Robot Mapping

EKF SLAM

Gian Diego Tipaldi, Wolfram Burgard

Simultaneous Localization and Mapping (SLAM)

- Building a map and locating the robot in the map at the same time
- Chicken-or-egg problem

Definition of the SLAM Problem

Given

- The robot's controls

$$u_{1:T} = \{u_1, u_2, u_3, \ldots, u_T\}$$

- Observations

$$z_{1:T} = \{z_1, z_2, z_3, \ldots, z_T\}$$

Wanted

- Map of the environment

$$m$$

- Path of the robot

$$x_{0:T} = \{x_0, x_1, x_2, \ldots, x_T\}$$

Three Main Paradigms

Kalman filter

Particle filter

Graph-based

Bayes Filter

- Recursive filter with prediction and correction step

- **Prediction**

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \; bel(x_{t-1}) \, dx_{t-1}$$

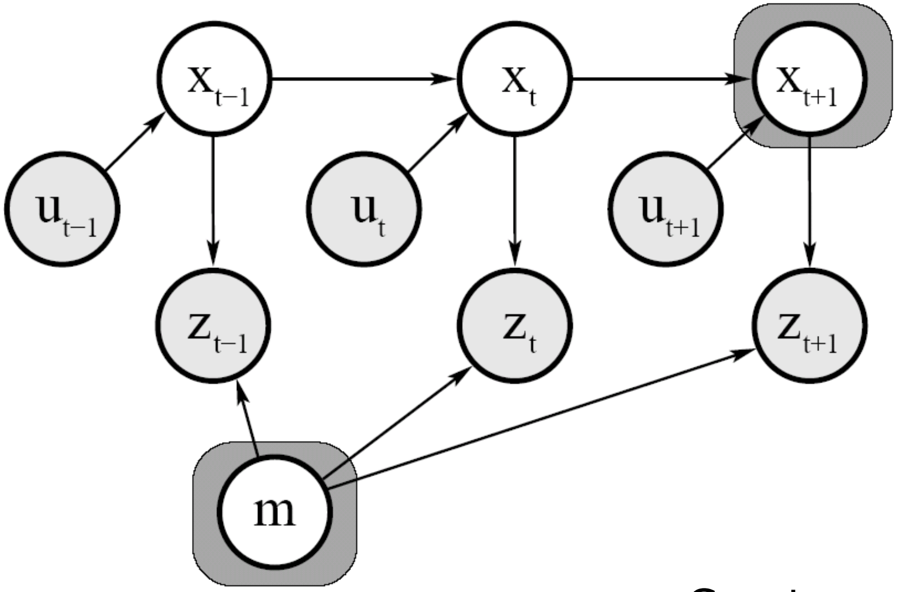
- **Correction**

$$bel(x_t) = \eta \, p(z_t \mid x_t) \; \overline{bel}(x_t)$$

EKF for Online SLAM

- We consider here the Kalman filter as a solution to the online SLAM problem

$$p(x_t, m \mid z_{1:t}, u_{1:t})$$



Courtesy: Thrun, Burgard, Fox

Extended Kalman Filter Algorithm

Extended_Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$
$$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$$

$$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$
$$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$$
$$return \; \mu_t, \Sigma_t$$

EKF SLAM

- Application of the EKF to SLAM
- Estimate robot's pose and locations of landmarks in the environment
- Assumption: known correspondences
- State space (for the 2D plane) is

$$x_t = (\underbrace{x, y, \theta}_{\text{robot's pose}}, \underbrace{m_{1,x}, m_{1,y}}_{\text{landmark 1}}, \ldots, \underbrace{m_{n,x}, m_{n,y}}_{\text{landmark n}})^T$$

EKF SLAM: State Representation

- Map with *n* landmarks: (3+2*n*)-dimensional Gaussian
- Belief is represented by

$$
\underbrace{\begin{pmatrix} x \\ y \\ \theta \\ m_{1,x} \\ m_{1,y} \\ \vdots \\ m_{n,x} \\ m_{n,y} \end{pmatrix}}_{\mu}
\quad
\underbrace{\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xm_{1,x}} & \sigma_{xm_{1,y}} & \cdots & \sigma_{xm_{n,x}} & \sigma_{xm_{n,y}} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{ym_{1,x}} & \sigma_{ym_{1,y}} & \cdots & \sigma_{m_{n,x}} & \sigma_{m_{n,y}} \\
\sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta m_{1,x}} & \sigma_{\theta m_{1,y}} & \cdots & \sigma_{\theta m_{n,x}} & \sigma_{\theta m_{n,y}} \\
\sigma_{m_{1,x}x} & \sigma_{m_{1,x}y} & \sigma_{\theta} & \sigma_{m_{1,x}m_{1,x}} & \sigma_{m_{1,x}m_{1,y}} & \cdots & \sigma_{m_{1,x}m_{n,x}} & \sigma_{m_{1,x}m_{n,y}} \\
\sigma_{m_{1,y}x} & \sigma_{m_{1,y}y} & \sigma_{\theta} & \sigma_{m_{1,y}m_{1,x}} & \sigma_{m_{1,y}m_{1,y}} & \cdots & \sigma_{m_{1,y}m_{n,x}} & \sigma_{m_{1,y}m_{n,y}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{m_{n,x}x} & \sigma_{m_{n,x}y} & \sigma_{\theta} & \sigma_{m_{n,x}m_{1,x}} & \sigma_{m_{n,x}m_{1,y}} & \cdots & \sigma_{m_{n,x}m_{n,x}} & \sigma_{m_{n,x}m_{n,y}} \\
\sigma_{m_{n,y}x} & \sigma_{m_{n,y}y} & \sigma_{\theta} & \sigma_{m_{n,y}m_{1,x}} & \sigma_{m_{n,y}m_{1,y}} & \cdots & \sigma_{m_{n,y}m_{n,x}} & \sigma_{m_{n,y}m_{n,y}}
\end{pmatrix}}_{\Sigma}
$$

EKF SLAM: State Representation

- More compactly

$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: State Representation

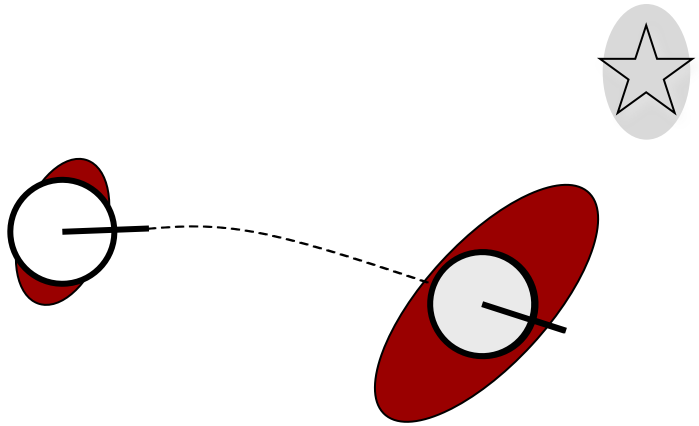
- Even more compactly (note: $x_R \rightarrow x$)

$$\underbrace{\begin{pmatrix} x \\ m \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Filter Cycle

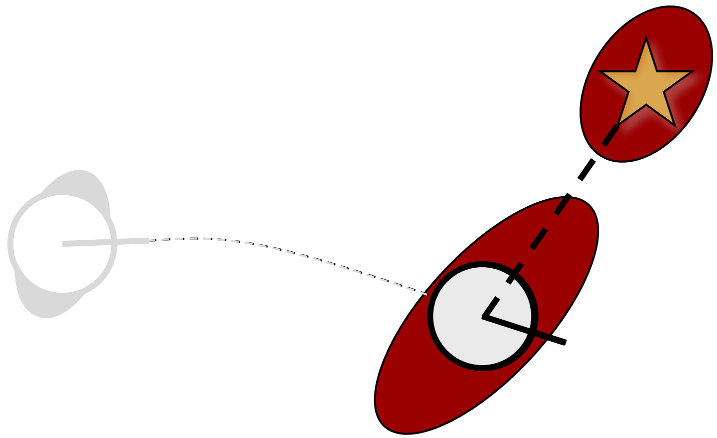
1. State prediction
2. Measurement prediction
3. Measurement
4. Data association
5. Update

EKF SLAM: State Prediction



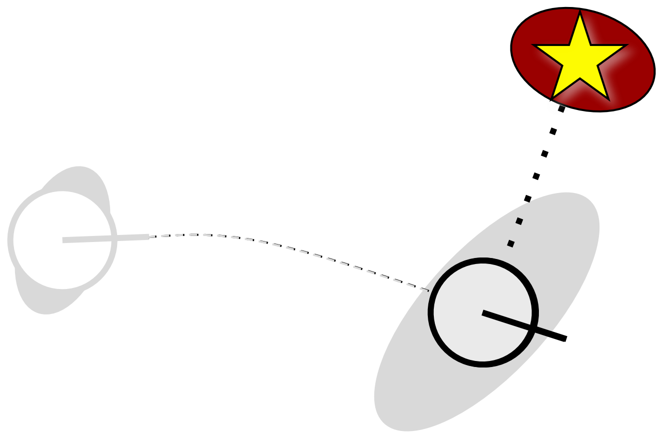
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Measurement Prediction



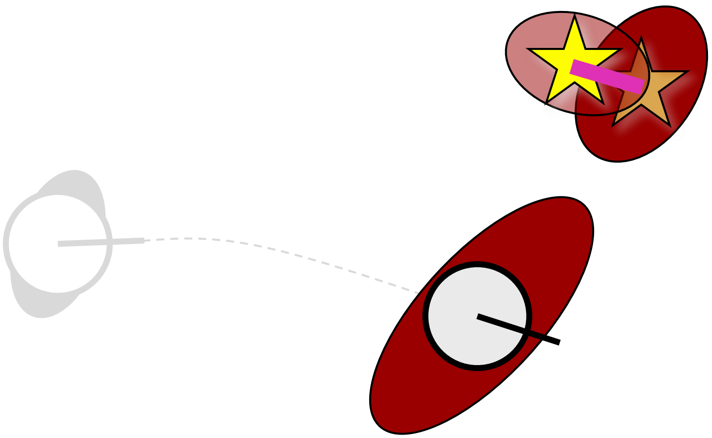
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Obtained Measurement



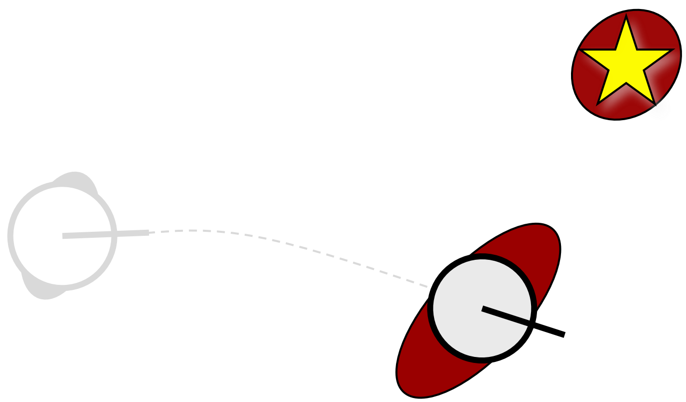
$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Data Association and Difference Between h(x) and z



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Update Step



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Concrete Example

Setup

- Robot moves in the 2D plane
- Velocity-based motion model
- Robot observes point landmarks
- Range-bearing sensor
- Known data association
- Known number of landmarks

Initialization

- Robot starts in its own reference frame (all landmarks unknown)
- 2N+3 dimensions

$$\mu_0 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \end{pmatrix}^T$$

$$\Sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \infty & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \infty \end{pmatrix}$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$

$$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$$

$$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$$

$$return \ \mu_t, \Sigma_t$$

Prediction Step (Motion)

- Goal: Update state space based on the robot's motion
- Robot motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}}_{g_{x,y,\theta}(u_t,(x,y,\theta)^T)}$$

- How to map that to the 2N+3 dim space?

Update the State Space

- From the motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}$$

- to the 2N+3 dimensional space

$$\begin{pmatrix} x' \\ y' \\ \theta' \\ \vdots \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \\ \vdots \end{pmatrix} + \underbrace{\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0\dots0 \\ 0 & 1 & 0 & 0\dots0 \\ 0 & 0 & 1 & \underbrace{0\dots0}_{2N cols} \end{pmatrix}^T}_{F_x^T} \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}}_{g(u_t,x_t)}$$

Extended Kalman Filter Algorithm

1: $\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$ **DONE**

3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

7: $return \; \mu_t, \Sigma_t$

Update Covariance

- The function g only affects the robot's motion and not the landmarks

Jacobian of the motion (3x3)

$$G_t = \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix}$$

Identity (2N x 2N)

Jacobian of the Motion

$$G_t^x = \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta+\omega_t\Delta t) \\ \omega_t\ \Delta t \end{pmatrix}\right]$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta+\omega_t\Delta t) \\ \omega_t\ \Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta+\omega_t\Delta t) \\ \omega_t\ \Delta t \end{pmatrix}
\end{aligned}
$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix} \\
&= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix} \\
&= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 1 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

This Leads to the Update

1: $\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$ **Apply & DONE**

3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

$$
\begin{aligned}
\bar{\Sigma}_t &= G_t \Sigma_{t-1} G_t^T + R_t \\
&= \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix} \begin{pmatrix} (G_t^x)^T & 0 \\ 0 & I \end{pmatrix} + R_t \\
&= \begin{pmatrix} G_t^x \Sigma_{xx} (G_t^x)^T & G_t^x \Sigma_{xm} \\ (G_t^x \Sigma_{xm})^T & \Sigma_{mm} \end{pmatrix} + R_t
\end{aligned}
$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$$ **DONE**

$$\cancel{\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t}$$ **DONE**

$$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$$

$$return \ \mu_t, \Sigma_t$$

EKF SLAM:Prediction Step

EKF_SLAM_Prediction$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, R_t)$**:**

2: $$F_x = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \end{pmatrix}$$

3: $$\bar{\mu}_t = \mu_{t-1} + F_x^T \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$

4: $$G_t = I + F_x^T \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & 0 \end{pmatrix} F_x$$

5: $$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + \underbrace{F_x^T \, R_t^x \, F_x}_{R_t}$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$$ **DONE**

$$\cancel{\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t}$$ **Apply & DONE**

$$K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$$

$$return\ \mu_t, \Sigma_t$$

EKF SLAM: Correction Step

- Known data association
- $c_t^i = j$: i-th measurement at time t observes the landmark with index j
- Initialize landmark if unobserved
- Compute the expected observation
- Compute the Jacobian of h
- Proceed with computing the Kalman gain

Range-Bearing Observation

- Range-Bearing observation $z_t^i = (r_t^i, \phi_t^i)^T$
- If landmark has not been observed

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \ \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \ \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

observed location of landmark j

estimated robot's location

relative measurement

Expected Observation

- Compute expected observation according to the current estimate

$$
\begin{aligned}
\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\
q &= \delta^T \delta \\
\hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \mathrm{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \\
&= h(\bar{\mu}_t)
\end{aligned}
$$

Jacobian for the Observation

- Based on

$$\begin{aligned} \delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \end{aligned}$$

- Compute the Jacobian

$$^{\text{low}}H_t^i = \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t}$$

low-dim space $(x, y, \theta, m_{j,x}, m_{j,y})$

Jacobian for the Observation

- Based on

$$\begin{aligned}
\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\
q &= \delta^T \delta \\
\hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}
\end{aligned}$$

- Compute the Jacobian

$$\begin{aligned}
{}^{\text{low}}H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\
&= \begin{pmatrix} \frac{\partial \sqrt{q}}{\partial x} & \frac{\partial \sqrt{q}}{\partial y} & \dots \\ \frac{\partial \operatorname{atan2}(\dots)}{\partial x} & \frac{\partial \operatorname{atan2}(\dots)}{\partial y} & \dots \end{pmatrix}
\end{aligned}$$

The First Component

- Based on

$$\begin{aligned}\delta &= \begin{pmatrix}\delta_x\\ \delta_y\end{pmatrix} = \begin{pmatrix}\bar{\mu}_{j,x} - \bar{\mu}_{t,x}\\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y}\end{pmatrix}\\ q &= \delta^T\delta\\ \hat{z}_t^i &= \begin{pmatrix}\sqrt{q}\\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta}\end{pmatrix}\end{aligned}$$

- We obtain (by applying the chain rule)

$$\begin{aligned}\frac{\partial\sqrt{q}}{\partial x} &= \frac{1}{2}\frac{1}{\sqrt{q}}\,2\,\delta_x\,(-1)\\ &= \frac{1}{q}\,(-\sqrt{q}\delta_x)\end{aligned}$$

Jacobian for the Observation

- Based on

$$\begin{aligned} \delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \end{aligned}$$

- Compute the Jacobian

$$\begin{aligned} {}^{\text{low}}H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\ &= \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix} \end{aligned}$$

Jacobian for the Observation

- Use the computed Jacobian

$$ {}^{\text{low}}H_t^i = \frac{1}{q}\begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix} $$

- map it to the high dimensional space

$$ H_t^i = {}^{\text{low}}H_t^i \, F_{x,j} $$

$$ F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\ 0 & 1 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\ 0 & 0 & 1 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\ 0 & 0 & 0 & 0\cdots 0 & 1 & 0 & 0\cdots 0 \\ 0 & 0 & 0 & \underbrace{0\cdots 0}_{2j-2} & 0 & 1 & \underbrace{0\cdots 0}_{2N-2j} \end{pmatrix} $$

Next Steps as Specified...

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$ **DONE**

3: $\cancel{\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t}$ **DONE**

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

7: $return \ \mu_t, \Sigma_t$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\cancel{\bar{\mu}_t = g(u_t, \mu_{t-1})}$$ **DONE**

$$\cancel{\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t}$$ **DONE**

$$\cancel{K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}}$$ **Apply & DONE**

$$\cancel{\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))}$$ **Apply & DONE**

$$\cancel{\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t}$$ **Apply & DONE**

$$return\ \mu_t, \Sigma_t$$

EKF SLAM – Correction (1/2)

EKF_SLAM_Correction

$$Q_t = \begin{pmatrix} {\sigma_r}^2 & 0 \\ 0 & {\sigma_\phi}^2 \end{pmatrix}$$

for all observed features $z_t^i = (r_t^i, \phi_t^i)^T$ *do*

$$j = c_t^i$$

if landmark j *never seen before*

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \, \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \, \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

endif

$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$

$$q = \delta^T \delta$$

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \mathrm{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

EKF SLAM – Correction (2/2)

$$F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 1 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 0 & 0\cdots0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 0 & \underbrace{0\cdots0}_{2j-2} & 0 & 1 & \underbrace{0\cdots0}_{2N-2j} \end{pmatrix}$$

$$H_t^i = \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & +\delta_x \end{pmatrix} F_{x,j}$$

$$K_t^i = \bar{\Sigma}_t\, H_t^{iT} (H_t^i\, \bar{\Sigma}_t\, H_t^{iT} + Q_t)^{-1}$$

$$\bar{\mu}_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$$

$$\bar{\Sigma}_t = (I - K_t^i\, H_t^i)\, \bar{\Sigma}_t$$

endfor

$$\mu_t = \bar{\mu}_t$$

$$\Sigma_t = \bar{\Sigma}_t$$

return μ_t, Σ_t

Implementation Notes

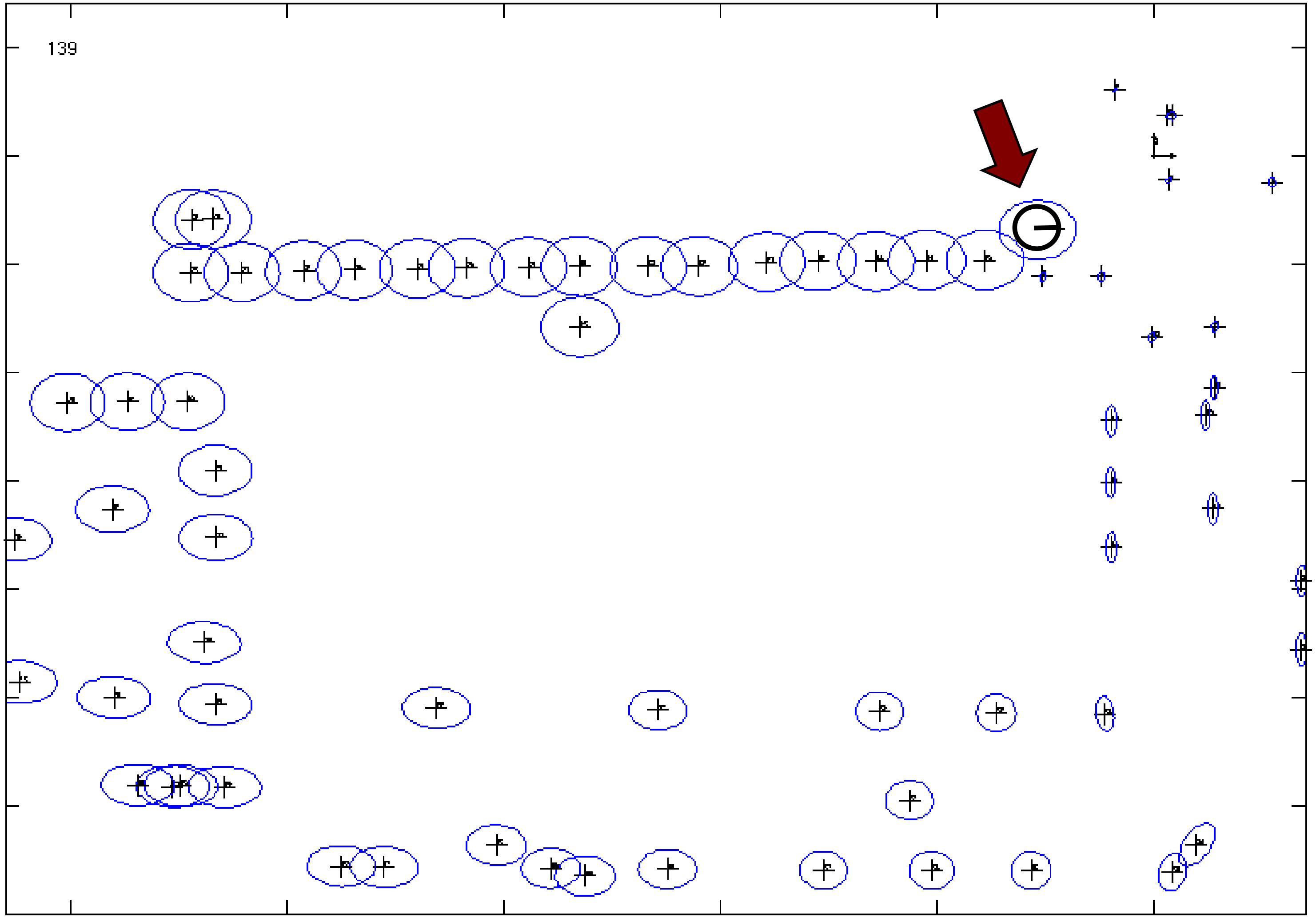
- Measurement update in a single step requires only one full belief update
- Always normalize the angular components
- You may not need to create the F matrices explicitly (e.g., in Octave)

Done!

Loop Closing

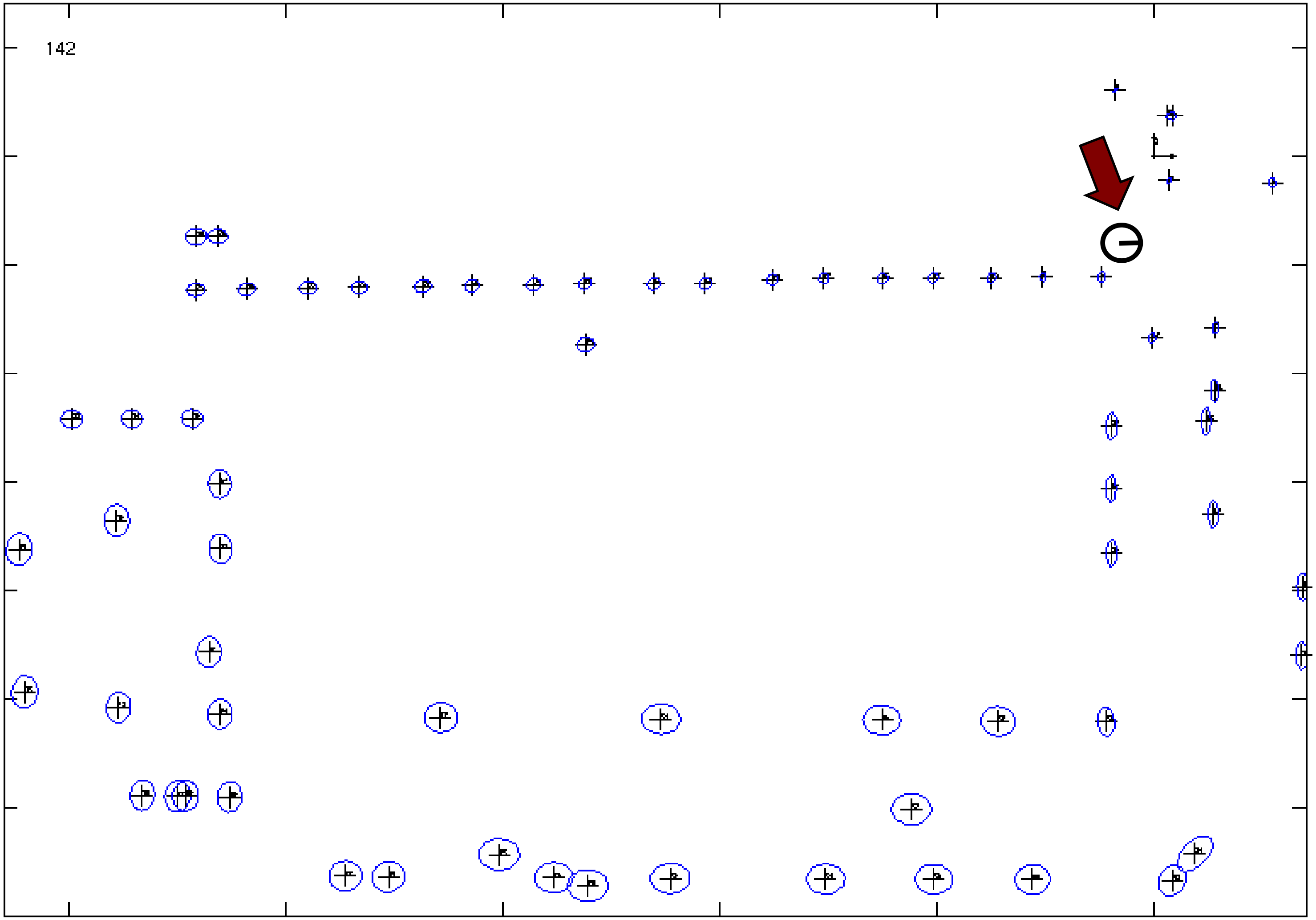
- Loop closing means recognizing an already mapped area
- Data association under
  - high ambiguity
  - possible environment symmetries
- Uncertainties **collapse** after a loop closure (whether the closure was correct or not)

Before the Loop Closure



Courtesy: K. Arras

After the Loop Closure



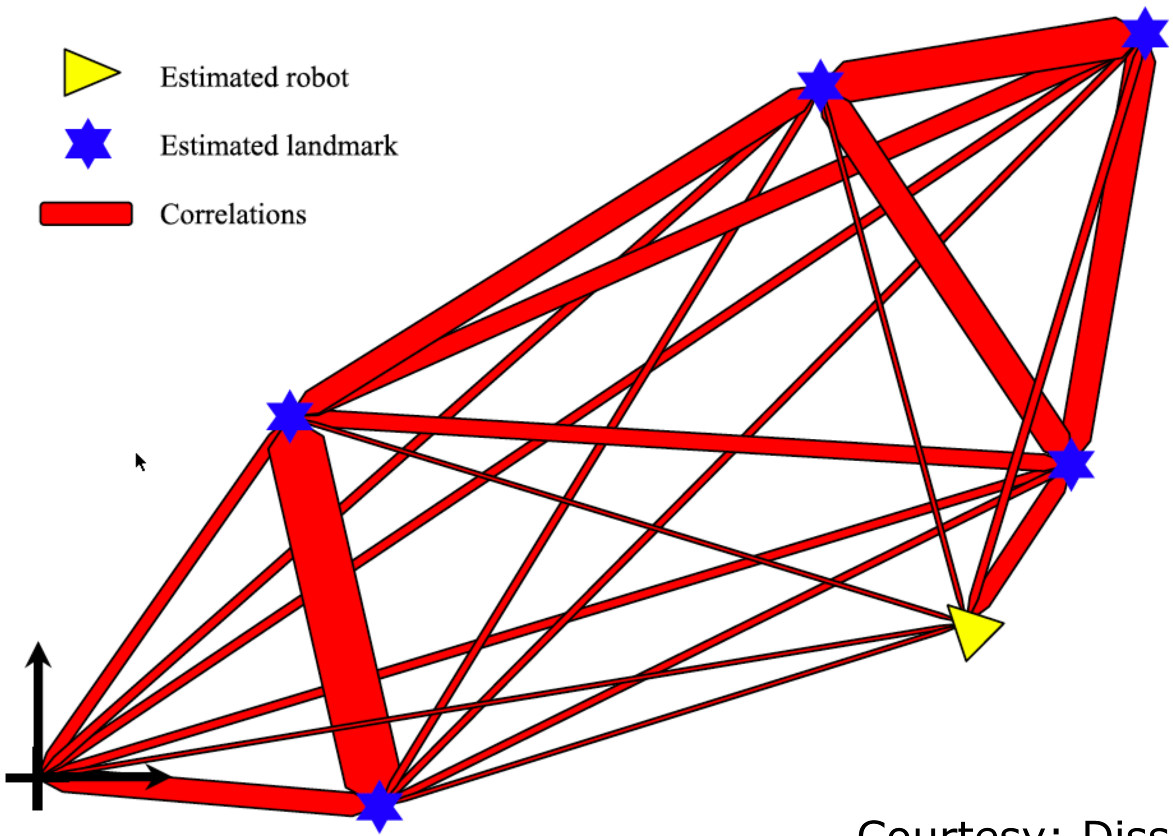
Courtesy: K. Arras

Loop Closures in SLAM

- Loop closing **reduces** the uncertainty in robot and landmark estimates
- This can be exploited when exploring an environment for the sake of better (e.g. more accurate) maps
- **Wrong loop closures lead to filter divergence**

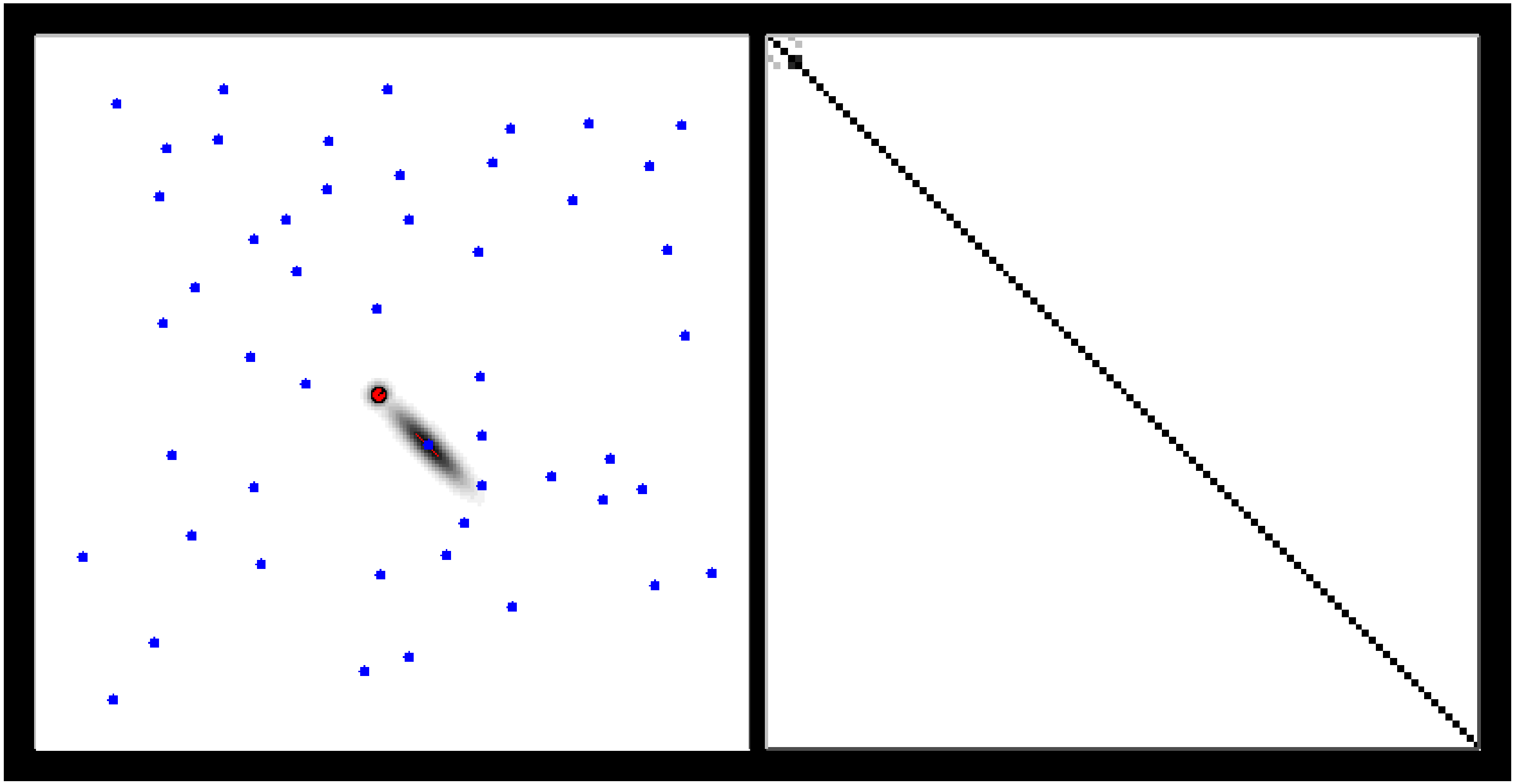
EKF SLAM Correlations

- In the limit, the landmark estimates become **fully correlated**



Courtesy: Dissanayake

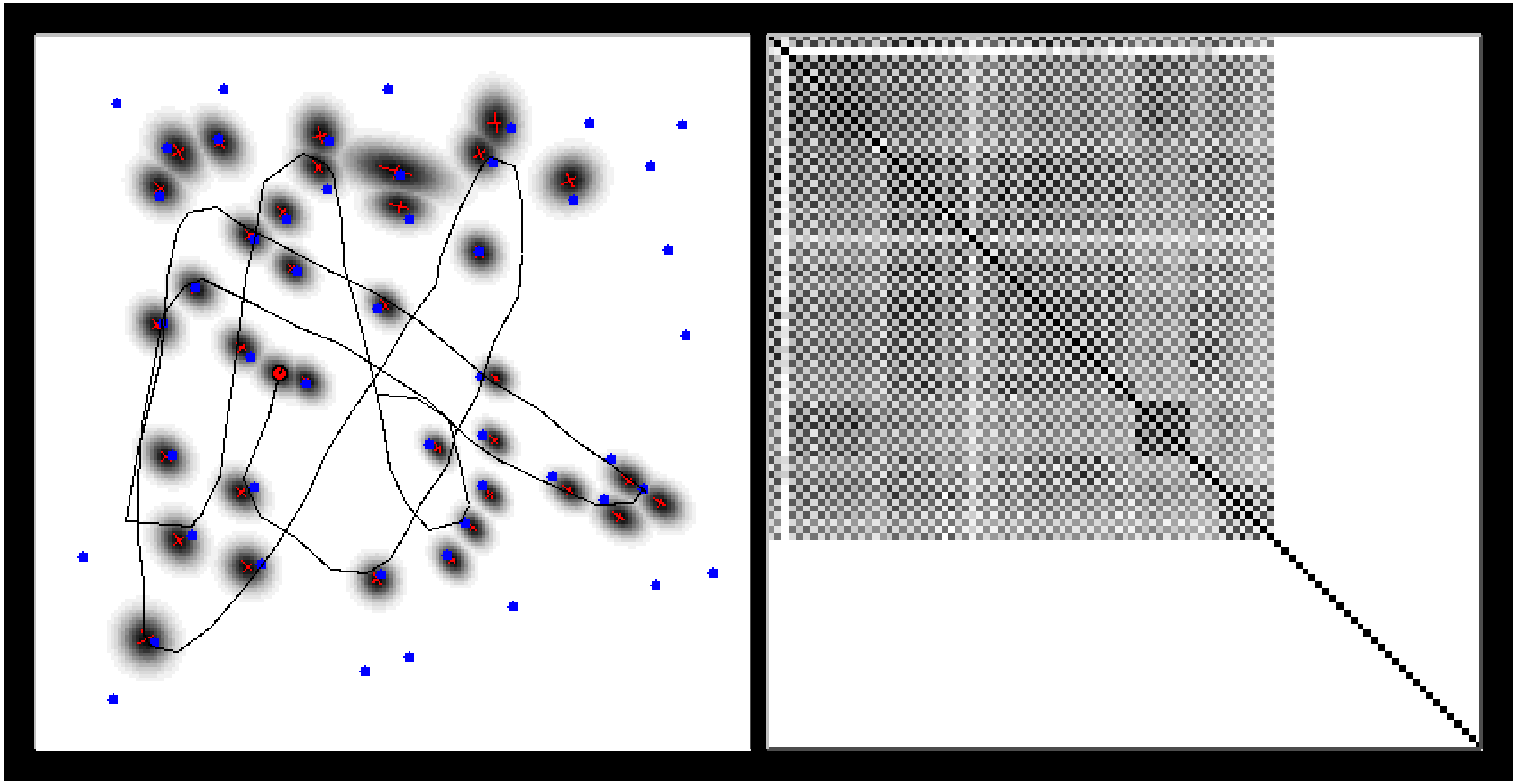
EKF SLAM Correlations



Map Correlation matrix

Courtesy: M. Montemerlo

EKF SLAM Correlations

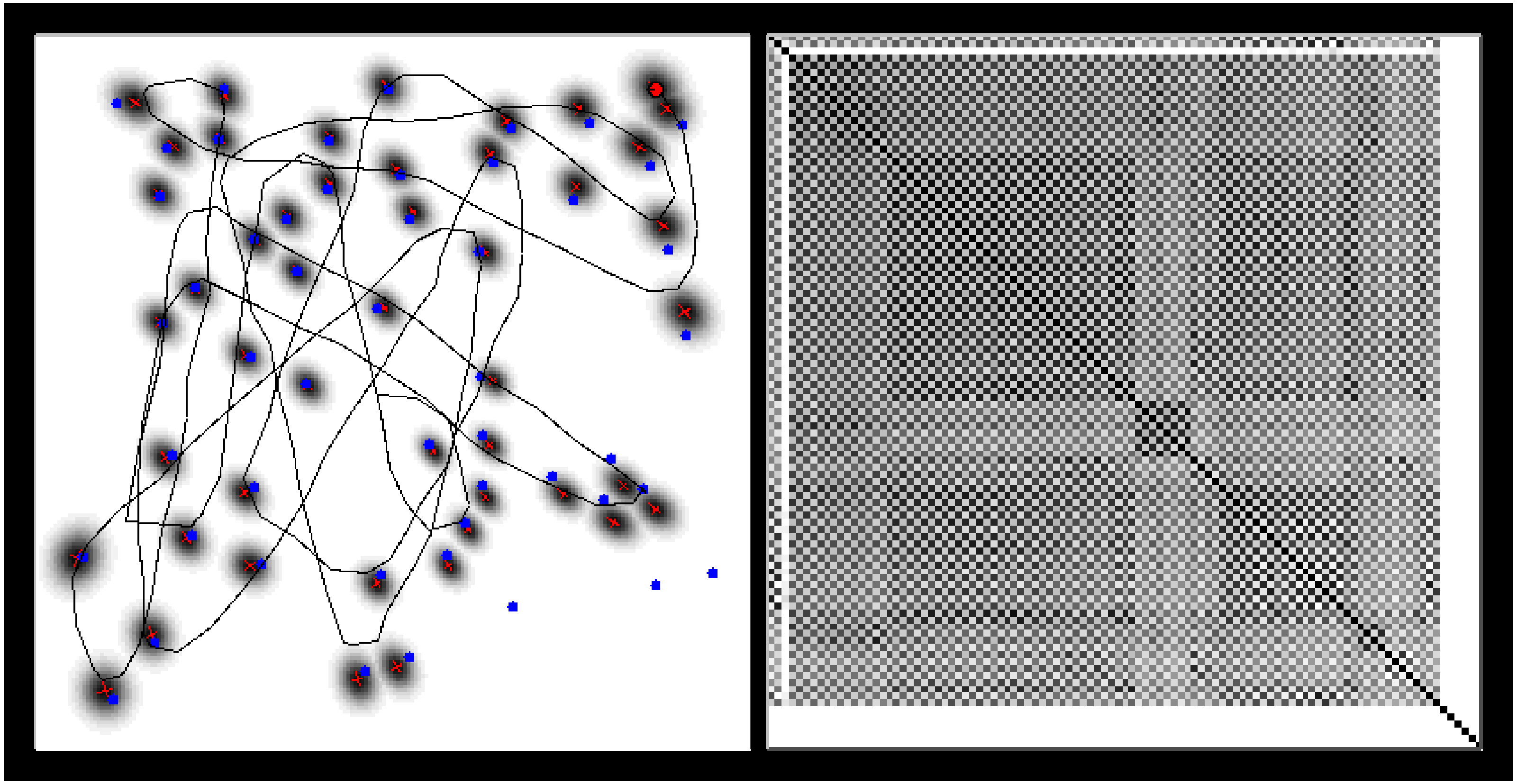


Map

Correlation matrix

Courtesy: M. Montemerlo

EKF SLAM Correlations



Map Correlation matrix

Courtesy: M. Montemerlo

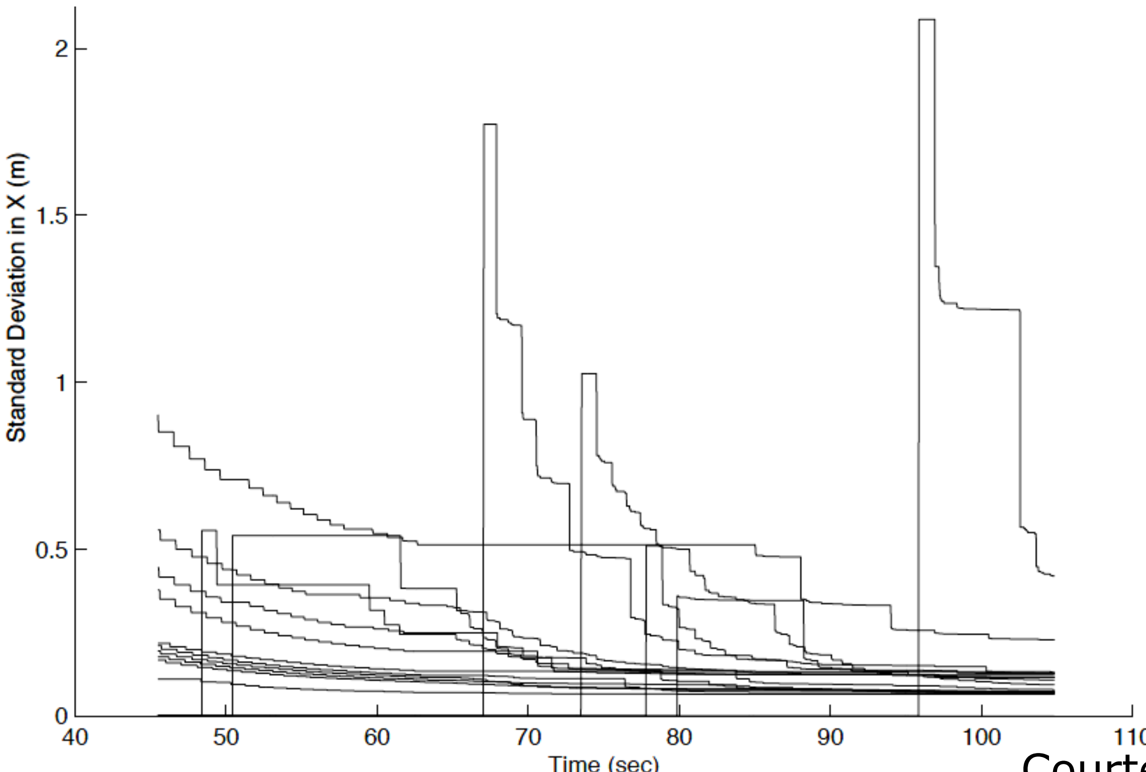
EKF SLAM Correlations

- The correlation between the robot's pose and the landmarks **cannot** be ignored
- Assuming independence generates too optimistic estimates of the uncertainty

Courtesy: J.M. Castellanos

EKF SLAM Uncertainties

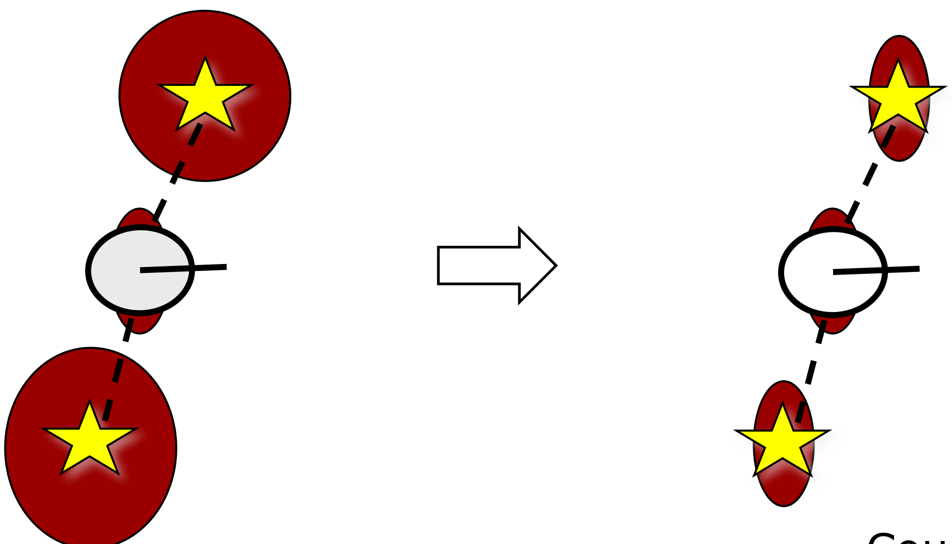
- The **determinant** of any sub-matrix of the map covariance matrix **decreases monotonically**
- New landmarks are initialized with **maximum uncertainty**



Courtesy: Dissanayake

EKF SLAM in the Limit

- In the limit, the covariance associated with any single landmark location estimate is determined only by the initial covariance in the vehicle location estimate.



Courtesy: Dissanayake

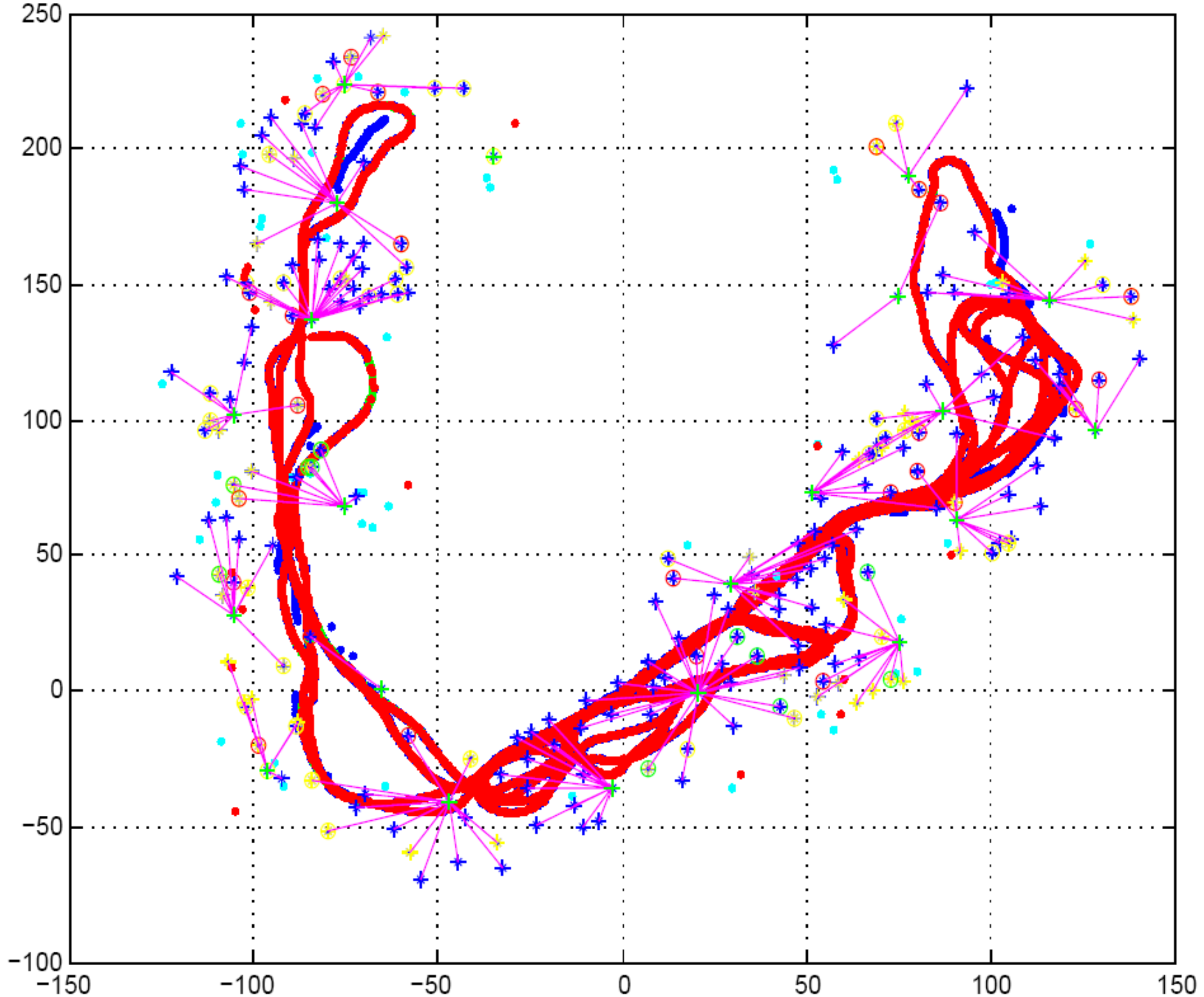
Example: Victoria Park Dataset

Courtesy: E. Nebot

Victoria Park: Data Acquisition

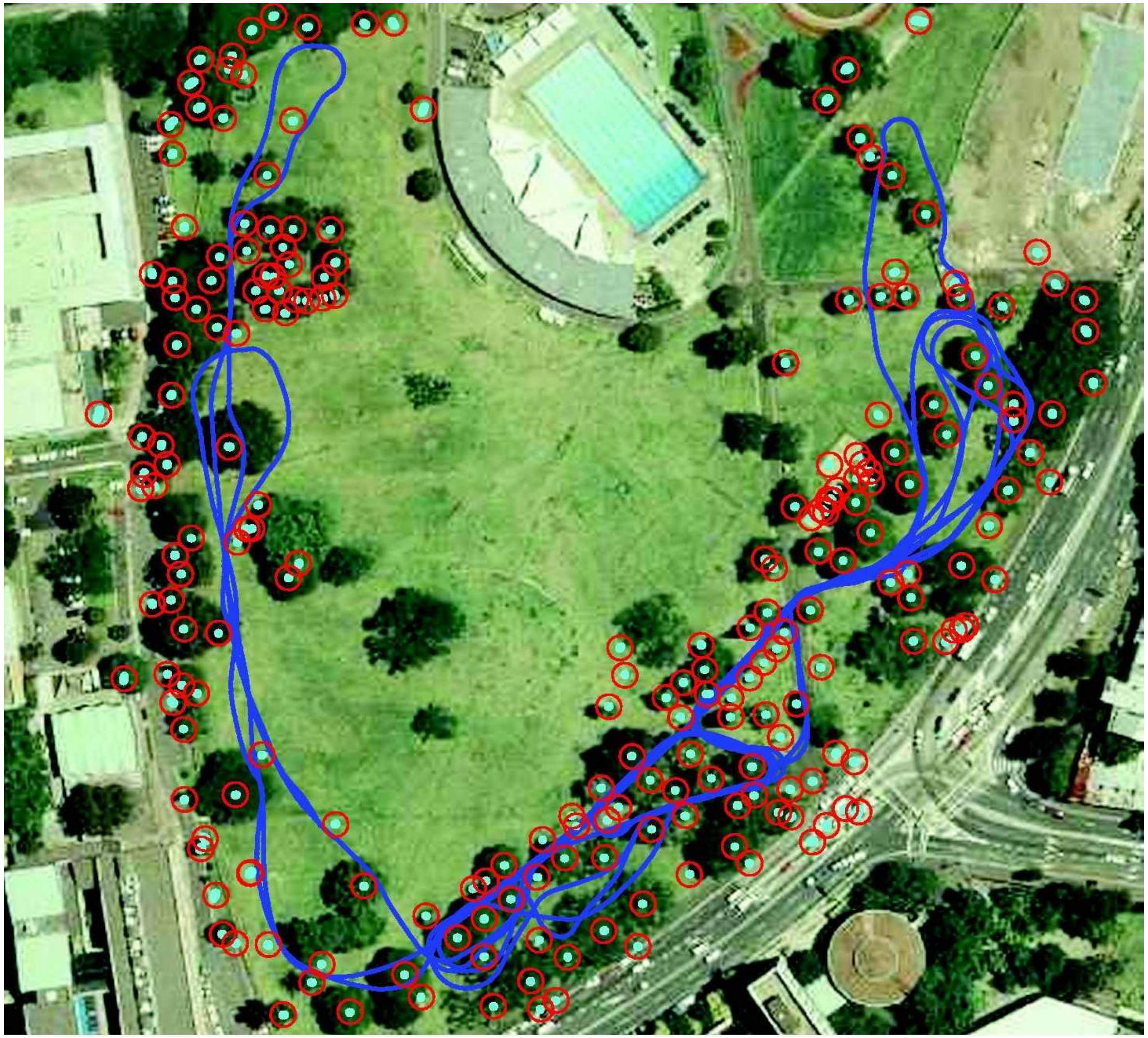
Courtesy: E. Nebot

Victoria Park: EKF Estimate



Courtesy: E. Nebot

Victoria Park: Landmarks

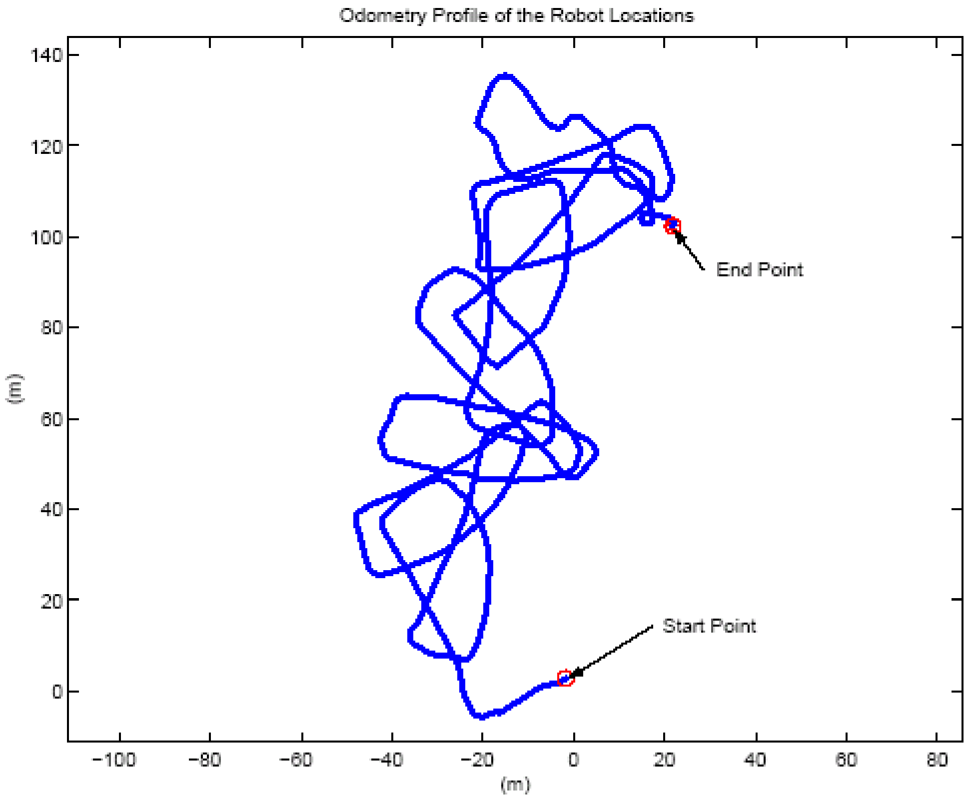


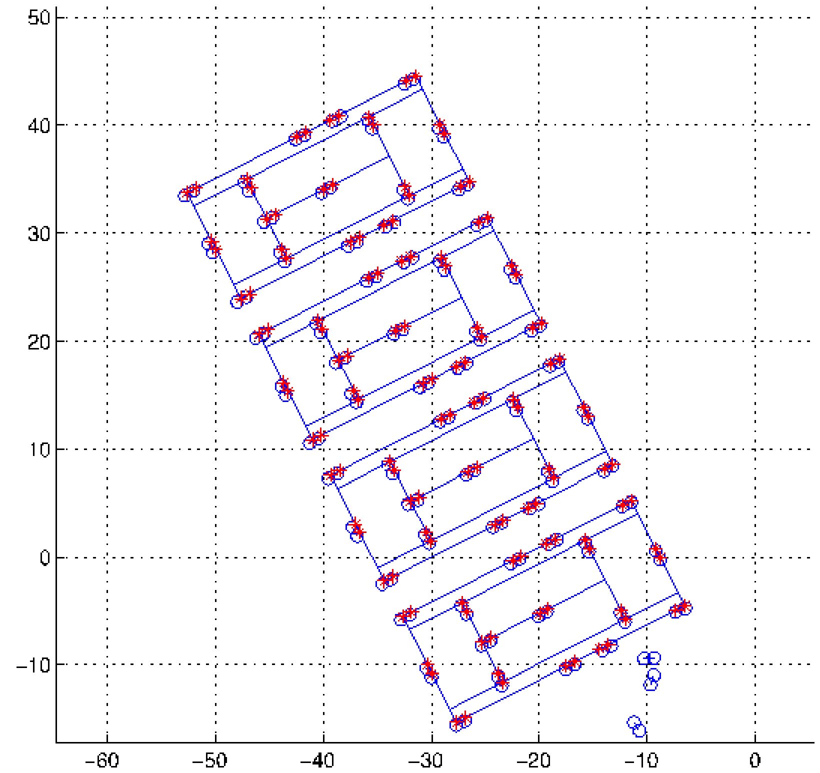
Courtesy: E. Nebot

Example: Tennis Court Dataset

Courtesy: J. Leonard and M. Walter

EKF SLAM on a Tennis Court





odometry

estimated trajectory

Courtesy: J. Leonard and M. Walter

EKF SLAM Complexity

- Cubic complexity depends only on the measurement dimensionality
- Cost per step: dominated by the number of landmarks: $O(n^2)$
- Memory consumption: $O(n^2)$
- The EKF becomes computationally intractable for large maps!

EKF SLAM Summary

- The first SLAM solution
- Convergence proof for the linear Gaussian case
- Can diverge if non-linearities are large (and the reality is non-linear...)
- Can deal only with a single mode
- Successful in medium-scale scenes
- Approximations exists to reduce the computational complexity

Literature

EKF SLAM

- Thrun et al.: “Probabilistic Robotics”, Chapter 10

Slide Information

- These slides have been created by Cyrill Stachniss as part of the robot mapping course taught in 2012/13 and 2013/14. I created this set of slides partially extending existing material of Edwin Olson, Pratik Agarwal, and myself.
- I tried to acknowledge all people that contributed image or video material. In case I missed something, please let me know. If you adapt this course material, please make sure you keep the acknowledgements.
- Feel free to use and change the slides. If you use them, I would appreciate an acknowledgement as well. To satisfy my own curiosity, I appreciate a short email notice in case you use the material in your course.
- My video recordings are available through YouTube:
  http://www.youtube.com/playlist?list=PLgnQpQtFTOGQrZ4O5QzbIHgl3b1JHimN_&feature=g-list

Cyrill Stachniss, 2014
cyrill.stachniss@igg.uni-bonn.de