right\}
\end{aligned}$$

The first line of the above set of equations represents a re-writing of the expectation of the definition as an iterated expectation. The second line uses the fact that we don't change the second term on the right side if we take expectations over a variable upon which nothing in that second term depends. Upon explicitly writing out the inner expectation in the third equation, we have

$$\Delta W = \lim_{N \rightarrow \infty} \frac{1}{N} E_{X_N} \left\{ \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} \varphi(\mathbf{m}_n) d\mathbf{m}_n \int_{-\infty}^{\infty} f(\mathbf{v}_n | X_n, \mathbf{m}_n) \log_2 \left[\frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} \right] d\mathbf{v}_n \right\}$$

We can re-write the argument of the logarithm above as

$$\log_2 \left[\frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} \right] = \log_2 \left[\frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} \frac{g(\mathbf{v}_n | X_n)}{f(\mathbf{v}_n | X_n, \mathbf{m}_n)} \frac{f(\mathbf{v}_n | X_n, \mathbf{m}_n)}{g(\mathbf{v}_n | X_n)} \right],$$

from which it follows that the integrals in the above sum can be re-written as

$$\begin{aligned} & \int_{-\infty}^{\infty} \varphi(\mathbf{m}_n) d\mathbf{m}_n \int_{-\infty}^{\infty} f(\mathbf{v}_n | X_n, \mathbf{m}_n) \log_2 \left[\frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} \frac{g(\mathbf{v}_n | X_n)}{f(\mathbf{v}_n | X_n, \mathbf{m}_n)} \frac{f(\mathbf{v}_n | X_n, \mathbf{m}_n)}{g(\mathbf{v}_n | X_n)} \right] d\mathbf{v}_n \\ &= \int_{-\infty}^{\infty} \varphi(\mathbf{m}_n) d\mathbf{m}_n \int_{-\infty}^{\infty} f(\mathbf{v}_n | X_n, \mathbf{m}_n) \log_2 \left[\frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} \frac{g(\mathbf{v}_n | X_n)}{f(\mathbf{v}_n | X_n, \mathbf{m}_n)} \right] d\mathbf{v}_n + \\ & \quad \int_{-\infty}^{\infty} \varphi(\mathbf{m}_n) d\mathbf{m}_n \int_{-\infty}^{\infty} f(\mathbf{v}_n | X_n, \mathbf{m}_n) \log_2 \left[\frac{f(\mathbf{v}_n | X_n, \mathbf{m}_n)}{g(\mathbf{v}_n | X_n)} \right] d\mathbf{v}_n \end{aligned}$$

The second term in the right of the last equation is the conditional pragmatic information of X_n and μ_n , given \mathbf{X}_n . For the first term in the above equation, Jensen's inequality allows us to write

$$\begin{aligned} & \int_{-\infty}^{\infty} f(\mathbf{v}_n | \mathbf{m}_n) \log_2 \left[\frac{\mathbf{b}_n^T \mathbf{x}_n}{\mathbf{c}_n^T \mathbf{x}_n} \frac{g(\mathbf{v}_n | \mathbf{X}_n)}{f(\mathbf{v}_n | \mathbf{X}_n, \mathbf{m}_n)} \right] d\mathbf{v}_n \leq \log_2 \left[\int_{-\infty}^{\infty} f(\mathbf{v}_n | \mathbf{X}_n, \mathbf{m}_n) \frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} \frac{g(\mathbf{v}_n | \mathbf{X}_n)}{f(\mathbf{v}_n | \mathbf{X}_n, \mathbf{m}_n)} d\mathbf{v}_n \right] \\ &= \log_2 \left[\int_{-\infty}^{\infty} g(\mathbf{v}_n | \mathbf{X}_n) \frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} d\mathbf{v}_n \right] \end{aligned}$$

Recall that \mathbf{c}_n is the optimum allocation for the n^{th} period if the side information is unavailable and that $g(\mathbf{v}_n)$ is the conditional density of X_n , regardless of side information, a situation in which \mathbf{b}_n is *not necessarily* optimal. As a result,

$$\int_{-\infty}^{\infty} g(\mathbf{v}_n | \mathbf{X}_n) \frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} d\mathbf{v}_n \leq 1$$

and

$$\log_2 \left[\int_{-\infty}^{\infty} g(\mathbf{v}_n | \mathbf{X}_n) \frac{\mathbf{b}_n^T \mathbf{v}_n}{\mathbf{c}_n^T \mathbf{v}_n} d\mathbf{v}_n \right] \leq 0$$

We conclude that

$$\begin{aligned} \Delta W &\leq \lim_{N \rightarrow \infty} E_{\mathbf{X}_N} \left\{ \frac{1}{N} \sum_{n=1}^N \mathcal{I}(X_n; \mu_n | \mathbf{X}_{n-1}) \right\} \\ &= \lim_{N \rightarrow \infty} E_{\mathbf{X}_N} \left\{ \frac{\mathcal{I}(\mathbf{X}_N; \mu_N)}{N} \right\} \\ &= i(\mathbf{X}; \mu), \end{aligned}$$

as claimed.

An Application to Market Efficiency

As suggested previously, pragmatic information can be a useful tool in the study of financial market efficiency. A physical system in equilibrium is characterized by being in a state of maximum entropy. Similarly, markets are in equilibrium when there is no possibility of arbitrage, since the pragmatic information of the future, given the past is zero. As Figure 1 makes clear, a well defined equilibrium price, $P_i(t)$, exists for a security only at those moments when the best bid is at least as large as the best offer, and, even then, that price exists only for the exact position size that was traded. The rest of the time, the best that can be said is that the existence of a bid/offer spread implies that prices, and thus equilibria, are “noisy”, in a sense we now make precise.

The previous discussion shows that the per-symbol rate

$$h(\mathbf{P}) - h(\mathbf{P} | \boldsymbol{\mu}) = i(\mathbf{P} ; \boldsymbol{\mu})$$

exists almost everywhere under the input-output measure, and that it represents the rate at which a market converts economically relevant information, $\boldsymbol{\mu}$, into prices, \mathbf{P} . We are therefore justified in identifying market efficiency with the statement that

$$h(\mathbf{P}) - h(\mathbf{P} | \boldsymbol{\mu}^-) = i(\mathbf{P} ; \boldsymbol{\mu}^-) = 0.$$

Here, $\boldsymbol{\mu}^-$ represents economically relevant information known in the “tradable past”, *i.e.* the past separated from the present by the (ever decreasing) lag between the time we acquire economically relevant information and the time we can utilize this information in a trade. If $i(\mathbf{P} ; \boldsymbol{\mu}^-) = 0$, then $h(\mathbf{P}) = h(\mathbf{P} | \boldsymbol{\mu}^-)$, implying that a knowledge of the “tradable past” tells us nothing worth knowing about present prices. In fact, this is demonstrably not true, as we now show.

In comparing their multifractal model to GARCH(1,1) price dynamics, Calvet and Fisher (2008) state that “GARCH(1,1) is often viewed as a standard benchmark that is very difficult to outperform in forecasting exercises.” Another, even more widely used benchmark for financial modeling is the normal distribution of returns. Accordingly, we choose a GARCH(1,1) process with normally distributed returns for a study of $i(\mathbf{P} ; \boldsymbol{\mu}^-)$.

Cover and Thomas (1991) give the formula $\mathcal{H}(\mathbf{P}) = \frac{1}{2} \log_2 (2\pi e \sigma_0^2)$ for the entropy (in bits) of the unconstrained distribution of returns, where e has its usual meaning as the base of natural logarithms and σ_0 is the unconstrained volatility of returns. The appearance of σ_0 in this formula should not be surprising, because it is a reasonable proxy for uncertainty in returns on an absolute scale⁶. Similarly, $\mathcal{H}(\mathbf{P} | \boldsymbol{\mu}^-)$, the entropy of returns, conditional on the tradable past, is $\frac{1}{2} \log_2 (2\pi e \sigma_n^2)$, given that

$$\sigma_n^2 = \alpha \sigma_{n-1}^2 + \beta \sigma_0^2 + \gamma R_n^2 .$$

Since we must have $\alpha + \beta + \gamma = 1$ to ensure that σ_n^2 remains positive and finite,

$$\begin{aligned} i &= \frac{1}{2} \log_2 (2\pi e \sigma_0^2) - \frac{1}{2} \log_2 (2\pi e \sigma_n^2) \\ &= \frac{1}{2} \log_2 \left\{ \frac{\sigma_0^2}{\alpha \sigma_{n-1}^2 + (1-\alpha-\gamma) \sigma_0^2 + \gamma R_{n-1}^2} \right\} \\ &= -\frac{1}{2} \log_2 \left\{ 1 + \alpha \left[\left(\frac{\sigma_{n-1}}{\sigma_0} \right)^2 - 1 \right] + \gamma \left[\left(\frac{R_{n-1}}{\sigma_0} \right)^2 - 1 \right] \right\} \end{aligned}$$

If R_n is a stochastic process, then both σ_n and the above expression must also be stochastic. It therefore makes sense to take expectations. We thus obtain

⁶ as opposed to the logarithmic scale used by entropy.

$$\begin{aligned}
E[i] &= -\frac{1}{2} E \left\langle \log_2 \left\{ 1 + \alpha \left[\left(\frac{\sigma_{n-1}}{\sigma_0} \right)^2 - 1 \right] + \gamma \left[\left(\frac{R_{n-1}}{\sigma_0} \right)^2 - 1 \right] \right\} \right\rangle \\
&\geq -\frac{1}{2} \log_2 \left\{ E \left\langle 1 + \alpha \left[\left(\frac{\sigma_{n-1}}{\sigma_0} \right)^2 - 1 \right] + \gamma \left[\left(\frac{R_{n-1}}{\sigma_0} \right)^2 - 1 \right] \right\rangle \right\} \\
&= -\frac{1}{2} \log_2 \left\{ 1 + \alpha E \left[\left(\frac{\sigma_{n-1}}{\sigma_0} \right)^2 - 1 \right] + \gamma E \left[\left(\frac{R_{n-1}}{\sigma_0} \right)^2 - 1 \right] \right\} \\
&= -\frac{1}{2} \log_2 \{1\} \\
&= 0,
\end{aligned}$$

where the inequality in the second line follows from Jensen's inequality, and the expectations are zero by the assumption of stationarity. Because $h(\mathbf{P}) \geq h(\mathbf{P} | \boldsymbol{\mu})$, we must have $E[i] \geq 0$ in any case. However, Jensen's inequality is strict except for the unlikely scenario in which $\sigma_n = R_n = \sigma_0$ almost surely. In all other scenarios, we conclude that $E[i]$ is strictly positive always, and thus i itself must be strictly positive, at least some of the time.

But the real advantage of the present formulation of market efficiency is the possibility of actually quantifying the amount of inefficiency and investigating where significant sources of inefficiency arise. For example, in most estimates of the GARCH(1,1) model, α is significantly larger than γ , which corresponds to the intuition that the predictability of volatility plays a greater role in any market inefficiency than the predictability of price because volatility is the more abstract quantity. More comprehensive studies of the observed pragmatic information in various markets, possibly gleaned from the compressibility of various price series, might provide additional details about how real markets approximate the ideal of efficiency.

The Noisy Coding Theorem as a Phase Transition and What that Means for Finance

As indicated in the Introduction, the primary goal of this paper is to identify the appropriate pragmatic information theoretic parallels to the Noiseless Coding Theorem of standard information theory. In this section, however, we briefly consider the most significant result in information theory, namely, Shannon's Noisy Coding Theorem (Cover and Thomas, 1991; Gray, 2009). According to this theorem, there exists a precise “channel capacity”, C , such that information can be transmitted at any rate $R < C$ with arbitrarily small probability of error. In contrast, if $R > C$, the probability of error is bounded away from zero. In general, C will depend on both the channel and ψ , the probability measure of the input signals. The ψ dependence is eliminated by defining the “channel capacity”, C , as

$$C = \sup_{\psi(Y)} [\mathcal{H}(Y) - \mathcal{H}(Y|Z)],$$

where the supremum is taken over all probability measures, $\psi(Y)$, of input signals, Y , and the entropy rates $\mathcal{H}(Y)$ and $\mathcal{H}(Y|Z)$ are given by

$$\mathcal{H}(Y) = \lim_{n \rightarrow \infty} \frac{1}{n} [\mathcal{H}(Y_1 \dots Y_n)]$$

and

$$\mathcal{H}(Y|Z) = \lim_{n \rightarrow \infty} \frac{1}{n} [\mathcal{H}(Y_1 \dots Y_n | Z_1 \dots Z_n)],$$

respectively.

The proof of the Noisy Coding Theorem usually assumes that the channel is memoryless, in that the output of the channel depends on neither previous inputs nor previous outputs. If the supremum in the definition of the channel capacity is over all ergodic or stationary sources, this assumption is justified, because it can be shown that feedback from previous inputs does not increase the channel capacity. However, if only normally distributed signals are input, feedback does increase channel capacity. Perhaps feedback or a channel with memory would increase channel capacity for sources restricted either to GARCH processes or multi-fractal processes. It is certainly true that a receiver can alter the statistical characteristics of an input signal. For example, the output of a finite state machine (Hopcroft, *et. al.*, 2006) that receives a sequence of independent input random variables is a Markov Chain, as the probability of a specific output will, in general, depend on the previous state of the receiver.

The significance of the Noisy Coding Theorem in its original context is that the channel capacity can be deduced from physical properties of the channel, such as its signal to noise ratio, etc. Physical and psychological considerations might well determine a “channel capacity” in the financial context, as well, in that there is surely a limit to the rate at which investors can respond to changing market conditions. What seems to be unique to the financial context is that investors are constantly re-evaluating what constitutes “signal” and what constitutes “noise”. Perhaps more importantly, investors sometimes willfully ignore the realities of the market, as we saw so tragically in the run-up to the current financial crisis. We can treat such willful ignorance as an effective lowering of the channel capacity.

In physics, phase transitions (sudden, qualitative changes in a physical system with small changes in temperature, pressure, or other state variable) are characterized by a discontinuity in the physical entropy of the system or its derivative with respect to that state variable (Thompson, 1979). The Noisy Coding Theorem is effectively the statement that the mutual information has a discontinuous derivative with respect to the input

entropy rate at the channel capacity. Since the input entropy rate to a communications channel is the analogue of the physical entropy, the Noisy Coding Theorem is therefore also the statement that there is a phase transition of the mutual information of the channel. In fact, papers such as (Kabashima and Saad, 1999) study a class of error correcting codes using the detailed formalism developed to describe physical phase transitions. These kinds of results are of interest because wide classes of systems seem to exhibit qualitatively and sometimes even quantitatively similar behaviors, irrespective of their detailed composition. For example, the so-called quasispecies model of evolution (Eigen, 1971, Eigen and Schuster, 1979, Eigen, Schuster, and McCaskill, 1988) has an “error threshold”, a mutation rate above which selective advantages cannot be passed on to future generations. This error threshold, which is effectively a phase transition (Campos and Fontanari, 1999), can also be viewed as an attempt by nature to transmit genetic information from one generation to the next at a rate that is faster than a genetic “channel capacity”.

Summary and Conclusion

As indicated in the Introduction, this paper is part of an ongoing investigation of the notion of “pragmatic information.” One goal of this investigation, and one of the first goals of this paper was to establish that the mutual information between the input to a decision making process and the decisions that are output is a sensible definition of pragmatic information. What seems novel about this view is that the actions made as the result of a decision are also part of the communications channel that transmitted the information to the decision maker. This paper has shown that this kind of analysis can be extended to systems involving information rates.

While the mathematics underlying mutual information is well known, the pragmatic information paradigm seems capable of providing new insight to apparently diverse phenomena. In a previous paper (Weinberger, 2002), it was shown that a pragmatic information could be associated with a biological population and that the rate of its increase could be identified with the rate at which the population was evolving. In this paper, we associated pragmatic information with optimal doubling rates of financial market returns and with a measure of market inefficiency. We also saw that the run-up to a financial bubble can be characterized as a lowering of a pragmatic informational channel capacity. Whether these ideas themselves contain any pragmatic information will only become clear as the result of further research.

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“Messages” to price discovery process, including previous price history, economic fundamentals, etc.

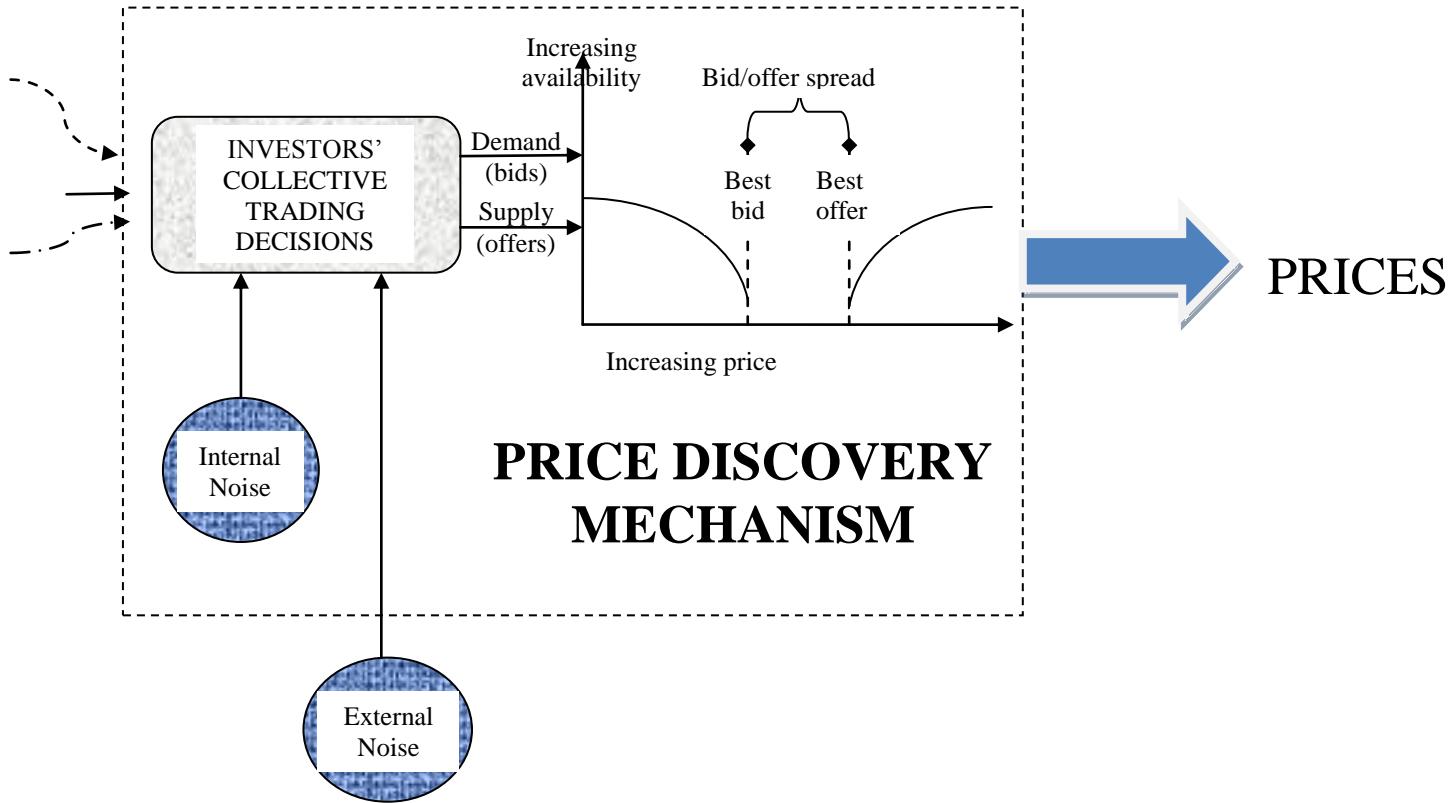


FIGURE 1. From market conditions to prices via countless individual trading decisions, resulting in the complex ecosystem of bid/s and offers that constitutes the price discovery process. Note that both the “internal noise” of market rumors, bad judgment, etc. and the “external noise” of errors in economic data measurements, structural limitations in markets, etc. can both limit the precision of price discovery