Shortest paths

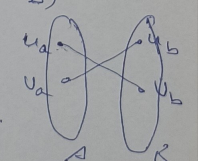
Q1) ~~Cont~~ Let $G=(V,E)$ be the graph & $w: E \to \mathbb{Z}^+$ be the amt of fuel used in traversing edge e. We will assume $w(e)$ is integer for all e.
Also C — capacity of tank is integer.

Create a graph whose vertices are (v,i) $v \in V$ & $0 \le i \le C$, $i \in \mathbb{Z}^+$
For every edge (u,v) in G create edges $(u,i) \longrightarrow (v, i-w(u,v))$ for ~~$w(u,v)$~~ $w(u,v) \le i \le C$. All these edges have length 0.
In addition we include edges $(u,i) \to (u,i+1)$ $\quad 0 \le i \le C-1$
& each of these edges has a length C_u.

Claim The shortest path from (A,C) to (B,i) for $0 \le i \le C$ is the cheapest way to travel from A to B.

Proof sketch: The vertex (u,i) represents that we are at city u with i litres of fuel in the tank. When we go from u to v, we expend $w(u,v)$ units of fuel & hence we have edges from (u,i) to $(v, i-w(u,v))$. Adding fuel costs money & so edge (u,i) to $(u,i+1)$ which corresponds to adding 1 unit of fuel at city u has a length C_u. Length of a path corresponds to money spent & so we want to find the shortest path from (A,C) to any vertex (B,i) $0 \le i \le C$.

2) ~~Create a bipartite graph $H=(A,B,F)$ where $A=B=V$~~
Assume $G=(V,E)$ is the given ~~d~~ undirected graph & we want to find a shortest path from s to t. Let $l: E \to \mathbb{R}^+$ be the edge lengths.
Create a bipartite graph $H=(A,B,F)$ where $A=B=V$; ie. for each vertex $v \in V$ we add a copy of v (say v_A) to A & another copy v_B to B.
For each edge $(u,v) \in E$ we add edges (u_a, v_b) & ~~t_B~~ (v_a, u_b) to F. Both these edges have the same length as edge u,v.



We now find the shortest path from s_a to t_a.
Any path from a vertex in A to another vertex in B has even length in the original graph. Further every path in G corresponds to a path in H & every path in H corresponds to a path in G. Thus
~~shortest path from s_a to t_a is the shortest even~~