Problem sheet 6

Saturday, 6 September 2025 10:43 PM

3a) 2, 4, 1, 2, 4, 1

3, 2, 6, 4, 5, 1 Hence $\omega = 3$

& $3+2+6+4+5+1 = 21 = 0 \bmod 7$

3b $M_6(\omega) = \begin{bmatrix} \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ \omega^2 & \omega^4 & \omega^6 & \omega^2 & \omega^4 & \omega^6 \\ \omega^3 & \omega^6 & \omega^3 & \omega^6 & \omega^3 & \omega^6 \\ \omega^4 & \omega^2 & \omega^6 & \omega^4 & \omega^2 & \omega^6 \\ \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega & \omega^6 \\ \omega^6 & \omega^6 & \omega^6 & \omega^6 & \omega^6 & \omega^6 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 & 4 & 5 & 1 \\ 2 & 4 & 1 & 2 & 4 & 1 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 4 & 2 & 1 & 4 & 2 & 1 \\ 5 & 4 & 6 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$M_6(\omega) \cdot v = \begin{bmatrix} 3 & 2 & 6 & 4 & 5 & 1 \\ 2 & 4 & 1 & 2 & 4 & 1 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 4 & 2 & 1 & 4 & 2 & 1 \\ 5 & 4 & 6 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 5 \\ 1 \\ 3 \end{bmatrix}$

c) $M_6^{-1}(\omega) = \begin{bmatrix} 5 & 4 & 6 & 2 & 3 & 1 \\ 4 & 2 & 1 & 4 & 2 & 1 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 2 & 4 & 1 & 2 & 4 & 1 \\ 3 & 2 & 6 & 4 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ $\quad 3^{-1} = 5,\ 2^{-1} = 4,\ 6^{-1} = 6,\ 4^{-1} = 2$

$5^{-1} = 3,\ 1^{-1} = 1$

and $\begin{bmatrix} 5 & 4 & 6 & 2 & 3 & 1 \\ 4 & 2 & 1 & 4 & 2 & 1 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 2 & 4 & 1 & 2 & 4 & 1 \\ 3 & 2 & 6 & 4 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 5 \\ 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 6 \\ 6 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ -5 \\ -2 \end{bmatrix}$

d) $x^2+x+1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = a$ & $x^3+2x-11 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} = b$

$M_6(\omega) \cdot a = \begin{bmatrix} 3 & 2 & 6 & 4 & 5 & 1 \\ 2 & 4 & 1 & 2 & 4 & 1 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 4 & 2 & 1 & 4 & 2 & 1 \\ 5 & 4 & 6 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} = p$

$M_6(\omega) \cdot b = \begin{bmatrix} 3 & 2 & 6 & 4 & 5 & 1 \\ 2 & 4 & 1 & 2 & 4 & 1 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 4 & 2 & 1 & 4 & 2 & 1 \\ 5 & 4 & 6 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 5 \\ 5 \\ 6 \end{bmatrix} = q$

pointwise product of $p, q = [6, 0, 0, 0, 1, 4] = z$

$M_6^{-1}(\omega) z = \begin{bmatrix} 5 & 4 & 6 & 2 & 3 & 1 \\ 4 & 2 & 1 & 4 & 2 & 1 \\ 6 & 1 & 6 & 1 & 6 & 1 \\ 2 & 4 & 1 & 2 & 4 & 1 \\ 3 & 2 & 6 & 4 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 6 \\ 6 \\ 4 \end{bmatrix} = -\begin{bmatrix} 5 \\ 5 \\ 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

and so the product polynomial is $5x+5x^2+3x^3+x^4+x^5+3$

Check

$(x^2+x+1)(x^3+2x-11) = x^5+x^4+x^3+2x^3+2x^2+2x$
$\qquad -11x^2-11x-11$

$= x^5+x^4+3x^3-9x^2-9x-11$

$= x^5+x^4+3x^3+5x^2+5x+3$

Q1) Let the alphabet be $\{a, b\}$. We create a polynomial $p(x) = (p_0, p_1 \dots p_{n-1})$ of degree $n-1$ where $p_i = 1$ iff $s[i] = a$, 0 if $s[i] = b$ (s is indexed from 0 to $n-1$). Similarly, we create a polynomial $q(x) = (q_0, q_1 \dots q_{m-1})$ of degree $m-1$ where $q_{m-i-1} = 1$ iff $t[i] = a$ & is 0 otherwise (t is also indexed from 0 to $m-1$)

Consider the product $p(x)q(x) = c(x) = (c_0, c_1 \dots c_{m+n-2})$

Note $c_{m+k-1} = \sum_{i=0}^{m-1} q_{m-i-1} * p_{k+i} = |\{i : t[i] = a \ \& \ s[k+i] = a\}|$

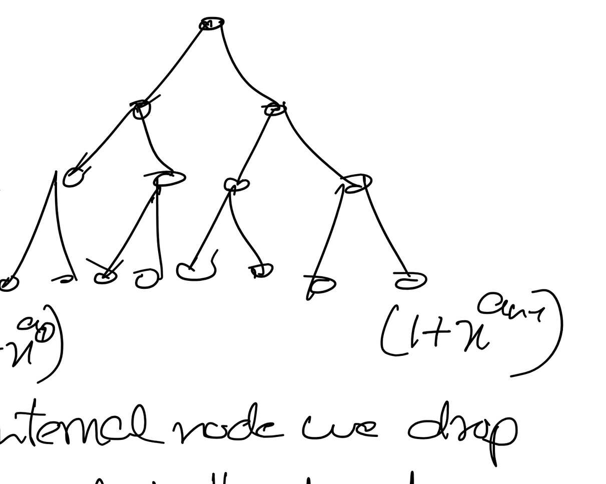
= no. of places where t and $s[k..(k+m-1)]$ match in an "a". We can similarly construct polynomials $u(x)$ & $v(x)$ corresponding to symbol "b" & let their product be $z(x)$. Then the coefficient of x^{m+k-1} in $z(x)+c(x)$ is the no. of places where t and $s[k...(k+m-1)]$ match.

The time required to compute $z(x)$ & $c(x)$ is $O(n\log n)$ (since $m < n$)

The sum $z(x)+c(x)$ can be computed in $O(n)$ time & hence the similarity between s, t at all positions k can be computed in $O(n\log n)$ time

Q2) We assume $a_i \le k \ \forall k$. As mentioned in the question we need the coefficient of x^k in $\prod_{i=0}^{n-1}(1+x^{a_i})$. We will use FFT to multiply polynomials but to do this efficiently the degree of the polynomials should be small.

Assume n is 2^k. We create a complete binary tree whose leaves correspond to the polynomials $(1+x^{a_i})$ $0 \le i \le n-1$



An internal node corresponds to a polynomial which is the product of the polynomials corresponding to the leaves of the subtree rooted at that node. Further at each internal node we drop the monomials with degree $> k$. Thus each polynomial at any node in the tree has degree at most k & so multiplying these require $O(k\log k)$ time. Since the total no. of multiplications is $n-1$ the total time is $O(nk\log k)$

Q4) Let G_1 be the graph which only includes edges with weight less than $w(e)$. Determine the connected components of G_1 using BFS in $O(n+m)$ time. If the endpoints of e are in different connected components of G_1 then e is in the MST else it is not. The proof follows from the fact that in Kruskal's algorithm the forest picked at the point edge e was considered, spans the connected components of G_1. Hence if inclusion of edge e did not form a cycle, it would have been picked by Kruskal & hence would be in the MST (there is a unique MST since edge weights are distinct)

Q5) In class we learnt that for any cut $(S, \bar{S})$ in G, the minimum edge in the cut is included in every MST of G.

Let $C_1^i, C_2^i \dots C_k^i$ be the connected components formed by the edges picked by Boruvka's algorithm in iterations $1 \dots i-1$. In iteration i the algorithm picks the minimum weight edge incident to each component C_j^i. This is also the minimum weight edge in the cut $(C_j^i, V \setminus C_j^i)$ & hence would be included in any MST. Thus the edges picked by Boruvka's algorithm are edges from a minimum spanning tree & so it gives us the MST.