Problem Sheet 8: Shortest Paths

1. You want to travel from city A to city B. There are N cities and M bidirectional roads connecting cities. Your car can store up to C litres of fuel, and the tank is initially full. Each road (*i*, *j*) has a value w_{ij} that represents the amount of fuel in litres required to cross this road. Additionally, in every city, you can purchase fuel at a price of C_i dollars per litre. You must compute the minimum amount of dollars spent on fuel to travel from A to B.
2. You are given an unusual undirected graph, and you want to find the shortest path from A to B, but there is an additional constraint: you must use an even number of edges.
3. **Monotonic shortest path.** Given an edge-weighted digraph, find a *monotonic* shortest path from s to every other vertex. A path is monotonic if the weight of every edge on the path is either strictly increasing or strictly decreasing.
4. To get in shape, you have decided to start running to work. You want a route that goes entirely uphill and then entirely downhill so that you can work up a sweat going uphill and then get a nice breeze at the end of your run as you run faster downhill. Your run will start at home and end at work, and you have a map detailing the roads with m road segments (any existing road between two intersections) and n intersections. Each road segment has a positive length, and each intersection has a distinct elevation.
    a. Assuming that every road segment is either uphill or downhill, give an efficient algorithm to find the shortest route that meets your specifications.
    b. Give an efficient algorithm to solve the problem if some roads may be level (i.e., both intersections at the end of the road segments are at the same elevation) and therefore can be taken at any point.
5. Let $G = (V, E)$ be a directed graph with edge weights $w: E \to R$. Define the mean-weight of a cycle, C, as $\sum_{e \in C} w(c)/|C|$. Give an efficient algorithm to find a cycle with the smallest mean-weight.