

THE LIMITS OF LEVERAGE

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When trading incurs proportional costs, leverage can scale an asset's return only up to a maximum multiple, which is sensitive to its volatility and liquidity. In a model with one safe and one risky asset, with constant investment opportunities and proportional costs, we find strategies that maximize long term returns given average volatility. As leverage increases, rising rebalancing costs imply declining Sharpe ratios. Beyond a critical level, even returns decline. Holding the Sharpe ratio constant, higher asset volatility leads to superior returns through lower costs.

KEYWORDS: leverage, transaction costs, portfolio choice.

MATHEMATICS SUBJECT CLASSIFICATION (2010): 91G10, 91G80

JEL CLASSIFICATION: G11, G12

1. INTRODUCTION

If trading is costless, leverage can scale returns without limits. Using the words of Sharpe (2011):

"If an investor can borrow or lend as desired, any portfolio can be leveraged up or down. A combination with a proportion k invested in a risky portfolio and $1 - k$ in the riskless asset will have an expected excess return of k [times the excess return of the risky portfolio] and a standard deviation equal to k times the standard deviation of the risky portfolio. Importantly, the Sharpe Ratio of the combination will be the same as that of the risky portfolio."

In theory, this insight implies that the efficient frontier is linear, that efficient portfolios are identified by their common maximum Sharpe ratio, and that any of them spans all the other ones. Also, if leverage can deliver any expected return, then risk-neutral portfolio choice is meaningless, as it leads to infinite leverage.

In practice, hedge funds and high-frequency trading firms employ leverage to obtain high returns from small relative mispricing of assets. A famous example is Long Term Capital Management, which used leverage of up to 40 times to increase returns from

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Volatility (σ)	Bid-Ask Spread (ε)		
	0.01%	0.10%	1.00%
10%	71.85 (71.22)	23.15 (22.58)	7.72 (7.12)
20%	50.88 (50.36)	16.45 (15.92)	5.56 (5.04)
50%	32.30 (31.85)	10.54 (10.07)	3.66 (3.18)

TABLE 1. Leverage multiplier (maximum factor by which a risky asset's return can be scaled) for different asset volatilities and bid-ask spreads, holding the Sharpe ratio at the constant level of 0.5. Multipliers are obtained from numerical solutions of (3.1), while their approximations from (1.1) are in brackets.

convergence trades between on-the-run and off-the-run treasury bonds (for example, see Edwards (1999)).

This paper shows that trading costs undermine these classical properties of leverage and set sharp theoretical limits to its applications. We start by characterizing the set of portfolios that maximize long term expected returns for given average volatility, extending the familiar efficient frontier to a market with one safe and one risky asset, where both investment opportunities and relative bid-ask spreads are constant. Figure 1 plots this frontier: expectedly, trading costs decrease returns, with the exception of a fully safe investment (the axes origin) or a fully risky investment (the attachment point with unit coordinates), which lead to static portfolios without trading, and hence earn their frictionless returns.¹

But trading costs do not merely reduce expected returns below their frictionless benchmarks. Unexpectedly, in the leverage regime (to the right of the full-investment point) rebalancing costs rise so quickly with volatility that returns cannot increase beyond a critical factor, the leverage *multiplier*. This multiplier depends on the relative bid-ask spread ε , the expected excess return μ and volatility σ , and approximately equals

$$0.3815 \left(\frac{\mu}{\sigma^2} \right)^{1/2} \varepsilon^{-1/2}. \quad (1.1)$$

Table 1 shows that even a modest bid-ask spread of 0.10% implies a multiplier of 23 for an asset with 10% volatility and 5% expected return (similar to a long-term bond), while the multiplier declines to 10 for an asset with equal Sharpe ratio, but volatility of 50% (similar to an individual stock). Leverage opportunities are much more limited for more illiquid assets with a spread of 1%: the multiplier declines from less than 8 for 10% volatility to less than 4 for 50% volatility. Importantly, these limits on leverage hold even allowing for continuous trading, infinite market depth (any quantity trades at the bid or ask price), and zero capital requirements.

Our results have two broad implications. First, with a positive bid-ask spread even a risk-neutral investor who seeks to maximize expected long-run returns takes finite leverage, and in fact a rather low leverage ratio in an illiquid market – risk-neutral portfolio choice is meaningful. The resulting multiplier sets an endogenous level of risk that the

¹As we focus on long term investments, we neglect the one-off costs of set up and liquidation, which are negligible over a long holding period.

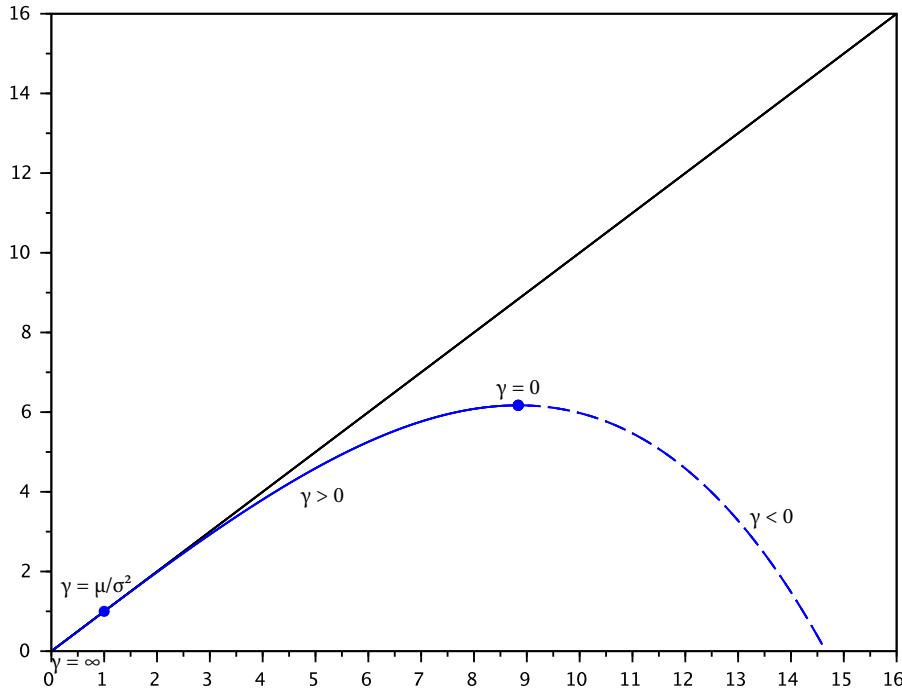


FIGURE 1. Efficient Frontier with trading costs, as expected excess return (vertical axis, in multiples of the asset's return) against standard deviation (horizontal axis, in multiples of the asset's volatility). The asset has expected excess return $\mu = 8\%$, volatility $\sigma = 16\%$, and bid-ask spread of 1%. Each point on the curve represents the performance of the optimal portfolio with risk aversion γ . The upper line denotes the classical efficient frontier, with no transaction costs. The maximum height of the curve ($\gamma = 0$) corresponds to the leverage multiplier. As γ increases, leverage, return, and volatility all decrease, reaching the asset's own performance $(1, 1)$ at $\gamma = \mu/\sigma^2$. As γ increases further, exposure to the asset declines below one, eventually vanishing at the origin ($\gamma = \infty$). The dashed frontier ($\gamma < 0$) is not “efficient” in that such returns are maximal for given volatility, but can be achieved with lower volatility in the solid frontier ($\gamma > 0$).

investor chooses not to exceed regardless of risk aversion, simply to avoid reducing returns with trading costs. In this context, margin requirements based on volatility (such as value at risk and its variations) are binding only when they reduce leverage below the multiplier, and are otherwise redundant. In addition, the multiplier shows that an exogenous increase in trading costs, such as a proportional Tobin tax on financial transactions, implicitly reduces the maximum leverage that any investor who seeks return is willing to take, regardless of risk attitudes.

Second, two assets with the same Sharpe ratio do not generate the same efficient frontier with trading costs, and more volatility leads to a superior frontier. For example (Table

1) with a 1% spread, the maximum leveraged return on an asset with 10% volatility and 5% return is $7.72 \times 5\% \approx 39\%$. By contrast, an asset with 50% volatility and 25% return (equivalent to the previous one from a classical viewpoint, as it has the same Sharpe ratio 0.5), leads to a maximum leveraged return of $3.66 \times 25\% \approx 92\%$. The reason is that a more volatile asset requires a lower leverage ratio (hence lower rebalancing costs) to reach a certain return. Thus, an asset with higher volatility spans an efficient frontier that achieves higher returns through lower costs.

This paper bears on the established literature on portfolio choice with frictions. The effect of transaction costs on portfolio choice is first studied by Magill and Constantinides (1976), Constantinides (1986), and Davis and Norman (1990), who identify a wide no-trade region, and derive the optimal trading boundaries through numerical procedures. While these papers focus on the maximization of expected utility from intertemporal consumption on an infinite horizon, Taksar, Klass and Assaf (1988), and Dumas and Luciano (1991) show that similar strategies are obtained in a model with terminal wealth and a long horizon – time preference has negligible effects on trading policies. This paper adopts the same approach of a long horizon, both for the sake of tractability, and because it focuses on the trade-off between return, risk, and costs, rather than consumption.

Our asymptotic results for positive risk aversion are similar in spirit to the ones derived by Shreve and Soner (1994), Rogers (2004), Gerhold, Guasoni, Muhle-Karbe and Schachermayer (2014), and Kallsen and Muhle-Karbe (2015), whereby transaction costs imply a no-trade region with width of order $O(\varepsilon^{1/3})$ and welfare costs of order $O(\varepsilon^{2/3})$. We also find that the trading boundaries obtained from a local mean-variance criterion are equivalent at the first order to the ones obtained from power utility.

The risk-neutral expansions and the limits of leverage of order $O(\varepsilon^{-1/2})$ are new, and are qualitatively different from the risk-averse case. These results are not regular perturbations of a frictionless analogue, which is ill-posed. They are rather singular perturbations, which display the speed at which the frictionless problem becomes ill-posed as the crucial friction parameter vanishes.

Finally, this paper connects to the recent work of Frazzini and Pedersen (2012) on embedded leverage. If different investors face different leverage constraints, they find that in equilibrium assets with higher factor exposures trade at a premium, thereby earning a lower return. Frazzini and Pedersen (2014) confirm this prediction across a range of markets and asset classes, and Asness et al. (2012) use it to explain the performance risk-parity strategies. With exogenous asset prices, we find that assets with higher volatility generate a superior efficient frontier by requiring lower rebalancing costs for the same return. This observation suggests that the embedded leverage premium may be induced by rebalancing costs in addition to leverage constraints, and should be higher for more illiquid assets.

The paper is organized as follows: section 2 introduces the model and the optimization problem. Section 3 contains the main results, which characterize the efficient frontier in the risk-averse (Theorem 3.1) and risk-neutral (Theorem 3.2) cases. Section 4 discusses the implications of these results for the efficient frontier, the trading boundaries of optimal policies and the embedded leverage effect. The section includes two supporting results, which show that the risk-neutral solutions arise as limits of their risk-averse counterparts for low risk-aversion (Theorem 4.1), and that the risk-neutral solutions are not constrained by the solvency condition (section 4.2). Section 5 offers a derivation of the main free-boundary problems from heuristic control arguments, and concluding remarks are in section 6. All proofs are in the appendix.

2. MODEL

The market includes one safe asset earning a constant interest rate of $r \geq 0$ and a risky asset with ask (buying) price S_t that follows

$$\frac{dS_t}{S_t} = (\mu + r)dt + \sigma dB_t, \quad S_0, \sigma, \mu > 0,$$

where B is a standard Brownian motion. The risky asset's bid (selling) price is $(1 - \varepsilon)S_t$, which implies a constant relative bid-ask spread of $\varepsilon > 0$, or, equivalently, constant proportional transaction costs.

We investigate the trade-off between a portfolio's average return against its realized variance. Denoting by w_t the portfolio value at time t , for an investor who observes returns with frequency $\Delta t = T/n$ in the time-interval $[0, T]$, the average return and its continuous-time approximation are²

$$\frac{1}{n\Delta t} \sum_{k=1}^n \left(\frac{w_{k\Delta t}}{w_{(k-1)\Delta t}} - 1 \right) \approx \frac{1}{T} \int_0^T \frac{dw_t}{w_t}.$$

In the familiar setting of no trading costs, $\frac{1}{T} \int_0^T \frac{dw_t}{w_t} = r + \frac{1}{T} \int_0^T \mu \pi_t dt + \frac{1}{T} \int_0^T \sigma \pi_t dB_t$, where π_t is the portfolio weight of the risky asset, hence the average return equals the average risky exposure times its excess return, plus the safe rate.

Likewise, the average squared volatility on $[0, T]$ is obtained by the usual variance estimator applied to returns, and has the continuous-time approximation

$$\frac{1}{n\Delta t} \sum_{k=1}^n \left(\frac{w_{k\Delta t}}{w_{(k-1)\Delta t}} - 1 \right)^2 \approx \frac{1}{T} \int_0^T \frac{d\langle w \rangle_t}{w_t^2}$$

reducing to $\frac{\sigma^2}{T} \int_0^T \pi_t^2 dt$ in the absence of trading costs.

With these definitions, the mean-variance trade-off is captured by maximizing

$$\frac{1}{T} \mathbb{E} \left[\int_0^T \frac{dw_t}{w_t} - \frac{\gamma}{2} \left\langle \int_0^T \frac{dw_t}{w_t} \right\rangle_T \right], \quad (2.1)$$

where the parameter $\gamma > 0$ is interpreted as a proxy for risk-aversion.

This objective nests several familiar problems. Without trading costs it reduces to

$$\frac{1}{T} \mathbb{E} \left[\int_0^T \left(\mu \pi_t - \frac{\gamma}{2} \sigma^2 \pi_t^2 \right) dt \right] \quad (2.2)$$

which is maximized by the optimal constant-proportion portfolio $\pi = \frac{\mu}{\gamma\sigma^2}$ dating back to Markowitz and Merton, and confirms that in a geometric Brownian motion market with costless trading, the objective considered here is equivalent to utility-maximization with constant relative risk aversion. With or without transaction costs, the risk-neutral objective $\gamma = 0$ boils down to the average annualized return over a long horizon, while $\gamma = 1$ reduces to logarithmic utility.

Trading costs make (2.1) lower than (2.2), as they hinder continuous portfolio rebalancing and make constant-proportion strategies unfeasible. The reason is that it is costly to keep the exposure to the risky asset high enough to achieve the desired return, and low enough to limit the level of risk – trading costs reduce returns and increase risk.

²All discrete statistics on this section converge in probability to their continuous-time counterparts. The budget equation and the definition of admissible strategies are in appendix A below.

To neglect the spurious, non-recurring effects of portfolio set-up and liquidation, we focus on the Equivalent Safe Rate³

$$\text{ESR} := \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \frac{dw_t}{w_t} - \frac{\gamma}{2} \left\langle \int_0^T \frac{dw_t}{w_t} \right\rangle_T \right] \quad (2.3)$$

which is akin to the one used by Dumas and Luciano (1991) in the context of utility maximization.

3. MAIN RESULTS

3.1. Risk aversion and efficient frontier. The first result characterizes the optimal solution to the main objective in (2.3) in the usual case of a positive aversion to risk ($\gamma > 0$). In this setting, the next theorem shows that trading costs create a no-trade region around the frictionless portfolio $\pi_* = \frac{\mu}{\gamma\sigma^2}$, and states the asymptotic expansions of the resulting average return and standard deviation⁴, thereby extending the familiar efficient frontier to account for trading costs.

Theorem 3.1. *Let $\frac{\mu}{\gamma\sigma^2} \neq 1$.*

- (i) *For any $\gamma > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there is a unique solution (W, ζ_-, ζ_+) , with $\zeta_- < \zeta_+$, for the free boundary problem*

$$\frac{1}{2}\sigma^2\zeta^2W''(\zeta) + (\sigma^2 + \mu)\zeta W'(\zeta) + \mu W(\zeta) - \frac{1}{(1+\zeta)^2} \left(\mu - \gamma\sigma^2 \frac{\zeta}{1+\zeta} \right) = 0, \quad (3.1)$$

$$W(\zeta_-) = 0 \quad (3.2)$$

$$W'(\zeta_-) = 0, \quad (3.3)$$

$$W(\zeta_+) = \frac{\varepsilon}{(1+\zeta_+)(1+(1-\varepsilon)\zeta_+)}, \quad (3.4)$$

$$W'(\zeta_+) = \frac{\varepsilon(\varepsilon-2(1-\varepsilon)\zeta_+-2)}{(1+\zeta_+)^2(1+(1-\varepsilon)\zeta_+)^2} \quad (3.5)$$

- (ii) *The trading strategy that buys at $\pi_- := \zeta_-/(1+\zeta_-)$ and sells at $\pi_+ := \zeta_+/(1+\zeta_+)$ as little as to keep the risky weight π_t within the interval $[\pi_-, \pi_+]$ is optimal.*

- (iii) *The maximum performance is*

$$\max_{\varphi \in \Phi} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \left(\mu\pi_t - \frac{\gamma\sigma^2}{2}\pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} \right] = \mu\pi_- - \frac{\gamma\sigma^2}{2}\pi_-^2, \quad (3.6)$$

where Φ is the set of admissible strategies in Definition A.1 below, $\varphi_t = \pi_t w_t / S_t$ is the number of shares held at time t , and φ_t^\downarrow is the cumulative number of shares sold up to time t .

- (iv) *The trading boundaries π_- and π_+ have the asymptotic expansions*

$$\pi_\pm = \pi_* \pm \left(\frac{3}{4\gamma} \pi_*^2 (\pi_* - 1)^2 \right)^{1/3} \varepsilon^{1/3} - \frac{(1-\gamma)\pi_*}{\gamma} \left(\frac{\gamma\pi_*(\pi_*-1)}{6} \right)^{1/3} \varepsilon^{2/3} + O(\varepsilon). \quad (3.7)$$

³In this equation the lim sup is used merely to guarantee a good definition a priori. A posteriori, optimal strategies exist in which the limit superior is a limit, hence the similar problem defined in terms of lim inf leads to the same solution.

⁴The exact formulae for average return, standard deviation, and average trading costs are in Appendix C.

The long-run mean (\hat{m}), standard deviation (\hat{s}), Sharpe ratio ($(\hat{m} - r)\hat{s}$), average trading costs (ATC) and equivalent safe rate (ESR) have expansions⁵

$$\hat{m} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{dw_t}{w_t} = r + \frac{\mu^2}{\gamma\sigma^2} - \frac{\sigma^2\pi_*(5\pi_*-3)}{2} \left(\frac{\gamma\pi_*(\pi_*-1)}{6} \right)^{1/3} \varepsilon^{2/3} + O(\varepsilon), \quad (3.8)$$

$$\hat{s} := \lim_{T \rightarrow \infty} \sqrt{\frac{1}{T} \left\langle \int_0^T \frac{dw_t}{w_t} \right\rangle_T} = \frac{\mu}{\gamma\sigma} - \frac{\sigma(7\pi_*-3)}{4\gamma} \left(\frac{\gamma\pi_*(\pi_*-1)}{6} \right)^{1/3} \varepsilon^{2/3} + O(\varepsilon), \quad (3.9)$$

$$\text{SR} := \frac{\hat{m} - r}{\hat{s}} = \frac{\mu}{\sigma} + \frac{3}{4 \cdot 6^{1/3}} (\pi_* - 1) (\gamma\pi_*(1 - \pi_*))^{1/3} \varepsilon^{2/3} + O(\varepsilon) \quad (3.10)$$

$$\text{ATC} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} = \frac{3\sigma^2}{\gamma} \left(\frac{\gamma\pi_*(\pi_*-1)}{6} \right)^{4/3} \varepsilon^{2/3} + O(\varepsilon), \quad (3.11)$$

$$\text{ESR} = r + \frac{\gamma\sigma^2}{2} \pi_*^2 - \frac{\gamma\sigma^2}{2} \left(\frac{3}{4\gamma} \pi_*^2 (\pi_* - 1)^2 \right)^{2/3} \varepsilon^{2/3} + O(\varepsilon). \quad (3.12)$$

Proof. The proof of the main part of this theorem is divided into Propositions B.1, B.4 and B.6 in Appendix B. The proof of the asymptotic results is in section C.4. \square

3.2. Risk neutrality and limits of leverage. In contrast to the risk-averse objective considered above, the risk-neutral objective leads to a solution which does not have a frictionless analogue: for small trading costs, both the optimal policy and its performance become unbounded as the optimal leverage increases arbitrarily. The next result describes the solution to the risk-neutral problem, identifying the approximate dependence of the leverage multiplier and its performance on the asset's risk, return and liquidity.

Theorem 3.2. Let $\gamma = 0$.

- (i) There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the free boundary problem (3.1)–(3.5) has a unique solution (W, ζ_-, ζ_+) with $\zeta_- < \zeta_+$.
- (ii) The trading strategy $\hat{\varphi}$ that buys at $\pi_- := \zeta_-/(1 + \zeta_-)$ and sells at $\pi_+ := \zeta_+/(1 + \zeta_+)$ as little as to keep the risky weight π_t within the interval $[\pi_-, \pi_+]$ is optimal.
- (iii) The maximum expected return is

$$\max_{\varphi \in \Phi} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{dw_t}{w_t} = r + \mu\pi_- . \quad (3.13)$$

- (iv) The trading boundaries have the series expansions

$$\pi_- = (1 - \kappa) \kappa^{1/2} \left(\frac{\mu}{\sigma^2} \right)^{1/2} \varepsilon^{-1/2} + 1 + O(\varepsilon^{1/2}), \quad (3.14)$$

$$\pi_+ = \kappa^{1/2} \left(\frac{\mu}{\sigma^2} \right)^{1/2} \varepsilon^{-1/2} + 1 + O(\varepsilon^{1/2}), \quad (3.15)$$

where $\kappa \approx 0.5828$ is the unique solution to

$$f(\xi) := \frac{3}{2}\xi + \log(1 - \xi) = 0, \quad \xi \in (0, 1). \quad (3.16)$$

Proof. See Appendix D below. \square

The next section discusses how these results modify the familiar intuition about risk, return, and performance evaluation in the context of trading costs.

⁵We are using the convention $a^{1/n} = \text{sign}(a) |a|^{1/n}$ for any $a \in \mathbb{R}$ and odd integer n , and $a^{2/n} = (a^2)^{1/n}$.

4. IMPLICATIONS AND APPLICATIONS

4.1. Efficient frontier. Theorem 3.1 extends the familiar efficient frontier to account for trading costs. Compared to the linear frictionless frontier, average returns decline because of rebalancing losses. Average volatility increases because more risk becomes necessary to obtain a given return net of trading costs.

To better understand the effect of trading costs on return and volatility, consider the dynamics of the portfolio weight in the absence of trading, which is

$$d\pi_t = \pi_t(1 - \pi_t)(\mu - \sigma^2\pi_t)dt + \sigma\pi_t(1 - \pi_t)dB_t. \quad (4.1)$$

The central quantity here is the portfolio weight volatility $\sigma\pi_t(1 - \pi_t)$, which vanishes for the single-asset portfolios $\pi_t = 0$ or $\pi_t = 1$, remains bounded above by $\sigma/4$ in the long-only case $\pi_t \in [0, 1]$, and rises quickly with leverage ($\pi_t > 1$). This quantity is important because it measures the extent to which a portfolio, left to itself, strays from its initial composition in response to market shocks and, by reflection, the quantity of trading that is necessary to keep it within some region. In the long-only case, the portfolio weight volatility decreases as the no-trade region widens to span $[0, 1]$, which means that a portfolio tends to spend more time near the boundaries. By contrast, with leverage portfolio weight volatility increases, which means that a wider boundary does not necessarily mitigate trading costs.

Consistent with this intuition, equation (3.8) shows that the impact of trading costs is small on long-only portfolios, but rises quickly with leverage, reducing returns at the order of $\varepsilon^{2/3}$ for $\pi_* > 1$. Of course, this expansion is valid for small ε while holding the value of γ fixed. As γ declines to zero, both the expected return and volatility diverge, but so does the impact of trading costs, making the asymptotics for $\gamma > 0$ uninformative for the risk-neutral limit $\gamma = 0$.

The performance (3.12) coincides at the first order with the equivalent safe rate from utility maximization with constant relative risk aversion γ (Gerhold et al., 2014, Equation (2.4)), supporting the interpretation of γ as a risk-aversion parameter, and confirming that, for asymptotically small costs, the efficient frontier captures the risk-return trade-off faced by a utility maximizer.

Figure 2 displays the effect of trading costs on the efficient frontier. As the bid-ask spread declines, the frontier increases to the linear frictionless frontier, and the asymptotic results in the theorem become more accurate. However, if the spread is held constant as leverage (hence volatility) increases, the asymptotic expansions become inaccurate, and in fact the efficient frontier ceases to increase at all after the leverage multiplier is reached.

4.2. Trading boundaries. Each point in the efficient frontier corresponds to a rebalancing strategy that is optimal for some value of the risk-aversion parameter γ . For small trading costs, equation (3.7) implies that the trading boundaries corresponding to the efficient frontier depart from the ones arising in utility maximization, which are (Gerhold et al., 2014)

$$\pi_{\pm} = \pi_* \pm \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{1/3} \varepsilon^{1/3} + O(\varepsilon). \quad (4.2)$$

The term of order $\varepsilon^{2/3}$ vanishes for $\gamma = 1$ because this case coincides with the maximization of logarithmic utility. For high levels of leverage ($\gamma < 1$ and $\pi_* > 1$), this term implies that the trading boundaries that generate the efficient frontier are lower than the trading boundaries that maximize utility. In Figure 3, $\gamma \rightarrow \infty$ corresponds to the safe portfolio in the origin $(0,0)$, while $\gamma = \mu/\sigma^2$ to the risky investment $(1,1)$, which has

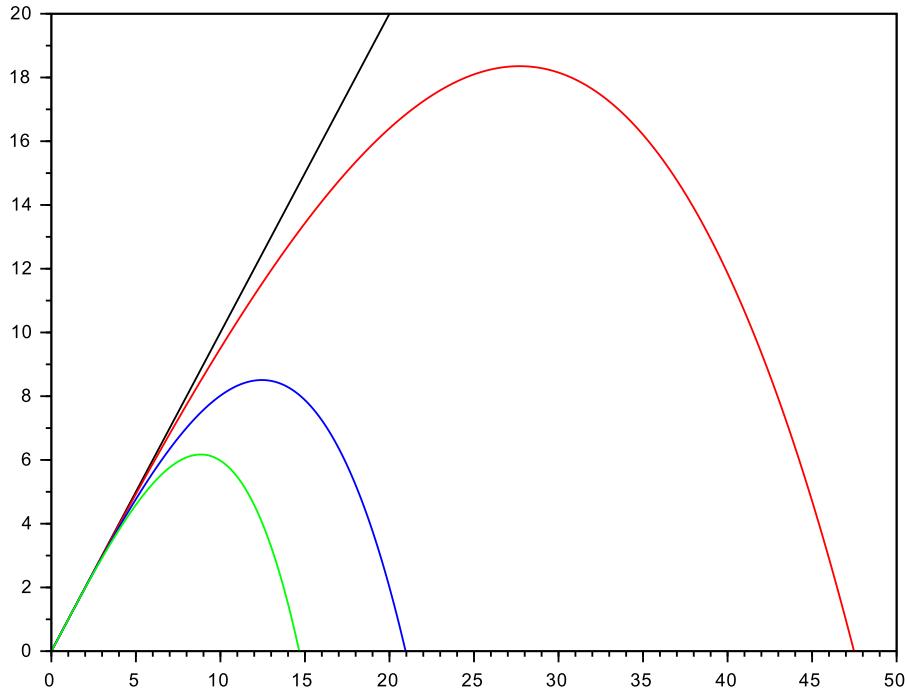


FIGURE 2. Efficient Frontier with trading costs, as expected excess return (vertical axis, in multiples of the asset's expected excess return) against standard deviation (horizontal axis, in multiples of the asset's volatility). The asset has expected excess return $\mu = 8\%$, volatility $\sigma = 16\%$, and bid-ask spread of 0.1%, 0.5%, 1%. The upper line is the frictionless efficient frontier. The maximum of each curve is the leverage multiplier.

by definition the same volatility and return as the risky asset. As γ declines to zero, the trading boundaries converge to the right endpoints, which correspond to the strategy that maximizes average return with no regard for risk, thereby achieving the multiplier.

As leverage increases, the sell boundary rises more quickly than the buy boundary (Figure 3). For example, the risk-neutral portfolio tolerates leverage fluctuations from approximately 6 to 14. The locations of these boundaries trade off the need to keep exposure to the risky asset high to maximize return while also keeping rebalancing costs low. Risk aversion makes boundaries closer to each other by penalizing the high realized variance generated by the wide risk-neutral boundaries.

Importantly, these boundaries remain finite even as the frictionless Merton portfolio $\mu/(\gamma\sigma^2)$ diverges to infinity with γ declining to zero. Thus the no-trade region is not symmetric around the frictionless portfolio, in contrast to the boundaries arising from utility maximization (Gerhold et al., 2014), which are always symmetric, and hence diverge when γ is low. The difference is that here the risk-neutral objective is to maximize the expected *return* of the portfolio, while a risk-neutral utility maximizer focuses on expected *wealth*. In a frictionless setting this distinction is irrelevant, and an investor

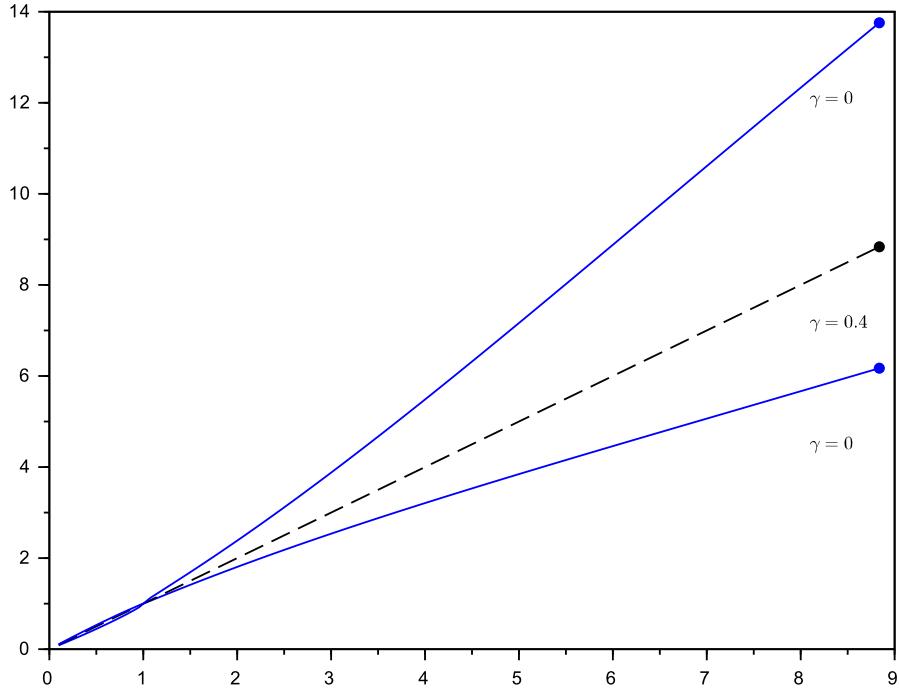


FIGURE 3. Trading boundaries π_{\pm} (vertical axis, outer curves, as risky weights) and implied Merton fraction (middle curve) against average portfolio volatility (horizontal axis, as multiples of σ). $\mu = 8\%$, $\sigma = 16\%$, and $\varepsilon = 1\%$.

can use a return-maximizing policy to maximize wealth instead. But trading costs drive a wedge between these two ostensibly equivalent risk-neutral criteria – maximizing expected return is not the same as maximizing expected wealth.

In the risk-neutral case (Theorem 3.2 (iv)) the optimal trading boundaries satisfy the approximate relation

$$\frac{\pi_-}{\pi_+} \approx 0.4172 \quad (4.3)$$

which is universal in that it holds for any asset, regardless of risk, return and liquidity. This relation means that an optimal risk-neutral rebalancing strategy should always tolerate wide variations in leverage over time, and that the maximum allowed leverage should be approximately 2.5 times the minimum. More frequent rebalancing cannot achieve the maximum return: it can be explained either by risk aversion or by elements that lie outside the model, such as price jumps.

Finally, note that the solvency constraint that wealth remain positive at all times implies that⁶ $\pi_t < \frac{1}{\varepsilon}$ for every admissible trading strategy. As $\pi_t \leq \pi_+$ for the optimal trading

⁶ Any potentially optimal strategy has positive exposure ($\pi_t > 0$), as the asset price has a positive risk premium (Remark A.4 below). Denoting by $X_t = w_t - \varphi_t S_t$ the safe position at time $t \geq 0$, where w_t is total portfolio wealth, the liquidation value is $w_t - \varepsilon \varphi_t S_t \geq 0$, which implies $1 - \varepsilon \frac{\varphi_t S_t}{w_t} > 0$ and thus the claim.

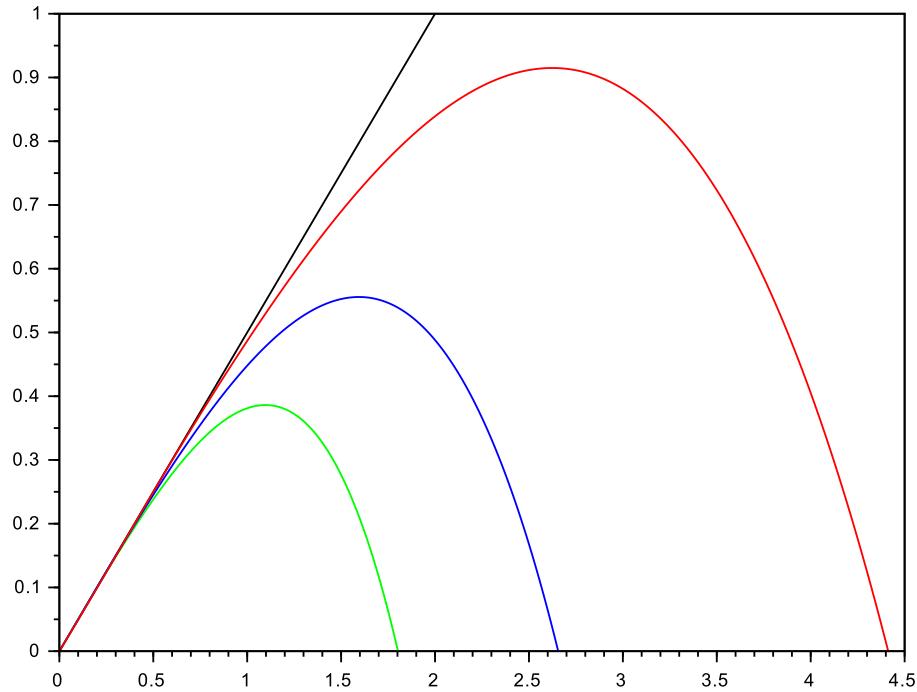


FIGURE 4. Efficient Frontier, as average expected excess return (vertical axis) against volatility (horizontal axis), for an asset with Sharpe ratio $\mu/\sigma = 0.5$, for various levels of asset volatility, from 10% (bottom), 20%, to 50% (top), for a bid-ask spread $\varepsilon = 1\%$. The straight line is the frictionless frontier.

strategy in Theorem 3.1 and Theorem 3.2, the upper bound $\pi_t \leq \frac{1}{\varepsilon}$ is never binding for realistic bid-ask-spreads.

4.3. Embedded leverage. In frictionless markets, two perfectly correlated assets with equal Sharpe ratio generate the same efficient frontier, and in fact the same payoff space. This equivalence fails in the presence of trading costs: as the more volatile asset has a proportionally higher return, it can be traded to generate higher returns with lower leverage ratios, resulting in an efficient frontier that dominates (for high returns) the one generated by the less volatile asset. Figure 4 (top of the three curves) displays this phenomenon: for example, a portfolio with an average return of 50% net of trading costs is obtained from an asset with 25% return and 50% volatility at a small cost, as an average leverage factor of 2 entails moderate rebalancing.

Achieving the same 50% return from an asset with 20% volatility (and 10% return) is more onerous: trading costs require leverage higher than 5, which in turn increases trading costs. Overall, the resulting portfolio needs about 120% rather than 100% volatility to achieve the desired 50% average return (middle curve in Figure 4).

From an asset with 10% volatility (and 5% return), obtaining a 50% return net of trading costs is impossible (bottom curve in Figure 4), because the leverage multiplier is less than 8 (Table 1, top right), and therefore the return can be scaled to less than 40%.

The intuition is clear: increasing leverage also increases trading costs, which in turn call for more leverage to increase return, but also increase costs. At some point, the marginal net return from more leverage becomes zero, and further increases are detrimental.

Because an asset with higher volatility is superior to another one, with equal Sharpe ratio and perfectly correlated, but with lower volatility, the model suggests that in equilibrium they cannot coexist, and that the asset with lower volatility should offer a higher return to be held by investors. Indeed, Frazzini and Pedersen (2012, 2014) document significant negative excess returns in assets with embedded leverage (higher volatility), and offer a theoretical explanation based on heterogeneous leverage constraints, which lead more constrained investors to bid up prices (and hence lower returns) of more volatile assets. Our results hint that the same phenomenon may arise even in the absence of constraints, as a result of rebalancing costs. In contrast to constraints-based explanations, our model suggests that the premium for embedded leverage should be higher for more illiquid assets.

4.4. From risk aversion to risk neutrality. Theorems 3.1 and 3.2 are qualitatively different: while Theorem 3.1 with positive risk aversion leads to a regular perturbation of the Markowitz-Merton solution, Theorem 3.2 with risk-neutrality leads to a novel result with no meaningful analogue in the frictionless setting – a singular perturbation. Furthermore, a close reading of the statement of Theorem 3.1 shows that the existence of a solution to the free-boundary problem, and the asymptotic expansions, hold for ε less than some threshold $\bar{\varepsilon}(\gamma)$ that depends on the risk aversion γ . In particular, if γ approaches zero while ε is held constant, Theorem 3.1 does not offer any conclusions on the convergence of the risk-averse to the risk-neutral solution. Still, if the risk-neutral result is to be accepted as a genuine phenomenon rather than an artifact, it should be clarified whether the risk averse trading policy and its performance converge to their risk neutral counterparts as risk aversion vanishes. The next result resolves this point under some parametric restrictions. Denote by

$$G(\zeta) := \frac{\varepsilon}{(1+\zeta)(1+(1-\varepsilon)\zeta)}, \quad h(\zeta) = \mu \left(\frac{\zeta}{1+\zeta} \right) - \frac{\gamma\sigma^2}{2} \left(\frac{\zeta}{1+\zeta} \right)^2$$

and associate to any solution $(W(\cdot; \gamma), \zeta_-(\gamma), \zeta_+(\gamma))$ of the free boundary problem (3.1) the function

$$\hat{W}(\zeta; \gamma) := \begin{cases} 0, & \zeta < \zeta_-(\gamma) \\ W(\zeta; \gamma), & \zeta \in [\zeta_-(\gamma), \zeta_+(\gamma)] \\ G(\zeta), & \zeta \geq \zeta_+(\gamma) \end{cases}$$

which naturally extends W to the left and right of the free-boundaries.

Theorem 4.1. *Let $\mu > \sigma^2$, $\bar{\varepsilon} > 0$, and $\bar{\gamma} > 0$, and assume that for any $\gamma \in [0, \bar{\gamma}]$ the free boundary problem (3.1) has a unique solution (W, ζ_-, ζ_+) satisfying $\zeta_+ < -1/(1 - \varepsilon)$ and that the function \hat{W} satisfies, for each $\gamma \in (0, \bar{\gamma}]$, the HJB equation*

$$\min \left(\frac{\sigma^2}{2} \zeta^2 \hat{W}' + \mu \zeta \hat{W} - h(\zeta) + h(\zeta_-), G(\zeta) - \hat{W}, \hat{W} \right) = 0. \quad (4.4)$$

Then, (4.4) is satisfied also for $\gamma = 0$, and for each $\gamma \in [0, \bar{\gamma}]$, the trading strategy that buys at $\pi_-(\gamma) = \frac{\zeta_-(\gamma)}{1+\zeta_-(\gamma)}$ and sells at $\pi_+(\gamma) = \frac{\zeta_+(\gamma)}{1+\zeta_+(\gamma)}$ to keep the risky weight π_t within the interval $[\pi_-(\gamma), \pi_+(\gamma)]$ is optimal. Furthermore, $\zeta_{\pm}(\gamma) \rightarrow \zeta_{\pm}(0)$ and $\hat{W}(\zeta; \gamma) \rightarrow \hat{W}(\zeta; 0)$ as $\gamma \downarrow 0$, each $\zeta \in \mathbb{R}$.

In summary, this result confirms that, as the risk-aversion parameter γ declines to zero, the risk-averse policy in Theorem 3.1 converges to the risk-neutral policy in Theorem 3.2, and that the corresponding mean-variance objective in Theorem 3.1 converges to the average return in Theorem 3.2.

5. HEURISTIC SOLUTION

This section offers a heuristic derivation of the HJB equation. Let $(\varphi_t^\uparrow)_{t \geq 0}$ and $(\varphi_t^\downarrow)_{t \geq 0}$ denote the cumulative number of shares bought and sold, respectively. The finite-horizon objective (2.1) reduces to the expression (compare eq. (A.8) in Lemma A.2 below)

$$\max_{\varphi \in \Phi} \mathbb{E} \left[\int_0^T \left(\mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} \right]. \quad (5.1)$$

From the outset, this objective is scale-invariant: doubling the initial number of risky shares and safe units, and also doubling the number of shares φ_t held at time t results in doubling also the number of safe units at time t (through the self-financing condition), thereby leaving $d\varphi_t/\varphi_t$, $\pi_t = S_t \varphi_t / X_t$, and hence the objective, unchanged. Thus, we conjecture that the residual value function V depends on the calendar time t and on the variable $\zeta_t = \pi_t / (1 - \pi_t)$, which denotes the number of shares held for each unit of the safe asset. In terms of this variable, the conditional value of the above objective at time t becomes:

$$F^\varphi(t) = \int_0^t \left(\mu \frac{\zeta_s}{1 + \zeta_s} - \frac{\gamma \sigma^2}{2} \frac{\zeta_s^2}{(1 + \zeta_s)^2} \right) ds - \varepsilon \int_0^t \frac{\zeta_s}{1 + \zeta_s} \frac{d\varphi_s^\downarrow}{\varphi_s} + V(t, \zeta_t). \quad (5.2)$$

By Itô's formula, the dynamics of F^φ is (henceforth the arguments of V are omitted for brevity)

$$dF^\varphi(t) = \left(\frac{\mu \zeta_t}{1 + \zeta_t} - \frac{\gamma \sigma^2}{2} \frac{\zeta_t^2}{(1 + \zeta_t)^2} \right) dt - \frac{\varepsilon \zeta_t}{1 + \zeta_t} \frac{d\varphi_t^\downarrow}{\varphi_t} + V_t dt + V_\zeta d\zeta_t + \frac{1}{2} V_{\zeta\zeta} d\langle \zeta \rangle_t,$$

where the subscripts of V denote partial derivatives. Recall now the self-financing condition for the safe position X_t and the risky position Y_t :

$$dX_t = r X_t dt - S_t d\varphi_t^\uparrow + (1 - \varepsilon) S_t d\varphi_t^\downarrow, \quad dY_t = S_t d\varphi_t^\uparrow - S_t d\varphi_t^\downarrow + \varphi_t dS_t,$$

which implies the dynamics for the risky-safe ratio ζ_t

$$\frac{d\zeta_t}{\zeta_t} = \mu dt + \sigma dW_t + (1 + \zeta_t) \frac{d\varphi_t}{\varphi_t} + \varepsilon \zeta_t \frac{d\varphi_t^\downarrow}{\varphi_t},$$

whence the dynamics of F^φ simplifies to

$$dF^\varphi(t) = \left(\mu \frac{\zeta_t}{1 + \zeta_t} - \frac{\gamma \sigma^2}{2} \frac{\zeta_t^2}{(1 + \zeta_t)^2} + V_t + \frac{\sigma^2}{2} \zeta_t^2 V_{\zeta\zeta} + \mu \zeta_t V_\zeta \right) dt \quad (5.3)$$

$$- \zeta_t \left(V_\zeta (1 + (1 - \varepsilon) \zeta_t) + \frac{\varepsilon}{1 + \zeta_t} \right) \frac{d\varphi_t^\downarrow}{\varphi_t} + \zeta_t (1 + \zeta_t) V_\zeta \frac{d\varphi_t^\uparrow}{\varphi_t} + \sigma \zeta_t V_\zeta dW_t. \quad (5.4)$$

Now, by the martingale principle of optimal control (Davis and Varaiya, 1973) the process $F^\varphi(t)$ above needs to be a supermartingale for any trading policy φ and a martingale for the optimal policy. As φ^\uparrow and φ^\downarrow are increasing processes, the supermartingale condition

implies⁷ the inequalities

$$-\frac{\varepsilon}{(1+\zeta)(1+(1-\varepsilon)\zeta)} \leq V_\zeta \leq 0, \quad (5.5)$$

and the martingale condition prescribes that the left (respectively, right) inequality becomes an equality at the points of increase of φ^\downarrow (resp. φ^\uparrow). Likewise, it follows that

$$\mu \frac{\zeta}{1+\zeta} - \frac{\gamma\sigma^2}{2} \frac{\zeta^2}{(1+\zeta)^2} + V_t + \frac{\sigma^2}{2} \zeta^2 V_{\zeta\zeta} + \mu\zeta V_\zeta \leq 0$$

with the inequality holding as an equality whenever both inequalities in (5.5) are strict. To achieve a stationary (that is, time-homogeneous) system, suppose that the residual value function is of the form $V(t, \zeta) = \lambda(T-t) - \int^\zeta W(z)dz$ for some λ to be determined, which represents the average optimal performance over a long period of time. Replacing this parametric form of the solution, the above inequalities become

$$0 \leq W(\zeta) \leq \frac{\varepsilon}{(1+\zeta)(1+(1-\varepsilon)\zeta)}, \quad (5.6)$$

$$\mu \frac{\zeta}{1+\zeta} - \frac{\gamma\sigma^2}{2} \frac{\zeta^2}{(1+\zeta)^2} - \lambda - \frac{\sigma^2}{2} \zeta^2 W'(\zeta) - \mu\zeta W(\zeta) \leq 0. \quad (5.7)$$

Assuming further that the first inequality holds over some interval $[\zeta_-, \zeta_+]$, with each inequality reducing to an equality at the respective endpoint, the optimality conditions become

$$\frac{\sigma^2}{2} \zeta^2 W'(\zeta) + \mu\zeta W(\zeta) - \mu \frac{\zeta}{1+\zeta} + \frac{\gamma\sigma^2}{2} \frac{\zeta^2}{(1+\zeta)^2} + \lambda = 0 \quad \text{for } \zeta \in [\zeta_-, \zeta_+], \quad (5.8)$$

$$W(\zeta_-) = 0, \quad W(\zeta_+) = \frac{\varepsilon}{(\zeta_+ + 1)(1 + (1-\varepsilon)\zeta_+)}, \quad (5.9)$$

which lead to a family of candidate value functions, each of them corresponding to a pair of boundaries (ζ_-, ζ_+) . The optimal boundaries are identified by the smooth-pasting conditions, formally derived by differentiating eqs. (5.9) with respect to their boundaries

$$W'(\zeta_-) = 0, \quad W'(\zeta_+) = \frac{\varepsilon(\varepsilon-2(1-\varepsilon)\zeta_+-2)}{(1+\zeta_+)^2(1+(1-\varepsilon)\zeta_+)^2}. \quad (5.10)$$

These conditions identify the value function. The four unknowns are the free parameter in the general solution to the ordinary differential equation (5.8), the free boundaries ζ_- and ζ_+ , and the optimal rate λ . These quantities are identified by the boundary and smooth-pasting conditions (5.9)–(5.10).

6. CONCLUSION

The costs of rebalancing a leveraged portfolio are substantial, and detract from its ostensible frictionless return. As leverage increases, such costs rise faster than the return, making it impossible for an investor to lever an asset's return beyond a certain multiple, net of trading costs.

In contrast to the frictionless theory, trading costs make the risk-return trade-off non-linear. An investor who seeks high return prefers an asset with high volatility to another one with equal Sharpe ratio but lower volatility, because higher volatility makes leverage cheaper to realize. A risk-neutral, return-maximizing investor does not take infinite leverage, but rather keeps it within a band that balances high exposure with low rebalancing costs.

⁷In particular, the coefficients of $\frac{d\varphi_t^\uparrow}{\varphi_t}$ and $\frac{d\varphi_t^\downarrow}{\varphi_t}$ need to be negative. As short positions are never optimal (cf. Remark A.3 and footnote 6), it follows that $0 < \pi_t < 1/\varepsilon$, whence only two cases arise: (a) $\zeta_t < -1/(1-\varepsilon)$, or (b) $\zeta_t > 0$. In both cases $\zeta(1+\zeta) > 0$ and $-\zeta(1+(1-\varepsilon)\zeta) < 0$, whence (5.5) follows.

APPENDIX A. ADMISSIBLE STRATEGIES

In view of transaction costs, only finite-variation trading strategies are consistent with solvency. Denote by X_t and Y_t the wealth in the safe and risky positions respectively, and by $(\varphi_t^\uparrow)_{t \geq 0}$ and $(\varphi_t^\downarrow)_{t \geq 0}$ the cumulative number of shares bought and sold, respectively. The self-financing condition prescribes that (X, Y) satisfy the dynamics

$$dX_t = rX_t dt - S_t d\varphi_t^\uparrow + (1 - \varepsilon)S_t d\varphi_t^\downarrow, \quad dY_t = S_t d\varphi_t^\uparrow - S_t d\varphi_t^\downarrow + \varphi_t dS_t. \quad (\text{A.1})$$

A strategy is admissible if it is nonanticipative and solvent, up to a small increase in the spread:

Definition A.1. Let $x > 0$ (the initial capital) and let $(\varphi_t^\uparrow)_{t \geq 0}$ and $(\varphi_t^\downarrow)_{t \geq 0}$ be continuous, increasing processes, adapted to the augmented natural filtration of B . Then $(x, \varphi_t = \varphi_t^\uparrow - \varphi_t^\downarrow)$ is an admissible trading strategy if

- (i) its liquidation value is strictly positive at all times: There exists $\varepsilon' > \varepsilon$ such that the discounted asset $\tilde{S}_t := e^{-rt}S_t$ satisfies

$$x - \int_0^t \tilde{S}_s d\varphi_s + \tilde{S}_t \varphi_t - \varepsilon' \int_0^t \tilde{S}_s d\varphi_s^\downarrow - \varepsilon' \varphi_t^\downarrow \tilde{S}_t > 0 \quad \text{a.s. for all } t \geq 0. \quad (\text{A.2})$$

- (ii) The following integrability conditions hold⁸

$$\mathbb{E} \left[\int_0^t |\pi_u|^2 du \right] < \infty, \quad \mathbb{E} \left[\int_0^t \pi_u \frac{d\|\varphi_u\|}{\varphi_u} \right] < \infty \quad \text{for all } t \geq 0, \quad (\text{A.3})$$

where $\|\varphi_t\|$ denotes the total variation of φ on $[0, t]$.

The family of admissible trading strategies is denoted by Φ .

The following lemma describes the dynamics of the wealth process w_t , the risky weight π_t , and the risky-safe ratio ζ_t .

Lemma A.2. For any admissible trading strategy φ , ⁹

$$\frac{d\zeta_t}{\zeta_t} = \mu dt + \sigma dB_t + (1 + \zeta_t) \frac{d\varphi_t^\uparrow}{\varphi_t} - (1 + (1 - \varepsilon)\zeta_t) \frac{d\varphi_t^\downarrow}{\varphi_t}, \quad (\text{A.4})$$

$$\frac{dw_t}{w_t} = rdt + \pi_t(\mu dt + \sigma dB_t - \varepsilon \frac{d\varphi_t^\downarrow}{\varphi_t}), \quad (\text{A.5})$$

$$\frac{d\pi_t}{\pi_t} = (1 - \pi_t)(\mu dt + \sigma dB_t) - \pi_t(1 - \pi_t)\sigma^2 dt + \frac{d\varphi_t^\uparrow}{\varphi_t} - (1 - \varepsilon\pi_t) \frac{d\varphi_t^\downarrow}{\varphi_t}. \quad (\text{A.6})$$

For any such strategy, the functional

$$F_T(\varphi) := \frac{1}{T} \mathbb{E} \left[\int_0^T \frac{dw_t}{w_t} - \frac{\gamma}{2} \left\langle \int_0^T \frac{dw_t}{w_t} \right\rangle_T \right] \quad (\text{A.7})$$

equals to

$$F_T(\varphi) = r + \frac{1}{T} \mathbb{E} \left[\int_0^T \left(\mu\pi_t - \frac{\gamma\sigma^2}{2}\pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} \right]. \quad (\text{A.8})$$

⁸ Note that $\frac{\pi_t}{\varphi_t} = \frac{S_t}{w_t}$, therefore on the set $\{(\omega, t) : \varphi_t = 0\}$ the quantity $\frac{\pi_t}{\varphi_t}$ is well-defined.

⁹The notation $\frac{dx_t}{x_t} = dy_t$ means $x_t = x_0 + \int_0^t x_s dy_s$, hence the SDEs are well defined even for null x_t .

Proof. The self-financing conditions (A.1) imply that

$$\frac{dX_t}{X_t} = rdt - \zeta_t \frac{d\varphi_t^\uparrow}{\varphi_t} + (1 - \varepsilon)\zeta_t \frac{d\varphi_t^\downarrow}{\varphi_t}, \quad (\text{A.9})$$

$$\frac{dY_t}{Y_t} = \frac{d\varphi_t^\uparrow}{\varphi_t} - \frac{d\varphi_t^\downarrow}{\varphi_t} + \frac{dS_t}{S_t}, \quad (\text{A.10})$$

$$\frac{d(Y_t/X_t)}{Y_t/X_t} = \frac{dY_t}{Y_t} - \frac{dX_t}{X_t} + \frac{d\langle X \rangle_t}{X_t^2} - \frac{d\langle X, Y \rangle_t}{X_t Y_t} = \frac{dY_t}{Y_t} - \frac{dX_t}{X_t}. \quad (\text{A.11})$$

Equation (A.4) follows from the last equation, and (A.5) holds in view of equations (A.9) and (A.10). Equation (A.6) follows from the identity $\pi_t = 1 - \frac{1}{1+\zeta_t}$ and (A.4). The expression in (A.8) for the objective functional follows from equation (A.5). \square

The following lemma shows that, without loss of generality, it is enough to consider trading strategies which do not take short positions in the risky asset.

Lemma A.3. *If $\varphi \in \Phi$ is optimal for (2.3), then also the strategy $\hat{\varphi}_t := \varphi_t 1_{\{\varphi_t \geq 0\}}$ is optimal.*

Proof. Due to Lemma A.2, the objective functional has the equivalent form (A.8), (letting $T \rightarrow \infty$). It is clear that $\hat{\varphi}$ is an admissible trading strategy if φ is. Furthermore, as $\mu \geq 0$, $\mu\hat{\pi}_t \geq \mu\pi_t$ at all times t , and $\hat{\pi}_t = 0$ whenever $\varphi_t < 0$, whence $F_T(\hat{\varphi}) \geq F_T(\varphi)$ for each $T > 0$. \square

Remark A.4. *In view of this Lemma and admissibility, it suffices to consider trading strategies which satisfy $0 \leq \pi_t \leq 1/\varepsilon$, or, in terms of the risky-safe ratio, $\zeta_t < -1/(1-\varepsilon)$ or $\zeta_t \geq 0$.*

APPENDIX B. RISK AVERSION AND EFFICIENT FRONTIER

This section contains a series of propositions that lead to the proof of Theorem 3.1 (i)–(iii). Part (iv) of the theorem is postponed to Appendix C. Set

$$G(\zeta) := \frac{\varepsilon}{(1 + \zeta)(1 + (1 - \varepsilon)\zeta)} \quad \text{and} \quad h(\zeta) := \mu \left(\frac{\zeta}{1 + \zeta} \right) - \frac{\gamma\sigma^2}{2} \left(\frac{\zeta}{1 + \zeta} \right)^2. \quad (\text{B.1})$$

Defining $H := h'$, the free boundary problem (3.1)–(3.5) reduces to

$$\frac{1}{2}\sigma^2\zeta^2W''(\zeta) + (\sigma^2 + \mu)\zeta W'(\zeta) + \mu W(\zeta) - H(\zeta) = 0, \quad (\text{B.2})$$

$$W(\zeta_-) = 0, \quad (\text{B.3})$$

$$W'(\zeta_-) = 0, \quad (\text{B.4})$$

$$W(\zeta_+) = G(\zeta_+), \quad (\text{B.5})$$

$$W'(\zeta_+) = G'(\zeta_+). \quad (\text{B.6})$$

Proposition B.1. *Let $\gamma > 0$ and $\pi_* \neq 1$. For sufficiently small ε , the free boundary problem (B.2)–(B.6) has a unique solution (W, ζ_-, ζ_+) , with $\zeta_- < \zeta_+$. The free boundaries have the asymptotic expansion*

$$\zeta_\pm = \frac{\pi_*}{1-\pi_*} \pm \left(\frac{3}{4\gamma} \right)^{1/3} \left(\frac{\pi_*}{(\pi_*-1)^2} \right)^{2/3} \varepsilon^{1/3} - \frac{(5-2\gamma)\pi_*}{2\gamma(\pi_*-1)^2} \left(\frac{\gamma\pi_*(\pi_*-1)}{6} \right)^{1/3} \varepsilon^{2/3} + O(\varepsilon). \quad (\text{B.7})$$

Proof of Proposition B.1. Note that (B.2) is equivalent to the ODE

$$\left(\frac{\sigma^2 \zeta^2}{2} W'(\zeta) + \mu \zeta W(\zeta) - h(\zeta) \right)' = 0$$

and thus, the initial conditions (B.3), (B.4) imply that W satisfies

$$\frac{\sigma^2 \zeta^2}{2} W'(\zeta) + \mu \zeta W(\zeta) = h(\zeta) - h(\zeta_-), \quad W(\zeta_-) = 0.$$

By the variation of constants method, and as $\zeta_- \notin \{-1, 0\}$, any solution of the initial value problem (B.2)–(B.4) is thus of the form

$$\widetilde{W}(\zeta_-, \zeta) := \frac{2}{(\sigma \zeta)^2} \int_{\zeta_-}^{\zeta} (h(y) - h(\zeta_-)) \left(\frac{y}{\zeta} \right)^{2\gamma\pi_*-2} dy. \quad (\text{B.8})$$

Suppose (W, ζ_-, ζ_+) is a solution of (B.2)–(B.6). In view of (B.8), $W(\cdot) \equiv \widetilde{W}(\zeta_-, \cdot)$. Let

$$J(\zeta_-, \zeta) := \frac{\sigma^2 \zeta^{2\gamma\pi_*}}{2} \widetilde{W}(\zeta_-, \zeta). \quad (\text{B.9})$$

By the terminal conditions (B.5)–(B.6) at ζ_+ , and setting $\delta = \varepsilon^{1/3}$, (ζ_-, ζ_+) satisfy the following system of algebraic equations,

$$\Psi_1(\zeta_-, \zeta_+) := \widetilde{W}(\zeta_-, \zeta_+) - \frac{\delta^3}{(1 + \zeta_+)(1 + (1 - \delta^3)\zeta_+)} = 0, \quad (\text{B.10})$$

$$\Psi_2(\zeta_-, \zeta_+) := \frac{2(h(\zeta_+) - h(\zeta_-))}{\sigma^2 \zeta_+^2} - \frac{2\gamma\pi_*}{\zeta_+} \widetilde{W}(\zeta_-, \zeta_+) - \frac{(1 - \delta^3)^2}{(1 + (1 - \delta^3)\zeta_+)^2} + \frac{1}{(1 + \zeta_+)^2} = 0. \quad (\text{B.11})$$

Conversely, if (ζ_-, ζ_+) solve (B.10)–(B.11), then the triplet $(\zeta \mapsto \widetilde{W}(\zeta_-, \zeta), \zeta_-, \zeta_+)$ provides a solution to the free boundary problem (B.2)–(B.6). Therefore, to provide a unique solution of the free boundary problem, it suffices to provide a unique solution of (B.10)–(B.11).

To obtain a guess for the asymptotic expansions of the prospective solutions ζ_{\pm} , expand $\Psi_{1,2}$ around

$$\zeta_- = \zeta_* + B_1 \delta + O(\delta^2), \quad \zeta_+ = \zeta_* + B_2 \delta + O(\delta^2), \quad \text{where } \zeta_* = \frac{\pi_*}{1 - \pi_*}, \quad (\text{B.12})$$

which yields

$$\Psi_1(\zeta_{\pm}(\delta)) = -\frac{\gamma(1 - \pi_*)^6}{3\pi_*^2} \left(2B_1^3 - 3B_1^2 B_2 + B_2^3 + \frac{3\pi_*^2}{\gamma(1 - \pi_*)^4} \right) \delta^3 + O(\delta^4), \quad (\text{B.13})$$

$$\Psi_2(\zeta_{\pm}(\delta)) = \frac{(B_1 - B_2)(B_1 + B_2)\gamma(\pi_* - 1)^6}{\pi_*^2} \delta^2 + O(\delta^3). \quad (\text{B.14})$$

Equating the coefficients of the leading order terms to zero yields¹⁰

$$2B_1^3 - 3B_1^2 B_2 + B_2^3 + \frac{3\pi_*^2}{\gamma(1 - \pi_*)^4} = 0, \quad (\text{B.15})$$

$$B_1 + B_2 = 0, \quad (\text{B.16})$$

whence $B_1 = -B_2$ and solves $B_1^3 = -\frac{3}{4\gamma} \frac{\pi_*^2}{(1 - \pi_*)^4} = 0$, and thus

$$B_1 = -\left(\frac{3}{4\gamma} \right)^{1/3} \left(\frac{\pi_*}{(1 - \pi_*)^2} \right)^{2/3}. \quad (\text{B.17})$$

¹⁰The coefficient in (B.14) vanishes also for $B_1 = B_2$, but (B.13) does not, excluding such a case.

With the change of variables

$$\eta_{\pm} := \frac{\zeta_{\pm} - \zeta_*}{\delta} \quad (\text{B.18})$$

and the notation

$$\Phi_1(\eta_-, \eta_+) := \Psi_1(\zeta_-(\eta_-), \zeta_+(\eta_+)), \quad \Phi_2(\eta_-, \eta_+) := \Psi_2(\zeta_-(\eta_-), \zeta_+(\eta_+)) \quad (\text{B.19})$$

the system (B.10)–(B.11) for ζ_{\pm} reduces to

$$\Phi(\eta_-, \eta_+) = (\Phi_1(\eta_-, \eta_+), \Phi_2(\eta_-, \eta_+)) = 0 \quad (\text{B.20})$$

in the unknowns η_{\pm} . Because of (B.17), the guess (B.12) takes the explicit form

$$\zeta_{\pm} = \zeta_* \pm \left(\frac{3}{4\gamma} \right)^{1/3} \left(\frac{\pi_*}{(1 - \pi_*)^2} \right)^{2/3} \delta + O(\delta^2), \quad (\text{B.21})$$

which suggests that the solution (η_-, η_+) is around $(B_1, B_2 = -B_1)$. Proposition B.2 below indeed guarantees the existence a unique solution around $(B_1, B_2 = -B_1)$ for sufficiently small $\delta > 0$, which is analytic in δ . Hence, also the original system $\Psi(\zeta_-, \zeta_+) = 0$ has a unique solution (ζ_-, ζ_+) for small δ , with the first order proxies (B.21). This implies that the free boundary problem (B.2)–(B.6) has a unique solution for sufficiently small ε .

To derive the higher order terms of (B.7), it is useful to rewrite the integral (B.9) as¹¹

$$J(\zeta_-, \zeta_+) = \underbrace{\frac{h(\zeta_-)(\zeta_-^{2\gamma\pi_*-1} - \zeta_+^{2\gamma\pi_*-1})}{2\gamma\pi_* - 1}}_{=:I_1} + \underbrace{\int_{\zeta_-}^{\zeta_+} h(y)y^{2\gamma\pi_*-2}dy}_{=:I_2}. \quad (\text{B.22})$$

The derivative of I_2 with respect to δ equals

$$\frac{dI_2}{d\delta} = h(\zeta_+)\zeta_+^{2\gamma\pi_*-2}\frac{d\zeta_+}{d\delta} - h(\zeta_-)\zeta_-^{2\gamma\pi_*-2}\frac{d\zeta_-}{d\delta}. \quad (\text{B.23})$$

Now, expanding the right-hand side as a power series in δ , and integrating with respect to δ yields an asymptotic expansion of I_2 .

To obtain these expansions, guess a solution of equations (B.10)–(B.11) of the form

$$\zeta_{\pm} = \frac{\pi_*}{1 - \pi_*} \pm \left(\frac{3}{4\gamma} \right)^{1/3} \left(\frac{\pi_*}{(1 - \pi_*)^2} \right)^{2/3} \delta + A_{\pm}\delta^2 + O(\delta^3),$$

for some unkowns A_{\pm} , and substitute it into equations (B.10)–(B.11), thereby using (B.22) and (B.23). Comparing the coefficients in the asymptotic expansion of the two equations reveals that

$$A_- = A_+ = \left(\frac{(5 - 2\gamma)\pi_*}{2\gamma(1 - \pi_*)^2} \right) \left(\frac{\gamma\pi_*(1 - \pi_*)}{6} \right)^{1/3},$$

and therefore (B.7) holds. \square

Proposition B.2. *Let $\gamma > 0$ and $\pi_* \neq 1$, and recall $B_1 = -B_2$ from (B.17). For sufficiently small $\delta > 0$, the system (B.20), where $\Phi = (\Phi_1, \Phi_2)$ is defined by (B.19), has a unique solution $(\eta_-(\delta), \eta_+(\delta))$ satisfying $\eta_-(0) = B_1$, $\eta_+(\delta) = B_2$, and $\delta \mapsto \eta_{\pm}(\delta)$ are analytic functions.*

¹¹For $\pi_* = 1/(2\gamma)$, $I_1 = h(\zeta_-)(\log \zeta_- - \log \zeta_+)$ and $I_2 = \int_{\zeta_-}^{\zeta_+} \frac{h(y)}{y} dy$.

Proof. Consider first the “general” case $\mu/\sigma^2 \neq 1/2$: Introduce the rescaled functions $\tilde{\Phi}_{1,2}$ and $\tilde{\Phi} := (\tilde{\Phi}_1, \tilde{\Phi}_2)$ defined as

$$\tilde{\Phi}_1 := \frac{\Phi_1}{\delta^l}, \quad \tilde{\Phi}_2 := \frac{\Phi_2}{\delta^m}, \quad (\text{B.24})$$

where $l = 3$ and $m = 2$. By scaling, the function $\tilde{\Phi}$ depends on three arguments, and for the sake of clarity henceforth it is denoted by

$$\tilde{\Phi} = \tilde{\Phi}(\eta_-, \eta_+, \delta)$$

Let $D\tilde{\Phi}$ be the Frechet differential of $\tilde{\Phi}$. As shown next, the Jacobian satisfies,

$$\det(D\tilde{\Phi})(\eta_- = B_1, \eta_+ = B_2, \delta = 0) = \frac{6\gamma(1 - \pi_*)^8(2\gamma\pi_* - 1)}{\pi_*^2} \neq 0, \quad (\text{B.25})$$

hence the implicit function theorem for analytic functions (Gunning and Rossi, 2009, Theorem I.B.4) ensures that for sufficiently small δ there exists a unique solution (η_-, η_+) of $\tilde{\Phi}(\eta_-, \eta_+) = 0$ around (B_1, B_2) which is analytic in δ .

It remains to prove (B.25). By construction,

$$\Psi_2(\zeta_-, \zeta_+) = \frac{\partial \Psi_1(\zeta_-, \zeta_+)}{\partial \zeta_+},$$

whence

$$\begin{aligned} \frac{\partial \tilde{\Phi}_1}{\partial \eta_+}(\eta_{\pm}) &= \frac{1}{\delta^l} \frac{\partial \Phi_1}{\partial \eta_+}(\eta_{\pm}) = \frac{1}{\delta^l} \frac{\partial \Psi_1(\zeta_{\pm}(\eta_{\pm}))}{\partial \zeta_+} \frac{\partial \zeta_+}{\partial \eta_+} \\ &= \frac{1}{\delta^{l-1}} \frac{\partial \Psi_1(\zeta_{\pm}(\eta_{\pm}))}{\partial \zeta_+} = \frac{\Psi_2(\zeta_{\pm}(\eta_{\pm}))}{\delta^{l-1}} \end{aligned}$$

and thus inserting the definition of η_{\pm} (cf. (B.18)) into equation (B.11) and, letting $\delta \rightarrow 0$, in view of (B.15)–(B.16) and their solution (B.17) it follows that

$$\frac{\partial \tilde{\Phi}_1}{\partial \eta_+} \Big|_{(B_1, B_2, 0)} = 0.$$

Thus the determinant of the Jacobian is simply

$$\det(D\tilde{\Phi})(B_1, B_2, 0) = \frac{\partial \tilde{\Phi}_1(\eta_-, \eta_+)}{\partial \eta_-} \Big|_{(B_1, B_2, 0)} \times \frac{\partial \tilde{\Phi}_2(\eta_-, \eta_+)}{\partial \eta_+} \Big|_{(B_1, B_2, 0)}.$$

Because

$$\frac{\partial \Psi_1}{\partial \zeta_-} = -\frac{2h'(\zeta_-)}{\sigma^2 \zeta_+^{2\mu/\sigma^2}} \left(\frac{\zeta_+^{2\mu/\sigma^2-2}}{2\mu/\sigma^2 - 1} - \frac{\zeta_-^{2\mu/\sigma^2-2}}{2\mu/\sigma^2 - 1} \right)$$

and by the chain rule

$$\frac{\partial \tilde{\Phi}_1(\eta_-, \eta_+)}{\partial \eta_-} = \frac{1}{\delta_3} \frac{\partial \Psi_1}{\partial \zeta_-} \times \delta,$$

it follows that

$$\frac{\partial \tilde{\Phi}_1(\eta_-, \eta_+)}{\partial \eta_-} \Big|_{(B_1, B_2, 0)} = \frac{6^{2/3}(1 - \pi_*)^3(\gamma\pi_*(1 - \pi_*))^{1/3}(1 - 2\gamma\pi_*)}{\pi_*}.$$

Similarly,

$$\frac{\partial \tilde{\Phi}_2(\eta_-, \eta_+)}{\partial \eta_+} \Big|_{(B_1, B_2, 0)} = -\frac{6^{1/3}(1 - \pi_*)^4(\gamma(1 - \pi_*)\pi_*)^{2/3}}{\pi_*^2},$$

from which (B.25) and hence the assertion in the proposition follows.

For the “singular” case $\mu/\sigma^2 = 1/2$ one needs to set $l = 5, m = 3$ in (B.24), then the right side of (B.25) equals $3(1 - 1/\pi_*)^8\pi_*^5 \neq 0$, and therefore similar arguments as in the general case apply. \square

Definition B.3. A solution of the HJB equation is a pair (V, λ) , where V is a twice continuously differentiable function, which satisfies

$$\min(\mathcal{A}V(x) - h(x) + \lambda, G(x) - V'(x), V''(x)) = 0, \quad x \in \left(-\infty, -\frac{1}{1-\varepsilon}\right) \cup (0, \infty), \quad (\text{B.26})$$

where $\mathcal{A} : \mathcal{C}^2(\mathbb{R}) \mapsto \mathcal{C}^2(\mathbb{R})$ is the differential operator

$$\mathcal{A}f(x) := \frac{\sigma^2}{2}x^2f''(x) + \mu xf'(x).$$

Note that the restriction $x \in (-\infty, -\frac{1}{1-\varepsilon}) \cup (0, \infty)$ is motivated by Remark A.4.

Proposition B.4. Let (W, ζ_-, ζ_+) be the solution of the free boundary problem (B.5)–(B.6) (provided by Proposition B.1) with asymptotic expansion (B.7). For sufficiently small ε , the pair

$$V(\cdot) := \int_0^\cdot \hat{W}(\zeta)d\zeta, \quad \lambda := h(\zeta_-),$$

where

$$\hat{W}(\zeta) := \begin{cases} 0 & \text{for } \zeta < \zeta_-, \\ W(\zeta) & \text{for } \zeta \in [\zeta_-, \zeta_+], \\ G(\zeta) & \text{for } \zeta \geq \zeta_+, \end{cases} \quad (\text{B.27})$$

is a solution of the HJB equation (B.26).

Proof of Proposition B.4. To check that (V, λ) solves the HJB equation (B.26), consider separately the domains $[\zeta_-, \zeta_+]$, $\zeta < \zeta_-$ and $\zeta > \zeta_+$. From the decompositions

$$G(\zeta) = \frac{1}{1+\zeta} - \frac{1-\varepsilon}{1+(1-\varepsilon)\zeta} \quad \text{and} \quad G'(\zeta) = \left(\frac{1-\varepsilon}{1+(1-\varepsilon)\zeta}\right)^2 - \frac{1}{(1+\zeta)^2},$$

note first that on $[\zeta_-, \zeta_+]$, by construction it holds that

$$(\mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-))' = \frac{1}{2}\sigma^2\zeta^2W''(\zeta) + (\sigma^2 + \mu)\zeta W'(\zeta) + \mu W(\zeta) - H(\zeta) = 0.$$

Furthermore, in view of the initial conditions (B.3)–(B.4),

$$(\mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-))|_{\zeta=\zeta_-} = \mathcal{A}V(\zeta)|_{\zeta=\zeta_-} = 0,$$

whence

$$\mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-) \equiv 0, \quad \zeta \in [\zeta_-, \zeta_+].$$

To see that $0 \leq V' \leq G$ on all of $[\zeta_-, \zeta_+]$, observe that

$$(h(\zeta) - h(\zeta_-))' = h'(\zeta) = H(\zeta) = \frac{\mu}{\pi_*(1+\zeta)^2} \left(\pi_* - \frac{\zeta}{1+\zeta}\right). \quad (\text{B.28})$$

Note that for $\zeta_- < \zeta \leq \zeta^*$, where $\zeta^*/(1+\zeta^*) = \pi_*$, $V'(\zeta) = W(\zeta) > 0$. It is shown that also $W(\cdot) \geq 0$ on all of $[\zeta_-, \zeta_+]$. This is equivalent to showing non-negativity of

$$w(\zeta) := 2\sigma^2\zeta^{2\gamma\pi_*}W(\zeta) = \int_{\zeta_-}^{\zeta} (h(x) - h(\zeta_-))x^{2\gamma\pi_*-2}dx. \quad (\text{B.29})$$

Now $w'(\zeta) = (h(\zeta) - h(\zeta_-))\zeta^{2\gamma\pi_*-2} = 0$ if and only if $h(\zeta_-) = h(\zeta)$. Hence, either $\zeta = \zeta_-$ or $\zeta = \bar{\zeta}$, where

$$\pi(\bar{\zeta}) = \frac{\bar{\zeta}}{1 + \bar{\zeta}} = 2\pi_* - \pi_-.$$

By the first-order asymptotics of (B.7), one obtains $\bar{\zeta} \notin [\zeta_-, \zeta_+]$ for sufficiently small ε . Therefore $w' > 0$ on $(\zeta_-, \zeta_+]$, and by (B.29) it follows that $V' \geq 0$ on all of $[\zeta_-, \zeta_+]$. To conclude the validity of the HJB equation on $[\zeta_-, \zeta_+]$, it only remains to show the inequality $V' \leq G$. To this end, notice that $\Psi_1(\zeta) = W(\zeta) - G(\zeta)$, (this is the function defined in (B.10), with fixed ζ_-) satisfies

$$\Psi_1(\zeta_-) = -G(\zeta_-) = -\frac{\varepsilon}{(1 + \zeta_-)(1 + (1 - \varepsilon)\zeta_-)} = -(1 - \pi_*)^2\varepsilon + O(\varepsilon^{4/3}),$$

hence for sufficiently small ε , $\Psi_1(\zeta) < 0$ on some interval $[\zeta_-, \bar{\zeta}]$, and $\Psi_1(\bar{\zeta}) = 0$. Therefore, $\bar{\zeta} \leq \zeta_+$. As $\Psi_1(\zeta_+) = 0$ by construction, it suffices to show that $\bar{\zeta} = \zeta_+$ to prove non-negativity of Ψ_1 on $[\zeta_-, \zeta_+]$. Suppose, by contradiction, that there exists a sequence $\delta_k \downarrow 0$ such that for each $k \geq 1$, $\Psi_1(\bar{\zeta}(\delta_k)) = 0$, and that $\zeta_-(\delta_k) < \bar{\zeta}(\delta_k) < \zeta_+(\delta_k)$. Now, change variable to $u = \frac{\zeta - \zeta_*}{\delta}$, and introduce the notation $u_\pm = \frac{\zeta_\pm - \zeta_*}{\delta}$, $\bar{u} = \frac{\bar{\zeta} - \zeta_*}{\delta}$. Up to a subsequence, without loss of generality assume that $\bar{u}(\delta_k)$ converges, whence it satisfies

$$\lim_{k \rightarrow \infty} \bar{u}(\delta_k) =: B_0 \in [B_1, B_2],$$

where B_1 is defined in (B.17), and $B_2 = -B_1$. The calculations leading to (B.17) therefore entail that B_0 must satisfy (B.15) in place of B_2 , i.e.

$$2B_1^3 - 3B_1^2B_0 + B_0^3 + \frac{3\pi_*^2}{\gamma(1 - \pi_*)^4} = 0. \quad (\text{B.30})$$

With B_1 from (B.17) and the change of variable $\xi = -B_0/B_1$ implies $2 - 3\xi + \xi^3 = 0$ which has the only solutions 1 and -2 . Therefore, (B.30) has the only relevant solution

$$B_0 = -B_1 = B_2.$$

By intertwining $u_+(\delta)$ and $\bar{u}(\delta_k)$, one can introduce

$$\bar{u}^*(\delta) = \begin{cases} \bar{u}(\delta_k), & k \in \mathbb{N} \\ u_+(\delta), & \text{otherwise} \end{cases}.$$

Hence $(u_-(\delta), u^*(\delta))$ satisfies $\Phi(u_-, u_+) = 0$ near (B_1, B_2) , for sufficiently small δ . By Proposition B.2, $u^*(\delta) = u_+(\delta)$, which contradicts our assumption $\bar{\zeta} \neq \zeta_+$.

Consider now $\zeta \leq \zeta_-$. V solves the HJB equation, if

$$\mathcal{A}V - h(\zeta) + h(\zeta_-) = h(\zeta_-) - h(\zeta) \geq 0, \quad G(\zeta) \geq 0.$$

As $h(\zeta) - h(\zeta_-) = 0$ for $\zeta = \zeta_-$, it suffices to show that h' is non-negative to obtain the first inequality. To this end, the explicit formula (B.28) for the derivative is used. Now for small ε clearly $\pi_- < \pi_*$, hence for $\zeta = \zeta_-$ (B.28) is indeed strictly positive, hence, upon integration, one obtains the first inequality for any $\zeta < \zeta_-$. To settle the second inequality, recall that either $\zeta < -1/(1 - \varepsilon)$ or $\zeta > 0$. On these domains, G is clearly a strictly positive function. Hence it is proved that V satisfies the HJB equation for $\zeta \leq \zeta_-$.

Finally, consider $\zeta \geq \zeta_+$. As $G = W$, it suffices to show

$$L(\zeta) := \mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-) \geq 0, \quad G(\zeta) \geq 0. \quad (\text{B.31})$$

As $G(\zeta)$ is strictly positive, the second inequality holds. For the first inequality in (B.31), note that

$$L(\zeta) = \frac{\sigma^2 \zeta^2}{2} G'(\zeta) + \mu \zeta G(\zeta) - h(\zeta) + h(\zeta_-)$$

and $L(\zeta_+) = 0$, because of (B.2), (B.5) and (B.6). Therefore it suffices to show L has no zeros on $[\zeta_+, -1/(1-\varepsilon)]$, besides ζ_+ .

Consider first, $\gamma = 1$. Using the transformation $z = \frac{\zeta}{1+\zeta}$ one can rewrite L in terms of z , denoting it by $F(z, \varepsilon) := L(\zeta(z))$. As $F(\pi_+) = 0$, polynomial division by $(z - \pi_+)$ yields

$$F(z, \varepsilon) = \frac{(z - \pi_+)}{(1 - \varepsilon z)^2} g(z), \quad (\text{B.32})$$

and $g(z) = \frac{1}{2}(g_0 + g_1 z)$, where

$$\begin{aligned} g_0 &= 2\mu(-1 + (1 - 2\pi_- + \pi_+)\varepsilon - (1 - \pi_-)\pi_+\varepsilon^2 \\ &\quad + \sigma^2(\pi_+ + 2(\pi_-^2 - \pi_+)\varepsilon + \pi_+(1 - \pi_-^2)\varepsilon^2), \\ g_1 &= 1 - (1 - \pi_-)\varepsilon(\sigma^2 + \varepsilon(2\mu - (1 + \pi_-)\sigma^2)). \end{aligned}$$

Therefore, the following asymptotic expansions hold

$$g(\pi_+) = \sigma^2 \left(\frac{3}{4} \left(\frac{\mu}{\sigma^2} \right)^2 \left(1 - \frac{\mu}{\sigma^2} \right)^2 \right)^{1/3} \varepsilon^{1/3} + O(\varepsilon^{2/3}), \quad g(1/\varepsilon) = \frac{\sigma^2}{2\varepsilon} + O(1).$$

It follows that g has no zeros on $[\pi_+, 1/\varepsilon]$, for sufficiently small ε . Hence $F(z) > 0$ for $z \in (\pi_+, 1/\varepsilon)$.

Next, consider $\gamma \neq 1$. Using the transformation $z = \frac{\zeta}{1+\zeta}$ one can rewrite, similar to the case $\gamma = 1$, L in terms of z , obtaining the function $F(z, \varepsilon) = L(\zeta(z))$. It is proved next that F has no zeros on $(\pi_+, 1/\varepsilon)$.

As $F(\pi_+) = 0$, polynomial division by $(z - \pi_+)$ yields (B.32), where the third order polynomial g has derivative

$$g' = a_0 + a_1 z + a_2 z^2,$$

where the coefficients a_0, a_1 and a_2 are complex, yet explicit, functions of the parameters and the relative bid-ask spread ε .

In view of (B.32), it is enough to show that g has no zeros on $[\pi_+, 1/\varepsilon]$. First, note the following asymptotic expansions,

$$g(\pi_+) = \left(\frac{3}{4\gamma} \pi_*^2 (\pi_* - 1)^2 \right)^{1/3} \varepsilon^{1/3} + O(\varepsilon^{2/3}), \quad g(1/\varepsilon) = \frac{\sigma^2}{2\varepsilon} + O(1). \quad (\text{B.33})$$

Therefore, for sufficiently small ε , $g > 0$ on both endpoints of $[\pi_+, 1/\varepsilon]$. It remains to show that any local minimum of g in $[\pi_+, 1/\varepsilon]$ is non-negative. The local extrema z_\pm , where $g'(z_\pm) = 0$, have asymptotic expansions $z_\pm = \frac{2}{3\varepsilon} \pm \frac{1}{3\varepsilon} \sqrt{\frac{\gamma-4}{\gamma-1}} + O(1)$. Obviously, there are no local extrema in $[\pi_+, 1/\varepsilon]$ whenever $\gamma \in [1, 4)$. Therefore $g > 0$ on all of $[\pi_+, 1/\varepsilon]$, and thus $F(z) \geq 0$ on $[\pi_+, 1/\varepsilon]$. The non-trivial case $\gamma \notin [1, 4)$ remains:

For $0 < \gamma < 1$ it holds that $\frac{4-\gamma}{1-\gamma} > 4$, hence $z_\pm \notin [\pi_+, 1/\varepsilon]$. It follows that g' has no zeros in this interval and thus $g > 0$ on $[\pi_+, 1/\varepsilon]$.

Next, consider $\gamma \geq 4$: The local minimum z_- of a third order polynomial with negative leading coefficient satisfies $z_- < z_+$ and $g(z_-) < g(z_+)$. In view of (B.33), it remains

to show $g(z_-) > 0$. It holds that $g(z_-) = \frac{3\gamma+(\gamma-4)(2+\gamma+\sqrt{(\gamma-4)(\gamma-1)})}{27(\gamma-1)\varepsilon} + O(1)$, whence $g(z_-) > 0$ for sufficiently small ε . Hence $g > 0$ on $[\pi_+, 1/\varepsilon]$ is shown. \square

Lemma B.5. *Let $\eta_- < \eta_+$ be such that either $\eta_+ < -1/(1-\varepsilon)$ or $\eta_- > 0$. Then there exists an admissible trading strategy $\hat{\varphi}$ such that the risky-safe ratio η_t satisfies SDE (A.4). Moreover, $(\eta_t, \hat{\varphi}_t^\uparrow, \hat{\varphi}_t^\downarrow)$ is a reflected diffusion on the interval $[\eta_-, \eta_+]$. In particular, η_t has stationary density equals*

$$\nu(\eta) := \frac{\frac{2\mu}{\sigma^2} - 1}{\eta_+^{\frac{2\mu}{\sigma^2}-1} - \eta_-^{\frac{2\mu}{\sigma^2}-1}} \eta^{\frac{2\mu}{\sigma^2}-2}, \quad \eta \in [\eta_-, \eta_+], \quad (\text{B.34})$$

when $\eta_- > 0$, and otherwise equals

$$\nu(\eta) := \frac{\frac{2\mu}{\sigma^2} - 1}{|\eta_-|^{\frac{2\mu}{\sigma^2}-1} - |\eta_+|^{\frac{2\mu}{\sigma^2}-1}} |\eta|^{\frac{2\mu}{\sigma^2}-2}, \quad \eta \in [\eta_-, \eta_+]. \quad (\text{B.35})$$

Proof. By the solution of the Skorohod problem for two reflecting boundaries (Kruk et al., 2007), there exists a well-defined reflected diffusion (η_t, L_t, U_t) satisfying $\frac{d\eta_t}{\eta_t} = \mu dt + \sigma dB_t + dL_t - dU_t$, where B is a standard Brownian motion. If $\eta_- > 0$, L (resp. U) is a non-decreasing processes which increases only on the set $\{\eta = \eta_-\}$ (resp. $\{\eta = \eta_+\}$)¹². Also, $\eta_- > 0$ or $\eta_+ < -1/(1-\varepsilon)$ implies that $\eta_t > 0$ or $\eta_t < -1/(1-\varepsilon)$ for all t , almost surely. Hence for each $t > 0$ the coefficients $(1 + (1-\varepsilon)\eta_t)$ and $(1 + \eta_t)$ are invertible, almost surely. Define the increasing processes $(\hat{\varphi}^\uparrow, \hat{\varphi}^\downarrow)$ by $\frac{d\hat{\varphi}_t^\uparrow}{\hat{\varphi}_t} = (1 + \eta_t)^{-1} dL_t$, $\frac{d\hat{\varphi}_t^\downarrow}{\hat{\varphi}_t} = (1 + (1-\varepsilon)\eta_t)^{-1} dU_t$. The associated measures $d\hat{\varphi}^\uparrow, d\hat{\varphi}^\downarrow$ are supported on $\eta_t = \eta_-$ and $\eta_t = \eta_+$, respectively. Hence $\hat{\varphi}$ is a trading strategy, which by Lemma A.2 yields a risky-safe satisfying precisely the stochastic differential equation (A.4). The admissibility of the trading strategy is clear, as $\hat{\varphi}$ is a continuous, finite variation trading strategy, and it satisfies $\pi_+ < 1/\varepsilon$, which implies that there exists $\varepsilon' > \varepsilon$ such that $\pi_t < 1/\varepsilon'$, for all $t > 0$, a.s..

Write the infinitesimal generator of $(\eta_t)_{t \geq 0}$ in the general form

$$\mathcal{A}f(\eta) = \frac{\sigma^2}{2}\eta^2 f''(\eta) + \mu\eta f'(\eta) =: \frac{1}{2}a^2(\eta)f''(\eta) + b(\eta)f'(\eta).$$

The speed measure is $m(d\eta) = \left(\frac{2}{a^2(\eta)} e^{\int_{\eta_-}^{\eta} \frac{2b(y)}{a^2(y)} dy} \right) d\eta$, and as $m([\eta_-, \eta_+]) < \infty$, $(\eta_t)_{t \geq 0}$ is positively recurrent and its invariant density ν is

$$\nu(\eta)d\eta = \frac{m(d\eta)}{m([\eta_-, \eta_+])}, \quad (\text{B.36})$$

(see (Borodin and Salminen, 2002, II.9, II. 12, and II. 36)). Distinguishing the cases $\eta_+ < 0$ or $\eta_- > 0$, the probability densities (B.34) and (B.35) follow. \square

The following constitutes the verification of optimality of the trading strategy of Lemma B.5 with the trading boundaries in Proposition B.1:

Proposition B.6. *Let ζ_\pm be the free boundaries as derived in Proposition B.1, and set $\pi_\pm := \zeta_\pm / (1 + \zeta_\pm)$. Denote by $\hat{\varphi}$ the trading strategy of Lemma B.5 associated with these free boundaries. Then for all $t > 0$, the fraction of wealth π_t invested in the risky asset lies in the interval $[\pi_-, \pi_+]$, almost surely, entails no trading whenever $\pi \in (\pi_-, \pi_+)$ (the*

¹²If $\eta_+ < 0$, the terms “decreasing” and “increasing” are exchanged.

no-trade region) and engages in trading only at the boundaries π_{\pm} . For sufficiently small ε , $\hat{\varphi}$ is optimal, and the value function is

$$\begin{aligned} F_{\infty}(\hat{\varphi}) &= r + \max_{\varphi \in \Phi} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T (\mu \pi_t - \frac{\gamma}{2} \sigma^2 \pi_t^2) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t^{\downarrow}}{\varphi_t} \right] \\ &= r + \mu \pi_- - \frac{\gamma \sigma^2}{2} \pi_-^2. \end{aligned} \quad (\text{B.37})$$

Proof of Proposition B.6. Recall from Proposition B.4 that $\lambda = h(\zeta_-)$ and (V, λ) , defined from the unique solution of the free boundary problem, is a solution of the HJB equation (B.26). For the verification, the proportion π_t of wealth in the risky asset is used instead of the risky-safe ratio ζ_t . The change of variable $\zeta = -1 + \frac{1}{1-\pi}$ amounts to a compactification of the real line, such that the two intervals $[-\infty, -1/(1-\varepsilon))$ and $(0, \infty]$ are mapped onto the connected interval $[0, 1/\varepsilon]$. Denote by \mathcal{L} the differential operator

$$(\mathcal{L}f)(\pi) := \frac{\sigma^2}{2} f''(\pi) \pi^2 (1-\pi)^2 + f'(\pi) (\mu - \sigma^2 \pi) \pi (1-\pi).$$

Set $\hat{h}(\pi) = h(\zeta(\pi)) = \mu \pi - \frac{\gamma \sigma^2}{2} \pi^2$. The function $\hat{V}(\pi) := V(\zeta(\pi))$ satisfies the HJB equation

$$\min(\mathcal{L}\hat{V}(\pi) - \hat{h}(\pi) + \lambda, \hat{V}'(\pi), \varepsilon/(1-\varepsilon\pi) - \hat{V}'(\pi)) = 0, \quad 0 \leq \pi < 1/\varepsilon. \quad (\text{B.38})$$

First, note that $F_{\infty}(\varphi) \leq \lambda + r$ for any admissible trading strategy φ : By Lemma A.3 and Remark A.4, without loss of generality assume $\pi_t \geq 0$ almost surely for all $t \geq 0$. An application of Itô's formula to the stochastic process $\hat{V}(\pi_t)$, where \hat{V} is the solution of the HJB equation (B.38), yields

$$\hat{V}(\pi_T) - \hat{V}(\pi_0) = \int_0^T \hat{V}'(\pi_t) d\pi_t + \frac{1}{2} \int_0^T \hat{V}''(\pi_t) d\langle \pi \rangle_t \quad (\text{B.39})$$

$$= \int_0^T \left(\mathcal{L}\hat{V}(\pi) - \hat{h}(\pi_t) + \lambda \right) dt + \int_0^T (\hat{h}(\pi_t) - \lambda) dt \quad (\text{B.40})$$

$$+ \int_0^T \hat{V}'(\pi_t) \pi_t (1-\pi_t) \sigma dB_t \quad (\text{B.41})$$

$$- \int_0^T \hat{V}'(\pi_t) (1-\varepsilon\pi_t) \pi_t \frac{d\varphi_t^{\downarrow}}{\varphi_t} \quad (\text{B.42})$$

$$+ \int_0^T \hat{V}'(\pi_t) \pi_t \frac{d\varphi_t^{\downarrow}}{\varphi_t}. \quad (\text{B.43})$$

The first term in line (B.40) is non-negative, in view of (B.38). Furthermore, (A.2) implies the existence of $\varepsilon' > \varepsilon$ such that $\pi_t < 1/\varepsilon' < 1/\varepsilon$, for all t , a.s. Using (B.38) one thus obtains

$$\hat{V}'(\pi_t) \leq \frac{\varepsilon \varepsilon'}{\varepsilon' - \varepsilon}, \quad \text{a.s. for all } t \geq 0. \quad (\text{B.44})$$

Hence (B.41) is a martingale with zero expectation. Again, (B.38) implies that

$$\hat{V}'(\pi_t) \pi_t (1-\varepsilon\pi_t) \leq \varepsilon\pi_t,$$

whence (B.42) satisfies

$$- \int_0^T \hat{V}'(\pi_t) (1-\varepsilon\pi_t) \pi_t \frac{d\varphi_t^{\downarrow}}{\varphi_t} \geq -\varepsilon \int_0^T \pi_t \frac{d\varphi_t^{\downarrow}}{\varphi_t}.$$

Finally, (B.43) is non-negative, because $\hat{V}' \geq 0$ due to (B.38).

Taking the expectation of (B.39) yields the estimate

$$\frac{1}{T} \mathbb{E}[\hat{V}(\pi_T) - \hat{V}(\pi_0)] \geq -\lambda + \frac{1}{T} \mathbb{E}[\int_0^T \hat{h}(\pi_t) dt] - \varepsilon \frac{1}{T} \int_0^T \pi_t \frac{d\varphi_t^{\downarrow}}{\varphi_t}. \quad (\text{B.45})$$

By eq. (B.44)

$$|\hat{V}(\pi_t) - \hat{V}(\pi_0)| \leq |\pi_T - \pi_0| \sup_{0 < u \leq 1/\varepsilon'} |\hat{V}'(u)| \leq \frac{\varepsilon}{\varepsilon' - \varepsilon},$$

therefore $\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\hat{V}(\pi_T) - \hat{V}(\pi_0)] = 0$. Hence letting $T \rightarrow \infty$ in (B.45) implies that for any admissible strategy φ one has $F_\infty(\varphi) \leq \lambda + r$. Finally, this bound is attained by the admissible trading strategy $\hat{\varphi}$ defined by Lemma (B.5) in terms of the free boundaries (ζ_-, ζ_+) : Let ζ_t be the corresponding risky-safe ratio. Using Itô's formula, one has $dV(\zeta_t) = V'(\zeta_t)\zeta_t \sigma dB_t + 0 - \varepsilon \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} + (h(\zeta_t) - \lambda)dt$. Division by T yields, in view of (A.8),

$$\frac{1}{T} \mathbb{E} \left[\int_0^T \left(\mu \pi_t - \frac{\gamma}{2} \sigma^2 \pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} \right] = \lambda + \frac{1}{T} \mathbb{E}[\hat{V}(\pi_T) - \hat{V}(\pi_0)].$$

Letting $T \rightarrow \infty$, one obtains $F_\infty(\hat{\varphi}) = \lambda + r$. \square

B.1. Proof of Theorem 3.1 (i)–(iii). Theorem 3.1 (i) is proved in Proposition B.1, and Theorem 3.1 (ii) & (iii) are proved in Proposition B.6.

APPENDIX C. PERFORMANCE AND ASYMPTOTICS

In this section, ergodicity arguments are used to derive closed-form expressions for average trading costs (ATC) and long-run mean and long-run variance of the optimal trading strategy. These formulae in turn yield the asymptotic expansions of Theorem 3.1 (iv).

C.1. The frictionless contribution. Let ζ_-, ζ_+ be the free boundaries obtained in Proposition B.1. In view of Remark A.4, assume that either $\zeta_- < \zeta_+ < -1$ (leveraged case) or $\zeta_- > \zeta_+ > 0$ throughout (non-leveraged case), and define the integral

$$I := \frac{1}{c} \int_{\zeta_-}^{\zeta_+} h(\zeta) |\zeta|^{2\gamma\pi_*-2} d\zeta, \quad (\text{C.1})$$

where the normalizing constant is

$$c := \int_{\zeta_-}^{\zeta_+} |\zeta|^{2\gamma\pi_*-2} d\zeta = \text{sgn}(\zeta_-) \frac{|\zeta_+|^{2\gamma\pi_*-1} - |\zeta_-|^{2\gamma\pi_*-1}}{2\gamma\pi_* - 1}. \quad (\text{C.2})$$

Lemma C.1.

$$I = h(\zeta_-) + \frac{\sigma^2(2\gamma\pi_* - 1)}{2} \left(\frac{G(\zeta_+) \zeta_+}{1 - \left(\frac{\zeta_-}{\zeta_+} \right)^{2\gamma\pi_*-1}} \right). \quad (\text{C.3})$$

Proof. From equations (B.8) and (B.10) it follows that

$$\int_{\zeta_-}^{\zeta_+} h(\zeta) |\zeta|^{2\gamma\pi_*-2} d\zeta = h(\zeta_-) \text{sgn}(\zeta_-) \frac{|\zeta_+|^{2\gamma\pi_*-1} - |\zeta_-|^{2\gamma\pi_*-1}}{2\gamma\pi_* - 1} + \frac{\sigma^2 \zeta_+^{2\gamma\pi_*}}{2} G(\zeta_+).$$

By normalizing, (C.3) follows. \square

C.2. Transaction costs. For the optimal trading policy, the risky-safe ratio ζ is a geometric Brownian motion with parameters (μ, σ) , reflected at ζ_-, ζ_+ respectively, see Lemma B.5. Hence the following ergodic result (Gerhold et al., 2014, Lemma C.1) applies:

Lemma C.2. *Let η_t be a diffusion on an interval $[l, u]$, $0 < l < u$, reflected at the boundaries, i.e.*

$$d\eta_t = b(\eta_t)dt + a(\eta_t)^{1/2}dB_t + dL_t - dU_t,$$

where the mappings $a(\eta) > 0$ and $b(\eta)$ are both continuous, and the continuous, non-decreasing processes L_t and U_t satisfy $L_0 = U_0 = 0$ and increase only on $\{L_t = l\}$ and $\{U_t = u\}$, respectively. Denoting by $\nu(\eta)$ the invariant density of η_t , the following almost sure limits hold:

$$\lim_{T \rightarrow \infty} \frac{L_T}{T} = \frac{a(l)\nu(l)}{2}, \quad \lim_{T \rightarrow \infty} \frac{U_T}{T} = \frac{a(u)\nu(u)}{2}.$$

The next formula evaluates trading costs.

Lemma C.3. *The average trading costs for the optimal trading policy are*

$$\text{ATC} := \varepsilon \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} = \frac{\sigma^2(2\gamma\pi_* - 1)}{2} \left(\frac{G(\zeta_+)\zeta_+}{1 - \left(\frac{\zeta_-}{\zeta_+}\right)^{2\gamma\pi_*-1}} \right). \quad (\text{C.4})$$

Proof. Note that $\varepsilon \int_0^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} = G(\zeta_+) \frac{U_T}{T}$. Applying Lemma C.2 to $\eta := \zeta$ (setting $l := \zeta_-, u = \zeta_+$) and using the stationary density of ζ_t (Lemma B.5) (C.4) follows. \square

Remark C.4. An alternative proof of Lemma C.3 follows from Lemma A.2, by rewriting the objective functional as $F_\infty(\varphi) = r + \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T h(\zeta_t)dt - \text{ATC}$. By the ergodic theorem (Borodin and Salminen, 2002, II.35 and II.36), $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T h(\zeta_t)dt = I$ hence using Lemma C.1 and Proposition B.6 it follows that

$$\text{ATC} = -F_\infty(\varphi) + r + I = \frac{\sigma^2(2\gamma\pi_* - 1)}{2} \left(\frac{G(\zeta_+)\zeta_+}{1 - \left(\frac{\zeta_-}{\zeta_+}\right)^{2\gamma\pi_*-1}} \right).$$

which is in agreement with the formula in Proposition B.6.

C.3. Long-run mean and variance. Set

$$I_\mu := \int_{\zeta_-}^{\zeta_+} \left(\frac{\zeta}{1 + \zeta} \right) |\zeta|^{2\gamma\pi_*-2} d\zeta, \quad I_{s^2} := \int_{\zeta_-}^{\zeta_+} \left(\frac{\zeta}{1 + \zeta} \right)^2 |\zeta|^{2\gamma\pi_*-2} d\zeta.$$

In view of the ergodic theorem (Borodin and Salminen, 2002, II.35 and II.36), the long-run mean and long-run variance satisfy

$$\begin{aligned} \hat{m} &= r + \mu \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \pi_t dt \right] - \text{ATC} = r + \frac{\mu}{c} I_\mu - \text{ATC}, \\ \hat{s}^2 &= \sigma^2 \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \pi_t^2 dt \right] = \frac{\sigma^2}{c} I_{s^2}, \end{aligned}$$

whence the following decomposition holds:

$$I = \frac{1}{c} \left(\mu I_\mu - \frac{\gamma\sigma^2}{2} I_{s^2} \right) = \frac{\pi_*}{\pi_*} (\hat{m} - r + \text{ATC}) - \frac{\gamma}{2} \hat{s}^2 = h(\zeta_-) + \text{ATC}. \quad (\text{C.5})$$

Integration by parts yields

$$I_\mu = \int_{\zeta_-}^{\zeta_+} \frac{\zeta}{1 + \zeta} |\zeta|^{2\gamma\pi_*-2} d\zeta = \frac{|\zeta_+|^{2\gamma\pi_*}}{2\gamma\pi_*(1 + \zeta_+)} - \frac{|\zeta_-|^{2\gamma\pi_*}}{2\gamma\pi_*(1 + \zeta_-)} + \frac{I_{s^2}}{2\gamma\pi_*}. \quad (\text{C.6})$$

Plugging (C.6) into (C.5) yields $I = \frac{\sigma^2}{2c} \frac{\pi_*}{\pi_*} \left(\frac{|\zeta_+|^{2\gamma\pi_*}}{1 + \zeta_+} - \frac{|\zeta_-|^{2\gamma\pi_*}}{1 + \zeta_-} + (1 - \frac{\gamma\pi_*}{\pi_*}) I_{s^2} \right)$. Except for the singular case $\gamma = 1$, one can extract I_{s^2} , and thus (C.6) and (C.4) yield a formula for \hat{s}^2 . Therefore, the right side of equation (C.5) gives a formula for \hat{m} in terms of \hat{s} :

Lemma C.5. When $\gamma \neq 1$, the following identities hold:

$$\hat{s}^2 = \frac{2}{1-\gamma} (h(\zeta_-) + \text{ATC}) - \frac{\sigma^2}{c(1-\gamma)} \left(\frac{|\zeta_+|^{2\gamma\pi_*}}{1+\zeta_+} - \frac{|\zeta_-|^{2\gamma\pi_*}}{1+\zeta_-} \right), \quad (\text{C.7})$$

$$\hat{m} = r + \frac{\gamma}{2} \hat{s}^2 + h(\zeta_-). \quad (\text{C.8})$$

C.4. Proof of Theorem 3.1 (iv).

Proof. The asymptotic expansion (3.7) for the trading boundaries π_{\pm} is derived by expanding $\frac{\zeta_{\pm}}{1+\zeta_{\pm}}$ into a power series, thereby using the asymptotic expansions (B.7) of ζ_{\pm} .

The long-run mean \hat{m} , variance \hat{s}^2 , Sharpe ratio $((\hat{m} - r)/\hat{s})$, average trading costs ATC, and value function λ have closed form expressions in terms of the free boundaries ζ_- , ζ_+ (see equations (C.8), (C.7), and equations (C.4) and (B.37)). Using these formulas in combination with the asymptotic expansions (B.7) of the free boundaries, the assertion follows. \square

APPENDIX D. FROM RISK AVERSION TO RISK NEUTRALITY

In this section the free boundary problem (3.1)–(3.5) for $\gamma = 0$ is solved for sufficiently small ε , it is shown that the solution (W, ζ_-, ζ_+) allows to construct a solution of the corresponding HJB equation and, similarly to the case $\gamma > 0$, a verification argument yields the strategy's optimality.

Numerical experiments using $\gamma > 0$ indicate that the trading boundaries π_{\pm} (hence the leverage multiplier) satisfy $\lim_{\varepsilon \downarrow 0} \varepsilon^{1/2} \pi_{\pm} = 1/A_{\pm}$ for two constants $A_- > A_+ > 0$. This entails that the free boundaries have the approximation $\zeta_{\pm} \approx -1 - A_{\pm}\varepsilon^{1/2}$, thereby suggesting that ζ_{\pm} are analytic in $\delta := \varepsilon^{1/2}$. The system (B.10)–(B.11) is rewritten by using the new parameter $\delta := \varepsilon^{1/2}$ and by multiplying the second equation by δ :

$$W(\zeta_-, \zeta_+) - \frac{\delta^2}{(1+\zeta_+)(1+(1-\delta^2)\zeta_+)} = 0, \quad (\text{D.1})$$

$$\delta \left(\frac{2(h(\zeta_+)-h(\zeta_-))}{\sigma^2 \zeta_+^2} - \frac{2\mu/\sigma^2}{\zeta_+} W(\zeta_-, \zeta_+) - \frac{(1-\delta^2)^2}{(1+(1-\delta^2)\zeta_+)^2} + \frac{1}{(1+\zeta_+)^2} \right) = 0. \quad (\text{D.2})$$

Using the transformation $u = \frac{-1-\zeta}{\delta}$ and noting that $|\zeta| = 1 + \delta u$, it follows that

$$\Xi(u_-, u_+) := W(-1 - u_- \delta, -1 - u_+ \delta) = \frac{2\mu}{\sigma^2 (1+u\delta)^2} \int_{u_-}^{u_+} \left(\frac{1}{u_-} - \frac{1}{\xi} \right) \left(\frac{1+\xi\delta}{1+u\delta} \right)^{\frac{2\mu}{\sigma^2}-2} d\xi.$$

Accordingly, the system (D.1)–(D.2) transforms into

$$\Xi(u_-, u_+) - \frac{1}{u_+((1-\delta^2)u_+-\delta)} = 0, \quad (\text{D.3})$$

$$\frac{2\mu}{\sigma^2} \left(\frac{1}{u_+} - \frac{1}{u_-} + \frac{\delta}{1+u_+\delta} \Xi(u_-, u_+) \right) - \frac{2(1-\delta^2)u_+-\delta}{u_+^2(\delta+(\delta^2-1)u_+)^2} = 0. \quad (\text{D.4})$$

Letting $\delta \rightarrow 0$ in (D.3)–(D.4), one obtains a system of equations for (A_-, A_+) ,

$$\frac{2\mu}{\sigma^2} \left(\log(A_-/A_+) - \frac{A_- - A_+}{A_-} \right) - \frac{1}{A_+^2} = 0, \quad \frac{\mu}{\sigma^2} \left(\frac{1}{A_+} - \frac{1}{A_-} \right) - \frac{1}{A_+^3} = 0. \quad (\text{D.5})$$

Lemma D.1. The unique solution (A_-, A_+) of the system (D.5) is

$$A_- = \frac{\kappa^{-1/2}}{1-\kappa} \sqrt{\frac{\sigma^2}{\mu}}, \quad A_+ = \kappa^{-1/2} \sqrt{\frac{\sigma^2}{\mu}}, \quad (\text{D.6})$$

where $\kappa \approx 0.5828$ is the unique solution of (3.16).

Proof. The second equation in (D.5) gives

$$A_- = \frac{\mu A_+^3}{\mu A_+^2 - \sigma^2}. \quad (\text{D.7})$$

Hence substituting (D.7) into the first equation of (D.5) gives the well-posed problem

$$-\frac{3}{A_+^2} + \frac{2\mu \log\left(\frac{\mu A_+^2}{\mu A_+^2 - \sigma^2}\right)}{\sigma^2} = 0, \quad A_+ > 0. \quad (\text{D.8})$$

Therefore it is enough to establish that the unique solution of (D.8) is as in the second equation in line (D.6); the formula for A_- then follows from (D.7). To this end, substitute $\xi := \sigma^2/(\mu A_+^2)$ into (D.8) to obtain equation (3.16). Note that $f(0) = 0$, $f' > 0$ on $(0, 1/3)$ and $f' < 0$ on $(1/3, 1)$, while $f(\xi) \downarrow -\infty$ as $\xi \rightarrow 1$. This implies that f has a single zero κ on $(1/3, 1)$ and thus the claim concerning A_+ is proved. \square

Proposition D.2. *For sufficiently small δ , there exists a unique solution (u_+, u_-) of (D.3)–(D.4) near (A_-, A_+) . This solution is analytic in δ and satisfies the asymptotic expansion $u_\pm = A_\pm + O(\delta)$, where A_\pm are in (D.6).*

Proof. Denote the left sides of (D.3)–(D.4), by $F_i((u_-, u_+), \delta)$, $i = 1, 2$ and $F = (F_1, F_2)$. By Lemma D.1, $F((A_-, A_+), 0) = 0$. As

$$\frac{\partial \Xi}{\partial u_-}((A_-, A_+), 0) = \frac{2\mu}{\sigma^2} \left(\frac{A_- - A_+}{A_-^2} \right), \quad \frac{\partial \Xi}{\partial u_+}((A_-, A_+), 0) = \frac{2\mu}{\sigma^2} \left(\frac{A_+ - A_-}{A_- A_+} \right),$$

one obtains at (A_\pm) , $\frac{\partial F_1}{\partial u_-} = \frac{2\mu}{\sigma^2} \left(\frac{A_- - A_+}{A_-^2} \right)$, $\frac{\partial F_2}{\partial u_+} = \frac{6}{A_+^4} - \frac{2\mu}{\sigma^2 A_+^2}$ and

$$\frac{\partial F_1}{\partial u_+} = \frac{2}{A_+^3} + \frac{2\mu}{\sigma^2} \left(\frac{A_+ - A_-}{A_- A_+} \right) = 0,$$

where the last equality follows from the second equation in (D.5). Therefore, as $\kappa \in (1/3, 1)$, the Jacobian DF satisfies $\det(DF)((A_-, A_+), 0) = -4(\mu/\sigma^2)^{7/2}(\kappa - 1)\kappa^{5/2}(3\kappa - 1) \neq 0$. By the implicit function theorem for analytic functions (Gunning and Rossi, 2009, Theorem I.B.4) the assertion follows. \square

Lemma D.3. *Let κ be the solution of (3.16) and $\theta \in [0, 1]$. Then*

$$\log(1 - \kappa(1 - \theta)) + (1 - \theta)\kappa + \frac{1}{2} \frac{\kappa(1 - \kappa)^2}{(1 - \kappa(1 - \theta))^2} = 0 \quad (\text{D.9})$$

implies $\theta = 0$.

Proof. Clearly $f(0) = 0$ and also $f(1) = 1/2\kappa(1 - \kappa)^2 > 0$. There is a single local extremum of f , in $(0, 1)$, namely, $\theta_1 = \frac{0.5(3\kappa^2 + \sqrt{4\kappa^3 - 3\kappa^4 - 2\kappa})}{\kappa^2} \approx 0.7669$. Because $f'(0) = 0$ and $f''(0) = \frac{\kappa^2(\kappa(3\kappa^2 - 7\kappa + 5) - 1)}{(1 - \kappa)^4} > 0$, θ_1 must be the global maximum. Hence $f > 0$ on $(0, 1]$, whence $\theta = 0$, as claimed. \square

Lemma D.4. *Let A_- be as in (D.6). The only solution of*

$$\frac{2\mu}{\sigma^2} \left(\log(A_-/\xi) - \frac{A_- - \xi}{A_-} \right) - \frac{1}{\xi^2} = 0 \quad (\text{D.10})$$

on $[A_+, A_-]$ is $\xi = A_+$.

Proof. Let ξ be a solution of (D.10). There exists $\theta \in [0, 1]$ such that

$$\xi = \theta A_- + (1 - \theta)A_+ = A_+ \left(\frac{1 + \kappa(1 - \theta)}{1 - \kappa} \right).$$

Hence $A_+^*/A_- = 1 + \kappa(\theta - 1)$, and therefore (D.10) is rewritten as (D.9). An application of Lemma D.3 yields $\xi = A_+$. \square

D.1. Proof of Theorem 3.2.

Proof. Arguing similarly as in the Proof of Proposition B.1 for the case $\gamma > 0$, the solvability of the free boundary problem (3.1)–(3.5) for $\gamma = 0$ is equivalent to solvability of the non-linear system (D.1)–(D.2). This, in turn, is equivalent to solving the system (D.3)–(D.4) for $(u_+(\delta), u_-(\delta))$. A unique solutions of the transformed system (D.3)–(D.4) near (A_+, A_-) is provided by Proposition D.2, and one has $\zeta_\pm = -1 - u_\pm \delta$. In particular, one obtains

$$\zeta_\pm = -1 - A_\pm \varepsilon^{1/2} + O(1). \quad (\text{D.11})$$

The solution of (3.1)–(3.5) is

$$W(\zeta) := \frac{2\mu}{\sigma^2 |\zeta|^{\frac{2\mu}{\sigma^2}}} \int_{\zeta_-}^{\zeta} \left(\frac{y}{1+y} - \frac{\zeta_-}{1+\zeta_-} \right) |y|^{2\mu/\sigma^2 - 2} dy. \quad (\text{D.12})$$

One defines exactly as in (B.27) a candidate solution (V, λ) of the HJB equation (B.26). Next it is shown that (V, λ) solves the HJB equation (B.26) (for the intervals $[\zeta_-, \zeta_+]$, $(-\infty, \zeta_-]$ and finally for $[\zeta_+, \infty)$). In fact, the interval $[-1/(1-\varepsilon), 0)$ is excluded.

On $[\zeta_-, \zeta_+]$,

$$(\mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-))' = \frac{1}{2}\sigma^2 \zeta^2 W''(\zeta) + (\sigma^2 + \mu)\zeta W'(\zeta) + \mu W(\zeta) - \frac{\mu}{(1+\zeta)^2} = 0$$

by construction. Because of the initial conditions (3.2)–(3.3), $(\mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-))|_{\zeta=\zeta_-} = \mathcal{A}V(\zeta)|_{\zeta=\zeta_-} = 0$, and thus $\mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-) \equiv 0$ for $\zeta \in [\zeta_-, \zeta_+]$. Next it is shown that $0 \leq V' \leq G$ on all of $[\zeta_-, \zeta_+]$. As $(h(\zeta) - h(\zeta_-))' = h'(\zeta) = \frac{\mu}{(1+\zeta)^2}$ is strictly positive, $h(\zeta) - h(\zeta_-) > 0$ for $\zeta \in (\zeta_-, \zeta_+]$. From the explicit formula (D.12) it then follows that $V' = W \geq 0$ for $\zeta \in [\zeta_-, \zeta_+]$. It remains to show $V' \leq G$. As $V'(\zeta_+) - G(\zeta_+) = 0$, and $V'(\zeta_-) - G(\zeta_-) = -G(\zeta_-) < 0$, it suffices to rule out any zero ζ_+^* of $V'(\zeta) - G(\zeta)$ on (ζ_-, ζ_+) , for sufficiently small ε . This is equivalent to ruling out any zeros of

$$\kappa(u, \delta) := V'(\zeta(u)) - G(\zeta(u)), \quad u \in (u_+(\delta), u_-(\delta)),$$

where $\zeta(u) = -1 - u\delta$, for sufficiently small δ . Recall that $u_\pm(\delta)$ is implicitly defined by $\zeta_\pm = -1 - u_\pm(\delta)\delta$, $\lim_{\delta \rightarrow 0} u_\pm(\delta) = A_\pm$. Suppose, by contradiction, that there exists $\delta_k \downarrow 0$ and a sequence $u_+(\delta_k)$ satisfying $u_-(\delta_k) < u_+^*(\delta_k) < u_+(\delta_k)$ which is a solution of $\kappa(u_+^*(\delta_k), \delta_k) = 0$ for each $k \in \mathbb{N}$. By taking a subsequence, if necessary, one may without loss of generality assume $u_+^*(\delta_k) \rightarrow A_+^* \in [A_+, A_-]$ as $k \rightarrow \infty$. Suppose first that $A_+^* = A_+$ and define the map $\delta \mapsto u^*(\delta)$ by intertwining u_+ and u_+^* as follows:

$$u_+^*(\delta) = \begin{cases} u_+^*(\delta_k), & k \in \mathbb{N} \\ u_+(\delta), & \delta \neq \delta_k \end{cases}.$$

Then for sufficiently small δ , the pair $(u_-(\delta), u_+^*(\delta))$ solves (D.3)–(D.4) near (A_-, A_+) , hence by Proposition D.2, $u_+^* = u_+$, in contradiction to our previous assumption $\zeta_+^* \in (\zeta_-, \zeta_+)$. Second, consider the case $A_+^* \in (A_+, A_-]$: By equation (D.3)

$$\frac{2\mu}{\sigma^2} \left(\log(A_-/A_+^*) - \frac{A_- - A_+^*}{A_-} \right) - \frac{1}{(A_+^*)^2} = 0.$$

Lemma D.4 states $A_+^* = A_+$, which is also impossible. Hence $V'(\zeta) - G(\zeta)$ has no zeroes on (ζ_-, ζ_+) , and thus V solves the HJB equation on $[\zeta_-, \zeta_+]$.

Consider now $\zeta \leq \zeta_-$. V solves the HJB equation, if

$$\mathcal{A}V - h(\zeta) + h(\zeta_-) = h(\zeta_-) - h(\zeta) \geq 0, \quad G(\zeta) \geq 0.$$

The first inequality is clearly fulfilled. Also, as $\zeta < -1/(1-\varepsilon)$ or $\zeta > 0$, G is a strictly positive function on $[-\infty, \zeta_-]$, which finishes the proof for $\zeta \leq \zeta_-$.

Finally, consider $\zeta \geq \zeta_+$. As $G = W$, it suffices to show that

$$L(\zeta) := \mathcal{A}V(\zeta) - h(\zeta) + h(\zeta_-) \geq 0, \quad G(\zeta) \geq 0. \quad (\text{D.13})$$

The second inequality has now been proved, and it remains to establish the first inequality in (D.13). As $h_1(\zeta) = \mu \frac{\zeta}{1+\zeta} - \frac{\sigma^2}{2} \left(\frac{\zeta}{1+\zeta} \right)^2$ it follows that

$$L(\zeta) = \frac{\sigma^2 \zeta^2}{2} G'(\zeta) + \mu \zeta G(\zeta) - h(\zeta) + h(\zeta_-) = h(\zeta_-) - h_1((1-\varepsilon)\zeta) - \frac{\sigma^2}{2} \left(\frac{\zeta}{1+\zeta} \right)^2.$$

Therefore, by the boundary conditions at ζ_+ , and as W solves the free boundary problem on $[\zeta_-, \zeta_+]$,

$$L(\zeta_+) = \frac{\sigma^2 \zeta^2}{2} W'(\zeta_+) + \mu \zeta W(\zeta_+) + h(\zeta_-) - h(\zeta_+) = 0.$$

To show that $L(\zeta) \geq 0$ for all ζ , it suffices to show that there are no solutions of the equation $L(\zeta) = 0$ on $\zeta \geq \zeta_+$ except ζ_+ . The transformation $z = \frac{\zeta}{1+\zeta}$ introduces $F(z, \varepsilon) := L(\zeta(z))$. As $F(\pi_+) = 0$, polynomial division by $(z - \pi_+)$ yields (B.32), where the third order polynomial g has derivative $g' = a_0 + a_1 z + a_2 z^2$ with certain, relatively complex but explicit coefficients a_0, a_1, a_2 . By the second formula of (D.6)

$$g(\pi_+) = -\mu + \frac{3\sigma^2}{A_+^2} + O(\varepsilon^{1/2}) \quad (\text{D.14})$$

is strictly positive for sufficiently small ε because $\kappa > 1/3$. The zeros z_\pm of g' are $z_- = -\frac{1}{2A_+ \varepsilon^{1/2}} + O(1)$, $z_+ = \frac{4}{3\varepsilon} + O(1)$. For sufficiently small ε the first one is negative, and the second solution is larger than $1/\varepsilon$, hence both are irrelevant. Also, $g'(1/\varepsilon) = \sigma^2/2 + O(\varepsilon^{1/2})$ and thus $g'(z) > 0$ on all of $[\pi_+, 1/\varepsilon]$. Together with (D.14), it follows that $g > 0$ on $[\pi_+, 1/\varepsilon]$. Hence $F(z) > 0$ for all $z > \pi_+$ which proves that (V, λ) solves the HJB equation (B.26).

Using the proof of Proposition B.6, one can obtain assertion (ii) and (iii). Finally, the expansions of the trading boundaries claimed in (iv) follow from the asymptotic expansions of the free boundaries ζ_-, ζ_+ in (D.11). \square

APPENDIX E. CONVERGENCE

Lemma E.1. *Let $\mu > \sigma^2$. There exists $\delta_0 > 0$ such that for all $\delta \leq \delta_0$ and for all $0 \leq \gamma \leq \gamma_0 := \frac{\mu}{\sigma^2}$, the objective functional for a trading strategy φ which only engages in buying at $\pi_- = 1 + \delta$ and selling at $\pi_+ = (1 - \delta)/\varepsilon > \pi_-$ outperforms a buy and hold strategy. More precisely, for all $\gamma \leq \gamma_0$ and for all $\delta \leq \delta_0$*

$$F_\infty(\varphi) \geq r + \mu - \frac{\gamma \sigma^2}{2} + \left(\frac{\mu - \gamma \sigma^2}{2} \right) \delta.$$

Proof. As $\varepsilon \in (0, 1)$ and $\tilde{\pi} := \frac{\mu}{\gamma_0 \sigma^2} > 1$, there exists $\tilde{\delta} > 0$ such that $\pi_* \geq \pi_+$ for all $\delta \leq \tilde{\delta}$ and $\gamma \leq \gamma_0$.

Let $\rho(\pi)d\pi = \nu(\pi/(1-\pi)) \frac{d\pi}{(\pi-1)^2}$, where $\nu(d\zeta)$ is the stationary density of a reflected diffusion ζ on $[\zeta_-, \zeta_+]$ (Lemma B.5). As $\pi_* \geq \pi_+$, also $\mu\pi - \frac{\gamma\sigma^2}{2}\pi^2 \geq \mu\pi_- - \frac{\gamma\sigma^2}{2}\pi_-^2$

holds for all $\pi \in [\pi_-, \pi_+]$. Thus,

$$\begin{aligned} F_\infty(\varphi) &= r + \int_{\pi_-}^{\pi_+} \left(\mu\pi - \frac{\gamma\sigma^2}{2}\pi^2 \right) \rho(d\pi) - \text{ATC} \\ &\geq r + \mu(1 + \delta) - \frac{\gamma\sigma^2}{2}(1 + \delta)^2 - \frac{(\delta+1)(2\epsilon-1)^3(2\mu-\sigma^2)}{4\epsilon \left(\delta \left(\frac{-(\delta+1)\epsilon+\delta+1}{\delta} \right)^{\frac{2\mu}{\sigma^2}} + (\delta+1)(2\epsilon-1) \right)} \\ &\geq r + \mu - \frac{\gamma\sigma^2}{2} + (\mu - \gamma\sigma^2)\delta - O(\delta^{\min(2, \frac{2\mu}{\sigma^2}-1)}), \end{aligned} \quad (\text{E.1})$$

where Lemma C.3 has been invoked to calculate and estimate the average trading costs ATC. The asymptotic expansion holds for sufficiently small δ and, as $\mu > \gamma\sigma^2$, the exponent in the asymptotic formula (E.1) satisfies $2\mu/\sigma^2 - 2 > 1$. \square

E.1. Proof of Theorem 4.1.

Proof. As $\zeta_+ < -1/(1 - \varepsilon)$, the curves $(0, \bar{\gamma}] \rightarrow \mathbb{R} : \gamma \mapsto \pi_\pm(\gamma)$ range in a relatively compact set, namely $[1, \frac{1}{\varepsilon})$. Consider therefore a sequence γ_k , $k = 1, 2, \dots$ which satisfies $1 \leq \pi_-^0 := \lim_{i \rightarrow \infty} \pi_-(\gamma_k) \leq \lim_{i \rightarrow \infty} \pi_+(\gamma_k) =: \pi_+^0 \leq 1/\varepsilon$. Set $\zeta_\pm^k := \frac{\pi_\pm(\gamma_k)}{1 - \pi_\pm(\gamma_k)}$, for $k = 0, 1, 2, \dots$, and note that $-\infty \leq \zeta_-^0 \leq \zeta_+^0 \leq -\frac{1}{1-\varepsilon}$. For each k , $k = 1, 2, \dots$, by assumption the HJB equation (B.26) is satisfied with $\lambda = \lambda_k := h(\zeta_-^k)$. The verification arguments in the proof of Proposition B.6 yield that the trading strategies associated with the intervals $[\pi_-(\gamma_k), \pi_+(\gamma_k)]$ are optimal.

Next, three facts are proved. First $\pi_-^0 > 1$, which is equivalent to $\zeta_-^0 > -\infty$. Suppose, by contradiction, that $\pi_-^0 = 1$. Then $\pi_-(\gamma_k) \rightarrow 1$ and thus $\lambda_k \rightarrow \mu$, as $k \rightarrow \infty$. Hence, the objective functional eventually minorizes the uniform bound provided by Lemma E.1, a mere impossibility to optimality. Hence $\pi_-^0 > 1$. Second, $\pi_-^0 < \pi_+^0$: This holds due to the fact that, by observing limits for the initial and terminal conditions of zero order in (3.1), $W(\zeta_-^0) = 0 < G(\zeta_-^0)$. Third, $\pi_+^0 < \frac{1}{\varepsilon}$. Suppose, by contradiction, that $\pi_+^0 = \frac{1}{\varepsilon}$. Then $G(\zeta_+^k) \rightarrow \infty$, as $k \rightarrow \infty$, and, as $\zeta_-^0 < \zeta_+^0$, the average trading costs corresponding to γ_k satisfy (by Lemma C.3)

$$\text{ATC}(k) := \frac{\sigma^2 \left(\frac{2\mu}{\sigma^2} - 1 \right)}{2} \frac{G(\zeta_+^k)\zeta_+^k}{1 - \left(\frac{\zeta_-^k}{\zeta_+^k} \right)^{2\mu/\sigma^2-1}} \rightarrow \infty,$$

as $k \rightarrow \infty$. Denote by $\hat{\varphi}^k$ the trading strategy which only buys (resp. sells) at $\pi_-(\gamma_k)$ (resp. $\pi_+(\gamma_k)$). By the results of Appendix C the value function satisfies

$$\lim_{k \rightarrow \infty} F_\infty(\hat{\varphi}^k) = \lim_{k \rightarrow \infty} \int_{\pi_-(\gamma_k)}^{\pi_+(\gamma_k)} \left(\mu\pi - \frac{\gamma_k\sigma^2}{2}\pi^2 \right) \rho(d\pi) - \text{ATC}(k) \leq \frac{\mu}{\varepsilon} - \lim_{k \rightarrow \infty} \text{ATC}(k) = -\infty$$

as $k \rightarrow \infty$. In particular, for sufficiently large $k \geq k_0$, a buy-and-hold strategy φ satisfies $F_\infty(\varphi) = \mu - \frac{\gamma_k\sigma^2}{2} > F_\infty(\hat{\varphi}^k)$, which contradicts optimality of the trading strategy $[\pi_-(\gamma_k), \pi_+(\gamma_k)]$. Hence $\pi_+^0 < 1/\varepsilon$.

As the sequence ζ_-^k converges, by (Keller-Ressel et al., 2010, Lemma 9) the solutions of the initial value problem associated with (3.1) and γ_k , namely $W(\zeta; \zeta_-^k)$, converge to the solution of the initial value problem (3.1) (for $\gamma = 0$),

$$W^0(\zeta) = -\frac{2}{\sigma^2\zeta^2} \int_{\zeta_-^0}^{\zeta} \left(\mu \frac{\zeta}{1+\zeta} - \mu \frac{\zeta_-^0}{1+\zeta_-^0} \right) (\zeta/\zeta_-^0)^{2\mu/\sigma^2-2} d\zeta.$$

The terminal conditions are met by W^0 , because G is continuous on $(-\infty, -\frac{1}{1-\varepsilon})$. Also, for each k , $k = 1, 2, \dots$, by assumption the HJB equation (B.26) is satisfied. Non-negativity is preserved by taking limits, hence, $(\hat{W}(\zeta; 0), \lambda_0)$ satisfies the HJB equation as well. The verification arguments in the proof of Proposition B.6 imply that the trading strategies associated with the intervals $[\pi_-(\gamma), \pi_+(\gamma)]$ are not only optimal for risk-aversion levels $\gamma \in [0, \bar{\gamma}]$, but also $[\pi_-^0, \pi_+^0]$ is optimal for a risk-neutral investor.

$\zeta_-(\gamma)$ can have only one accumulation point for $\gamma \downarrow 0$, because $\lambda_0 = h(\zeta_-^0)$ is the value function. Uniqueness of ζ_-^0 is therefore clear and it follows that $\zeta_-^0 = \zeta_-(0)$. By assumption, the free boundary problem has a unique solution, hence it follows that $\pi_+(0) = \pi_+^0$. In particular, the curves $(0, \bar{\gamma}] \rightarrow \mathbb{R} : \gamma \mapsto \pi_{\pm}(\gamma)$ each have a unique limit π_{\pm}^0 as $\gamma \downarrow 0$, which equals $\pi_{\pm}(0)$, the solution of the free boundary problem. \square

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